On minimal subshifts of linear word complexity with slope less than $3/2$

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Abstract We prove that every infinite minimal subshift with word complexity $p(q)$ satisfying $\limsup p(q)/q < 3/2$ is measure-theoretically isomorphic to its maximal equicontinuous factor; in particular, it has measurably discrete spectrum. Among other applications, this provides a proof of Sarnak's conjecture for all subshifts with $\limsup p(q)/q < 3/2$ (which can be thought of as a much stronger version of zero entropy).

As in [CP23], our main technique is proving that all low-complexity minimal subshifts have a specific type of representation via a sequence $\{\tau_k\}$ of substitutions, usually called an S-adic decomposition. The maximal equicontinuous factor is the product of an odometer with a rotation on an abelian one-dimensional nilmanifold with adelic structure, for which we can give an explicit description in terms of the substitutions τ_k. We also prove that all such odometers and nilmanifolds may appear for minimal subshifts with $\limsup p(q)/q = 1$, demonstrating that lower complexity thresholds do not further restrict the possible structure.

1 Introduction

In this work, we demonstrate some surprising connections between symbolically defined dynamical systems called **subshifts** and algebraic number theory. Our main result (see Section 1.3) shows that every minimal subshift with word complexity function (see Section 1.1) of very slow growth is measurably isomorphic to a specific rotation of a compact abelian group called its **maximal equicontinuous factor** or **MEF** (which can be defined as the character group of its eigenvalue group; see Section 1.2).

What's more, the group in question has a very specific structure as the product of an odometer and a number-theoretic object known as an adelic 1-step (i.e. abelian) one-dimensional nilmanifold. Sections 1.5 and 1.6 give more details about the former object, but for context we mention that a simple example is irrational rotation of the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$, and all adelic 1-step one-dimensional nilmanifolds can be thought of as rotations on p-adic extensions of S^1.

In this introduction, we will describe our main results and consequences/connections to several disparate areas, including Sarnak's conjecture (see Section 1.4), Pisot's substitution conjecture and so-called S-adic representations of subshifts (see Section 1.7), and word complexity thresholds (see Section 1.8). To explain these, we need some brief definitions/background; for full formal definitions, see Section 2.

1.1 Topological dynamics, subshifts, and word complexity

A **topological dynamical system (TDS)** is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a homeomorphism. A TDS is **transitive** if there exists $x \in X$ for which $X = \overline{\{T^n x\}}$; every TDS throughout this work will be assumed transitive to avoid degenerate examples such as disjoint unions, for which it is impossible to give a unified structure. A TDS is called **minimal** if it does not properly contain any nonempty TDS.

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A **subshift** is a TDS given by a finite set A (called the **alphabet**) and a set $X \subset A^{\mathbb{Z}}$ which is closed (thereby compact) in the product topology and invariant under all powers of the left shift σ (i.e. any shift of a sequence in X must also be in X). Every subshift is endowed with the action by the homeomorphism σ, and so we generally refer to a subshift as X instead of (X, σ).

The **word complexity function** $p(q)$ of a subshift X simply counts the number of q-letter contiguous strings which appear within at least one $x \in X$. For instance, if X is the so-called golden mean shift, consisting of 0-1 sequences without consecutive 1s, then it's not hard to check that $p(1) = 2$, $p(2) = 3$, $p(3) = 5$, and in fact $p(q)$ is the Fibonacci sequence.

The well-known Morse-Hedlund theorem ([MH38]) states that if there exists any q for which $p(q) \leq q$, then X is a finite set of periodic sequences, i.e. any infinite X must have $p(q) \geq q + 1$ for all q. This means that for infinite X, word complexity grows at least linearly. Ferenczi ([Fer96]) proved that minimal subshifts with linear complexity have a recursive structure given by a sequence τ_k of **substitutions** (such subshifts are now called **S-adic**); see Section 2 for formal definitions.

This structure already restricts the dynamical behavior of such a subshift significantly; for instance X must have finite topological rank ([DDMP21]), finitely many ergodic invariant measures ([Bos92], [CK19], [DOP22]), and cannot support a strongly mixing measure ([Fer96]). (All σ-invariant measures on subshifts are assumed to be Borel probability measures.)

In [CP23], we showed that any minimal subshift X with $\limsup p(q)/q < 4/3$ has a quite restrictive S-adic structure, which implies (measurable) **discrete spectrum**, meaning that X is measurably isomorphic to the rotation of a compact abelian group. In this work, we improve that result in multiple ways. First, we increase the threshold from 4/3 to 3/2, which is optimal since [CP23] also contains an example with $\limsup p(q)/q = 3/2$ which is (measurably) **weakly mixing**, which is antithetical to discrete spectrum. Second, we describe the exact group in question (in terms of the S-adic decomposition) and show that the canonical projection of X to the group rotation is a measure-theoretic isomorphism; this is in a sense showing that X is as close as possible to a group rotation (since an infinite subshift cannot be topologically isomorphic to a rotation).

We note for future reference that $\limsup p(q)/q < 3$ is known to imply uniqueness of invariant measure for minimal subshifts ([Bos92]), and so when we refer to 'the measure' on such a subshift X, there is no ambiguity.

1.2 Eigenvalues, characters, MEFs, and Sturmian and Toeplitz subshifts

We say that $f \in C(X)$ is a **continuous eigenfunction** of the TDS (X, T) with **continuous additive eigenvalue** γ if $f(Tx) = e^{2\pi i \gamma} f(x)$ for all $x \in X$. The continuous additive eigenvalues form a subgroup E_X of $(\mathbb{R}, +)$ containing $\mathbb{Z}$; the **continuous multiplicative eigenvalues** $\mathcal{E}_X = \{\exp(2\pi i \gamma) : \gamma \in E\}$ form a subgroup of the unit circle $(S^1, \cdot)$ in the complex plane, which is always isomorphic to $E_X/\mathbb{Z}$.

For any minimal TDS, its MEF is a rotation of the **dual group** or **character group** of the continuous multiplicative eigenvalues, i.e. the group $\widehat{\mathcal{E}_X}$ (under pointwise multiplication) of homomorphisms from $\mathcal{E}_X$ to S^1, see [BK13].

Two well-studied classes of subshifts with known MEF are the **Sturmian subshifts** and **Toeplitz subshifts**. Sturmian subshifts are particularly relevant for our purposes since they are subshifts of minimal word complexity, i.e. $p(q) = q+1$ for all q. In addition, any Sturmian subshift S has $E_S = \mathbb{Z}\alpha + \mathbb{Z}$ for some $\alpha \notin \mathbb{Q}$, and so $\mathcal{E}_S = \mathbb{Z}\alpha$, which is isomorphic to $\mathbb{Z}$. The MEF of a Sturmian shift is therefore $\widehat{\mathbb{Z}}$, which is an (irrational) rotation on the unit circle.

A subshift T is Toeplitz if it is an almost 1-1 extension of its MEF and has E_T a subgroup of $(\mathbb{Q}, +)$ (which must contain $\mathbb{Z}$). Then $\mathcal{E}_T$ is isomorphic to $E_T/\mathbb{Z}$, and so the MEF of T is given by $\widehat{E_T/\mathbb{Z}}$. One way of viewing any such dual group is as an **odometer**, which is defined as coordinatewise addition by 1 in an inverse limit of the form $\varprojlim_k \mathbb{Z}/o_1 o_2 \ldots o_k \mathbb{Z}$. (In a slight abuse of notation, we sometimes also use the term 'odometer' to refer to the group itself and not the rotation; since the only rotation ever considered on such groups in this work is the coordinatewise addition by 1, we hope this will not cause ambiguity.)

Unlike Sturmian subshifts, Toeplitz subshifts need not have low word complexity (it can even grow exponentially) and may have many invariant measures; in fact one of the most celebrated results about this class ([Dow91]) is that their measure-theoretic structure can be that of an arbitrary Choquet simplex, and so measure-theoretically Toeplitz subshifts are no more restrictive than general topological dynamical systems.

As will be seen in the next section, we prove that every minimal subshift X with sufficiently low word complexity is a combination of Sturmian and Toeplitz subshifts in the sense that their MEFs are a combination of an adelic 1-step one-dimensional nilsystem and an odometer and such nilsystems generalize and factor onto circle rotations; see Theorem 6.4.

1.3 Our main results

Theorem A. *Let X be an infinite minimal subshift with $\limsup p(q)/q < 3/2$. Then, if (M, η) is the maximal equicontinuous factor of (X, σ) and $\phi : (X, \sigma) \to (M, \eta)$ is the associated factor map,*

- *ϕ is a measure-theoretic isomorphism with respect to the unique invariant measures on (X, σ) and (M, η) (Proposition 6.5),*
- *the additive continuous eigenvalue group of (X, σ) is $E_X = \{q\alpha + \sum\{qe_p\}_p : q \in Q\} + R$ for some $\alpha \notin \mathbb{Q}$, $e_p \in \mathbb{Q}_p$ and Q, R subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$ (Theorem 5.3),*
- *(M, η) is isomorphic to a product of a (possibly finite) odometer $\mathcal{O}_X$ (controlling the rational continuous eigenvalues) and a rotation of an adelic 1-step one-dimensional nilmanifold $\mathcal{M}_X$ (Theorem 6.4), and*
- *for every odometer $\mathcal{O}$ and adelic 1-step one-dimensional nilmanifold $\mathcal{M}$ which can appear in such an MEF, there exists a minimal subshift with $\limsup p(q)/q = 1$ for which the MEF is the product of a rotation on $\mathcal{O}$ and $\mathcal{M}$. This limsup may take any prescribed value in $[1, 3/2]$ as long as either $\mathcal{O}$ is infinite or $\mathcal{M}$ is not a finite extension of a circle (Theorem 8.1).*

In particular, X has measurably discrete spectrum for its unique invariant measure (Theorem 4.1), factors onto an irrational circle rotation (Corollary 5.9), and every eigenfunction is continuous (Theorem 5.3).

Also, any two such subshifts are orbit equivalent iff they are strong orbit equivalent iff they have the same additive eigenvalue group.

1.4 The Sarnak conjecture

The celebrated **Sarnak conjecture** states that for any zero entropy TDS (X, T), any $f \in C(X)$, and any $x \in X$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x_0)\mu(n) \to 0, \tag{1}$$

where μ is the Möbius function. A simple application of Theorem A is the following.

Corollary 1.1. *For any subshift (X, σ) with $\limsup p(q)/q < 3/2$, any $f \in C(X)$, and any $x_0 \in X$, (1) holds.*

Proof. Assume that (X, σ) has $\limsup p(q)/q < 3/2$, and consider $f \in C(X)$ and $x_0 \in X$. The subsystem $X_0 = \overline{\{\sigma^n x_0\}}$ is transitive by definition. If X_0 is finite, then it is a finite union of periodic orbits, and it is a simple consequence of the Prime Number Theorem that (1) holds for x_0 in this case. If X_0 is infinite, transitive, and non-minimal, then by [OP19], x_0 is eventually periodic in both directions, in which case (1) holds for x_0 for the same reason.

Finally, if X is infinite and minimal, then by Theorem A, there exists a factor π from (X_0, σ) to a minimal equicontinuous (M, η) which is a measure-theoretic isomorphism for the unique invariant measures on both systems . But now the fact that (1) holds for x_0 follows from two theorems from [DK15]: Theorem 4.2 (which shows that Sarnak's conjecture holds for minimal equicontinuous systems) and Theorem

4.1, which shows that Sarnak's conjecture is preserved under any topological extension which is also a measure-theoretic isomorphism between a pair of minimal TDS with unique invariant measures. □

Corollary 1.1 is, to our knowledge, the first result to show that sufficiently low word complexity (which is just a stronger version of zero entropy) implies the conclusion of Sarnak's conjecture.

1.5 Adelic groups

The MEF of a low-complexity minimal subshift can be interpreted in purely algebraic terms as a group rotation, a viewpoint which relates to both class field theory and Lie theory. In this framework, the MEF is a so-called (abelian) nilsystem, placing these objects in a greater context of recent work in ergodic theory.

Before discussing the general case, a (relatively) simple example may be helpful. Let $\mathcal{M}_2 = S^1 \times \mathbb{Z}_2$ (here and elsewhere, $\mathbb{Z}_2$ are the 2-adic integers) and consider a distinguished element $(\alpha, a_2) \in \mathcal{M}_2$. One can define a skew product action on $\mathcal{M}_2$ by

$$(\theta, z) \mapsto \begin{cases} (\theta + \alpha, z + a_2) & \text{if } \theta + \alpha < 1 \\ (\theta + \alpha - 1, z + a_2 + 1) & \text{otherwise} \end{cases}$$

This action is not a rotation on $\mathcal{M}_2$ viewed as a product group, but it can be viewed as the restriction of the rotation by (α, a_2) on $(\mathbb{R} \times \mathbb{Q}_2)/\mathbb{Z}[1/2]$ to its natural fundamental domain $\mathcal{M}_2 = S^1 \times \mathbb{Z}_2$. We note that the projection to the real coordinate is precisely the factor map onto S^1 under rotation by α.

The **ring of adeles** $\mathbb{A}$ over $\mathbb{Q}$ is $\mathbb{R} \times \prod_p \mathbb{Q}_p$ where p ranges over the primes and $\mathbb{Q}_p$ are the p-adic numbers, restricted to elements where all but finitely many terms lie in $\mathbb{Z}_p$). The field $\mathbb{Q}$ sits naturally as a lattice (discrete co-compact subgroup) diagonally in $\mathbb{A}$, and its character group is $\widehat{\mathbb{Q}} = \mathbb{A}/\mathbb{Q}$.

The eigenvalue groups of the low complexity subshifts in Theorem A involve arbitrary subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$, and describing the MEF via their character groups requires more refined techniques. It's not hard to check that such subgroups are in one-to-one correspondence with sequences (ℓ_p) in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ indexed by primes p, where $Q_{(\ell_p)} := \{\frac{m}{n} : n = \prod_p p^{t_p} \text{ such that } 0 \leq t \leq \ell_p\}$. This case (where infinitely many ℓ_p are allowed to be nonzero) is often called the **adelic** case in the literature.

Adapting relevant proofs to the adelic setting requires a bit of care since $Q_{(\ell_p)}$ generally does not sit as a lattice in $\mathbb{A}$ (being of infinite covolume). We can define a natural substitute $\mathbb{A}_{(\ell_p)}$ for $\mathbb{A}$, and we then verify that $\widehat{Q_{(\ell_p)}} = \mathbb{A}_{(\ell_p)}/Q_{(\ell_p)}$. This also explains why adelic subgroups arise naturally in connection to odometers (Proposition 6.9): for any odometer $\mathcal{O} = \varprojlim \mathbb{Z}/o_k\mathbb{Z}$, if $\ell_p = \sup\{t : p^t \text{ divides } o_k \text{ for some } k\}$, then $\mathcal{O} \simeq \widehat{Q_{(\ell_p)}/\mathbb{Z}}$.

Theorem A shows that the character groups in question cannot be purely p-adic, i.e. must have nontrivial real component. The phenomenon of nontrivial real component being more 'natural' in the study of Lie groups/lattices is not new; one example is the generalization of the Margulis Arithmeticity Theorem ([Mar91]) proved in ([Oh01]).

1.6 Nilsystems

A quotient G/Γ of a nilpotent Lie group G (above $\mathbb{A}_{(\ell_p)}$) by a lattice Γ (above $Q_{(\ell_p)}$) is called a **nilmanifold**, introduced by Mal'cev. When a nilmanifold is equipped with a translation action, it is called a **nilsystem**. Nilsystems were introduced in breakthrough work by Host-Kra [HK05] and Bergelson-Host-Kra [BHK05] to prove convergence of nonconventional ergodic averages, and since then they have proved invaluable in ergodic theory and dynamical systems (e.g. [Lei05], [Zie07], [GTZ12], [BLM12], [Wal12], [Zie14], [HK18], [BL18]).

Not every compact abelian group rotation (such as the MEF of a TDS) is a nilsystem, though all are inverse limits of 1-step nilsystems. However, in our case we prove that the MEF is itself a (1-step) one-dimensional nilsystem. Given the heuristic that nilsystems are the 'simplest' type of dynamical systems, the phenomenon that irrational eigenvalues yield lower complexity than rational alone makes sense;

when all eigenvalues are rational, the character group is purely p-adic, and so cannot have nilmanifold structure.

Our Lie group can contain p-adic parts for infinitely many primes, and we refer to this case as an **adelic nilsystem**. This is itself a generalization of the so-called 'S-adic' theory of nilmanifolds, in which p-adic parts can exist for finitely many primes, and which was studied for instance in [BG21] and in [SY17] in connection with solenoids.

1.7 Substitutions and the Pisot conjecture

As mentioned earlier, Ferenczi proved that all minimal subshifts of linear complexity have a so-called S-adic structure, meaning that all $x \in X$ have a recursive structure coming from a sequence (ρ_k) of substitutions. As was done in [CP23], the main component of the proof of Theorem A is a proof (Corollary 3.3) that low word complexity implies a very special type of S-adic structure, where all substitutions (denoted by τ_{m_k,n_k,r_k} in the proofs) have a very particular form.

Connections between substitutive structure and discrete spectrum have been known for many years, and the most famous such connection is the so-called **Pisot conjecture**. A full treatment is beyond our scope here, but informally it states that if X is defined by a single substitution (i.e. all ρ_k are the same in the description above) and if that substitution has the Pisot property (this means that its associated adjacency matrix has Perron eigenvalue which is a Pisot-Vijarayaghavan number) and is algebraically irreducible, then X has measurably discrete spectrum. The conjecture remains open, though there has been substantial progress, including a complete solution for X for alphabet of size 2 ([BD02], [HS03]).

Much more difficult is the general S-adic case; even finding a proper plausible formulation seems quite difficult. There have been multiple impressive recent results in this direction, including a version of S-adic Pisot for two-letter alphabets ([BMST16]). However, their result includes several hypotheses which cannot hold even for all Sturmian subshifts, most notably recurrence, which means that for every m, there exists n so that $\rho_k = \rho_{n+k}$ for $1 \leq k \leq m$. In particular, their results seemingly cannot be used to verify discrete spectrum under any complexity hypothesis alone.

Another hypothesis required for previous versions of the S-adic Pisot conjecture is that such subshifts are **balanced on words**, see Section 2 for a definition. This property is often difficult to verify, but we do so (Theorem 7.1) in the course of finding the **dimension group** for minimal subshifts of low word complexity (Theorem 7.2), which can be used to characterize **orbit equivalence** and **strong orbit equivalence** for these shifts.

Our proof of Theorem A can then be, at least in part, thought of as a direct verification of a version of the S-adic Pisot conjecture for only the restricted class of substitutions appearing in our S-adic decomposition. In fact, the eigenvalue group and the nilmanifold $\mathcal{M}$ and odometer $\mathcal{O}$ appearing in the MEF can be explicitly defined in terms of these substitutions; this is too technical to describe here, but is done in Theorems 5.3 and 6.4. This allows us to explicitly define some simple examples (including traditional substitutions rather than S-adic) which have certain MEFs which, to our knowledge, haven't appeared in the literature. (See Section 8 for proofs.)

Example 1.2. If ρ is the substitution on $\{0,1\}$ defined by $\rho(0) = 001$ and $\rho(1) = 00001$, and we define $X = \overline{\{\sigma^n x\}}$ by $x = \lim_k \rho^k(0)$, then its MEF is a rotation of the adelic 1-step nilmanifold $\mathcal{M}_2$ as in Section 1.5.

Example 1.3. If ρ is the substitution on $\{0,1\}$ defined by $\rho(0) = 00000011$ and $\rho(1) = 0000000011$, and we define $X = \overline{\{\sigma^n x\}}$ by $x = \lim_k \rho^k(0)$, then its MEF is the product of the binary odometer with a rotation of $\mathcal{M}_2$ as in Section 1.5.

Example 1.4. Let π be the substitution on $\{0,1\}$ defined by $\pi(0) = a$ and $\pi(1) = ab$, let ω_1 be the substitution on $\{0,1\}$ defined by $\omega_1(0) = 001$ and $\omega_1(1) = 00001$, and let ω_2 be the substitution on $\{0,1\}$ defined by $\omega_2(0) = 00001$ and $\omega_2(1) = 0000001$. Define a sequence of substitutions $\rho_k \in \{\omega_1, \omega_2\}$ by $\rho_k = \omega_1$ if 2^{k+2} divides the length of $(\pi \circ \rho_0 \circ \cdots \circ \rho_{k-1})(1)$, and ω_2 otherwise. (For instance, ρ_0 is ω_2 since 2^2 does not divide the length 2 of $\pi(1) = ab$, and ρ_1 is ω_1 since 2^3 does divide the length 8 of

$(\pi \circ \rho_0)(1) = bbbbbbab$.) If we define $X = \overline{\{\sigma^n x\}}$ by $x = \lim_k (\pi \circ \rho_0 \circ \cdots \circ \rho_k)(0)$, then its MEF is the product of the binary odometer and an irrational circle rotation.

1.8 3/2 as a threshold

Several recent works have demonstrated that $\limsup p(q)/q = 3/2$ is an important threshold for several different types of dynamical properties. First, Theorems 1.2 and 1.3 of [OP19] imply that if a subshift X is transitive and nonminimal and has $\limsup p(q)/q < 3/2$, then it is the orbit closure of a sequence which is eventually periodic in both directions. (In particular, this means that Theorem A automatically applies to all transitive shifts not of this degenerate form.) We can rewrite as the following threshold result.

Theorem 1.5.

$3/2 = \min\{\limsup p(q)/q \ : \ X \textit{ is transitive, not minimal, and contains a non-eventually periodic sequence}\}$.

In [Cre23], it was shown (Theorem C) that every aperiodic rank-one subshift satisfies $\limsup p(q)/q \geq 3/2$, and an example was given there (Theorem D) of an aperiodic rank-one subshift with $\limsup p(q)/q = 3/2$. This immediately implies the following.

Corollary 1.6.

$$3/2 = \min\{\limsup p(q)/q \ : \ X \textit{ is an aperiodic rank-one subshift}\}.$$

Theorem A implies similar results for different dynamical properties. Theorem A, when combined with the weakly mixing example from [CP23] with $\limsup p(q)/q = 3/2$ mentioned above, yields the following result, which shows that any bound on $\limsup p(q)/q$ which implies existence of eigenvalues automatically implies discrete spectrum.

Corollary 1.7.

$$3/2 = \sup\{\limsup p(q)/q \ : \ X \textit{ has discrete spectrum}\} = \min\{\limsup p(q)/q \ : \ X \textit{ is weakly mixing}\}.$$

Surprisingly, the same number is also the complexity threshold for Toeplitz subshifts. Our results already show that $\limsup p(q)/q < 3/2$ precludes X being Toeplitz; all Toeplitz shifts are minimal, and have no irrational continuous eigenvalues, so cannot have the structure of Theorem A. In the other direction, [Sel20] gives word complexity estimates for a subclass called simple Toeplitz subshifts, and those estimates show that there exist simple Toeplitz subshifts with $\limsup p(q)/q = 3/2$ (this happens whenever the parameter sequence (n_k) from that paper is unbounded). We now have the following.

Theorem 1.8.

$$3/2 = \min\{\limsup p(q)/q \ : \ X \textit{ is Toeplitz}\}.$$

A nearly identical proof shows that 3/2 is a threshold for irrational continuous eigenvalues.

Corollary 1.9.

$$3/2 = \min\{\limsup p(q)/q \ : \ X \textit{ is minimal, infinite, and has no irrational continuous eigenvalue}\}.$$

1.9 Summary

Section 2 contains definitions not fully given in the introduction. In Section 3, we describe and prove the S-adic structure for minimal subshifts of low complexity. Sections 4, 5, 6, 7, 8 contain, respectively, proofs of discrete spectrum, the eigenvalue group, the structure of the MEF, classification of orbit equivalence and strong orbit equivalence, and realization of all possible $\mathcal{M}, \mathcal{O}$ for all $\limsup p(q)/q \leq 1.5$.

2 Definitions

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}$; the **full shift** is the set $\mathcal{A}^{\mathbb{Z}}$ equipped with the product topology and σ is the left shift homeomorphism on $\mathcal{A}^{\mathbb{Z}}$. A **subshift** is a closed σ-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}}$. The **orbit** of $x \in X$ is the set $\{\sigma^n x\}_{n\in\mathbb{Z}}$. In a slight abuse of notation, we sometimes define X as the orbit closure of a one-sided sequence $y \in \mathcal{A}^{\mathbb{N}}$; this can be interpreted in the obvious way using natural extensions.

A **word** is any element of $\mathcal{A}^n$ for some $n \in \mathbb{N}$, referred to as its **length** and denoted by $|w|$. For any word v, the number of occurrences of v as a subword of w is denoted $|w|_v$. We say $\mathcal{A}^* = \bigcup_{n\geq 1} \mathcal{A}^n$ and represent the concatenation of words $w_1, w_2, \dots, w_n$ by $w_1 w_2 \dots w_n$.

The **language** of a subshift X on $\mathcal{A}$, denoted $L(X)$, is the set of all finite words appearing as subwords of points in X. For any $q \in \mathbb{N}$, we denote $L_q(X) = L(X) \cap \mathcal{A}^q$, the set of q-letter words in $L(X)$, and define the **word complexity function** of X to be $p(q) := |L_q(X)|$. For a subshift X and a word $w \in L(X)$ we denote by $[w]$ the clopen subset in X consisting of all $x \in X$ such that $x_0 \dots x_{|w|-1} = w$.

A **substitution** (sometimes called a morphism) is a map $\tau : \mathcal{A} \to \mathcal{B}^*$ for finite alphabets $\mathcal{A}$ and $\mathcal{B}$. Substitutions can be composed when viewed as homomorphisms on the monoid of words under composition, i.e. if $\tau : \mathcal{A} \to \mathcal{B}^*$ and $\pi : \mathcal{B} \to \mathcal{C}^*$, then $\pi \circ \tau : \mathcal{A} \to \mathcal{C}^*$ can be defined by $(\pi \circ \tau)(a) = \pi(b_1)\pi(b_2)\dots\pi(b_k)$, where $\tau(a) = b_1 \dots b_k$. When a sequence of substitutions $\tau_k : \mathcal{A} \to \mathcal{A}^*$ shares the same alphabet, and when there exists $a \in \mathcal{A}$ for which $\tau_k(a)$ begins with a for all k, clearly $(\tau_1 \circ \dots \circ \tau_k)(a)$ is a prefix of $(\tau_1 \circ \dots \circ \tau_{k+1})(a)$ for all k. In this situation one may then speak of the (right-infinite) limit of $(\tau_1 \circ \dots \circ \tau_k)(a)$.

For any subshift X, there is a convenient way to represent the n-language and possible transitions between words in points of X by the **Rauzy graphs**: the nth Rauzy graph of X is the directed graph $G_{X,n}$ with vertex set $L_n(X)$ and directed edges from $w_1 \dots w_n$ to $w_2 \dots w_{n+1}$ for all $w_1 \dots w_{n+1} \in L_{n+1}(X)$. Then, a vertex with multiple outgoing edges corresponds to a word $w = w_1 \dots w_n$ which is **right-special**, meaning that there exist letters $a \neq b$ for which $wa, wb \in L(X)$. **Left-special** words are defined similarly, and a word is **bi-special** if it is both left- and right-special. The reader is referred to Section 1 of [CP23] for more details.

We will sometimes endow a subshift X with a measure μ; any such μ is understood to be a Borel probability measure which is invariant under σ. A subshift has (measurably) discrete spectrum with respect to a measure μ when the eigenfunctions span $L^2(X, \mu)$. It is well-known, e.g. ([Wal82], Theorem 3.4), that:

Theorem 2.1. *An ergodic transformation on a standard probability space with discrete spectrum is measure-theoretically isomorphic to the space of characters of its (multiplicative) eigenvalue group, endowed with the Haar measure, under the ergodic "rotation" of multiplication by the identity homomorphism.*

A subshift X is **balanced on words** when for every word $v \in \mathcal{L}(X)$, there exists $C_v > 0$ such that for any $w, w' \in \mathcal{L}(X)$ with $|w| = |w'|$, $||w|_v - |w'|_v| < C_v$, i.e. the number of occurrences of v in any two words of the same length differs by less than C_v. We say that X is **balanced on letters** if the above holds whenever v has length 1.

3 Substitutive structure of minimal subshifts with low complexity

This section is devoted to establishing the following proposition, which establishes a substitutive structure for minimal subshifts with complexity $\limsup \frac{p(q)}{q} < 1.5$.

Proposition 3.1. *Let X be an infinite minimal subshift with $\limsup \frac{p(q)}{q} < 1.5$. Then there exist $\nu > 0$ and words u_k and v_k for $k \geq 0$ such that, writing p_k for the maximal common prefix of u_k and v_k and s_k for the maximal common suffix of v_k^∞ and $v_k^\infty u_k$, the following hold:*

- *every $x \in X$ is uniquely decomposable as a concatenation of v_k and u_k;*
- *$|v_k| < |u_k|$ and v_k is a suffix of u_k;*

- $|p_k| + |s_k| < |v_k| + |u_k|$;
- *for* $k \geq 1$, *also* $|p_k| + |s_k| < 2|v_k| + |v_{k-1}| < 3|v_k|$; *and*
- $p(q) < (1.5 - \nu)q$ *for all* $q \geq |v_0|$.

For each k, *exactly one of the following holds:*

- *there exist positive integers* $m_k < n_k$ *such that*

$$v_{k+1} = v_k^{m_k-1} u_k \quad \text{and} \quad u_{k+1} = v_k^{n_k-1} u_k; \text{ or}$$

- *there exist positive integers* $r_k < m_k < n_k$ *such that*

$$v_{k+1} = v_k^{m_k-1} u_k v_k^{r_k-1} u_k \quad \text{and} \quad u_{k+1} = v_k^{n_k-1} u_k v_k^{r_k-1} u_k.$$

Notation 3.2. For k such that r_k does not exist, set $r_k = 0$ and for all k set

$$\mathbb{1}_{r_k} := \begin{cases} 1 & r_k > 0 \\ 0 & r_k = 0 \end{cases}.$$

The substitutive structure can be explicitly stated as follows:

Corollary 3.3. *For integers* $0 \leq r < m < n$, *define the substitutions* $\tau_{m,n,r} : \{0,1\} \to \{0,1\}^*$ *by*

$$\tau_{m,n,0} : \begin{cases} 0 \mapsto & 0^{m-1}1 \\ 1 \mapsto & 0^{n-m}0^{m-1}1 \end{cases} \qquad \tau_{m,n,r} : \begin{cases} 0 \mapsto & 0^{m-1}10^{r-1}1 \\ 1 \mapsto & 0^{n-m}0^{m-1}10^{r-1}1 \end{cases} \quad \text{when } r > 0.$$

Then X *is the orbit closure of* $\lim \pi \circ \tau_{m_0,n_0,r_0} \circ \cdots \circ \tau_{m_k,n_k,r_k}(0)$ *for* $\pi : \{0,1\} \to \mathcal{A}^*$ *for some finite* $\mathcal{A}$.

Proof. Define $\pi(0) = v_0$ and $\pi(1) = u_0$. Write $\xi_k = \pi \circ \tau_{m_0,n_0,r_0} \circ \cdots \tau_{m_{k-1},n_{k-1},r_{k-1}}$.
It follows immediately from Proposition 3.1 that if $v_k = \xi_k(0)$ and $u_k = \xi_k(1)$ then, when $r_k > 0$,

$$v_{k+1} = v_k^{m_k-1} u_k v_k^{r_k-1} u_k = (\xi_k(0))^{m_k-1} \xi_k(1) (\xi_k(0))^{r_k-1} \xi_k(1) = \xi_k \circ \tau_{m_k,n_k,r_k}(0) = \xi_{k+1}(0)$$

and similarly for u_{k+1}. Similar reasoning applies when $r_k = 0$. The claim then follows by induction. □

We first collect several several basic facts established in previous work of the authors.

Definition 3.4. A word v is a **root** of w if $|v| \leq |w|$ and w is a suffix of the left-infinite word v^∞.

Lemma 3.5 ([Cre22] Lemma 5.7)**.** *If* w *and* v *are words with* $|v| \leq |w|$ *such that* wv *has* w *as a suffix then* v *is a root of* w.

Lemma 3.6 ([CP23] Lemma 2.5)**.** *Let* u *and* v *be words with* $|v| < |u|$ *and let* s *be the maximal common suffix of* v^∞ *and* $v^\infty u$. *If* $|s| \geq |vu|$ *then* u *and* v *are multiples of the same word.*

Lemma 3.7 ([CP23] Lemma 2.6)**.** *Let* v *and* u *be words with* $|v| < |u|$ *which are not multiples of the same word and where* v *is a suffix of* u. *Let* s *be the maximal common suffix of* v^∞ *and* $v^\infty u$. *Then* s *is a suffix of any left-infinite concatenation of* u *and* v.

Lemma 3.8 ([CP23] Lemma 2.7)**.** *Let* v *and* u *be words and* s *be the maximal common suffix of* v^∞ *and* $v^\infty u$. *Let* y *and* z *be suffixes of some (possibly distinct) concatenations of* u *and* v, *both of length at least* $|s|$. *Then for any word* w, *the maximal common suffix of* yvw *and* zuw *is* sw.

Lemma 3.9 ([CP23] Lemma 1.4)**.** *Let* X *be a subshift on alphabet* $\mathcal{A}$, *for all* n *let* $RS_n(X)$ *denote the set of right-special words of length* n *in the language of* X, *and for all right-special* w, *let* $F(w)$ *denote the set of letters which can follow* w, *i.e.* $\{a \in \mathcal{A} : wa \in L(X)\}$. *Then, for all* $q > r$,

$$p(q) = p(r) + \sum_{i=r}^{q-1} \sum_{w \in RS_i(X)} (|F(w)| - 1).$$

Lemma 3.10. *Let w and y be right-special words with $|w| \leq |y|$ and maximal common suffix s. Then*

$$\frac{p(|w||)}{|w|} \geq 1 + \frac{|w| - |s|}{|w|}.$$

Proof. For all $|s| < q \leq |w|$, the suffixes of w and y of length q are distinct and are both right-special so $|\{w \in RS_q(X)\}| \geq 2$. By Lemma 3.9, then $p(|w|) \geq p(|s|) + \sum_{q=|s|+1}^{|w|} 2 = p(|s|) + 2(|w| - |s|)$. □

Lemma 3.11 ([CP23] Lemma 2.8)**.** *If $p(q+1) - p(q) = 1$ then there exists a bi-special word which has length in $[q, q + p(q)]$, has exactly two successors, and is the unique right-special word of its length and also the unique left-special word of its length.*

The starting off point for our construction of the words v_k and u_k is the following lemma.

Lemma 3.12. *Let X be an infinite minimal subshift satisfying $\liminf \frac{p(q)}{q} < 2$. For any $Q > 0$, there exist words u and v with $|v| \geq Q$ such that, writing s for the maximal common suffix of v^∞ and $v^\infty u$,*

- *u and v begin with different letters;*
- *v is a proper suffix of u;*
- *s is the unique left-special and unique right-special word of its length;*
- *every word which has s as a suffix is a suffix of a concatenation of u and v; and*
- *every $x \in X$ can be written in exactly one way as a concatenation of u and v.*

Proof. Since $\limsup \frac{p(q)}{q} < 2$, there exist infinitely many q such that $p(q+1) - p(q) = 1$ and eventually $p(q) < 2q$. By Lemma 3.11, there are then infinitely many q such that there exists a word w_q which is the unique left-special and unique right-special word of length q (which will have exactly two successors).

Let y_q and z_q, with $|y_q| \leq |z_q|$, be the two shortest return words for w_q which will be the labels of the two paths from w_q to itself in the Rauzy graph $G_{X,|w_q|}$. Then y_q and z_q begin with different letters and every $x \in X$ can be written in exactly one way as a concatenation of y_q and z_q since every $x \in X$ must label a path in the Rauzy graph. Since $2|y_q| \leq |y_q| + |z_q| \leq p(|w_q|) + 1 < 2|w_q| + 1$, we have $|y_q| \leq |w_q|$ so by Lemma 3.5, y_q is a root of w_q. Since $w_q z_q$ has w_q as a suffix, it has y_q as a suffix and as $|y_q| \leq |z_q|$ then y_q is a suffix of z_q.

If $|y_q| = |z_q|$ then $|z_q| \leq |w_q|$ so both y_q and z_q are suffixes of w_q of the same length which would imply $y_q = z_q$ so $|y_q| < |z_q|$. Since y_q is a suffix of w_q, it is also a suffix of $w_{q'}$ for $q' > q$. Then y_q must be a suffix of $y_{q'}$ since $y_{q'}$ is a return word for $w_{q'}$ hence for w_q so $|y_q| \leq |y_{q'}|$. Suppose $|y_q|$ is bounded. Then $y_q = y_Q$ for some fixed Q eventually but that would make y_Q a root of w_q for arbitrarily long w_q making $v_Q^\infty \in X$ which contradicts that X is infinite and minimal.

Let s_q be the maximal common suffix of y_q^∞ and $y_q^\infty z_q$. Since y_q and z_q are return words for w_q, then $w_q y_q^t z_q$ has w_q as a suffix for all $t \geq 0$ so w_q is a suffix of $y_q^\infty z_q$. Since y_q is a root of w_q, then w_q is a suffix of s_q. Since w_q is left-special, the two words $y_q w_q$ and $z_q w_q$ differ on the letter prior to w_q. Therefore $s_q = w_q$. Then any word which has s_q as a suffix must be a suffix of a concatenation of y_q and z_q as those are the labels of the two return paths. □

The inductive step in the construction of v_k and u_k comes from our next lemma.

Lemma 3.13. *Let X be an infinite minimal subshift, and let u and v be words where all $x \in X$ can be written as a concatenation of u and v, $|v| < |u|$, v is a suffix of u but not a prefix of u, and $p(q)/q < 1.5$ for all $q \geq |v|$. Let p be the maximal common prefix of u and v and s be the maximal common suffix of v^∞ and $v^\infty u$. Provided that $|p| + |s| < |u| + |v|$, exactly one of the following holds:*

- *there exist positive integers $m < n$ such that every $x \in X$ can be written as a concatenation of $v^{m-1}u$ and $v^{n-1}u$; or*
- *there exist positive integers $r < m < n$ such that every $x \in X$ can be written as a concatenation of $v^{m-1}uv^{r-1}u$ and $v^{n-1}uv^{r-1}u$.*

Proof. For brevity, we will use 'concatenation' to refer to a concatenation of u and v corresponding to some $x \in X$. Consider the set $S = \{t \geq 0 : uv^t u \text{ appears in a concatenation}\}$. If $|S| = 1$ then the subshift would be periodic by minimality contradicting that X is infinite. If $|S| = 2$ then the first of the two possible conclusions hold by setting $m = \min(S) + 1$ and $n = \max(S) + 1$ since every concatenation is of the form $\dots uv^{i_1}uv^{i_2}uv^{i_3}u\dots$ where all i_j are either $m-1$ or $n-1$.

So we may assume that $uv^x u, uv^y u, uv^z u$ for $x < y < z$ all appear in the concatenations and take x to be the minimal such value and y to be the next smallest value.

Suppose that $uv^x uv^x u$ appears in a concatenation. Then, using Lemma 3.6, $sv^x uv^x uv^x p$ is right-special as $uv^x uv^x u$ must be preceded by v^x due to the minimality of x and the $v^x ux^x u$ pattern cannot continue forever (by minimality) and when it is broken we see $v^x uv^y$. Also $sv^x uv^y p$ is right-special due to x being minimal and $z > y$. By Lemma 3.8, the maximal common suffix of $sv^x uv^x uv^x p$ and $sv^x uv^y p$ is $sv^x p$.

In the case when $|sv^x uv^y p| \leq |sv^x uv^x uv^x p|$, since $x \leq y - 1$ and $|p| + |s| < |v| + |u|$, by Lemma 3.10,

$$\begin{aligned}\frac{p(|sv^x uv^y p|)}{|sv^x uv^y p|} &\geq 1 + \frac{|u| + y|v|}{|p| + |s| + |u| + (x+y)|v|} > 1 + \frac{|u| + y|v|}{2|u| + (x+y+1)|v|} \\ &\geq 1 + \frac{|u| + y|v|}{2|u| + (y-1+y+1)|v|} = \frac{3}{2}.\end{aligned}$$

In the case when $|sv^x uv^x uv^x p| < |sv^x uv^y p|$, since $|u| > |v|$, by Lemma 3.10,

$$\begin{aligned}\frac{p(|sv^x uv^x uv^x p|)}{|sv^x uv^x uv^x p|} &\geq 1 + \frac{2|u| + 2x|v|}{|p| + |s| + 2|u| + 3x|v|} \\ &> 1 + \frac{2|u| + 2x|v|}{3|u| + (3x+1)|v|} > 1 + \frac{2|u| + 2x|v|}{4|u| + 3x|v|} \geq \frac{3}{2}.\end{aligned}$$

Since $p(q) < 1.5q$ for $q \geq |v|$, this is a contradiction and therefore $uv^x uv^x u$ never appears in a concatenation.

Now suppose that $v^y uv^y$ appears in a concatenation. Then $sv^y uv^x p$ is right-special as $uv^x u$ must be preceded by v^y as y is the next smallest value (and $uv^x uv^x u$ does not appear). Also $sv^x uv^y p$ is right-special as $uv^z u$ must be preceded by v^x by minimality of x. By Lemma 3.8, the maximal common suffix of $sv^y uv^x p$ and $sv^x uv^y p$ is $sv^x p$. Therefore, as $x \leq y - 1$, by Lemma 3.10,

$$\begin{aligned}\frac{p(|sv^y uv^x p|)}{|sv^y uv^x p|} &\geq 1 + \frac{|u| + y|v|}{|p| + |s| + |u| + (x+y)|v|} \\ &> 1 + \frac{|u| + y|v|}{2|u| + (x+y+1)|v|} \geq 1 + \frac{|u| + y|v|}{2|u| + (y-1+y+1)|v|} = \frac{3}{2}\end{aligned}$$

which again contradicts that $p(q) < 1.5q$ for $q \geq |v|$; therefore $v^y uv^y$ never appears. Then every appearance of $uv^w u$ for $w > x$ appears as part of $uv^x uv^w uv^x u$. As $uv^x uv^x u$ never appears, then every occurrence of $uv^w u$ appears as part of $v^y uv^x uv^w uv^x uv^y$ by the minimality of y as the second smallest possible value. By Lemma 3.7 then $uv^w u$ for $w > x$ always appears as part of $sv^y uv^x uv^w uv^x uv^y p$.

Since $z > x$, then $sv^y uv^x uv^z uv^x uv^y p$ appears in a concatenation. That word has $sv^y uv^x uv^y v$ as a prefix (as $z > y$) and $sv^y uv^x uv^y uv^x uv^y p$, which also appears, has $sv^y uv^x uv^y u$ as a prefix so $sv^y uv^x uv^y p$ is right-special. If $uv^w u$ for $w > z$ also appears then by the same reasoning, $sv^y uv^x uv^z p$ is right-special. By Lemma 3.8, the maximal common suffix of $sv^y uv^x uv^y p$ and $sv^y uv^x uv^z p$ is $sv^y p$ so we would have, by Lemma 3.10,

$$\begin{aligned}&\frac{p(|sv^y uv^x uv^y p|)}{|sv^y uv^x uv^y p|} \geq 1 + \frac{(x+y)|v| + 2|u|}{(x+2y)|v| + 2|u| + |p| + |s|} \\ &> 1 + \frac{(x+y)|v| + 2|u|}{(x+2y+1)|v| + 3|u|} = \frac{3}{2} + \frac{(\frac{1}{2}x - \frac{1}{2})|v| + \frac{1}{2}|u|}{(x+2y+1)|v| + 3|u|} \geq \frac{3}{2} + \frac{\frac{1}{2}|u| - \frac{1}{2}|v|}{(x+2y+1)|v| + 3|u|} > \frac{3}{2}\end{aligned}$$

contradicting that $p(q) < 1.5q$ for $q \geq |v|$; therefore $|S| = 3$. Therefore every concatenation is of the form

$$\dots uv^x uv^{i_0} uv^x uv^{i_1} uv^x ux^{i_2} uv^x uv^{i_3} uv^x u \dots$$

where i_j are all either y or z. Setting $r = x + 1$ and $m = y + 1$ and $n = z + 1$ then proves the claim. □

We are now prepared to prove Proposition 3.1.

Proof of Proposition 3.1. Since $\limsup \frac{p(q)}{q} < 1.5$, there exists $\nu > 0$ and q_0 so that $p(q) < (1.5 - \nu)q$ for all $q \geq q_0$. Let u_0 and v_0 be the words guaranteed by Lemma 3.12 such that $|v_0| \geq q_0$. Then p_0 is empty as u_0 and v_0 begin with different letters. Lemma 3.6 implies $|s_0| < |u_0 v_0|$ so $|p_0|+|s_0| = |s_0| < |u_0|+|v_0|$.

Proceed by induction assuming we have constructed the words u_k and v_k. By Lemma 3.13, there either exist positive integers $m_k < n_k$ such that every $x \in X$ can be written as a concatenation of $v_{k+1} := v_k^{m_k-1} u_k$ and $u_{k+1} := v_k^{n_k-1} u_k$ or there exist positive integers $r_k < m_k < n_k$ such that every $x \in X$ can be written as a concatenation of $v_{k+1} := v_k^{m_k-1} u_k v_k^{r_k-1} u_k$ and $u_{k+1} := v_k^{n_k-1} u_k v_k^{r_k-1} u_k$.

Since v_{k+1} has $v_k^{m_k-1} u_k$ as a prefix and u_{k+1} has $v_k^{m_k-1} v_k$ as a prefix (as $m_k < n_k$), $p_{k+1} = v_k^{m_k-1} p_k$. Since v_{k+1}^∞ has $u_k v_{k+1}$ as a suffix and $u_{k+1} = v_k^{n_k-m_k} v_{k+1}$ has $v_k v_{k+1}$ as a suffix, by Lemmas 3.7 and 3.8, we have $s_{k+1} = s_k v_{k+1}$. Therefore

$$|p_{k+1}| + |s_{k+1}| = (m_k - 1)|v_k| + |v_{k+1}| + |p_k| + |s_k| < (m_k - 1)|v_k| + |v_{k+1}| + |u_k| + |v_k| = 2|v_{k+1}| + |v_k|$$

and as $|u_{k+1}| \geq |v_{k+1}| + |v_k|$, then $|p_{k+1}| + |s_{k+1}| < |u_{k+1}| + |v_{k+1}|$.

By induction on k, each $x \in X$ can be decomposed uniquely into words v_k and u_k. For $k = 0$, this follows from Lemma 3.12 since v_0 and u_0 were constructed using that lemma. If x can be uniquely represented as a concatenation of v_k and u_k then the same must be true of $v_k^{m_{k-1}} u_k$ and $v_k^{n_k-1} u_k$, or of both followed immediately by $v_k^{r_k-1} u_k$ for k for which r_k exists. □

Remark 3.14. For all k, $p_{k+1} = v_k^{m_k-1} p_k$ and $s_{k+1} = s_k v_{k+1}$ as shown in the proof of Proposition 3.1.

3.1 Complexity estimates

Having established the substitutive structure of low complexity minimal subshifts, we can now determine what their right-special words are, which allows us to estimate word complexity using Lemma 3.9.

Proposition 3.15. *Let X be an infinite minimal subshift satisfying the conclusions of Proposition 3.1. For the words $\{u_k\}$ and $\{v_k\}$, the following hold:*

- *the left-infinite word $p_\infty = \lim s_k p_k = \lim s_0 v_1 \cdots v_{k+1} v_k^{m_k-1} v_{k-1}^{m_{k-1}-1} \cdots v_0^{m_0-1}$ is right-special;*
- *for each k, the word $s_k v_k^{n_k-2} p_k$ is right-special and the maximal common suffix of it and p_∞ is $s_k v_k^{m_k-1} p_k$; and*
- *for k such that $r_k > 0$, the word $s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k$ is right-special and the maximal common suffix of it and p_∞ and of it and $s_k v_k^{n_k-2} p_k$ is $s_k v_k^{r_k-1} p_k$.*

Proof. Clearly p_k is right-special as $u_k \neq v_k$ and p_k must be followed by different letters in each due to maximality so Lemma 3.7 implies $s_k p_k$ is right-special. By Remark 3.14, $s_{k+1} p_{k+1} = s_k v_{k+1} v_k^{m_k-1} p_k$, and as this, by Lemma 3.7, has $s_k p_k$ as a suffix, p_∞ exists and is right-special.

Since $v_k^{n_k-1} u_k$ appears in a concatenation (if not then u_{k+1} never appears so the subshift would be periodic) and is preceded by a concatenation of u_k and v_k, by Lemma 3.7, $s_k v_k^{n_k-1} u_k$ appears. Since $s_k v_k^{n_k-2} v_k$ is a prefix of that word and $s_k v_k^{n_k-2} u_k$ is a suffix of it, $s_k v_k^{n_k-2} p_k$ is right-special.

Since $s_{k+1} p_{k+1} = s_k v_{k+1} v_k^{m_k-1} p_k$ has $s_k u_k v_k^{m_k-1} p_k$ as a suffix, by Lemma 3.8, the maximal common suffix of p_∞ and $s_k v_k^{n_k-2} p_k$ is then $s_k v_k^{m_k-1} p_k$.

For k such that $r_k > 0$, the word $v_{k+1} = v_k^{m_k-1} u_k v_k^{r_k-1} u_k$ appears in a concatenation preceded by a concatenation of u_k and v_k showing that $s_k v_k^{r_k-1} u_k v_k^{r_k-1} u_k$ appears (as $r_k < m_k$). The word $u_k v_k^{r_k-1} u_k v_k^{m_k-1}$ appears in a concatenation (in fact with the second u_k being a suffix of any v_{k+1} or u_{k+1} that appears in a $(k+1)$-concatenation) showing that $s_k v_k^{r_k-1} u_k v_k^{r_k-1} v_k$ appears (as $r_k < m_k$). Therefore $s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k$ is right-special.

As both p_∞ and $s_k v_k^{n_k-2} p_k$ have $v_k^{m_k-1} p_k$ as a suffix and $r_k < m_k$, both have $v_k v_k^{r_k-1} p_k$ as a suffix. By Lemma 3.8, the maximal common suffix of $s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k$ and either of them is then $s_k v_k^{r_k-1} p_k$. □

Lemma 3.16. *Under the hypotheses of Proposition 3.15, every right-special word of length at least $|s_0|$ is a suffix of one of those from that proposition, and so for $q > |s_0|$,*

$$p(q+1) - p(q) = 1 + \sum_{k=0}^{\infty} \mathbb{1}_{(|s_k v_k^{m_k-1} p_k|, |s_k v_k^{n_k-2} p_k|]}(q) + \sum_{k=0}^{\infty} \mathbb{1}_{r_k} \mathbb{1}_{(|s_k v_k^{r_k-1} p_k|, |s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k|]}(q).$$

Proof. By Lemma 3.12, s_0 is the unique right-special word of its length. Therefore every right-special word of length at least $|s_0|$ has s_0 as a suffix. Lemma 3.12 also implies every right-special word of length at least $|s_0|$ is a suffix of a concatenation of u_0 and v_0. Assume that every right-special word of length at least $s_k p_k$ is a suffix of a concatenation of u_k and v_k followed by p_k. Let w be a right-special word of length at least $|s_{k+1} p_{k+1}|$. Then w is a suffix of a concatenation of u_k and v_k followed by p_k so has $s_k p_k$ as a suffix.

By Remark 3.14, $s_k p_k = s_{k-1} v_k p_k$ so w has $v_k p_k$ as a suffix and also $|w| \geq |s_{k+1} p_{k+1}| = |s_k v_{k+1} v_k^{m_k-1} p_k|$. Since $v_k u_k$ only appears in a concatenation of v_k and u_k as a suffix either of v_{k+1} or of $v_{k+1} v_k^{m_k-1} u_k$, then w either has $v_{k+1} v_k^{m_k-1} p_k$ as a suffix or has $v_{k+1} p_k$ as a suffix. Since $v_{k+1} u_k$ only appears when $m_k = 1$ and since w is right-special, in both cases w has $v_{k+1} v_k^{m_k-1} p_k$ as a suffix. As the v_{k+1} is preceded by a concatenation of u_{k+1} and v_{k+1} of length at least $|s_k|$, by Remark 3.14, then w has $s_k v_{k+1} v_k^{m_k-1} p_k = s_{k+1} p_{k+1}$ as a suffix. By induction, every right-special word with length at least $|s_k p_k|$ has $s_k p_k$ as a suffix and is a suffix of a concatenation of u_k and v_k followed by p_k.

Let w be any right-special word with $|w| \geq |s_0| = |s_0 p_0|$ which is not a suffix of p_∞. Let k maximal such that $|w| \geq |s_k p_k|$. Then $w = y p_k$ where y is a suffix of a concatenation of u_k and v_k of length at least s_k. Since w is right-special, $y u_k$ and $y v_k$ must both appear in a concatenation. So y must share a suffix either with $v_k^{m_k-1}$ or with $v_k^{r_k-1} u_k v_k^{r_k-1}$ (in which case $r_k > 0$).

When y shares a suffix with $v_k^{m_k-1}$, as w is not a suffix of p_∞, then y has $s_k v_k^{m_k-1}$ as a proper suffix and shares a suffix with $s_k v_k^{n_k-1}$. If $|y| > |s_k v_k^{n_k-2}|$ then w being right-special would force $s_k v_k^{n_k-1} p_k$ or $s_k u_k v_k^{n_k-1} p_k$ to be right-special but $s_k v_k^{n_k}$ never appears in a concatenation. So when y shares a suffix with $v_k^{m_k-1}$, w is a suffix of $s_k v_k^{n_k-2} p_k$.

When y shares a suffix with $v_k^{r_k-1} u_k v_k^{r_k-1}$, since w is not a suffix of p_∞ it must be that y has $s_k v_k^{r_k-1}$ as a proper suffix. If w is not a suffix of $s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k$ then y must have either $u_k v_k^{r_k-1} u_k v_k^{r_k-1}$ or $v_k^{r_k} u_k v_k^{r_k-1}$ as a suffix. In the first case w being right-special would force $u_k v_k^{r_k-1} u_k v_k^{r_k-1} u_k$ to appear, which is impossible, and in the second case it would force $v_k^{r_k} u_k v_k^{r_k}$ to appear in a concatenation, also impossible. So in the case y shares a suffix with $v_k^{r_k-1} u_k v_k^{r_k-1}$, w is a suffix of $s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k$.

The right-special words of any length $n > |s_0|$ are then: the suffix of length n of p_∞, the suffix of length n of some $s_k v_k^{n_k-2} p_k$ (which exists and is different from the first word iff $n \in (|s_k v_k^{m_k-1} p_k|, |s_k v_k^{n_k-2} p_k|])$, and the suffix of length n of some $(|s_k v_k^{r_k-1} p_k|, |s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k|]$ for which $r_k > 0$ (which exists and is different from the first and second words iff $n \in (|s_k v_k^{r_k-1} p_k|, |s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k|])$. The complexity difference formula is now an immediate consequence of Lemma 3.9, along with the observation that there is no overlap between the right-special words for distinct k, since by Lemma 3.8, the maximal common suffix of $s_k v_k p_k$ and $s_{k+1} v_{k+1} p_{k+1}$ is $s_k p_k$ (recall that v_{k+1} has u_k as a suffix). □

Knowing the set of right-special words, we can write an explicit formula for the complexity function at some specific lengths, which determine $\limsup p(q)/q$.

Corollary 3.17. *Let X be an infinite minimal subshift satisfying the conclusions of Proposition 3.1. Set $C = p(|s_0|) - |s_0|$. Then for every k, writing $\ell_k = \begin{cases} \min(|v_k^{n_k-r_k-1}|, |v_k^{r_k-1}u_k|) & \text{if } r_k > 0 \\ 0 & \text{otherwise} \end{cases}$,*

$$p(|s_k v_k^{n_k-2} p_k|) = |s_k v_k^{n_k-2} p_k| + \mathbb{1}_{r_k}\ell_k + \sum_{j=0}^{k}(n_j - m_j - 1)|v_j| + \sum_{j=0}^{k-1}((r_j - 1)|v_j| + |u_j|)\mathbb{1}_{r_j} + C$$

and for k such that $r_k > 0$,

$$p(|s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k|) = |s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k| + \sum_{j=0}^{k-1}(n_j - m_j - 1)|v_j| + \sum_{j=0}^{k}((r_j - 1)|v_j| + |u_j|)\mathbb{1}_{r_j}$$
$$+ \left|(|s_k v_k^{m_k-1} p_k|, |s_k v_k^{n_k-2} p_k|] \cap (1, |s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k|]\right| + C$$

and $\limsup \frac{p(q)}{q}$ *is attained along some subsequence of these values.*

Proof. This is a fairly immediate corollary of Lemmas 3.9 and 3.16; we note only that ℓ_k is, when $r_k > 0$, the number of elements of $(|s_k v_k^{r_k-1} p_k|, |s_k v_k^{r_k-1} u_k v_k^{r_k-1} p_k|]$ which are less than $|s_k v_k^{n_k-2} p_k|$.

The limsup must be attained along a subsequence of the indicated sequences since they are the right endpoints of the intervals in the characteristic functions from Lemma 3.16. □

Remark 3.18. We could make a similar formulation of $\liminf p(q)/q$ using the left endpoints of the intervals from Lemma 3.16, but since we do not have need of that in this work, we do not do so here.

3.2 Restrictions on the substitutions

By Corollary 3.3, the complexity hypothesis $\limsup \frac{p(q)}{q} < 1.5$ ensures that X is defined by substitutions τ_{m_k,n_k,r_k}. In this section, we give some restrictions on how these integers are related.

Throughout this section, X is an infinite minimal subshift with $\limsup \frac{p(q)}{q} < 1.5$ and $\nu > 0$ and the sequences of words $\{u_k\}, \{v_k\}, \{p_k\}, \{s_k\}$ and integers $\{m_k\}, \{n_k\}, \{r_k\}$ are from Proposition 3.1.

Lemma 3.19. *For all k,*

$$|s_k| + |p_k| < (m_{k-3} + 2)|v_{k-3}| + m_{k-2}|v_{k-2}| + m_{k-1}|v_{k-1}| + |v_k|;$$
$$|s_k| + |p_k| < (m_{k-2} + 2)|v_{k-2}| + m_{k-1}|v_{k-1}| + |v_k|; \text{ and}$$
$$|s_k| + |p_k| < (m_{k-1} + 2)|v_{k-1}| + |v_k|.$$

Proof. By Remark 3.14 applied three times and that $|s_{k-3}| + |p_{k-3}| < 3|v_{k-3}|$,

$$\begin{aligned} |s_k| + |p_k| &= |s_{k-1}| + |v_k| + |p_{k-1}| + (m_{k-1} - 1)|v_{k-1}| \\ &= |s_{k-2}| + |v_{k-1}| + |v_k| + |p_{k-2}| + (m_{k-2} - 1)|v_{k-2}| + (m_{k-1} - 1)|v_{k-1}| \\ &= |s_{k-3}| + (m_{k-3} - 1)|v_{k-3}| + (m_{k-2} - 1)|v_{k-2}| + m_{k-1}|v_{k-1}| + |v_k| + |p_{k-3}| + |v_{k-2}| \\ &< (m_{k-3} + 2)|v_{k-3}| + m_{k-2}|v_{k-2}| + m_{k-1}|v_{k-1}| + |v_k|. \end{aligned}$$

Since $|s_{k-1}| + |p_{k-1}| < 3|v_{k-1}|$, $|s_{k-1}| + (m_{k-1} - 1)|v_{k-1}| + |p_{k-1}| + |v_k| < (m_{k-1} + 2)|v_{k-1}| + |v_k|$ and since $|s_{k-2}| + |p_{k-2}| < 3|v_{k-2}|$, $|s_{k-2}| + (m_{k-2} - 1)|v_{k-2}| + m_{k-1}|v_{k-1}| + |p_{k-2}| + |v_k| < (m_{k-2} + 2)|v_{k-2}| + m_{k-1}|v_{k-1}| + |v_k|$. □

Proposition 3.20. *For $k \geq 2$ such that $n_k > 2m_k$, exactly one of the following holds:*

(i) $n_k = 2m_k + 2$, $n_{k-1} = m_{k-1} + 1$, $n_{k-2} \leq \frac{4}{3}m_{k-2} + 1$ *and* $r_k = 0$ *and* $r_{k-1} = 0$*;*

(ii) $n_k = 2m_k + 1$, $n_{k-1} \leq 2m_{k-1}$ *and* $r_k = 0$*; or*

(iii) $n_k = 2m_k + 1$, $m_{k-1} = 1$, $n_{k-1} = 3$, $n_{k-2} = m_{k-2} + 1$ *and* $r_k = r_{k-1} = r_{k-2} = 0$.

Proof. Let k such that $n_k > 2m_k$. Since $|p_k| + |s_k| < 3|v_k|$, Corollary 3.17 implies

$$\frac{p(|s_k v_k^{n_k-2} p_k|)}{|s_k v_k^{n_k-2} p_k|} \geq 1 + \frac{(n_k - m_k - 1 + \mathbb{1}_{r_k})|v_k|}{(n_k-2)|v_k| + |p_k| + |s_k|} > 1 + \frac{n_k - m_k - 1 + \mathbb{1}_{r_k}}{n_k - 2 + 3} = \frac{3}{2} + \frac{\frac{1}{2}n_k - m_k - \frac{3}{2} + \mathbb{1}_{r_k}}{n_k + 1}$$

and therefore $n_k - 2m_k - 3 + 2 \cdot \mathbb{1}_{r_k} < 0$. So if $r_k > 0$ then $n_k < 2m_k + 1$, a contradiction, and if not then $n_k < 2m_k + 3$.

By Corollary 3.17 and Lemma 3.19,

$$\begin{aligned} \frac{p(|s_k v_k^{n_k-2} p_k|)}{|s_k v_k^{n_k-2} p_k|} &\geq 1 + \frac{(n_k - m_k - 1)|v_k| + (n_{k-1} - m_{k-1} - 1 + \mathbb{1}_{r_{k-1}})|v_{k-1}|}{(n_k - 2)|v_k| + |p_k| + |s_k|} \\ &> 1 + \frac{(n_k - m_k - 1)|v_k| + (n_{k-1} - m_{k-1} - 1 + \mathbb{1}_{r_{k-1}})|v_{k-1}|}{(n_k - 2)|v_k| + |v_k| + (m_{k-1} + 2)|v_{k-1}|} \\ &= \frac{3}{2} + \frac{(\frac{1}{2}n_k - m_k - \frac{1}{2})|v_k| + (n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}})|v_{k-1}|}{(n_k - 1)|v_k| + (m_{k-1} + 2)|v_{k-1}|} \end{aligned}$$

and therefore $(\frac{1}{2}n_k - m_k - \frac{1}{2})|v_k| + (n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}})|v_{k-1}| < 0$.

If $n_k = 2m_k + 2$ then $\frac{1}{2}|v_k| + (n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}})|v_{k-1}| < 0$ and since, $|v_k| = |v_{k-1}^{m_{k-1}-1} u_{k-1}| > m_{k-1}|v_{k-1}|$, then $n_{k-1} - m_{k-1} - 2 + \mathbb{1}_{r_{k-1}} < 0$ so $r_{k-1} = 0$ and $n_{k-1} = m_{k-1} + 1$. By Corollary 3.17 and Lemma 3.19,

$$\begin{aligned} \frac{p(|s_k v_k^{n_k-2} p_k|)}{|s_k v_k^{n_k-2} p_k|} &\geq 1 + \frac{(m_k + 1)|v_k| + (n_{k-2} - m_{k-2} - 1)|v_{k-2}|}{(n_k - 2)|v_k| + |v_k| + |v_{k-1}| + (m_{k-2} + 2)|v_{k-2}|} \\ &= \frac{3}{2} + \frac{\frac{1}{2}|v_k| - \frac{1}{2}|v_{k-1}| + (n_{k-2} - \frac{3}{2}m_{k-2} - 2)|v_{k-2}|}{(2m_k + 1)|v_k| + |v_{k-1}| + (m_{k-2} + 2)|v_{k-2}|}. \end{aligned}$$

Since $|v_k| - |v_{k-1}| \geq |u_{k-1}| - |v_{k-1}| = (n_{k-2} - m_{k-2})|v_{k-2}|$, this implies $\frac{3}{2}n_{k-2} - 2m_{k-2} - 2 < 0$ meaning that $n_{k-2} \leq \frac{4}{3}m_{k-2} + 1$, putting us in case (i).

So we may assume from here on that $n_k = 2m_k + 1$. Since $r_k = 0$, if $n_{k-1} \leq 2m_{k-1}$ then we are in case (ii). So we may assume from here on that $n_{k-1} = 2m_{k-1} + a + 1$ for some $a \geq 0$. The above gives that $n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}} < 0$ so $\frac{1}{2}m_{k-1} + a - 1 + \mathbb{1}_{r_{k-1}} < 0$. Then $r_{k-1} = 0$ and $\frac{1}{2}m_{k-1} + a < 1$ meaning that $m_{k-1} = 1$ and $a = 0$ so $n_{k-1} = 3$. By Corollary 3.17 and Lemma 3.19,

$$\begin{aligned} \frac{p(|s_k v_k^{n_k-2} p_k|)}{|s_k v_k^{n_k-2} p_k|} &\geq 1 + \frac{m_k|v_k| + (n_{k-1} - m_{k-1} - 1)|v_{k-1}| + (n_{k-2} - m_{k-2} - 1 + \mathbb{1}_{r_{k-2}})|v_{k-2}|}{(n_k - 2)|v_k| + |p_k| + |s_k|} \\ &> 1 + \frac{m_k|v_k| + |v_{k-1}| + (n_{k-2} - m_{k-2} - 1 + \mathbb{1}_{r_{k-2}})|v_{k-2}|}{(2m_k - 1)|v_k| + (m_{k-2} + 2)|v_{k-2}| + |v_{k-1}| + |v_k|} \\ &= \frac{3}{2} + \frac{\frac{1}{2}|v_{k-1}| + (n_{k-2} - \frac{3}{2}m_{k-2} - 2 + \mathbb{1}_{r_{k-2}})|v_{k-2}|}{2m_k|v_k| + |v_{k-1}| + (m_{k-2} + 2)|v_{k-2}|} \end{aligned}$$

and, since $|v_{k-1}| > m_{k-2}|v_{k-2}|$, therefore $n_{k-2} - m_{k-2} - 2 + \mathbb{1}_{r_{k-2}} < 0$. Then $r_{k-2} = 0$ and $n_{k-2} = m_{k-2} + 1$, putting us in case (iii). ☐

Proposition 3.21. *For $k \geq 2$ such that $r_{k+1} > 0$, $n_k \leq \frac{4}{3}m_k + 1$ and exactly one of the following holds:*

(i) $n_k \leq \frac{3}{2}m_k$;

(ii) $m_k = 3$, $n_k = 5$, $n_{k-1} = m_{k-1} + 1$ *and* $r_k = 0$ *and* $r_{k-1} = 0$;

(iii) $m_k = 1$, $n_k = 2$, $n_{k-1} \leq 2m_{k-1}$ *and* $r_k = 0$; *or*

(iv) $m_k = 1$, $n_k = 2$, $m_{k-1} = 1$, $n_{k-1} = 3$, $n_{k-2} = m_{k-2} + 1$ *and* $r_k = 0$ *and* $r_{k-1} = 0$.

Proof. Let $k \geq 2$ such that $r_{k+1} > 0$. For brevity, we will write

$$P_k := \frac{p(|s_{k+1}v_{k+1}^{r_{k+1}-1}u_{k+1}v_{k+1}^{r_{k+1}-1}p_{k+1}|)}{|s_{k+1}v_{k+1}^{r_{k+1}-1}u_{k+1}v_{k+1}^{r_{k+1}-1}p_{k+1}|}.$$

By Corollary 3.17 and Lemma 3.19,

$$\begin{aligned} P_k &> 1 + \frac{(r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + (n_k - m_k - 1 + \mathbb{1}_{r_k})|v_k|}{2(r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + |v_{k+1}| + (m_k+2)|v_k|} \\ &= \frac{3}{2} + \frac{\frac{1}{2}|u_{k+1}| - \frac{1}{2}|v_{k+1}| + (n_k - \frac{3}{2}m_k - 2 + \mathbb{1}_{r_k})|v_k|}{(2r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + (m_k+2)|v_k|} \end{aligned}$$

and, as $|u_{k+1}| - |v_{k+1}| = (n_k - m_k)|v_k|$, therefore $\frac{3}{2}n_k - 2m_k - 2 + \mathbb{1}_{r_k} < 0$. As n_k is an integer, then $3n_k + 2 \cdot \mathbb{1}_{r_k} \leq 4m_k + 3$ so in particular $n_k \leq \frac{4}{3}m_k + 1$.

Assume that $n_k \geq \frac{3m_k+1}{2}$ as otherwise we are in case (i). Then $\frac{9m_k+3}{2} + 2 \cdot \mathbb{1}_{r_k} \leq 4m_k + 3$ meaning that $\frac{m_k}{2} + 2 \cdot \mathbb{1}_{r_k} \leq \frac{3}{2}$. So $r_k = 0$ and $m_k \leq 3$. The only possibilities for (m_k, n_k) are then $(3,5)$ or $(1,2)$ since $3n_k \leq 4m_k + 3$ (and $(2,3)$ is ruled out by the assumption that $n_k \geq \frac{1}{2}(3m_k+1)$).

We now estimate using Corollary 3.17 again, knowing that $r_k = 0$. By Corollary 3.17 and Lemma 3.19,

$$\begin{aligned} P_k &> 1 + \frac{(r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + (n_k - m_k - 1)|v_k| + (n_{k-1} - m_{k-1} - 1 + \mathbb{1}_{r_{k-1}})|v_{k-1}|}{2(r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + |v_{k+1}| + m_k|v_k| + (m_{k-1}+2)|v_{k-1}|} \\ &= \frac{3}{2} + \frac{\frac{1}{2}|u_{k+1}| - \frac{1}{2}|v_{k+1}| + (n_k - \frac{3}{2}m_k - 1)|v_k| + (n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}})|v_{k-1}|}{(2r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + m_k|v_k| + (m_{k-1}+2)|v_{k-1}|} \end{aligned}$$

and, as $|u_{k+1}| - |v_{k+1}| = (n_k - m_k)|v_k|$, then

$$\left(\frac{3}{2}n_k - 2m_k - 1\right)|v_k| + (n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}})|v_{k-1}| < 0. \tag{†}$$

Consider first when $(m_k, n_k) = (3,5)$. Then (†) gives that $\frac{1}{2}|v_k| + (n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}})|v_{k-1}| < 0$ and, as $|v_k| > m_{k-1}|v_{k-1}|$, then $n_{k-1} - m_{k-1} - 2 + \mathbb{1}_{r_{k-1}} < 0$ meaning $r_{k-1} = 0$ and $n_{k-1} = m_{k-1} + 1$ so we are in case (ii).

Assume from here on that $(m_k, n_k) = (1,2)$. If $n_{k-1} \leq 2m_{k-1}$ then we are in case (iii) so we may also assume $n_{k-1} = 2m_{k-1} + 1 + a$ for some $a \geq 0$. Then (†) gives that $n_{k-1} - \frac{3}{2}m_{k-1} - 2 + \mathbb{1}_{r_{k-1}} < 0$ meaning that that $\frac{1}{2}m_{k-1} + a - 1 + \mathbb{1}_{r_{k-1}} < 0$. Then $r_{k-1} = 0$ and $a = 0$ and $m_{k-1} = 1$ and so $n_{k-1} = 3$. By Corollary 3.17, as $n_k - m_k - 1 = 0$ and $n_{k-1} - m_{k-1} - 1 = 1$ and $r_k = 0$ and $r_{k-1} = 0$, and Lemma 3.19,

$$\begin{aligned} P_k &> 1 + \frac{(r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + |v_{k-1}| + (n_{k-2} - m_{k-2} - 1 + \mathbb{1}_{r_{k-2}})|v_{k-2}|}{2(r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + |v_{k+1}| + |v_k| + |v_{k-1}| + (m_{k-2}+2)|v_{k-2}|} \\ &= \frac{3}{2} + \frac{\frac{1}{2}|u_{k+1}| - \frac{1}{2}|v_{k+1}| - \frac{1}{2}|v_k| + \frac{1}{2}|v_{k-1}| + (n_{k-2} - \frac{3}{2}m_{k-2} - 2 + \mathbb{1}_{r_{k-2}})|v_{k-2}|}{(2r_{k+1}-1)|v_{k+1}| + |u_{k+1}| + |v_k| + |v_{k-1}| + (m_{k-2}+2)|v_{k-2}|} \end{aligned}$$

and, as $|u_{k+1}| - |v_{k+1}| = |v_k|$ and $|v_{k-1}| > m_{k-2}|v_{k-2}|$, then $n_{k-2} - m_{k-2} - 2 + \mathbb{1}_{r_{k-2}} < 0$. Therefore $r_{k-2} = 0$ and $n_{k-2} = m_{k-2} + 1$, putting us in case (iv). □

Remark 3.22. Any specific substitution of the form $\tau_{m,n,r}$ in Corollary 3.3, with parameters compatible with Propositions 3.20 and 3.21, can be used infinitely often in the construction of a subshift with $\limsup \frac{p(q)}{q} < 1.5$. Indeed, preceding that specific substitution by enough substitutions of the form $\tau_{m,m+1,0}$ for appropriate m will provide such a subshift; we do not elaborate further as we do not make use of this.

Remark 3.23. The above reasoning can also be used to show that certain substitutions are ruled out at various complexity cutoffs:

- $\tau_{m,2m+2,0}$ cannot occur when $\limsup \frac{p(q)}{q} < 1.4$;

- $\tau_{n,m,r}$, $r > 0$ cannot occur when $\limsup \frac{p(q)}{q} < \frac{4}{3}$ (c.f. [CP23]); and
- $\tau_{m,2m+1,0}$ cannot occur when $\limsup \frac{p(q)}{q} < 1.25$.

We likewise do not elaborate as we do not make use of this.

Our last fact regarding the substitutive structure is that $\limsup \frac{p(q)}{q} < 1.5$ imposes a bound on $\frac{n_k}{m_k}$. Recall that $\frac{p(q)}{q} < (1.5 - \nu)q$ for all $q \geq |v_0|$.

Proposition 3.24. *There exists $\delta > 0$ and $N \in \mathbb{N}$ such that for all $k \geq 2$ where $m_k \geq N$, if $r_{k+1} > 0$ then $n_k < \frac{3-\delta}{2} m_k$ and if $r_{k+1} = 0$ then $n_k < (2-\delta)m_k$.*

Proof. By Proposition 3.21, if $r_{k+1} > 0$ then $n_k \leq \frac{4}{3}m_k + 1$ so if $m_k \geq 8$ then $\frac{n_k}{m_k} \leq \frac{4}{3} + \frac{1}{8} = \frac{3}{2} - \frac{1}{24}$. Take $\frac{1}{24} > \delta > 0$ and $N \geq 8$ such that $\frac{\delta N + 3}{2(2-\delta)N+2} \leq \nu$.

Suppose that there exists k with $r_k = 0$ and $m_k \geq N$ and $n_k \geq (2-\delta)m_k$. Then, since $\frac{\frac{1}{2}n - m - \frac{3}{2}}{n+1}$ is increasing with n, by Corollary 3.17 and Remark 3.14,

$$\frac{p(|s_k v_k^{n_k-2} p_k|)}{|s_k v_k^{n_k-2} p_k|} > 1 + \frac{(n_k - m_k - 1)|v_k|}{(n_k-2)|v_k| + 3|v_k|} = \frac{3}{2} + \frac{\frac{1}{2}n_k - m_k - \frac{3}{2}}{n_k+1} \geq \frac{3}{2} - \frac{\delta m_k + 3}{2(2-\delta)m_k + 2}.$$

Therefore, as $\frac{\delta m + 3}{2(2-\delta)m+2}$ is decreasing with m, $\frac{p(|s_k v_k^{n_k-2} p_k|)}{|s_k v_k^{n_k-2} p_k|} > \frac{3}{2} - \frac{\delta N + 3}{2(2-\delta)N+2} \geq \frac{3}{2} - \nu$ contradicting that $p(q) < (1.5-\nu)q$ for all $q \geq |v_0|$. □

4 Discrete spectrum

The first consequence we derive from the substitutive structure and inequalities established in Section 3.2 is that infinite minimal low complexity subshifts have (measurably) discrete spectrum.

Theorem 4.1. *Every infinite minimal subshift with* $\limsup \frac{p(q)}{q} < 1.5$ *has discrete spectrum.*

(We remark that finite transitive subshifts have unique measure supported on a periodic orbit, and the same is true for infinite transitive subshifts with $\limsup p(q)/q < 1.5$ by [OP19], and so Theorem 4.1 in fact applies to all transitive subshifts.)

The key ingredient in this proof is the following proposition, which proves exponential decay of a sequence related to the substitutive structure, and which plays the same role in our analysis as exponential decay played in Host's [Hos86] proof of the existence of eigenfunctions for subshifts coming from certain single substitutions.

Proposition 4.2. *Let X be an infinite minimal subshift with* $\limsup \frac{p(q)}{q} < 1.5$*. Let m_k, n_k and r_k be the sequences from Proposition 3.1. Then there exists ϵ_k with $\sum_{k=0}^{\infty} \epsilon_k < \infty$ such that for all k,*

$$\frac{2^{\sum_{j=0}^{k} \mathbb{1}_{r_j}} \prod_{j=0}^{k-1}(n_j - m_j)}{|v_k|} < \epsilon_k.$$

The proof of Proposition 4.2 will first require a few technical lemmas. Throughout this section, let X be an infinite minimal subshift with $\limsup \frac{p(q)}{q} < 1.5$ and u_k and v_k be the words from Proposition 3.1.

We define some auxiliary sequences which will be crucial throughout the remainder of the paper. For all $k \geq 0$, define

$$a_{k+1} = 2^{\mathbb{1}_{r_{k+1}}}(n_k - m_k), \quad b_k = m_k + r_k, \text{ and } a_0 = 2^{\mathbb{1}_{r_0}}. \tag{2}$$

Also set $\beta_k = \frac{a_{k+1}|v_k|}{|v_{k+1}|} > 0$ for $k \geq 0$.

Lemma 4.3. *For all $k \geq 2$, $a_{k+1} \leq b_k + 2$. If $a_{k+1} = b_k + 2$ then $r_{k+1} = 0$ and $n_k = 2m_k + 2$.*

Proof. By Proposition 3.20, $n_k \leq 2m_k + 2$ so if $r_{k+1} = 0$ then $a_{k+1} = n_k - m_k \leq m_k + 2 \leq b_k + 2$.

If $r_{k+1} > 0$ then by Proposition 3.21, either $n_k \leq \frac{3}{2}m_k$ or $(m_k, n_k) = (1, 2)$ or $(m_k, n_k) = (3, 5)$, all of which preclude $a_{k+1} = b_k + 2$. If $n_k \leq \frac{3}{2}m_k$ then $a_{k+1} = 2(n_k - m_k) \leq m_k \leq b_k$. If $(m_k, n_k) = (1, 2)$ then $a_{k+1} = 2(2-1) = 2 = m_k + 1 \leq b_k + 1$. If $(m_k, n_k) = (3, 5)$ then $a_{k+1} = 2(5-3) = 4 = 3+1 \leq b_k + 1$. □

Lemma 4.4. *For all $k \geq 1$,*

$$|v_{k+1}| = b_k|v_k| + a_k|v_{k-1}|.$$

Proof. Since $|u_{k+1}| - |v_{k+1}| = (n_k - m_k)|v_k|$, if $r_k > 0$ then

$$\begin{aligned} |v_{k+1}| &= (m_k + r_k - 2)|v_k| + 2|u_k| = (m_k + r_k)|v_k| + 2(|u_k| - |v_k|) \\ &= (m_k + r_k)|v_k| + 2(n_{k-1} - m_{k-1})|v_{k-1}| = b_k|v_k| + a_k|v_{k-1}| \end{aligned}$$

and if $r_k = 0$ then $|v_{k+1}| = (m_k - 1)|v_k| + |u_k| = m_k|v_k| + (n_{k-1} - m_{k-1})|v_{k-1}| = b_k|v_k| + a_k|v_{k-1}|$. □

Lemma 4.5. *For $k \geq 1$,*

$$\beta_k = \frac{a_{k+1}}{b_k + \beta_{k-1}}.$$

Proof. $\beta_k = \frac{a_{k+1}|v_k|}{b_k|v_k| + a_k|v_{k-1}|} = \frac{a_{k+1}}{b_k + a_k\frac{|v_{k-1}|}{|v_k|}} = \frac{a_{k+1}}{b_k + \beta_{k-1}}$. □

The next several lemmas establish that the β_k or products of them are always less than one, the first step in establishing the desired exponential decay.

Lemma 4.6. *If $k \geq 1$ and $a_{k+1} \leq b_k$ then $\beta_k < 1$.*

Proof. Since $\beta_{k-1} > 0$, by Lemma 4.5, $\beta_k = \frac{a_{k+1}}{b_k + \beta_{k-1}} < \frac{a_{k+1}}{b_k} \leq 1$. □

Lemma 4.7. *If $k \geq 1$ and $a_{k+1} \leq b_k + 1$ and $\beta_{k-1} < 1$ then $\beta_k\beta_{k-1} < 1 - \frac{1-\beta_{k-1}}{2} < 1$.*

Proof. Since $\beta_{k-1} < 1$, $1 - \beta_{k-1} > 0$ and since $\frac{b(1-\beta_{k-1})}{b+1}$ is then increasing with b, by Lemma 4.5,

$$\beta_k\beta_{k-1} = \frac{a_{k+1}\beta_{k-1}}{b_k + \beta_{k-1}} \leq \frac{(b_k + 1)\beta_{k-1}}{b_k + \beta_{k-1}} = 1 - \frac{b_k(1 - \beta_{k-1})}{b_k + \beta_{k-1}} < 1 - \frac{b_k(1 - \beta_{k-1})}{b_k + 1} \leq 1 - \frac{1 - \beta_{k-1}}{2} < 1. \quad \square$$

Lemma 4.8. *If $k \geq 2$ and $n_k = 2m_k+1$ and $n_{k-1} \leq 2m_{k-1}$ then $a_k \leq b_{k-1}$ and $\beta_k\beta_{k-1} < 1-\frac{1-\beta_{k-1}}{2} < 1$.*

Proof. By Proposition 3.21, $r_{k+1} = 0$. By Proposition 3.20 (cases (ii) and (iii)), $r_k = 0$ so by Lemma 4.6, $\beta_{k-1} < 1$. As $a_{k+1} = n_k - m_k = m_k + 1 = b_k + 1$, Lemma 4.7 gives the claim. □

Lemma 4.9. *If $k \geq 2$ and $n_k = 2m_k + 1$ and $n_{k-1} > 2m_{k-1}$ then $a_{k-1} \leq b_{k-2}$ and $\beta_k\beta_{k-1}\beta_{k-2} < 1 - \frac{1-\beta_{k-2}}{2} < 1$.*

Proof. By Proposition 3.21, $r_{k+1} = 0$. By Proposition 3.20 case (iii), $r_k = 0$ and $m_{k-1} = 1$ and $n_{k-1} = 3$ and $r_{k-1} = 0$ and $n_{k-2} = m_{k-2} + 1$. So $a_{k+1} = b_k + 1$ and $a_k = 2$ and $b_{k-1} = 1$ and $a_{k-1} = 1$. By Lemma 4.6, then $\beta_{k-2} < 1$. Then by Lemma 4.5,

$$\beta_k\beta_{k-1}\beta_{k-2} = \frac{b_k + 1}{b_k + \frac{2}{1+\beta_{k-2}}}\frac{2}{1 + \beta_{k-2}}\beta_{k-2} = \frac{(2b_k + 2)\beta_{k-2}}{b_k + b_k\beta_{k-2} + 2} = 1 - \frac{(b_k + 2)(1 - \beta_{k-2})}{b_k + b_k\beta_{k-2} + 2}$$

and since $\beta_{k-2} < 1$ implies $b_k + 2 + b_k\beta_{k-2} < 2b_k + 2 < 2(b_k + 2)$, then $\beta_k\beta_{k-1}\beta_{k-2} < 1 - \frac{1-\beta_{k-2}}{2} < 1$. □

Lemma 4.10. *If $k \geq 2$ and $r_{k+1} > 0$, $n_k > \frac{3}{2}m_k$ and $n_{k-1} \leq 2m_{k-1}$ then $a_k \leq b_{k-1}$ and $\beta_k\beta_{k-1} < 1 - \frac{1-\beta_{k-1}}{2} < 1$.*

Proof. By Proposition 3.21, $r_k = 0$ and either $(m_k, n_k) = (1, 2)$ or $(m_k, n_k) = (3, 5)$. By Lemma 4.6, since $n_{k-1} \le 2m_{k-1}$, then $\beta_{k-1} < 1$. When $m_k = 1, n_k = 2$, we have $a_{k+1} = 2$ and $b_k = 1$ and when $m_k = 3, n_k = 5$, we have $a_{k+1} = 4$ and $b_k = 3$ so Lemma 4.7 gives the claim. □

Lemma 4.11. *If $k \ge 2$ and $r_{k+1} > 0$, $n_k > \frac{3}{2}m_k$ and $n_{k-1} > 2m_{k-1}$ then $a_{k-1} \le b_{k-2}$ and $\beta_k\beta_{k-1}\beta_{k-2} < 1 - \frac{3(1-\beta_{k-2})}{4} < 1$.*

Proof. By Proposition 3.21 case (iv), $m_k = 1$ and $n_k = 2$ and $m_{k-1} = 1$ and $n_{k-1} = 3$ and $n_{k-2} = m_{k-2} + 1$ and $r_k = 0$ and $r_{k-1} = 0$. So $a_{k+1} = 2$ and $b_k = 1$ and $a_k = 2$ and $b_{k-1} = 1$ and, by Lemma 4.6, $\beta_{k-2} < 1$. Then

$$\beta_k\beta_{k-1}\beta_{k-2} = \frac{2}{1 + \frac{2}{1+\beta_{k-2}}} \frac{2}{1 + \beta_{k-2}}\beta_{k-2} = \frac{4\beta_{k-2}}{3 + \beta_{k-2}} = 1 - \frac{3 - 3\beta_{k-2}}{3 + \beta_{k-2}}$$

and since $\beta_{k-2} < 1$ implies $3 + \beta_{k-2} < 4$, we have $\beta_k\beta_{k-1}\beta_{k-2} < 1 - \frac{3(1-\beta_{k-2})}{4} < 1$. □

Lemma 4.12. *If $k \ge 2$ and $n_k = 2m_k + 2$ then $a_{k-1} \le b_{k-2}$ and $\beta_k\beta_{k-1}\beta_{k-2} < 1 - \frac{2(1-\beta_{k-2})}{3} < 1$.*

Proof. By Proposition 3.21, $r_{k+1} = 0$. By Proposition 3.20 case (i), $r_k = 0$ and $r_{k-1} = 0$ and $n_{k-1} = m_{k-1} + 1$, so $a_k = 1$, and $n_{k-2} \le \frac{4}{3}m_{k-2} + 1$ so $n_{k-2} \le 2m_{k-2}$. Therefore $a_{k+1} = b_k + 2$ and $a_k = 1$ and $\beta_{k-2} < 1$ by Lemma 4.6. Observe that

$$\beta_k\beta_{k-1}\beta_{k-2} = \frac{b_k + 2}{b_k + \frac{1}{b_{k-1}+\beta_{k-2}}} \frac{1}{b_{k-1} + \beta_{k-2}}\beta_{k-2} = \frac{(b_k + 2)\beta_{k-2}}{b_{k-1}b_k + b_k\beta_{k-2} + 1}$$

which is decreasing in both b_k and b_{k-1} so

$$\beta_k\beta_{k-1}\beta_{k-2} \le \frac{(b_k + 2)\beta_{k-2}}{b_k + b_k\beta_{k-2} + 1} \le \frac{3\beta_{k-2}}{2 + \beta_{k-2}} = 1 - \frac{2(1 - \beta_{k-2})}{2 + \beta_{k-2}} < 1 - \frac{2(1 - \beta_{k-2})}{3}.$$ □

We now combine all of the above lemmas bounding β_k or products of them by 1 into a single statement.

Lemma 4.13. *For every $k \ge 2$ there exists $0 \le i_k \le 2$ such that $a_{k-i_k+1} \le b_{k-i_k}$ and $\prod_{j=k-i_k}^{k} \beta_j < 1 - \frac{1}{2}(1 - \beta_{k-i_k}) < 1$.*

Proof. For k such that $a_{k+1} \le b_k$, set $i_k = 0$. Lemma 4.6 gives that $\beta_k < 1$. Then $1 - \frac{1}{2}(1 - \beta_{k-i_k}) = \frac{1}{2} + \frac{\beta_k}{2} > \beta_k = \prod_{j=k-i_k}^{k} \beta_j$.

By Lemma 4.3, $a_{k+1} \le b_k + 2$ and they are only equal when $n_k = 2m_k + 2$ and $r_{k+1} = 0$. For k such that $a_{k+1} = b_k + 2$, set $i_k = 2$ and the claim follows from Lemma 4.12.

Let k such that $a_{k+1} = b_k + 1$. Consider first when $r_{k+1} > 0$. Then $2(n_k - m_k) = b_k + 1 \ge m_k + 1$ so $n_k > \frac{3}{2}m_k$. If $n_{k-1} \le 2m_{k-1}$ then set $i_k = 1$ and the claim follows from Lemma 4.10; if $n_{k-1} > 2m_{k-1}$ then set $i_k = 2$ and the claim follows from Lemma 4.11.

Now consider when $r_{k+1} = 0$ so $n_k - m_k = m_k + 1$. If $n_{k-1} \le 2m_{k-1}$ then set $i_k = 1$ and the claim follows from Lemma 4.8; if $n_{k-1} > 2m_{k-1}$ then set $i_k = 2$ and the claim follows from Lemma 4.9. □

Our next pair of lemmas reframes the bound on $\frac{n_k}{m_k}$ established in Proposition 3.24 in terms of a_k, b_k, and β_k.

Lemma 4.14. *For $k \ge 2$, if $a_{k+1} \le b_k$ and $\beta_k \ge 1 - \delta$ and $\beta_{k-1} \ge 1 - \delta$ then $b_k \ge \frac{(1-\delta)^2}{\delta}$.*

Proof. Since $1 - \delta \le \beta_k \le \frac{b_k}{b_k+\beta_{k-1}} \le \frac{b_k}{b_k+1-\delta}$, then $b_k(1 - \delta) + (1 - \delta)^2 \le b_k$ so $(1 - \delta)^2 \le \delta b_k$. □

Lemma 4.15. *There exists $\delta > 0$ such that for $k \ge 2$, if $a_{k+1} \le b_k$ then at least one of $\beta_k < 1 - \delta$ or $\beta_{k-1} < 1 - \delta$.*

Proof. By Proposition 3.24, there exists $\delta_0 > 0$ and N such that for $k \geq 2$ and $m_k \geq N$, if $r_{k+1} > 0$ then $n_k < \frac{3-\delta_0}{2} m_k$ in which case $a_{k+1} = 2(n_k - m_k) < (1-\delta_0)m_k \leq (1-\delta_0)b_k$ and if $r_k = 0$ then $n_k < (2-\delta_0)m_k$ in which case $a_{k+1} = n_k - m_k < (1-\delta_0)m_k \leq (1-\delta_0)b_k$. So for k such that $m_k \geq N$, by Lemma 4.5, $\beta_k = \frac{a_{k+1}}{b_k + \beta_{k-1}} < 1 - \delta_0$.

Let $0 < \delta \leq \delta_0$ such that $\frac{(1-\delta)^2}{2\delta} \geq N$. Let k such that $a_{k+1} \leq b_k$ and $\beta_{k-1} \geq 1-\delta$. By the above, if $m_k \geq N$ then $\beta_k < 1-\delta_0 \leq 1-\delta$. If $m_k < N$ then, as $r_k < m_k$, $b_k < 2m_k < 2N \leq \frac{(1-\delta)^2}{\delta}$ so, by Lemma 4.14, $\beta_k < 1-\delta$. $\square$

We are now ready to prove exponential decay of the β_k, from which Proposition 4.2 quickly follows.

Lemma 4.16. *There exists $0 < \kappa < 1$ and $C > 0$ so that for all k, we have $\prod_{j=0}^{k} \beta_j < C\kappa^k$.*

Proof. By Lemma 4.15, there exists $\delta > 0$ such that for $k \geq 2$, if $a_{k+1} \leq b_k$ then at least one of $\beta_k < 1-\delta$ or $\beta_{k-1} < 1-\delta$. Let $k \geq 3$ such that $\beta_k \geq 1-\delta$. By Lemma 4.13, there exists $0 \leq i_k \leq 2$ such that $\prod_{j=k-i_k}^{k} \beta_j < 1 - \frac{1}{2}(1-\beta_{k-i_k}) < 1$ and $a_{k-i_k+1} \leq b_{k-i_k}$.

If $i_k = 0$ then we have $a_{k+1} \leq b_k$ so $\beta_{k-1} < 1-\delta$ (since $\beta_k \geq 1-\delta$). By Lemma 4.6, we have $\beta_k < 1$ and therefore $\beta_k\beta_{k-1} < 1-\delta$.

If $i_k > 0$ then $a_{k-i_k+1} \leq b_{k-i_k}$ so at least one of $\beta_{k-i_k} < 1-\delta$ or $\beta_{k-i_k-1} < 1-\delta$ holds. If $\beta_{k-i_k} < 1-\delta$ then $\prod_{j=k-i_k}^{k} \beta_j < 1 - \frac{\delta}{2}$ and if $\beta_{k-i_k-1} < 1-\delta$ then, as Lemma 4.6 implies $\beta_{k-i_k} < 1$, we have $\prod_{j=k-i_k-1}^{k} \beta_j < \beta_{k-i_k-1} < 1-\delta$.

So for all $k \geq 5$, there exists $0 \leq i'_k \leq 3$ such that $\prod_{j=k-i'_k}^{k} \beta_j < 1 - \frac{\delta}{2}$. Set $\kappa_0 = 1 - \frac{\delta}{2}$.

Let $C > \max\{\prod_{j=0}^{k} \beta_j \kappa_0^{-k/4} : 0 \leq k \leq 5\}$. Then $\prod_{j=0}^{k} \beta_j < C\kappa_0^{k/4}$ for $0 \leq k \leq 5$.

Assume now that for some $k \geq 6$ we have $\prod_{j=0}^{k'} \beta_j < C\kappa_0^{k'/4}$ for all $k' < k$. Then, since $i'_k \leq 3$,

$$\prod_{j=0}^{k} \beta_j = \Big(\prod_{j=k-i'_k}^{k} \beta_j\Big)\Big(\prod_{j=0}^{k-i'_k-1} \beta_j\Big) < \kappa_0 \cdot C\kappa_0^{(k-i'_k-1)/4} = C\kappa_0^{(k+4-i'_k-1)/4} \leq C\kappa_0^{k/4}$$

so the claim follows by induction and setting $\kappa = \kappa_0^{1/4}$. $\square$

Proof of Proposition 4.2. Since $a_{k+1} = 2^{\mathbb{1}_{r_{k+1}}}(n_k - m_k)$,

$$2^{\sum_{j=0}^{k} \mathbb{1}_{r_j}} \prod_{j=0}^{k-1} (n_j - m_j) = \prod_{j=0}^{k-1} a_{j+1} = \frac{|v_k|}{|v_0|} \prod_{j=0}^{k-1} \frac{a_{j+1}|v_j|}{|v_{j+1}|} = \frac{|v_k|}{|v_0|} \prod_{j=0}^{k-1} \beta_j$$

so by Lemma 4.16, there exists $C > 0$ and $0 < \kappa < 1$ such that for all k,

$$\frac{2^{\sum_{j=0}^{k} \mathbb{1}_{r_j}} \prod_{j=0}^{k-1}(n_j - m_j)}{|v_k|} = \frac{1}{|v_0|} \prod_{j=0}^{k-1} \beta_j < \frac{C}{|v_0|} \kappa^{k-1}$$

meaning $\epsilon_k := \frac{C}{|v_0|}\kappa^{k-1}$ proves the claim. $\square$

The other ingredient needed to prove discrete spectrum is a bound on how much the words $v_k u_k$ and $u_k v_k$ differ.

Lemma 4.17. *For all $k \geq 0$, the words $u_k v_k$ and $v_k u_k$ differ on a number of locations less than*

$$2|u_0| 2^{\sum_{j=0}^{k-1} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1) \cdots (n_{k-1} - m_{k-1}).$$

Proof. Let d be the Hamming distance: the metric defined on pairs of words of the same length by $d(w, x) = |\{0 \leq t < |w| : w_t \neq x_t\}|$. Note that for words w and x of the same length and any words p and s, we have $d(pws, pxs) = d(w, x)$.

Since $|u_0 v_0| < 2|u_0|$, the claim is immediate for $k = 0$. Assume the claim holds for k.

Consider first the case when $r_k = 0$. Then, using the triangle inequality,

$$\begin{aligned}
d(v_{k+1}u_{k+1}, u_{k+1}v_{k+1}) &= d(v_k^{m_k-1}u_k v_k^{n_k-1}u_k, v_k^{n_k-1}u_k v_k^{m_k-1}u_k) \\
&= d(u_k v_k^{n_k-m_k}, v_k^{n_k-m_k}u_k) \\
&\le \sum_{j=0}^{n_k-m_k-1} d(v_k^j u_k v_k^{n_k-m_k-j}, v_k^{j+1}u_k v_k^{n_k-m_k-j-1}) \\
&= \sum_{j=0}^{n_k-m_k-1} d(u_k v_k, v_k u_k) \\
&< (n_k - m_k) 2|u_0| 2^{\sum_{j=0}^{k-1} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1)\cdots(n_{k-1} - m_{k-1}) \\
&= 2|u_0| 2^{\sum_{j=0}^{k} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1)\cdots(n_k - m_k).
\end{aligned}$$

Now consider the case when $r_k > 0$. Here

$$\begin{aligned}
d(v_{k+1}u_{k+1}, u_{k+1}v_{k+1}) &= d(v_k^{m_{k-1}}u_k v_k^{r_k-1}u_k v_k^{n_k-1}u_k v_k^{r_k-1}u_k, v_k^{n_k-1}u_k v_k^{r_k-1}u_k v_k^{m_k-1}u_k v_k^{r_k-1}u_k) \\
&= d(u_k v_k^{r_k-1}u_k v_k^{n_k-m_k}, v_k^{n_k-m_k}u_k v_k^{r_k-1}u_k) \\
&\le \sum_{j=0}^{n_k-m_k-1} \Big(d(v_k^j u_k v_k^{r_k-1}u_k v_k^{n_k-m_k-j}, v_k^j u_k v_k^{r_k}u_k v_k^{n_k-m_k-j-1}) \\
&\qquad + d(v_k^j u_k v_k^{r_k}u_k v_k^{n_k-m_k-j-1}, v_k^{j+1}u_k v_k^{r_k-1}u_k v_k^{n_k-m_k-j-1}) \Big) \\
&= \sum_{j=0}^{n_k-m_k-1} 2d(u_k v_k, v_k u_k) \\
&< 2(n_k - m_k) 2|u_0| 2^{\sum_{j=0}^{k-1} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1)\cdots(n_{k-1} - m_{k-1}) \\
&= 2|u_0| 2^{\sum_{j=0}^{k} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1)\cdots(n_k - m_k).
\end{aligned}$$

Therefore the claim follows by induction. □

We are now in a position to prove discrete spectrum.

Proof of Theorem 4.1. Let X be an infinite minimal subshift with $\limsup \frac{p(q)}{q} < 1.5$. Let v_k and u_k be the words from Proposition 3.1 with corresponding m_k, n_k and r_k.

A subshift (X, σ) is *mean almost periodic* if for all $\epsilon > 0$ and all $x \in X$, there exists a syndetic set S so that for all $s \in S$, x and $\sigma^s x$ differ on a set of locations with upper density[1] less than ϵ. Mean almost periodicity implies discrete spectrum; see e.g. Theorem 2.8 [LS09].

Let $x \in X$. Then x can be written as a bi-infinite concatenation of the words u_k and v_k. Without loss of generality, we may assume that x contains u_{k+1} starting at the origin, since for any i, j, the set of locations where $\sigma^i x$ and $\sigma^j(\sigma^i x)$ differ is just a shift of the set of locations where x and $\sigma^j x$ differ. Then, decomposing x into u_k and v_k, we have

$$\begin{aligned}
x &= \ldots v_k w_1 w_2 \ldots \\
\sigma^{|v_k|} x &= \ldots w_1 w_2 \ldots
\end{aligned}$$

where each $w_j \in \{u_k, v_k\}$.

By definition, x does not contain three consecutive u_k, i.e. there does not exist j so that $w_j = w_{j+1} = w_{j+2} = u_k$. We can then decompose x into blocks of the form $v_k^i u_k^j$, $i > 0$, $j \in \{1, 2\}$, and then each such block corresponds to a block $v_k^{i-1} u_k^j v_k$ (of the same length) within $\sigma^{|v_k|} x$. Therefore, these blocks occur at the same locations in x and $\sigma^{|v_k|} x$, and the set of locations at which x and $\sigma^{|v_k|} x$ differ is the union of

[1] The upper density of $D \subseteq \mathbb{N}$ is $\limsup_{N,M} \frac{1}{N} |D \cap \{M+1, M+2, \ldots, M+N\}|$.

such locations in these pairs of blocks. This number of differences in such a pair is $d(v_k^i u_k^j, v_k^{i-1} u_k^j v_k) = d(v_k u_k^j, u_k^j v_k)$, which is bounded from above by $4|u_0| 2^{\sum_{j=0}^{k-1} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1) \cdots (n_{k-1} - m_{k-1})$ by Lemma 4.17. Since each of u_{k+1} and v_{k+1} contains at most two occurrences of u_k, the density of locations where a u_k starts in x is bounded from above by $\frac{2}{|v_{k+1}|}$. Putting all of this together,

$$\overline{d}(\{t : x_t \neq (\sigma^{|v_k|} x)_t\}) \leq \frac{8|u_0| 2^{\sum_{j=0}^{k-1} \mathbb{1}_{r_j}} (n_0 - m_0)(n_1 - m_1) \cdots (n_{k-1} - m_{k-1})}{|v_{k+1}|}$$

so by Proposition 4.2,

$$\overline{d}(\{t : x_t \neq (\sigma^{|v_k|} x)_t\}) < \frac{8|u_0||v_k|}{|v_{k+1}|} \epsilon_k$$

where $\sum_{k=0}^{\infty} \epsilon_k < \infty$. Let

$$S_k = \Big\{ \sum_{i=k}^{\ell} p_i |v_i| : \ell > k, 0 \leq p_i < \frac{|v_{i+1}|}{|v_i|} \Big\}$$

and observe that for all $t \geq 1$, there exists $0 \leq m \leq |v_k|$ such that $t - m \in S_k$ so S_k is syndetic. Write $D_s = \{t : x_t \neq (\sigma^s x)_t\}$. For $s \in S_k$, by the subadditivity of $\overline{d}$,

$$\overline{d}(D_s) \leq \sum_{i=k}^{\ell} \overline{d}(D_{p_i |v_i|}) \leq \sum_{i=k}^{\ell} p_i \overline{d}(D_{|v_i|}) < \sum_{i=k}^{\ell} \frac{|v_{i+1}|}{|v_i|} \frac{8|u_0||v_i|}{|v_{i+1}|} \epsilon_i \leq 8|u_0| \sum_{i=k}^{\infty} \epsilon_i.$$

Since $\sum \epsilon_k < \infty$, then $\lim_k \sup_{s \in S_k} \overline{d}(D_s) = 0$ so X is mean almost periodic, and therefore has discrete spectrum. ☐

5 The additive eigenvalue group

In this section, we explicitly compute the additive continuous eigenvalue group for low complexity minimal subshifts in terms of the a_k and b_k defined in Section 4, which is the first step in characterizing the maximal equicontinuous factor. The main tools are the exponential decay already established and approximation arguments along a similar line of reasoning as in [CP23], though more complex.

Throughout this section, let X be an infinite minimal subshift with $\limsup p(q)/q < 1.5$, which therefore satisfies the conclusions of Propositions 3.1, 3.20, 3.21 and 4.2.

Let u_k and v_k be the words from Proposition 3.1 and (a_k) and (b_k) as in (2). Any reference to measure refers to the unique σ-invariant measure μ. By minimality, the measure of any nonempty open set is positive.

We first introduce the following notation for subgroups of $(\mathbb{Q}, +)$.

Definition 5.1. Let $0 \leq \ell_p \leq \infty$ for each prime $p \in \mathbb{P}$. The (ℓ_p)-subgroup of $\mathbb{Q}$ is

$$Q_{(\ell_p)} = \{q \in \mathbb{Q} : \exists p_1, \ldots, p_t \in \mathbb{P} \text{ such that } p_1 \cdots p_t q \in \mathbb{Z} \text{ and } |\{1 \leq i \leq t : p_i = p\}| \leq \ell_p \text{ for all } p \in \mathbb{P}\}$$

That $Q_{(\ell_p)}$ is a group under addition is easily verified.

The purpose of this section is to prove the following explicit description of the eigenvalue group. Our description requires introducing the following standard notation.

Notation 5.2. For a prime p and $a \in \mathbb{Q}_p$ a p-adic number, the ***p*-adic fractional part** is

$$\{a\}_p = \sum_{t=-m}^{-1} a_t p^t$$

where $a = \sum_{t=-m}^{\infty} a_t p^t$ is the p-adic expansion of a_p.

Note that $\{a + a'\}_p = \{a\}_p + \{a'\}_p \pmod{\mathbb{Z}}$ and that for $q \in \mathbb{Q}$, $q = \sum_p \{q\}_p \pmod{\mathbb{Z}}$.

Theorem 5.3. *For each prime $p \in \mathbb{P}$, let*

$$L_X(p) = \sup\{t \geq 0 : p^t \text{ divides } \frac{|v_0|a_0 \cdots a_k}{\gcd(|v_k|, |v_{k+1}|)} \text{ for some } k \geq 0\}$$
$$R_X(p) = \sup\{t \geq 0 : p^t \text{ divides } \gcd(|v_k|, |v_{k+1}|) \text{ for some } k \geq 0\}$$

and let Q_X be the $(L_X(p))$-subgroup of $\mathbb{Q}$ and R_X be the $(R_X(p))$-subgroup of $\mathbb{Q}$. Let

$$\alpha = \frac{\lambda}{|u_0|\lambda + |v_0|(1-\lambda)} \quad \text{where} \quad \lambda = \frac{a_0}{b_0 + \frac{a_1}{b_1 + \cdots}}.$$

Then there exist $e_p \in \mathbb{Q}_p$ for each prime p such that

$$E_X = \left\{ q\alpha + \sum_p \{qe_p\}_p + r : q \in Q_X, r \in R_X \right\}.$$

In addition, all measurable eigenfunctions are continuous.

Before proceeding, we establish that the $L_X(p)$ are integers.

Lemma 5.4. *For all $k \geq 0$, $\gcd(|v_k|, |v_{k+1}|)$ divides $|v_0|a_0 \cdots a_k$.*

Proof. Set $g_0 = \gcd(|v_0|, |u_0| - |v_0|)$ and $g_k = \gcd(|v_k|, |v_{k-1}|)$ for $k \geq 1$. Then g_0 divides $|v_0|$ and $g_{k+1} = g_k \gcd(b_k \frac{|v_k|}{g_k} + a_k \frac{|v_{k-1}|}{g_k}, \frac{|v_k|}{g_k}) = g_k \gcd(a_k \frac{|v_{k-1}|}{g_k}, \frac{|v_k|}{g_k})$ and since $\gcd(\frac{|v_k|}{g_k}, \frac{|v_{k-1}|}{g_k}) = 1$, then g_{k+1} divides $g_k a_k$ so by induction g_{k+1} divides $|v_0|a_0 \cdots a_k$ for all k. □

5.1 Additive continuous eigenvalues

Our first step is establishing the existence of a family of irrational additive continuous eigenvalues, all of which are explicitly defined in terms of generalized continued fractions using a_k and b_k.

Proposition 5.5. *λ and α as defined in Theorem 5.3 are irrational.*

Proof. Suppose that $\lambda \in \mathbb{Q}$ so $0 < \lambda = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $p, q > 0$. Define the sequence (p_k) by $p_{-2} = q$ and $p_{-1} = p$ and for $k \geq -1$, $p_{k+1} = -b_{k+1}p_k + a_{k+1}p_{k-1}$. By construction, $\lambda = \lambda_0 = \frac{p_{-1}}{p_{-2}}$. Assume that $\lambda_{k+1} = \frac{p_k}{p_{k-1}}$. Then, since $\lambda_{k+1} = \frac{a_{k+1}}{b_{k+1} + \frac{a_{k+2}}{b_{k+2} + \cdots}} = \frac{a_{k+1}}{b_{k+1} + \lambda_{k+2}}$,

$$\frac{p_{k+1}}{p_k} = -b_{k+1} + a_{k+1}\frac{p_{k-1}}{p_k} = -b_{k+1} + \frac{a_{k+1}}{\lambda_{k+1}} = \lambda_{k+2}$$

so by induction, $\lambda_k = \frac{p_k}{p_{k-1}}$ for all k. In particular, $p_k > 0$ for all k since $\lambda_k > 0$ and $p_{-1} > 0$ so if there were a minimal k such that $p_k < 0$ then that $\lambda_k < 0$).

Now observe that

$$\begin{aligned} p_{k+1}|v_{k+1}| + p_k|v_{k+2}| &= -b_{k+1}p_k|v_{k+1}| + a_{k+1}p_{k-1}|v_{k+1}| + p_k b_{k+1}|v_{k+1}| + p_k a_{k+1}|v_k| \\ &= a_{k+1}(p_k|v_k| + p_{k-1}|v_{k+1}|) \end{aligned}$$

and since $p_{-1}(|u_0| - |v_0|) + p_{-2}|v_0| = p|u_0| + (q-p)|v_0|$, by induction then

$$p_{k+1}|v_{k+1}| + p_k|v_{k+2}| = (p|u_0| + (q-p)|v_0|)a_0 a_1 \cdots a_{k+1} < (p|u_0| + (q-p)|v_0|)\epsilon_{k+1}|v_{k+1}|$$

where ϵ_{k+1} is as in Proposition 4.2. Since $p_k, p_{k+1} \geq 1$,

$$|v_{k+1}| < p_{k+1}|v_{k+1}| + p_k|v_{k+2}| < (p|u_0| + (q-p)|v_0|)\epsilon_{k+1}|v_{k+1}|$$

but then

$$\frac{1}{p|u_0| + (q-p)|v_0|} < \epsilon_{k+1} \to 0$$

which is impossible. Therefore $\lambda \notin \mathbb{Q}$ hence $\alpha \notin \mathbb{Q}$ (as $|u_0| > |v_0|$). □

Definition 5.6. For all $k \geq 0$, define

$$\lambda_k = \frac{a_k}{b_k + \frac{a_{k+1}}{b_{k+1}+\cdots}} \text{ and } \alpha_k = \frac{1}{|v_k| + |v_{k-1}|\lambda_k}$$

(where $|v_{-1}|$ is defined as $|u_0| - |v_0|$, obtained by reversing the recursion from Lemma 4.4).

The **eigenvalue family** is the set $\{\alpha_k\}_{k\geq 0}$.

We adapt the argument of Host [Hos86] using 'approximate eigenfunctions' and deduce convergence to an actual eigenfunction from the exponential decay of Proposition 4.2.

Proposition 5.7. *For $k \geq 0$, α_k is an additive continuous eigenvalue.*

Proof. Fix $k_0 \geq 0$ and let $k \geq k_0$. Every $x \in X$ can be written in a unique way as a concatenation of u_k and v_k; we will refer to such as a k-concatenation. Let B_k be the set of $x \in X$ such that x has u_k or v_k at the origin when written as a k-concatenation. Let $j(x,k)$ be the minimal $j \geq 0$ such that $\sigma^{-j}x \in B_k$.

Consider x which has u_k at the origin. If $r_k = 0$ then one of $\sigma^{-(m_k-1)|v_k|}x$ or $\sigma^{-(n_k-1)|v_k|}x$ is in B_{k+1}. If $r_k > 0$ then one of $\sigma^{-(m_k+r_k-2)|v_k|-|u_k|}x$ or $\sigma^{-(n_k+r_k-2)|v_k|-|u_k|}x$ is in B_{k+1}. For x that has v_k at the origin, there exists $1 \leq p < n_k + r_k$ such that $\sigma^{-p|v_k|}x$ or $\sigma^{-p|v_k|-|u_k|}x$ is in B_{k+1}. Therefore for every $x \in X$, we have $j(x,k+1) - j(x,k) = p|v_k|$ or $j(x,k+1) - j(x,k) = p|v_k| + |u_k|$ for some $1 \leq p < n_k + r_k$.

Since $|u_k| = |v_k| + (n_{k-1} - m_{k-1})|v_{k-1}|$, in the latter case, $j(x,k+1) - j(x,k) = (p+1)|v_k| + (n_{k-1} - m_{k-1})|v_{k-1}|$. Therefore

$$j(x,k+1) - j(x,k) = p|v_k| + p'|v_{k-1}| \tag{‡}$$

for some $1 \leq p \leq n_k + r_k$ and $p' = 0$ or $p' = n_{k-1} - m_{k-1} < n_{k-1}$.

Let $f_k(x) = \exp(2\pi i \alpha_{k_0} j(x,k))$. Each f_k is 'approximately' an eigenfunction: $f_k(\sigma x) = \exp(2\pi i \alpha_{k_0}) f_k(x)$ except when $\sigma x \in B_k$ and $\mu(B_k) \to 0$ (since $\sigma^i B_k$ are disjoint for at least $0 \leq i < |v_k|$ and $|v_k| \to \infty$). Observe that

$$\begin{aligned}
|f_k(x) - f_{k+1}(x)| &= |\exp(2\pi i \alpha_{k_0} j(x,k)) - \exp(2\pi i \alpha_{k_0} j(x,k+1))| \\
&= |\exp(2\pi i \alpha_{k_0} j(x,k))(1 - \exp(2\pi i \alpha_{k_0}(p|v_k| + p'|v_{k-1}|))| \\
&= |1 - \exp(2\pi i \alpha_{k_0}(p|v_k| + p'|v_{k-1}|)| \\
&\leq |1 - \exp(2\pi i \alpha_{k_0} p|v_k|)| + |\exp(2\pi i \alpha_{k_0} p|v_k|) - \exp(2\pi i \alpha_{k_0}(p|v_k| + p'|v_{k-1}|)| \\
&= |1 - \exp(2\pi i \alpha_{k_0} p|v_k|)| + |1 - \exp(2\pi i \alpha_{k_0} p'|v_{k-1}|)| \\
&\leq 2\pi \langle \alpha_{k_0} p|v_k| \rangle + 2\pi \langle \alpha_{k_0} p'|v_{k-1}| \rangle.
\end{aligned}$$

Suppose we knew that there exist $\epsilon'_k > 0$ with $\sum \epsilon'_k < \infty$ such that $\max_{1\leq p\leq n_k+r_k} \langle \alpha_{k_0} p|v_k| \rangle < \epsilon'_k$. Since $\langle \alpha_{k_0} p'|v_{k-1}| \rangle = 0$ when $p' = 0$ and $\langle \alpha_{k_0} p'|v_{k-1}| \rangle < \epsilon'_{k-1}$ when $p' > 0$, then we would have $\sum_{k=K}^{\infty} |f_{k+1}(x) - f_k(x)| < \sum_{k=K}^{\infty} (\epsilon'_k + \epsilon'_{k-1})$ which tends to zero uniformly over $x \in X$. So then the $f_k(x)$ are uniformly Cauchy in the sup norm, and as each $f_k(x)$ is continuous, they converge to a continuous limit $f(x)$. Since $f_k(\sigma x) = \exp(2\pi i \alpha_{k_0}) f_k(x)$ on sets approaching full measure (and the unique invariant measure necessarily has full support), by continuity $f(\sigma x) = \exp(2\pi i \alpha_{k_0}) f(x)$ for all x. We will now show that such ϵ'_k exist.

Set $d_k = |v_{k+k_0+1}|$ for $k \geq -2$ (if $k_0 = 0$ then set $d_{-2} = |u_0| - |v_0|$ as v_{-1} is undefined). Then $d_{k+1} = b_{k+k_0+1} d_k + a_{k+k_0+1} d_{k-1}$. Define sequences (c_k) and (e_k) by $c_{-2} = 1$, $c_{-1} = 0$, $e_{-2} = 0$, $e_{-1} = 1$ and the same recursion relation $c_{k+1} = b_{k+k_0+1} c_k + a_{k+k_0+1} c_{k-1}$ and $e_{k+1} = b_{k+k_0+1} e_k + a_{k+k_0+1} e_{k-1}$. Standard continued fraction theory shows that $\frac{c_k}{e_k} \to \lambda_{k_0}$. Since the sequences are all defined by the same linear recurrence relation, $d_k = d_{-2} c_k + d_{-1} e_k$ for all k. Then

$$\lim \frac{e_k}{d_k} = \lim \left(d_{-1} + d_{-2} \frac{c_k}{e_k} \right)^{-1} = (d_{-1} + d_{-2} \lambda_{k_0})^{-1} = \frac{1}{d_{-1} + d_{-2} \lambda_{k_0}} = \alpha_{k_0}$$

It is easily verified by induction that $e_{k+1}d_k - e_k d_{k+1} = (-1)^k |v_{k_0-1}| a_{k_0} a_{k_0+1} \cdots a_{k_0+k+1}$ for all k. Since $e_{k+1}d_k - e_k d_{k+1}$ alternates sign, $\frac{e_{2k}}{d_{2k}}$ approaches α_{k_0} from below and $\frac{e_{2k+1}}{d_{2k+1}}$ approaches α_{k_0} from above. Therefore

$$\left|\alpha_{k_0} - \frac{e_k}{d_k}\right| < \left|\frac{e_{k+1}}{d_{k+1}} - \frac{e_k}{d_k}\right| = \frac{|v_{k_0-1}| a_{k_0} \cdots a_{k_0+k+1}}{d_k d_{k+1}}$$

Then $|\alpha_{k_0} d_k - e_k| < \frac{|v_{k_0-1}| a_{k_0} \cdots a_{k_0+k+1}}{d_{k+1}}$ so for any p,

$$\langle p \alpha_{k_0} d_k \rangle < \frac{p |v_{k_0-1}| a_{k_0} \cdots a_{k+k_0+1}}{d_{k+1}}$$

By Proposition 4.2, there exists ϵ_k with $\sum \epsilon_k < \infty$ such that $\frac{a_0 \cdots a_{k+k_0+1}}{d_k} < \epsilon_{k+k_0}$.

For $p \leq n_{k+1} + r_{k+1} \leq 2m_{k+1} + 2 + r_{k+1} \leq 2b_{k+1} + 2$, we have $\frac{pd_k}{d_{k+1}} \leq \frac{(2b_{k+1}+2)d_k}{b_{k+1}d_k} \leq 4$. Then

$$\langle p \alpha_{k_0} d_k \rangle < \frac{pd_k |v_{k_0-1}| \epsilon_{k+k_0}}{d_{k+1} a_0 \cdots a_{k_0-1}} \leq 4 \frac{|v_{k_0-1}|}{a_0 \cdots a_{k_0-1}} \epsilon_{k+k_0}.$$

Setting $\epsilon'_k = 4|v_{k_0-1}|(a_0 \cdots a_{k_0-1})^{-1} \epsilon_{k+k_0}$ completes the proof. □

Corollary 5.8. *α is an additive continuous eigenvalue.*

Proof. Since $\lambda = \frac{a_0}{b_0 + \lambda_1}$,

$$\begin{aligned} \alpha &= \frac{\lambda}{|v_0| + (|u_0| - |v_0|)\lambda} = \frac{1}{\lambda^{-1}|v_0| + |u_0| - |v_0|} = \frac{1}{\frac{b_0+\lambda_1}{a_0}|v_0| + |u_0| - |v_0|} \\ &= \frac{a_0}{b_0|v_0| + \lambda_1|v_0| + a_0(|u_0| - |v_0|)} = \frac{a_0}{|v_1| + \lambda_1|v_0|} = a_0 \alpha_1. \end{aligned}$$ □

By Proposition 5.7, α_1 is a continuous additive eigenvalue so α is as well.

Corollary 5.9. *There is a continuous factor map $(X, \sigma) \to (S^1, R_\alpha)$ where R_α denotes rotation by $\exp(2\pi i \alpha)$. The same holds for (S^1, R_{α_k}) for each k.*

Proof. Let $f_\alpha : X \to S^1$ be a continuous eigenfunction for $\exp(2\pi i\alpha)$. Then $f_\alpha(\sigma x) = \exp(2\pi i\alpha) f_\alpha(x)$ so f_α is the factor map. The same reasoning applies to α_k. □

Next we prove that every element of $Q_X \alpha$ is, up to a rational, an element of the additive continuous eigenvalue group.

Proposition 5.10. *For all $q \in Q_X$ there exists $r_q \in \mathbb{Q}$ such that $q\alpha + r_q \in E_X$.*

Proof. Since $\lambda_k = \frac{a_k}{b_k + \lambda_{k+1}}$, we have that $\lambda_{k+1} = a_k \lambda_k^{-1} - b_k$. Therefore, for $k \geq 1$,

$$\alpha_{k+1} = \frac{1}{|v_{k+1}| + |v_k|\lambda_{k+1}} = \frac{1}{b_k|v_k| + a_k|v_{k-1}| + a_k|v_k|\lambda_k^{-1} - b_k|v_k|} = \frac{\lambda_k}{a_k(|v_k| + |v_{k-1}|\lambda_k)} = \frac{\alpha_k \lambda_k}{a_k}.$$

We claim now that $\alpha_k \lambda_k = (-1)^k \frac{|v_k|}{|v_0| a_0 \cdots a_{k-1}} \alpha + r_k$ for some $r_k \in \mathbb{Q}$. Clearly $\alpha_0 \lambda_0 = \alpha = \frac{|v_0|}{|v_0|}\alpha + 0$. Observe that

$$\alpha_1 \lambda_1 = \frac{\lambda_1}{|v_1| + |v_0|\lambda_1} = \frac{a_0 \lambda^{-1} - b_0}{b_0|v_0| + a_0(|u_0| - |v_0|) + a_0\lambda^{-1}|v_0| - b_0|v_0|} = \frac{a_0 - b_0\lambda}{a_0(|v_0| + (|u_0| - |v_0|)\lambda)}$$

so we have

$$\alpha_1 \lambda_1 + \frac{|v_1|}{|v_0| a_0}\alpha - \frac{1}{|v_0|} = \frac{|v_0|(a_0 - b_0\lambda) + |v_1|\lambda - a_0(|v_0| + (|u_0| - |v_0|)\lambda)}{a_0|v_0|(|v_0| + (|u_0| - |v_0|)\lambda)}$$

$$= \frac{-b_0\lambda|v_0| + b_0\lambda|v_0| + a_0\lambda(|u_0| - |v_0|) - a_0(|u_0| - |v_0|)\lambda}{a_0|v_0|(|v_0| + (|u_0| - |v_0|)\lambda)} = 0.$$

Assume that $\alpha_k\lambda_k = (-1)^k \frac{|v_k|}{|v_0|a_0\cdots a_{k-1}}\alpha + r_k$ and likewise for $k-1$. Then

$$\begin{aligned}
\alpha_{k+1}\lambda_{k+1} &= \frac{a_k\lambda_k^{-1} - b_k}{b_k|v_k| + a_k|v_{k-1}| + a_k|v_k|\lambda_k^{-1} - b_k|v_k|} = \frac{a_k - b_k\lambda_k}{a_k(|v_k| + |v_{k-1}|\lambda_k)} = \alpha_k - \frac{b_k\alpha_k\lambda_k}{a_k} \\
&= \frac{\alpha_{k-1}\lambda_{k-1}}{a_{k-1}} - \frac{b_k\alpha_k\lambda_k}{a_k} = (-1)^{k-1}\frac{|v_{k-1}|}{|v_0|a_0\cdots a_{k-1}}\alpha + \frac{r_{k-1}}{a_{k-1}} - (-1)^k\frac{b_k|v_k|}{|v_0|a_0\cdots a_k}\alpha - \frac{b_k r_k}{a_k} \\
&= (-1)^{k+1}\frac{a_k|v_{k-1}| + b_k|v_k|}{|v_0|a_0\cdots a_k}\alpha + \frac{r_{k-1}}{a_{k-1}} - \frac{b_k r_k}{a_k} = (-1)^{k+1}\frac{|v_{k+1}|}{|v_0|a_0\cdots a_k}\alpha + \frac{r_{k-1}}{a_{k-1}} - \frac{b_k r_k}{a_k}
\end{aligned}$$

so by induction, the claim holds. Then

$$\alpha_{k+1} = \frac{\alpha_k\lambda_k}{a_k} = (-1)^k\frac{|v_k|}{|v_0|a_0\cdots a_k}\alpha + \frac{r_k}{a_k}. \qquad \square$$

We now prove that all rationals with denominator an eventual common divisor of $|u_k|$ and $|v_k|$ are additive continuous eigenvalues.

Proposition 5.11. *A rational number m/n is an additive continuous eigenvalue if n eventually divides the lengths of both u_k and v_k, equivalently the lengths of both v_k and v_{k+1}.*

Proof. Assume that n divides the length of u_k and v_k for some k. Let B be the clopen set of $x \in X$ such that as a k-concatenation, $\sigma^{sn}x$ has v_k or u_k at the origin for some integer s. Then $\sigma^n B = B$, and so $e^{2\pi i/n}$ has continuous eigenfunction $\sum_{k=0}^{n-1}\chi_{\sigma^k B}e^{2\pi ik/n}$. Therefore n^{-1} is an additive continuous eigenvalue so m/n also is. $\square$

Proposition 5.12. *The group of additive continuous eigenvalues E_X contains $\{q\alpha + r_q + r \ : \ q \in Q_X, r \in R_X\}$.*

Proof. This is an immediate consequence of Propositions 5.10 and 5.11. $\square$

5.2 Additive measurable eigenvalues

We now prove that every additive measurable eigenvalue is contained in $Q_X\alpha + \mathbb{Q}$.

Lemma 5.13. *Define Rokhlin towers by, setting u'_k such that $u_k = u'_k v_k$,*

$$\begin{aligned}
B_k &= \{x \in X : x \text{ as a } (k+1)\text{-concatenation has } v_{k+1} \text{ at the origin, possibly as a suffix of } u_{k+1}\}, \\
B'_k &= \{x \in X : x \text{ as a } (k+1)\text{-concatenation has } u'_{k+1} \text{ at the origin, as a prefix of } u_{k+1}\},
\end{aligned}$$

and $T_k = \bigsqcup_{j=0}^{|v_{k+1}|-1}\sigma^j B_k$ and $T'_k = \bigsqcup_{j=0}^{|u_{k+1}|-|v_{k+1}|-1}\sigma^j B'_k$.
Then for all k, $T_k \sqcup T'_k = X$ and $\mu(T_k) \geq \frac{1}{4}$.

Proof. Every $x \in X$ is uniquely decomposable as a concatenation of u_{k+1} and v_{k+1} hence of u'_{k+1} and v_{k+1} so the levels of the towers are disjoint and union to the entire space. Since $n_{k+1} \leq 2m_{k+1} + 2$, $|u_{k+1}| - |v_{k+1}| = (n_k - m_k)|v_k| \leq (m_k + 2)|v_k|$ and $|v_{k+1}| = (m_k - 1)|v_k| + |u_k| > m_k|v_k|$, then $\frac{|u_{k+1}|-|v_{k+1}|}{|v_{k+1}|} < \frac{m_k+2}{m_k} \leq 3$. Therefore $\mu(T_k) \geq \frac{1}{4}\mu(T_k \sqcup T'_k)$. $\square$

Proposition 5.14. *Let γ be an additive measurable eigenvalue. Then there exists $q \in Q_X$ and $r \in \mathbb{Q}$ such that $\gamma = q\alpha + r$.*

Proof. Let f be a measurable eigenfunction with eigenvalue $\exp(2\pi i\gamma)$. Let B_k and B'_k as in Lemma 5.13. For each k, define

$$f_k(x) = \sum_{j=0}^{|v_{k+1}|-1} \frac{1}{\mu(B_k)} \left(\int_{\sigma^j B_k} f \, d\mu\right) \mathbb{1}_{\sigma^j B_k}(x) + \sum_{j=0}^{|u_{k+1}|-|v_{k+1}|-1} \frac{1}{\mu(B'_k)} \left(\int_{\sigma^j B'_k} f \, d\mu\right) \mathbb{1}_{\sigma^j B'_k}(x).$$

Let $\mathcal{F}_k$ be the σ-algebra generated by the sets $\sigma^j B_k$, $0 \leq j < |v_{k+1}|$, and $\sigma^j B'_k$, $0 \leq j < |u'_{k+1}|$. Since $\mu(T_k \sqcup T'_k) = 1$ and $\mu(B_k), \mu(B'_k) \to 0$ (since $|u_{k+1}| - |v_{k+1}| \geq |v_k| \to \infty$), the σ-algebras $\mathcal{F}_k$ converge to the σ-algebra of all measurable sets. Since each f_k is $\mathcal{F}_k$-measurable and $\mathbb{E}[f|\mathcal{F}_k] = \mathbb{E}[f_{k+1}|\mathcal{F}_k] = f_k$, by the Martingale Convergence Theorem, f_k converge almost everywhere to f.

For all $t \geq 0$, we have that $\sigma^{|v_{k+t+1}|}$ takes every occurrence of v_{k+t+1} to an occurrence of v_{k+t+1} except those immediately followed by an occurrence of u_{k+t+1}. Therefore $\sigma^{|v_{k+t+1}|}$ takes every occurrence of v_{k+1} in a v_{k+t+1} to an occurrence of v_{k+1} except for those in a v_{k+t+1} immediately followed by a u_{k+t+1}. Likewise, for $0 < i_{k+t+1}$, $\sigma^{i_{k+t+1}|v_{k+t+1}|}$ takes every occurrence of a v_{k+1} in a v_{k+t+1} to an occurrence of v_{k+1} except for those in a v_{k+t+1} less than i_{k+t+1} words prior to a u_{k+t+1}.

For any $(k+t+2)$-concatenation, since u_{k+t+2} has v_{k+t+2} as a suffix, the concatenation is a concatenation of $v_{k+t+1}^{m_{k+t+1}} u'_{k+t+1}$ and $v_{k+t+1}^{n_{k+t+1}} u'_{k+t+1}$ and, if $r_{k+t+1} > 0$, $v_{k+t+1}^{r_{k+t+1}} u'_{k+t+1}$ where u'_{k+t+1} is the prefix of u_{k+t+1} such that $u_{k+t+1} = u' v_{k+t+1}$. Since $n_{k+t+2} \leq 2m_{k+t+2} + 2$, then $|u'_{k+t+1}| \leq |v_{k+t+1}| + 2|v_{k+t}| < 3|v_{k+t+1}|$ so at least $\frac{1}{4}$ of the v_{k+1} appearing in a $(k+t+1)$-concatenation are in a v_{k+t+1}.

Let $\{i_k\}$ such that $0 < i_{k+t} \leq \max(1, 0.5b_{k+t})$. Write $d_k = |v_{k+1}|$. For $k+t$ such that $b_{k+t} > 1$, then

$$\mu(\sigma^{i_{k+t}d_{k+t}} B_k \cap B_k) \geq \frac{b_{k+t} - i_{k+t}}{b_{k+t}} \left(\frac{1}{4}\mu(B_k)\right) \geq \frac{1}{8}\mu(B_k).$$

For k such that $b_{k+t} = 1$, meaning $r_{k+t} = 0$ and $m_{k+t} = 1$, we have that $\sigma^{|v_{k+t+1}|} = \sigma^{|u_{k+t}|}$ takes every occurrence of v_{k+t} which precedes a u_{k+t} to the v_{k+t} which is a suffix of that u_{k+t}. Since $n_{k+t} \leq 4$, at least $\frac{1}{4}$ of the words in a $(k+t+1)$-concatenation are u_{k+t} so at least $\frac{1}{4}$ of the v_{k+t} are taken to a v_{k+t} by $\sigma^{|v_{k+t+1}|}$ (since u_{k+t} is always preceded by v_{k+t}, possibly as a suffix of another u_{k+t}). Then,

$$\mu(\sigma^{d_{k+t}} B_k \cap B_k) \geq \frac{1}{4}\mu(B_k).$$

Then $f_k(\sigma^{i_{k+t}d_{k+t}}x) = f_k(x)$ for a set of measure at least $\frac{1}{8}\mu(T_k) \geq \frac{1}{32}$. Since $f_k \to f$ almost everywhere, there is then a positive measure set such that for any sufficiently small $\epsilon > 0$ and almost every x in the set, there exists k so that for all t, $|f(\sigma^{i_{k+t}d_{k+t}}x) - f(x)| < \epsilon$. Therefore $\exp(2\pi i\gamma i_k d_k) \to 1$.

For large enough k (say $k \geq k_0$), $\langle i d_k \gamma \rangle < \frac{1}{160}$ for all $0 < i \leq \max(1, 0.5b_{k+1})$. Suppose that for all $c \in \mathbb{Z}$, we have $|c - d_k\gamma| \geq 0.05\frac{d_k}{d_{k+1}} \geq 0.05(2b_{k+1}+2)^{-1}$ (using that $d_{k+1} = b_{k+1}d_k + a_{k+1}d_{k-1} \leq b_{k+1}d_k + (b_{k+1}+2)d_{k-1}$). Then $|\max(1, \lfloor 0.5b_{k+1} \rfloor)c - \max(1, \lfloor 0.5b_{k+1} \rfloor)d_k\gamma| \geq 0.025\frac{\max(1, \lfloor 0.5(b_{k+1} \rfloor)}{b_{k+1}+1} \geq \frac{1}{160}$, a contradiction. This implies that for all $k \geq k_0$, there exists $c'_k \in \mathbb{Z}$, so that $\left|\gamma - \frac{c'_k}{d_k}\right| < 0.05(d_{k+1})^{-1}$.

We will prove that $c'_{k+1} = b_{k+1}c'_k + a_{k+1}c'_{k-1}$ for all $k > k_0$.

For $k > k_0$, let $c''_{k+1} = b_{k+1}c'_k + a_{k+1}c'_{k-1}$. By the above,

$$\left|\gamma - \frac{c'_{k-1}}{d_{k-1}}\right| < 0.05(d_k)^{-1} \text{ and } \left|\gamma - \frac{c'_k}{d_k}\right| < 0.05(d_{k+1})^{-1} \text{ so } \left|d_{k+1}\gamma - c'_k\frac{d_{k+1}}{d_k}\right| < 0.05.$$

Since $|d_{k-1}\gamma - c'_{k-1}| < 0.05d_{k-1}(d_k)^{-1}$,

$$\left|a_{k+1}d_{k-1}\gamma - a_{k+1}c'_{k-1}\right| < \frac{0.05a_{k+1}d_{k-1}}{d_k} \leq \frac{0.05(b_k+2)d_{k-1}}{d_k} < 0.05(3) = 0.15.$$

Similarly, since $|d_k\gamma - c'_k| < 0.05d_k(d_{k+1})^{-1}$,

$$\left|a_{k+1}d_{k-1}\gamma - c'_k a_{k+1}\frac{d_{k-1}}{d_k}\right| = \left|a_{k+1}d_{k-1}\left(\gamma - \frac{c'_{k-1}}{d_k}\right)\right| < \frac{0.05a_{k+1}d_{k-1}}{d_{k+1}} \leq \frac{0.05(b_k+2)d_{k-1}}{d_{k+1}} < 0.15.$$

Therefore,

$$\begin{aligned}\left|c'_k\frac{d_{k+1}}{d_k} - c''_{k+1}\right| &= \left|c'_k\left(b_{k+1} + \frac{a_{k+1}d_{k-1}}{d_k}\right) - b_{k+1}c'_k - a_{k+1}c'_{k-1}\right| = \left|c'_k a_{k+1}\frac{d_{k-1}}{d_k} - a_{k+1}c'_{k-1}\right| \\ &\leq \left|c'_k a_{k+1}\frac{d_{k-1}}{d_k} - a_{k+1}d_{k-1}\gamma\right| + \left|a_{k+1}d_{k-1}\gamma - a_{k+1}c'_{k-1}\right| < 0.3.\end{aligned}$$

Combining with $|d_k\gamma - c'_k| < 0.05d_k(d_{k+1})^{-1}$ via the triangle inequality yields

$$\left|d_{k+1}\gamma - c''_{k+1}\right| \leq \left|d_{k+1}\gamma - c'_k\frac{d_{k+1}}{d_k}\right| + \left|c'_k\frac{d_{k+1}}{d_k} - c''_k\right| < \frac{d_{k+1}}{d_k}|d_k\gamma - c'_k| + 0.3 < 0.35.$$

Recall that by definition,

$$\left|\gamma - \frac{c'_{k+1}}{d_{k+1}}\right| < 0.05(d_{k+2})^{-1}, \text{ and so } \left|d_{k+1}\gamma - c'_{k+1}\right| < 0.05\frac{d_{k+1}}{d_{k+2}} < 0.05.$$

This implies that $c'_{k+1} = c''_{k+1}$ (since they are both integers).

For $-2 \leq k < k_0$, define $c'_k \in \mathbb{Q}$ using the recursion relation $c'_{k+1} = b_{k+1}c'_k + a_{k+1}c'_{k-1}$ in reverse. Since the recurrence relations defining c_k, e_k, d_k and c'_k are the same linear relation, $c'_k = c'_{-2}c_k + c'_{-1}e_k$ so

$$\gamma = \lim\frac{c'_k}{d_k} = \lim\frac{c'_{-2}c_k + c'_{-1}e_k}{e_k}\frac{e_k}{d_{-2}c_k + d_{-1}e_k} = \frac{c'_{-2}\lambda + c'_{-1}}{d_{-2}\lambda + d_{-1}}.$$

Then

$$\begin{aligned}\gamma &= c'_{-2}\alpha + \frac{c'_{-1}}{d_{-2}\lambda + d_{-1}} = c'_{-2}\alpha + \frac{c'_{-1}}{d_{-1}}\frac{d_{-1}}{d_{-2}\lambda + d_{-1}} = c'_{-2}\alpha + \frac{c'_{-1}}{d_{-1}}\left(1 - \frac{d_{-2}\lambda}{d_{-2}\lambda + d_{-1}}\right) \\ &= c'_{-2}\alpha + \frac{c'_{-1}}{d_{-1}}(1 - d_{-2}\alpha) = \left(c'_{-2} - \frac{c'_{-1}d_{-2}}{d_{-1}}\right)\alpha + \frac{c'_{-1}}{d_{-1}}\end{aligned}$$

meaning that

$$d_{-1}\gamma = (c'_{-2}d_{-1} - c'_{-1}d_{-2})\alpha + c'_{-1}.$$

It is easily seen by induction that

$$\frac{c'_k d_{k+1} - c'_{k+1}d_k}{c'_{-2}d_{-1} - c'_{-1}d_{-2}} = (-1)^k a_0\cdots a_{k+1} = \frac{c'_k c_{k+1} - c'_{k+1}c_k}{c'_{-2}c_{-1} - c'_{-1}c_{-2}}$$

and therefore, as $c'_k \in \mathbb{Z}$ for $k \geq k_0$ and $c_{-1} = 0$ and $c_{-2} = 1$,

$$(c'_{-2}d_{-1} - c'_{-1}d_{-2})\frac{a_0\cdots a_{k_0+1}}{\gcd(d_{k_0}, d_{k_0+1})}, \; c'_{-1}a_0\cdots a_{k_0} \in \mathbb{Z}.$$

Then $\gamma = q\alpha + r$ for some $q \in \frac{\gcd(d_{k_0}, d_{k_0+1})}{d_{-1}a_0\cdots a_{k_0+1}}\mathbb{Z}$ and $r \in \mathbb{Q}$. □

5.3 Rational additive measurable eigenvalues

Next, we establish that the only rational additive measurable eigenvalues are those in R_X.

Proposition 5.15. *If a rational number m/n in lowest terms is an additive measurable eigenvalue then n eventually divides the lengths of both u_k and v_k, equivalently the lengths of both v_k and v_{k+1}.*

Proof. It suffices to prove the case when $m = 1$ and n is a prime power. Assume that p^{-r} is an additive eigenvalue for a prime p and integer $r \geq 1$. Then there exists a positive measure set A such that $\sigma^{p^r}A = A$ and $\sigma^{p^{r-1}}A$ is disjoint from A. Let B_k and B'_k as in Lemma 5.13. Since cylinder sets generate the algebra of measurable sets, there exists $S^v_k \subseteq \{0,\ldots,|v_{k+1}|-1\}$ and $S^u_k \subseteq \{0,\ldots,|u'_{k+1}|-1\}$ such that $A_k = \bigsqcup_{j\in S^v_k}\sigma^j B_k \sqcup \bigsqcup_{j\in S^u_k}\sigma^j B'_k$ has $\mu(A_k \triangle A)\to 0$. Since $\sigma^{p^r}A = A$, $\sup_{t\in\mathbb{Z}}\mu(A_k\triangle\sigma^{p^r t}A_k)\to 0$.

Set $A^v_k = A_k \cap T_k$. Since $\mu(T_k)\geq \frac{1}{4}$ and $\|\mathbb{1}_{\sigma T_k} - \mathbb{1}_{T_k}\|_2 \to 0$, by Lemma 3.6 [Dan16], $\liminf \mu(A\cap T_k) \geq \frac{1}{4}\mu(A)$. Then $\mu(A^v_k)$ is uniformly bounded above zero for sufficiently large k.

For $p^r \leq j < |v_{k+1}|$, if $j\in S^v_k$ and $j-p^r \notin S^v_k$ then $\sigma^j B_k \subseteq A^v_k$ and $\sigma^{j-p^r}B_k \cap A_k = \emptyset$ so $\sigma^j B_k \subseteq A^v_k \setminus \sigma^{p^r}A_k$. Therefore, since $\mu(A^v_k) = |S^v_k|\mu(B_k)$,

$$\frac{1}{|S^v_k|}\left|\{j<|v_{k+1}| : j\in S^v_k, j-p^r\notin S^v_k\}\right| \leq \frac{p^r}{|S^v_k|} + \frac{\mu(A^v_k\setminus\sigma^{p^r}A_k)}{\mu(A^v_k)} \leq \frac{p^r}{|S^v_k|} + \frac{\mu(A_k\setminus\sigma^{p^r}A_k)}{\mu(A^v_k)} \to 0$$

so $\frac{1}{|S^v_k|}|\{j\in S^v_k : j-p^r\in S^v_k\}| \to 1$.

Choose $t_k\in\mathbb{Z}$ such that $p^r t_k = |v_{k+1}| + \ell_k$ for some $0<\ell_k\leq p^r$. Then $\sigma^{p^r t_k}B_k \subseteq \sigma^{\ell_k}B_k \sqcup \sigma^{\ell_k}B'_k$. Since the set of $x\in X$ such that x has v_k^2 at the origin is positive measure (as otherwise every x would be a multiple of u_k), the same reasoning as above gives that $\frac{1}{|S^v_k|}|\{j\in S^v_k : j+\ell_k\in S^v_k\}|\to 1$.

Since $0<\ell_k\leq p^r$, there exists a constant $0<\ell\leq p^r$ such that $\ell_{k_i}=\ell$ for infinitely many k_i and we may assume ℓ is the minimal such constant. Let $0\leq z\leq r$ maximal such that p^z divides ℓ. Then there exist integers $a<0$ and $b>0$ such that $ap^r + b\ell = p^z$. As a and b are fixed and $\frac{1}{|S^v_{k_i}|}|\{j\in S^v_{k_i} : j+\ell, j-p^r\in S^v_{k_i}\}|\to 1$, we have $\frac{1}{|S^v_{k_i}|}|\{j\in S^v_{k_i} : j+p^z\in S^v_{k_i}\}|\to 1$.

Let $S'_{k_i} = \{j\in S^u_{k_i} : j = j_0 + |u'_{k_i+1}| - |v_{k_i+1}|$ for some $0\leq j_0 < |v_{k_i+1}| - p^z\}$ and $S''_{k_i} = \{j\in S^u_{k_i} : j = j_0 + |u'_{k_i+1}| - 2|v_{k_i+1}|$ for some $0\leq j_0<|v_{k_i+1}|-2p^z\}$. Since $|u_{k_i+1}| < 3|v_{k_i+1}|$ (Remark 3.14), $|S^u_{k_i}\setminus(S'_{k_i}\sqcup S''_{k_i})|\leq 3p^z$.

Since u'_{k_i+1}, when it appears at the start of a u_{k_i+1} in a concatenation, is always followed by v_{k_i+1}, $\sigma^{|u'_{k_i+1}|}B'_{k_i}\subseteq B_{k_i}$. For $j = j_0 + |u'_{k_i+1}| - |v_{k_i+1}| \in S'_{k_i}$, then $\sigma^{p^r t_{k_i}}\sigma^j B'_{k_i}\subseteq \sigma^{p^z+j_0}B_{k_i}$ which is a level in T_{k_i} (as $j_0 < |v_{k_i+1}| - p^z$) and for $j = j_0 + |u'_{k_i+1}| - 2|v_{k_i+1}|\in S''_{k_i}$, then $\sigma^{2p^r t_{k_i}}\sigma^j B'_{k_i}\subseteq\sigma^{2p^z+j_0}B_{k_i}$ which is also a level in T_{k_i}.

Since $\mu(\sigma^{p^r t_{k_i}}A_{k_i}\triangle A_{k_i})\to 0$, then $\frac{1}{|S'_{k_i}|}|\{j\in S'_{k_i} : p^r t_{k_i} + j - |u'_{k_i+1}|\in S^v_{k_i}\}|\to 1$ and $\frac{1}{|S''_{k_i}|}|\{j\in S''_{k_i} : 2p^r t_{k_i} + j - |u'_{k_i+1}|\in S^v_{k_i}\}|\to 1$. As $\frac{1}{|S^v_{k_i}|}|\{j\in S^v_{k_i} : j+p^z\in S^v_{k_i}\}|\to 1$, then $\frac{1}{|S'_{k_i}|}|\{j\in S'_{k_i} : j+p^z\in S'_{k_i}\}|\to 1$ and likewise for S''_{k_i} so $\frac{1}{|S^u_{k_i}|}|\{j\in S^u_{k_i} : j+p^z\in S^v_{k_i}\}|\to 1$.

Then $\mu(A_{k_i}\triangle\sigma^{p^z}A_{k_i})\to 0$ meaning that $\mu(A\triangle\sigma^{p^z}A) = 0$. By choice of A then $z=r$. Therefore $p^r t_{k_i} = |v_{k_i+1}| + p^r$ so p^r divides $|v_{k_i+1}|$. As ℓ was chosen minimally, then $p^r t_k = |v_{k+1}| + p^r$ for all sufficiently large k so p^r divides $|v_k|$ for all sufficiently large k. Since $|u_{k+1}| = |v_{k+1}| + (n_k - m_k)|v_k|$, then p^r divides $|u_k|$ for all sufficiently large k as well. □

5.4 The structure of the additive eigenvalue group

We are now ready to establish the relationship between Q_X, R_X and E_X.

Proposition 5.16. *There exists a homomorphism $\phi : Q_X \to \mathbb{Q}/R_X$ with $\phi(1) = \phi(0)$ such that $q\mapsto q\alpha + \phi(q)$ is an isomorphism $Q_X \to E_X/R_X$.*

Proof. By Proposition 5.10, for every $q\in Q_X$ there exists $r_q\in\mathbb{Q}$ such that $q\alpha + r_q\in E_X$. Let $\phi(q) = r_q + R_X$. If $r, r'\in\mathbb{Q}$ such that $q\alpha + r, q\alpha + r'\in E_X$ then $r - r'\in E_X\cap\mathbb{Q} = R_X$ so for every $r\in\mathbb{Q}$ such that $q\alpha + r\in E_X$, we have $r\in\phi(q)$. Since $\alpha\in E_X$, $\phi(1) = R_X = \phi(0)$. Since $r_{q+q'} - r_q - r_{q'} = (q+q')\alpha + r_{q+q'} - (q\alpha + r_q) - (q'\alpha + r_{q'})\in E_X\cap\mathbb{Q} = R_X$, ϕ is a homomorphism and therefore $q\mapsto q\alpha + \phi(q)$ is a homomorphism $Q_X\to E_X/R_X$.

By Proposition 5.14, every $\gamma\in E_X$ is of the form $q\alpha + r$ for some $q\in Q_X$ and $r\in\mathbb{Q}$ so $q\mapsto q\alpha+\phi(q)$ is onto. Since $\alpha\notin\mathbb{Q}$, the kernel of $q\mapsto q\alpha + \phi(q)$ is $\{0\}$ meaning $q\mapsto q\alpha+\phi(q)$ is an isomorphism. □

To characterize the structure of E_X, we need to establish the nature of such homomorphisms ϕ.

Proposition 5.17. *Let $0 \le \ell_p, r_p \le \infty$ and $\phi : Q_{(\ell_p)} \to \mathbb{Q}/Q_{(r_p)}$ be a homomorphism such that $\phi(1) = \phi(0)$. Then there exist $e_p \in \mathbb{Q}_p$ for each prime p such that for all $q \in Q_{(\ell_p)}$,*

$$\phi(q) = \sum_p \{qe_p\}_p + Q_{(r_p)}.$$

Proof. Since $\phi(1) = \phi(0)$, there exists a homomorphism $\tilde{\phi} : Q_{(\ell_p)}/\mathbb{Z} \to \mathbb{Q}/Q_{(r_p)}$ such that $\phi(q) = \tilde{\phi}(q+\mathbb{Z})$. As $Q_{(\ell_p)}/\mathbb{Z}$ is an abelian torsion group (since it is a subgroup of $\mathbb{Q}/\mathbb{Z}$), it is isomorphic to the direct sum of its p-power torsion groups. Concretely speaking, adopting the convention that $p^{-\infty}\mathbb{Z} = \mathbb{Z}[1/p]$, the map $i : Q_{(\ell_p)}/\mathbb{Z} \to \bigoplus_p p^{-\ell_p}\mathbb{Z}/\mathbb{Z}$ given by $i(q + \mathbb{Z}) = (\{q\}_p + \mathbb{Z})_p$ is an isomorphism with inverse map given by $(x_p + \mathbb{Z})_p \mapsto \sum_p x_p + \mathbb{Z}$. Likewise, $\mathbb{Q}/Q_{(r_p)}$ is a torsion group and $j : \mathbb{Q}/Q_{(r_p)} \to \bigoplus \mathbb{Z}[1/p]/p^{-r_p}\mathbb{Z}$ by $j(r + Q_{(r_p)}) = (\{r\}_p + p^{-r_p}\mathbb{Z})_p$ is the isomorphism. As p-power torsion elements must map to p-power torsion elements, there exist homomorphisms $\tilde{\phi}_p : p^{-\ell_p}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}[1/p]/p^{-r_p}\mathbb{Z}$ such that $\tilde{\phi} = j^{-1} \circ \left(\bigoplus \tilde{\phi}_p\right) \circ i$.

For p such that $r_p = \infty$, $\tilde{\phi}_p$ maps to the trivial group so $\tilde{\phi}_p(p^{-t}) = 0$ for all $t \le \ell_p$ and we set $e_p = 0$. For p such that $r_p < \infty$ and $\ell_p = \infty$, for each $n > 0$, let $c_{p,n} \in \tilde{\phi}_p(p^{-n} + \mathbb{Z})$. Then $p^n c_{p,n} \in \tilde{\phi}_p(\mathbb{Z}) = p^{-r_p}\mathbb{Z}$ and $p^{n+m}c_{p,n+m} - p^n c_{p,n} \in p^n(p^m\tilde{\phi}_p(p^{-n-m} + \mathbb{Z}) - \tilde{\phi}_p(p^{-n} + \mathbb{Z})) = p^n\tilde{\phi}_p(\mathbb{Z}) = p^{n-r_p}\mathbb{Z}$. Then $p^n c_{p,n}$ is a p-adic Cauchy sequence so $p^n c_{p,n} \to e_p \in \mathbb{Q}_p$ and since $p^n c_{p,n} \in p^{-r_p}\mathbb{Z}$, $e_p \in p^{-r_p}\mathbb{Z}_p$.

Let $e_p = \sum_{t=-r_p}^{\infty} e_{p,t}p^t$ and $p^n c_{p,n} = \sum_{t=-r_p}^{\infty} d_{p,n,t}p^t$ be the p-adic expansions. Since $p^{n+m}c_{p,n+m} - p^n c_{p,n} \in p^{n-r_p}\mathbb{Z}$, $e_{p,t} = d_{p,n,t}$ for $t \le n - r_p$ so $e_{p,t} = d_{p,n+r_p,t}$ for $t \le n$. Then $p^n\{p^{-n}e_{p,t}\}_p = \sum_{t=-r_p}^{n-1} e_{p,t}p^t = \sum_{t=-r_p}^{n-1} d_{p,n+r_p,t}p^t = p^n\{p^{-n}p^{n+r_p}c_{p,n+r_p}\}_p$ meaning that $\{p^{-n}e_p\}_p = \{p^{r_p}c_{p,n+r_p}\}_p$. Now $p^{r_p}c_{p,n+r_p} - c_{p,n} \in \tilde{\phi}_p(\mathbb{Z}) = p^{-r_p}\mathbb{Z}$ so $\{p^{r_p}c_{p,n+r_p}\}_p - c_{p,n} \in p^{-r_p}\mathbb{Z}$.

Since we have dealt with all possibilities for p, for every p and n we have $\{p^{-n}e_p\}_p \in \tilde{\phi}_p(p^{-n})$.

For p such that $\ell_p < \infty$ and $r_p < \infty$, let $c_{p,\ell_p} \in \tilde{\phi}_p(p^{-\ell_p})$ and set $e_p = p^{\ell_p}c_{p,\ell_p} \in p^{-r_p}\mathbb{Z}$. For $n \le \ell_p$, $\{p^{-n}e_p\}_p = \{p^{-n}p^{\ell_p}c_{p,\ell_p}\}_p = \{p^{\ell_p-n}c_{p,\ell_p}\}_p \in p^{\ell_p-n}\tilde{\phi}_p(p^{-\ell_p}) = \tilde{\phi}_p(p^{-n})$. Therefore, for all p, there exist $e_p \in p^{-r_p}\mathbb{Z}_p$ such that $\{p^{-n}e_p\}_p \in \tilde{\phi}_p(p^{-n})$ for all $n \le \ell_p$ meaning that $\tilde{\phi}_p(x + \mathbb{Z}) = \{xe_p\}_p + p^{-r_p}\mathbb{Z}$ for all $x \in p^{-\ell_p}\mathbb{Z}$. Therefore for all $q \in Q_{(\ell_p)}$,

$$\begin{aligned}\phi(q) = \tilde{\phi}(q + \mathbb{Z}) &= j^{-1} \circ \left(\bigoplus \tilde{\phi}_p\right) \circ i(q + \mathbb{Z}) \\ &= j^{-1} \circ \left(\bigoplus \tilde{\phi}_p\right)\left((\{q\}_p + \mathbb{Z})_p\right) = j^{-1}\left((\{\{q\}_p e_p\}_p + p^{-r_p}\mathbb{Z})_p\right).\end{aligned}$$

Since $q - \{q\}_p \in \mathbb{Z}_p$ and $\{e_p\}_p \in p^{-r_p}\mathbb{Z}$, $\{(q-\{q\}_p)e_p\}_p \in p^{-r_p}\mathbb{Z}$. As $\{qe_p\}_p = \{(q-\{q\}_p)e_p\}_p + \{\{q\}_p e_p\}_p \pmod{\mathbb{Z}}$, then $\{qe_p\}_p = \{\{q\}_p e_p\}_p \pmod{p^{-r_p}\mathbb{Z}}$. Therefore

$$\phi(q) = j^{-1}\left((\{\{q\}_p e_p\}_p + p^{-r_p}\mathbb{Z})_p\right) = j^{-1}\left((\{qe_p\}_p + p^{-r_p}\mathbb{Z})_p\right) = \sum\{qe_p\}_p + Q_{(r_p)}. \qquad \square$$

Now we are in a position to prove the explicit description of the eigenvalue group and verify that all eigenvalues are continuous.

Proof of Theorem 5.3. Consider any additive (measurable) eigenvalue γ. By Proposition 5.14, there exist $q \in Q_X$ and $r \in \mathbb{Q}$ so that $\gamma = q\alpha + r$. By Proposition 5.12, $q\alpha + r_q$ is an additive (even continuous) eigenvalue. Therefore, $r - r_q$ is a rational additive eigenvalue, which must be in R_X by Proposition 5.15, and so $\gamma = q\alpha + r_q + (r - r_q) \in \{q\alpha + r_q + r \ : \ q \in Q_X, r \in R_X\}$. Therefore, by Proposition 5.12, γ is also an additive continuous eigenvalue. Since all eigenspaces are one-dimensional by ergodicity of the unique σ-invariant measure μ, all eigenfunctions of γ are continuous.

By Proposition 5.16, $E_X = \{q\alpha + r : r \in \phi(q)\}$ for some homomorphism $\phi : Q_X \to E_X/R_X$. By Proposition 5.17, there exists $e_p \in \mathbb{Q}_p$ such that $\sum_p\{qe_p\}_p \in \phi(q)$ for all $q \in Q_X$. Therefore $E_X = \{q\alpha + \sum_p\{qe_p\}_p + r : q \in Q_X, r \in R_X\}$. $\square$

6 The maximal equicontinuous factor

In this section, we characterize the maximal equicontinuous factors of low complexity minimal subshifts as products of odometers and adelic nilsystems. We begin by describing the nilsystems and odometers in question.

Definition 6.1. Let $\mathbb{A}_X = \{(a_\infty, (a_p)) \in \mathbb{A} : a_p \in p^{-L_X(p)}\mathbb{Z}_p\}$ with the convention that $p^{-\infty}\mathbb{Z}_p = \mathbb{Q}_p$ and identify Q_X with its diagonal embedding $q \mapsto (q, (-q))$ as a lattice in $\mathbb{A}_X$. The **adelic nilsystem associated to** X is the adelic nilmanifold $\mathcal{M}_X = \mathbb{A}_X/Q_X$ equipped with the action of translation by an element of $\mathcal{M}_X$ equivalent to translation by the adele $(\alpha, (e_p))$ where α, e_p are as in Theorem 5.3.

Remark 6.2. The simplest example is when $L_X(p) = 0$ for all primes p, which for instance happens for any Sturmian subshift. Here $\mathbb{A}_X = \mathbb{R} \times \prod_p \mathbb{Z}_p$ and $Q_X = \mathbb{Z}$ so upon quotienting, $\mathcal{M}_X = \mathbb{A}_X/Q_X = \mathbb{R}/\mathbb{Z} = S^1$ and so the MEF is an irrational circle rotation.

An example of a p-adic MEF is when $L_X(2) = \infty$ and $L_X(p) = 0$ for $p \neq 2$. Here $\mathbb{A}_X = \mathbb{R} \times \mathbb{Q}_2 \times \prod_{p>2} \mathbb{Z}_p$ and $Q_X = \mathbb{Z}[1/2]$. Upon quotienting, $\prod_{p>2} \mathbb{Z}_p$ disappears, and so $\mathcal{M}_X = \mathbb{A}_X/Q_X = (\mathbb{R} \times \mathbb{Q}_2)/\mathbb{Z}[1/2] = \mathcal{M}_2$. Therefore, the MEF is a rotation of $\mathcal{M}_2$ as described in Section 1.5. This MEF structure occurs for Example 1.2, but could also occur for a subshift where $r_k = 1$ and $n_k = m_k + 1$ for all k.

Definition 6.3. The **odometer associated to** X is

$$\mathcal{O}_X = \varprojlim_{k\to\infty} \mathbb{Z} \Big/ \gcd(|v_{k+1}|, |v_k|)\mathbb{Z}$$

under the natural (coordinatewise) $+1$ action where v_k and v_{k+1} are the words from Proposition 3.1.

Theorem 6.4. *Let X be an infinite minimal subshift with $\limsup p(q)/q < 1.5$. Then X is measurably isomorphic to its maximal equicontinuous factor $\mathcal{M}_X \times \mathcal{O}_X$.*

We start by characterizing the MEF as the group of characters on the multiplicative eigenvalue group.

Proposition 6.5. *The maximal equicontinuous factor of X is $\widehat{\mathcal{E}_X}$ equipped with Haar measure under the action of multiplication by the identity character.*

Moreover, (X, σ) is measurably isomorphic to $\widehat{\mathcal{E}_X}$ under the action of multiplication by the identity character.

Proof. By Theorem 2.21 in [BK13], the maximal equicontinuous factor is homeomorphic to $\widehat{\mathcal{E}_X}$ under multiplication by the identity character. Since X has discrete spectrum, Theorem 2.1 implies X is measurably isomorphic to $\widehat{\mathcal{E}_X}$ under that action. □

Next we establish that the space of characters is a direct product of the spaces of characters on Q_X and $R_X/\mathbb{Z}$. By slight abuse of notation, for $\chi \in \widehat{\mathcal{E}_X}$ and $\gamma \in E_X$, we will write $\chi(\gamma)$ to mean $\chi(\exp(2\pi i\gamma))$ and treat χ as a character on E_X which maps $\mathbb{Z}$ to 1.

Proposition 6.6. *The space of characters $\widehat{\mathcal{E}_X}$ is isomorphic as a topological group to $\widehat{Q_X} \times \widehat{R_X/\mathbb{Z}}$.*

Let $e_p \in \mathbb{Q}_p$ and α be as in Theorem 5.3. The action of multiplication by the identity character on $\widehat{\mathcal{E}_X}$ maps to the action of multiplication by $\exp(2\pi i(q\alpha + \sum_p \{qe_p\}_p))$ on $\widehat{Q_X}$ and multiplication by the identity character on $\widehat{R_X/\mathbb{Z}}$.

Proof. By Theorem 5.3, $E_X = \{q\alpha + \sum_p \{qe_p\}_p + r : q \in Q_X, r \in R_X\}$. Let $\chi \in \widehat{\mathcal{E}_X}$. For $q \in Q_X$, set $\chi_Q(q) = \chi(q\alpha + \sum_p \{qe_p\}_p)$. Since $\sum\{(q+q')e_p\}_p = \sum\{qe_p\}_p + \sum\{q'e_p\}_p \pmod{\mathbb{Z}}$, and since $\chi(1) = 1$,

$$\begin{aligned}\chi_Q(q+q') &= \chi\left(q\alpha + \sum\{qe_p\}_p\right)\chi\left(q'\alpha + \sum\{q'e_p\}_p\right)\chi\left(\sum\{(q+q')e_p\}_p - \sum\{qe_p\}_p - \sum\{q'e_p\}_p\right)\\ &= \chi_Q(q)\chi_Q(q')\ 1\end{aligned}$$

so $\chi_Q \in \widehat{Q_X}$. Therefore for any $q \in Q_X$ and $r \in R_X$,

$$\chi\left(q\alpha + \sum\{qe_p\}_p + r\right) = \chi_Q(q)\chi(r)$$

so $\chi \mapsto \chi_Q \cdot \chi\big|_{R_X}$ defines a homomorphism $\widehat{\mathcal{E}_X} \to \widehat{Q_X} \times \widehat{R_X/\mathbb{Z}}$. As every such product of characters defines a character on $\mathcal{E}_X$, the homomorphism is onto and it is easily seen to be continuous and have trivial kernel.

The action of multiplication by the identity character on $\mathcal{E}_X$ on $\chi = \chi_Q \cdot \chi\big|_{R_X}$ is

$$\begin{aligned}(\chi\iota)\left(q\alpha + \sum\{qe_p\}_p + r\right) &= \chi\left(q\alpha + \sum\{qe_p\}_p + r\right)\exp\left(2\pi i\left(q\alpha + \sum\{qe_p\}_p + r\right)\right) \\ &= \chi_Q(q)\exp\left(2\pi i\left(q\alpha + \sum\{qe_p\}_p\right)\right)\ \chi\big|_{R_X}(r)\exp(2\pi i r).\end{aligned}$$

□

Our next task then is to characterize the character groups of $Q_{(\ell_p)}$ and $Q_{(r_p)}/\mathbb{Z}$. We begin with an observation connecting such characters to p-adic integers.

Lemma 6.7. *Let $0 \le \ell_p \le \infty$ for each prime p and $\chi \in \widehat{Q_{(\ell_p)}}$. Then there exists a unique $\theta \in [0,1)$ and unique $z_p \in \mathbb{Z}_p$ with $0 \le z_p < p^{\ell_p}$ when $\ell_p < \infty$ such that for all $q \in Q_{(\ell_p)}$,*

$$\chi(q) = \exp\left(2\pi i\left(q\theta + \sum\{qz_p\}_p\right)\right).$$

Proof. Let $\theta \in [0,1)$ be the unique value such that $\chi(1) = \exp(2\pi i\theta)$ and let $\chi'(q) = \chi(q)/\exp(2\pi i q\theta)$. Then $\chi' \in \widehat{Q_{(\ell_p)}}$ and $\chi'(1) = 1$. For $0 < t \le \ell_p$,

$$\left(\chi'(p^{-t})\right)^{p^t} = \left(\frac{\chi(p^{-t})}{\exp(2\pi i p^{-t}\theta)}\right)^{p^t} = \frac{\chi(1)}{\exp(2\pi i\theta)} = 1$$

so there exist unique integers $0 \le z_{p,t} < p^t$ such that $\chi'(p^{-t}) = \exp(2\pi i p^{-t} z_{p,t})$. Since $(\chi'(p^{-t-1}))^p = \chi'(p^{-t})$, we have $z_{p,t+1} \pmod{p^t} = z_{p,t}$. For p such that $\ell_p < \infty$, set $z_{p,t} = z_{p,\ell_p}$ for $t > \ell_p$. Then $z_{p,t} \to z_p \in \mathbb{Z}_p$ and $0 \le z_p < p^{\ell_p}$ when $\ell_p < \infty$.

Since $\{p^{-t}z_p\} = p^{-t}z_{p,t}$, then $\chi'(p^{-t}) = \exp(2\pi i\{p^{-t}z_p\}_p)$ for all p and $t \le \ell_p$. Let $q \in Q_{(\ell_p)}$. For each prime p, q has p-adic expansion $\sum_{t=-m}^{\infty} q_{p,t}p^t$ for some $m \le \ell_p$ so

$$\chi'(\{q\}_p) = \prod_{t=-m}^{-1} \chi'(q_{p,t}p^t) = \prod_{t=-m}^{-1} \exp(2\pi i\{q_{p,t}p^t z_p\}_p) = \exp(2\pi i\{qz_p\}_p)$$

and as $q = \sum\{q\}_p \pmod{\mathbb{Z}}$,

$$\begin{aligned}\chi(q) &= \exp(2\pi i q\theta)\chi'(q) = \exp(2\pi i q\theta)\prod \chi'(\{q\}_p) \\ &= \exp(2\pi i q\theta)\prod \exp(2\pi i\{qz_p\}_p) = \exp\left(2\pi i\left(q\theta + \sum\{qz_p\}_p\right)\right).\end{aligned}$$

□

We can now characterize the character group of Q_X as an adelic nilmanifold.

Proposition 6.8. *Let $0 \le \ell_p \le \infty$ for each prime p. Let $\mathbb{A}_{(\ell_p)} = \{(a_\infty, (a_p)) \in \mathbb{A} : a_p \in p^{-\ell_p}\mathbb{Z}_p\}$ with the convention that $p^{-\infty}\mathbb{Z}_p = \mathbb{Q}_p$ and identify $Q_{(\ell_p)}$ with its diagonal embedding $q \mapsto (q, (-q))$ as a lattice in $\mathbb{A}_{(\ell_p)}$. Then there exists a topological group isomorphism*

$$\widehat{Q_{(\ell_p)}} \simeq \mathbb{A}_{(\ell_p)} \Big/ Q_{(\ell_p)}.$$

Proof. For $(a_\infty, (a_p)) \in \mathbb{A}_{(\ell_p)}$, let $\chi_{a_\infty,(a_p)} \in \widehat{Q_{(\ell_p)}}$ by

$$\chi_{a_\infty,(a_p)}(q) = \exp\left(2\pi i\left(q a_\infty + \sum\{q a_p\}_p\right)\right).$$

The mapping $\mathbb{A}_{(\ell_p)} \to \widehat{Q_{(\ell_p)}}$ is clearly a continuous homomorphism and by Lemma 6.7, it is onto.

Let $(a_\infty, (a_p)) \in \mathbb{A}_{(\ell_p)}$ such that $\chi_{a_\infty,(a_p)}$ is the trivial character. Then $\exp(2\pi i(a_\infty + \sum\{a_p\}_p)) = 1$ so $a_\infty = -\sum\{a_p\}_p \pmod{\mathbb{Z}}$ hence $a_\infty \in \mathbb{Q}$. Then $a_\infty = \sum\{a_\infty\}_p \pmod{\mathbb{Z}}$ so

$$\begin{aligned}\chi_{a_\infty,(a_p)}(q) &= \exp\left(2\pi i \left(qa_\infty + \sum\{qa_p\}_p\right)\right) = \exp\left(2\pi i \left(\sum\{qa_\infty\}_p + \sum\{qa_p\}_p\right)\right) \\ &= \exp\left(2\pi i \sum\{q(a_\infty + a_p)\}_p\right) = \chi_{0,(a_p+a_\infty)}(q).\end{aligned}$$

Now $a_\infty + a_p = a_p - \sum_{p'}\{a_{p'}\}_{p'} \pmod{\mathbb{Z}} = (a_p - \{a_p\}_p) + \sum_{p' \neq p}\{a_{p'}\}_{p'} \pmod{\mathbb{Z}}$ so, as $\{a_{p'}\}_{p'} \in \mathbb{Z}_p$ for $p' \neq p$, we have $a_\infty + a_p \in \mathbb{Z}_p$. By Lemma 6.7, there is a unique θ and z_p such that the trivial character is $\exp(2\pi i(q\theta + \sum\{qz_p\}_p))$ which clearly must all be zero. Then $a_p + a_\infty = 0$ for all p which is precisely the statement that $(a_\infty, (a_p)) = (a_\infty, (-a_\infty)) \in Q_{(\ell_p)}$ when embedded diagonally so the kernel of the map $\mathbb{A}_{(\ell_p)} \to \widehat{Q_{(\ell_p)}}$ is $Q_{(\ell_p)}$. □

Likewise, we can characterize the character group of $R_X/\mathbb{Z}$ as an odometer.

Proposition 6.9. *Let $0 \leq \ell_p \leq \infty$. Then $\widehat{Q_{(\ell_p)}/\mathbb{Z}}$ equipped with multiplication by the identity character is isomorphic as a topological dynamical system to the odometer*

$$\mathcal{O}_{(\ell_p)} = \varprojlim \mathbb{Z} \Big/ \textstyle\prod_{p \leq k} p^{\min(k,\ell_p)}\mathbb{Z}.$$

Proof. By Lemma 6.7, any $\chi \in \widehat{Q_{(\ell_p)}/\mathbb{Z}}$ corresponds uniquely to $\theta \in [0, 1)$ and $a_p \in \mathbb{Z}_p$. Since $\chi(1) = 1$, we have $\theta = 0$. The p-adic expansions $a_p = \sum_{t=0}^{\infty} a_{p,t}p^t$ have the property that $a_{p,t+1} \pmod{p^t} = a_{p,t}$ so the values $a_{p,t}$ uniquely determine a point $x \in \mathcal{O}_{(\ell_p)}$ via the Chinese Remainder Theorem.

Conversely, given $x \in \mathcal{O}_{(\ell_p)}$, if one defines $a_{p,t}$ as above, then $a_{p,t} \to a_p \in \mathbb{Z}_p$ which uniquely determine a character on $Q_{(\ell_p)}/\mathbb{Z}$. We have then described a one-one onto mapping from $\widehat{Q_{(\ell_p)}/\mathbb{Z}}$ to $\mathcal{O}_{(\ell_p)}$, which is easily checked to be continuous from the topology of pointwise convergence to the natural topology.

Let (a_p) correspond to χ and ι be the identity character. Then for $t \leq \ell_p$,

$$(\iota\chi)(p^{-t}) = \exp(2\pi i p^{-t})\chi(p^{-t}) = \exp(2\pi i p^{-t}) \exp(2\pi i\{p^{-t}a_p\}_p) = \exp(2\pi i\{p^{-t}(a_p + 1)\}_p).$$

As the natural action on $\mathcal{O}_{(\ell_p)}$ maps to the action $a_{p,t} \mapsto a_{p,t} + 1 \pmod{p^t}$, the claim follows. □

Finally we are in a position to prove the MEF has the claimed structure.

Proof of Theorem 6.4. By Proposition 6.5, X is measurably isomorphic to its maximal equicontinuous factor $\widehat{\mathcal{E}_X}$ under multiplication by the identity character. By Proposition 6.6, $\widehat{\mathcal{E}_X}$ under multiplication by the identity character is the direct product of $\widehat{Q_X}$ under multiplication by $\exp(2\pi i(q\alpha + \sum\{qe_p\}_p))$ and $\widehat{R_X/\mathbb{Z}}$ under multiplication by the identity character.

By Proposition 6.9, $\widehat{R_X/\mathbb{Z}}$ is isomorphic as a topological dynamical system to $\mathcal{O}_X$. By Proposition 6.8, $\widehat{Q_X}$ is isomorphic as a topological group to the adelic nilmanifold $M = \mathbb{A}_{(L_X(p))}/Q_X$ and the action of multiplication by the identity character on $\widehat{\mathcal{E}_X}$ becomes multiplication by $\exp(2\pi i(q\alpha + \sum\{qe_p\}_p))$.

Set $q_0 = \sum\{e_p\}_p$. Since $\{e_{p'}\}_{p'} \in \mathbb{Z}_p$ for $p' \neq p$, we have $e_p - q_0 \in \mathbb{Z}_p$ and therefore $(\alpha + q_0, (e_p - q_0)) \in \mathbb{R} \times \prod' \mathbb{Z}_p \subseteq \mathbb{A}_X$. Since $\sum\{qe_p\}_p = \sum\{qq_0\}_p + \sum\{q(e_p - q_0)\}_p \pmod{\mathbb{Z}} = qq_0 + \sum\{q(e_p - q_0)\}_p \pmod{\mathbb{Z}}$, for $(a_\infty, (a_p)) \in M$, the action on the corresponding character $\chi_{a_\infty,(a_p)}$ is

$$\begin{aligned}\chi_{a_\infty,(a_p)}(q) \exp\left(2\pi i \left(q\alpha + \sum\{qe_p\}_p\right)\right) &= \exp\left(2\pi i \left(qa_\infty + \sum\{qa_p\}_p + q\alpha + \sum\{qe_p\}_p\right)\right) \\ &= \exp\left(2\pi i \left(q(a_\infty + \alpha + q_0) + \sum\{qa_p\}_p + \sum\{q(e_p - q_0)\}_p\right)\right) \\ &= \exp\left(2\pi i \left(q(a_\infty + \alpha + q_0) + \sum\{q(a_p + e_p - q_0)\}_p\right)\right) \\ &= \chi_{a_\infty+\alpha+q_0,(a_p+e_p-q_0)}(q).\end{aligned}$$

Therefore the action on M is $(a_\infty, (a_p)) \mapsto (a_\infty + \alpha + q_0, (a_p + e_p - q_0))$, i.e. translation by the element $(\alpha + q_0, (e_p - q_0)) \in \mathbb{A}_X$ which is equivalent as a $\mathbb{Q}$-adele to $(\alpha, (e_p))$. □

7 Orbit equivalence and strong orbit equivalence

Orbit equivalence and **strong orbit equivalence** are two weakened versions of isomorphism which are well-studied in dynamical systems. It was proved by Giordano, Putnam, and Skau in [GPS95] that for minimal TDS on a Cantor set, the so-called **dimension group** (a unital ordered group $K^0(X, \sigma)$) is a complete invariant for strong orbit equivalence, and the **reduced dimension group** (a unital ordered group $\widehat{K^0}(X, \sigma)$) is a complete invariant for orbit equivalence.

In this section, we will give a description of the dimension group for our class of subshifts, and prove that it is always equal to the reduced dimension group. As we do not make use of any nuanced properties of the dimension groups, we omit definitions and refer the reader to e.g. [BCBD+21] for definitions and details. The first step in characterizing the dimension groups is to show that our subshifts are balanced on words.

7.1 The balanced property

Theorem 7.1. *Any infinite minimal subshift X with $\limsup p(q)/q < 1.5$ is balanced on words.*

Proof. We apply our S-adic decomposition from Corollary 3.3 and Theorem 5.8 from [BD14], which gives a way to view balancedness for letters in terms of so-called incidence matrices of the substitutions.

For any substitution τ, the **incidence matrix** of τ is a square $|A| \times |A|$ matrix M with m_{ij} equal to $\tau(j)|_i$, the number of times i appears in $\tau(j)$. A subshift X has **uniform letter frequencies** if, for each letter $a \in A$, there exists $f(a)$ which is the uniform limit of the proportion of a letters in k-letter words in $L(X)$, uniformly in k.

Theorem 5.8, [BD14] states that if X is generated by a sequence (τ_k) of substitutions with incidence matrices (M_k), u has uniform letter frequencies with frequency vector f, and

$$\sum_k \|(M_0 M_1 \dots M_{k-1})^T\|_{f^\perp} \|M_k\| < \infty,$$

then X is balanced on letters. (Here $\|M\|_S = \sum_{v \in S^*} \frac{\|Mv\|}{\|v\|}$ represents the operator norm of M restricted to a subspace S.)

Let M_j be the incidence matrix for τ_{m_j,n_j,r_j} and M_{-1} be the incidence matrix for π.

Let $d_k = |v_{k+1}|$ so that $d_k = b_k d_{k-1} + a_k d_{k-2}$ for all $k \geq 0$ (setting $d_{-2} = |u_0| - |v_0|$). Let $g_k = |v_{k+1}|_1$ and $g_{-2} = |u_0|_1 - |v_0|_1$ so that $g_k = b_k g_{k-1} + a_k g_{k-2}$. Let $c_{-2} = e_{-1} = 1$ and $c_{-1} = e_{-2} = 0$ and define c_k and e_k via the same recurrence relation. As shown in the proof of Proposition 5.7, $\frac{c_k}{d_k} \to \alpha$ and $\frac{e_k}{d_k} \to \alpha_0$. Then $\frac{g_k}{d_k} \to g_{-2}\alpha + g_{-1}\alpha_0$. Set $\alpha^\star = g_{-2}\alpha + g_{-1}\alpha_0$.

Therefore, the frequency of 1s in v_k approaches $\alpha^\star$, and so X has uniform letter frequencies given by $f = (1 - \alpha^\star, \alpha^\star)$. Then, $f^\perp$ is spanned by $(-\alpha^\star, 1 - \alpha^\star)^T$, meaning that $\|(M_{-1} M_0 \dots M_{k-1})^T\|_{f^\perp} \leq \|(M_{-1} \dots M_{k-1})^T (-\alpha^\star, 1 - \alpha^\star)^T\|$. It's easily checked by induction that $M_{-1} \dots M_{k-1}$ is $\begin{pmatrix} |v_k|_0 & |u_k|_0 \\ |v_k|_1 & |u_k|_1 \end{pmatrix}$. Therefore,

$$(M_{-1} \dots M_{k-1})^T (-\alpha^\star, 1 - \alpha^\star)^T = \begin{pmatrix} -|v_k|_0 \alpha^\star + |v_k|_1 (1 - \alpha^\star) \\ -|u_k|_0 \alpha^\star + |u_k|_1 (1 - \alpha^\star) \end{pmatrix} = \begin{pmatrix} |v_k|_1 - |v_k| \alpha^\star) \\ |u_k|_1 - |u_k| \alpha^\star) \end{pmatrix}.$$

The top entry is, using the language above, $|g_{k-1} - d_{k-1}\alpha^\star| = d_{k-1} \left| \frac{g_{k-1}}{d_{k-1}} - \alpha^\star \right|$. It is easily checked by induction that $g_{k-1} d_k - g_k d_{k-1} = (-1)^k a_0 a_1 \cdots a_k (g_{-2} d_{-1} - g_{-1} d_{-2})$. Set $C = |g_{-2} d_{-1} - g_{-1} d_{-2}|$. Then

$$|g_{k-1} - d_{k-1}\alpha^\star| = d_{k-1} \left| \frac{g_{k-1}}{d_{k-1}} - \alpha^\star \right| < d_{k-1} \left| \frac{g_{k-1}}{d_{k-1}} - \frac{g_k}{d_k} \right| = \frac{|g_{k-1} d_k - g_k d_{k-1}|}{d_k} = \frac{C a_0 a_1 \cdots a_k}{d_k}$$

$$= C\frac{d_{k-1}}{d_k}\frac{2^{\sum_{j=0}^{k}\mathbb{1}_{r_j}}\prod_{j=0}^{k-1}(n_j - m_j)}{|v_k|} < C\frac{d_{k-1}}{d_k}\epsilon_k$$

where ϵ_k is as in Proposition 4.2.

Finally, we note that $\|M_k\|$ is the largest entry of M_k, which is bounded by $2n_k$. Therefore,

$$\|(M_{-1}M_0 \dots M_{k-1})^T\|_{f^\perp}\|M_k\| \leq C\frac{2n_k d_{k-1}}{d_k}\epsilon_k < C\frac{2n_k d_{k-1}}{m_k d_{k-1}}\epsilon_k = C\frac{2n_k}{m_k}\epsilon_k \leq C\frac{4m_k+4}{m_k}\epsilon_k \leq 8C\epsilon_k$$

so this series is summable and so X is balanced on letters. Finally, since our substitutions are each right-proper, meaning that the image of every letter ends with the same letter, Corollary 4.3 from [PS22] implies that X is balanced on words. □

7.2 The dimension group is the eigenvalue group

We can now describe the dimension groups of any low-complexity infinite minimal subshift.

Theorem 7.2. *The dimension group and reduced dimension group are both equal to* $(E_X, E_X \cap \mathbb{R}^+, 1)$.

Proof. We claim first that for every word w, we have $\mu([w]) \in E_X$. If $w \notin \mathcal{L}(X)$ then $\mu([w]) = 0 \in E_X$ so assume $w \in \mathcal{L}(X)$ and let k_0 be minimal such that w is a subword of v_{k_0}. Let a_k, b_k, c_k, d_k, e_k be as in the proof of Proposition 5.7. Define $f_k = |v_{k+k_0+1}|_w$ for $k \geq -2$. Then $f_{k+1} = b_{k+1}f_k + a_{k+1}f_{k-1}$ and $f_{-2} = 0$ (since k_0 is minimal) and so $f_k = f_{-1}e_k$ for all k. Since $\frac{e_k}{d_k} \to \alpha_{k_0} \in E_X$, then $\frac{f_k}{d_k} \to |v_{k_0}|_w \alpha_{k_0} \in E_X$. Since (X, σ) is uniquely ergodic, $\frac{f_k}{d_k} = \frac{|v_{k+k_0+1}|_w}{|v_{k+k_0+1}|}$ converges to $\mu([w])$, and the claim is proved.

By section 2.4 of [BCBD+21], since X is minimal and uniquely ergodic, the dimension group $K^0(X,\sigma)$ and its group of infinitesimals $\mathrm{Inf}(K^0(X,\sigma))$ have the property that the reduced dimension group $\widehat{K^0}(X,\sigma) = K^0(X,\sigma)/\mathrm{Inf}(K^0(X,\sigma))$ is isomorphic to the image group $(I(X,\sigma), I(X,\sigma) \cap \mathbb{R}^+, 1)$. Proposition 2.6 in [BCBD+21] states that $I(X,\sigma) = \{\mu([w]) : w \in \mathcal{L}(X)\}$ so $I(X,\sigma) \subseteq E_X$. Since E_X is always a subgroup of $I(X,\sigma)$ (see e.g. [CDP16] Proposition 11), then $I(X,\sigma) = E_X$. By Theorem 7.1, X is balanced on words so Proposition 5.4 of [BCBD+21] implies there are no infinitesimals. Then $\widehat{K^0}(X,\sigma) = K^0(X,\sigma) = (E_X, E_X \cap \mathbb{R}^+, 1)$. □

The following corollary is now immediate, modulo the simple observation that if G, G' are additive subgroups of $\mathbb{R}$ containing 1, then $(G, G \cap \mathbb{R}^+, 1)$ and $(G', G' \cap \mathbb{R}^+, 1)$ are isomorphic as unital ordered groups iff $G = G'$.

Corollary 7.3. *Two minimal subshifts with complexity satisfying* $\limsup p(q)/q < 1.5$ *are orbit equivalent if and only if they are strong orbit equivalent if and only if they have the same additive eigenvalue group.*

8 Existence of low complexity minimal subshifts for every odometer

We here demonstrate that there are no restrictions on the adelic nilmanifolds $\mathcal{M}_X$ and odometers $\mathcal{O}_X$ which can appear in the MEF of an infinite low complexity subshift. Other than the case when $\mathcal{M}_X$ is a finite group extension of S^1 and $\mathcal{O}_X$ is finite, we show that $\limsup \frac{p(q)}{q}$ can take any value in $[1, 1.5)$ for subshifts with that MEF.

Theorem 8.1. *Let $\mathcal{O}$ be an odometer and $\mathcal{M}$ be an adelic 1-step one-dimensional nilmanifold. There exists a infinite minimal subshift with* $\lim p(q)/q = 1$ *which has maximal equicontinuous factor the product of $\mathcal{O}$ and a rotation on $\mathcal{M}$.*

If $\mathcal{M}$ is not a finite group extension of S^1 or $\mathcal{O}$ is infinite (or both), then for every $0 < \delta < \frac{1}{2}$, there exists a minimal uniquely ergodic subshift with $\limsup p(q)/q = 1 + \delta$ *which has maximal equicontinuous factor of the same type.*

Remark 8.2. We make two comments about Theorem 8.1. First of all, it's unavoidable that the second statement excludes the case where both $\mathcal{M}$ is a finite group extension of S^1 and $\mathcal{O}$ is finite; in that case, a_k is eventually 1, meaning that the substitutions are eventually of the form $\tau_{m_k,m_k+1,0}$ (say for $k > k_0$). In that case, X is the image of a Sturmian subshift under the substitution ρ_{k_0}. Such a subshift is called **quasisturmian**, and is known to have $p(q) \leq q + C$ for a constant C ([Cas97]), and so is forced to have $\limsup p(q)/q = 1$.

Secondly, we want to be clear that we are not characterizing the set of possible MEFs of infinite minimal low-complexity subshifts, since we only show that a single adele $(\alpha, (e_p))$ can occur together with a pair of a nilmanifold and an odometer; we do not currently know which rotations $(\alpha, (e_p))$ can be associated with a specific group $\mathcal{M} \times \mathcal{O}$.

Proof. Define a sequence (δ_k) of positive reals as follows: when $\delta > 0$, set $\delta_k = \delta$, and when $\delta = 0$, let δ_k be any sequence approaching 0. We first consider the case when $\mathcal{M}$ is not a finite group extension of S^1. Let $0 \leq x_p \leq \infty$ such that $\mathcal{M} = \mathbb{A}_{(x_p)}/Q_{(x_p)}$. Let $s_k = p_k^{q_k}$ for $p_k \in \mathbb{P} \cup \{1\}$ and q_k nonnegative integers such that for each prime p, $\sum_{k:p_k=p} = x_p$ and such that $s_k \to \infty$ (possible as $\mathcal{M}$ is not a finite extension of S^1). Let $y_k \in \mathbb{P} \cup \{1\}$ such that $\mathcal{O} = \varprojlim \mathbb{Z}/y_0 \cdots y_k\mathbb{Z}$.

We will define ℓ_k, m_k, t_k and j_k inductively. Set $\ell_{-1} = \ell_0 = 1$ and $j_0 = 0$ and $t_0 = t_1 = 1$ and $s_{-1} = 0$. For all $k \geq 1$, we will set $\ell_{k+1} = m_k\ell_k + t_k s_{k-1}\ell_{k-1}$. For ease of notation, write $g_k = t_0 \cdots t_k$.

Choose m_0 such that $m_0 > \lceil(\delta_0^{-1} - 1)t_1 s_0\rceil$ and p_0 does not divide m_0+1. Then $\ell_1 = m_0\ell_0 + t_0 s_{-1}\ell_{-1} = m_0 + 1$ so t_1 divides $\frac{\ell_1}{g_0}$ and p_0 does not divide $\frac{\ell_1}{g_1}$. Also $\gcd(\ell_1, \ell_0) = 1 = g_0$.

Assume that t_k divides $\frac{\ell_k}{g_{k-1}}$ and p_{k-1} does not divide $\frac{\ell_k}{g_k}$ and $\gcd(\ell_k, \ell_{k-1}) = g_{k-1}$. If y_{j_k} divides $\frac{\ell_k}{g_k}$ then set $t_{k+1} = 1$ and $j_{k+1} = j_k$. If not, set $t_{k+1} = y_{j_k}$ and $j_{k+1} = j_k+1$. Set $m'_k = \lceil(\delta_k^{-1}-1)t_{k+1}s_k\rceil > s_k t_{k+1}$. The map $m \mapsto m\frac{\ell_k}{g_k} + s_{k-1}\frac{\ell_{k-1}}{g_{k-1}} \mod t_{k+1}$ is a cyclic onto homomorphism since $\gcd(\frac{\ell_k}{g_k}, t_{k+1}) = 1$ (since t_{k+1} is a prime power or 1). So there exists $0 \leq i < t_{k+1}$ such that t_{k+1} divides $(m'_k - i)\frac{\ell_k}{g_k} + s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}$.

If $p_k t_{k+1}$ were to divide both $(m'_k - i)\frac{\ell_k}{g_k} + s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}$ and $(m'_k - i - t_{k+1})\frac{\ell_k}{g_k} + s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}$ then $p_k t_{k+1}$ divides $t_{k+1}\frac{\ell_k}{g_k}$ so p_k divides $\frac{\ell_k}{g_k}$ but then p_k divides both $\frac{\ell_k}{g_k}$ and $s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}$ which is impossible as $\gcd(\frac{\ell_k}{g_k}, s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}) = 1$. Therefore we may take m_k such that $m'_k - 2t_{k+1} \leq m_k < m'_k$ so that t_{k+1} divides $\frac{\ell_{k+1}}{g_k}$ and p_k does not divide $\frac{\ell_{k+1}}{g_{k+1}}$. We also have $\gcd(\ell_{k+1}, \ell_k) = \gcd(m_k\ell_k + t_k s_{k-1}\ell_{k-1}, \ell_k) = \gcd(t_k s_{k-1}\ell_{k-1}, \ell_k) = g_{k-1}\gcd(t_k s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}, t_k\frac{\ell_k}{g_k}) = g_k \gcd(s_{k-1}\frac{\ell_{k-1}}{g_{k-1}}, \frac{\ell_k}{g_k}) = g_k$.

Therefore the sequences exist by induction. Note that if $y_{j_k} > 1$ and $t_{k+1} = 1$ then necessarily $t_{k+2} = y_{j_k}$ as otherwise y_{j_k} divides $\frac{\ell_k}{g_k}$ and y_{j_k} divides $\frac{\ell_{k+1}}{g_{k+1}}$ but $\gcd(\ell_{k+1}, \ell_k) = g_k$. By the construction of j_k, the sequence (t_k) is just the sequence (y_j) with extra interspersed 1s, and so the sequence (t_k) induces the odometer $\mathcal{O}^r$. Set $n_k = m_k + t_{k+1}s_k$. Let X be the orbit closure of $\lim \pi \circ \tau_{m_0,n_0,0} \circ \cdots \circ \tau_{m_k,n_k,0}(0)$ where $\pi(0) = 0$ and $\pi(1) = 01$ so $|v_k| = \ell_k$. By Remark 3.14, $\sum_{j=0}^{k-1}(n_j - m_j - 1)|v_j| < \sum_{j=0}^{k-1}(m_j - 1)|v_j| = |p_k| < 3|v_k|$, so by Corollary 3.17,

$$\frac{p(|s_k v_k^{n_k-2}p_k|)}{|s_k v_k^{n_k-2}p_k|} \leq 1 + \frac{(n_k - m_k - 1)|v_k| + |p_k| + C}{(n_k - 2)|v_k|} \leq 1 + \frac{t_{k+1}s_k + 3 + \frac{C}{|v_k|}}{\delta_k^{-1}t_{k+1}s_k - 2t_{k+1} - 2} \to 1 + \delta$$

since $s_k \to \infty$. By Corollary 3.17 and Remark 3.14,

$$\frac{p(|s_k v_k^{n_k-2}p_k|)}{|s_k v_k^{n_k-2}p_k|} \geq 1 + \frac{n_k - m_k - 1}{n_k + 1} \geq 1 + \frac{t_{k+1}s_k - 1}{\delta_k^{-1}t_{k+1}s_k + 2} \to 1 + \delta.$$

Since $\limsup \frac{p(q)}{q}$ is attained along the sequence $|s_k v_k^{n_k-2}p_k|$, $\limsup \frac{p(q)}{q} = 1 + \delta$.

By construction, $\gcd(|v_{k+1}|, |v_k|) = t_1 \cdots t_k$, and so $\mathcal{O}_X = \mathcal{O}$. Similarly, $a_0 \cdots a_k = (n_0 - m_0) \cdots (n_{k-1} - m_{k-1}) = s_0 \cdots s_{k-1}t_1 \cdots t_k$ so $\frac{|v_0|a_0 \cdots a_k}{\gcd(|v_k|, |v_{k+1}|)} = s_0 \cdots s_{k-1}$, implying that $\mathcal{M}_X = \mathcal{M}$. Therefore by Theorem 6.4, the subshift X defined as the orbit closure of $\lim \pi \circ \tau_{m_0,n_0,0} \circ \cdots \circ \tau_{m_k,n_k,0}(0)$ has the claimed properties.

Now consider when $\mathcal{M}$ is a finite group extension of S^1 and $\mathcal{O}$ is infinite. Let t_k such that $\mathcal{O} = \varprojlim \mathbb{Z}/t_0 \cdots t_k\mathbb{Z}$ and let $\mathcal{M} = S^1 \times \mathbb{Z}/q\mathbb{Z}$. Let $\pi(0) = 0^q$ and $\pi(1) = 0^q1$. Let $j_0 = 0$ and $s_0 = 1$. Given j_k and $|v_k|$, choose $s_{k+1} = t_{j_k}t_{j_k+1}\cdots t_{j_{k+1}-1}$ such that $\frac{s_{k+1}}{s_k|v_k|} \geq k$. Choose m_{k+1} such that $\gcd(m_{k+1}, s_k|v_k|) = s_k$ and $0 \leq m_{k+1} - (\delta_{k+1}^{-1} - 1)s_{k+1} \leq s_k|v_k|$ and set $n_{k+1} = m_{k+1} + s_{k+1}$. Then, as above, since $\frac{s_{k-1}|v_{k-1}|}{s_k} \to 0$,

$$\frac{p(|s_k v_k^{n_k-2}p_k|)}{|s_k v_k^{n_k-2}p_k|} \leq 1 + \frac{(n_k - m_k - 1)|v_k| + |p_k| + C}{(n_k-2)|v_k|} \leq 1 + \frac{s_k + 3 + \frac{C}{|v_k|}}{\delta_k^{-1}s_k - 2} \to 1 + \delta \text{ and}$$
$$\frac{p(|s_k v_k^{n_k-2}p_k|)}{|s_k v_k^{n_k-2}p_k|} \geq 1 + \frac{n_k - m_k - 1}{n_k + 1} \geq 1 + \frac{s_k - 1}{\delta_k^{-1}s_k + s_{k-1}|v_{k-1}| + 1} \to 1 + \delta,$$

so $\limsup \frac{p(q)}{q} = 1 + \delta$. By construction, $|v_0|a_0 \cdots a_k = qs_0 \cdots s_{k-1}$ and $|v_{k+1}| = m_k|v_k| + (n_{k-1} - m_{k-1})|v_{k-1}| = s_{k-1}y_k|v_k| + s_{k-1}|v_{k-1}|$, where $\gcd(y_k, |v_{k-1}|) = 1$ since $\gcd(m_k, s_{k-1}|v_{k-1}|) = s_{k-1}$. Then $\gcd(|v_{k+1}|, |v_k|) = s_{k-1}\gcd(|v_k|, |v_{k-1}|)$ so by induction $\gcd(|v_{k+1}|, |v_k|) = s_0 \cdots s_{k-1}$. For X defined in the usual way, since the sequence of partial products of (s_k) is a subsequence of partial products of (t_k), $\mathcal{O}_X = \mathcal{O}$, and since $\frac{|v_0|a_0\cdots a_k}{\gcd(|v_k|,|v_{k+1}|)} = q$, $\mathcal{M}_X = \mathcal{M}$. Therefore, X has the claimed properties by Theorem 6.4.

Finally, consider when both $\mathcal{M} = S^1 \times \mathbb{Z}/q\mathbb{Z}$ and $\mathcal{O} = \mathbb{Z}/r\mathbb{Z}$. Let $\pi(0) = 0^{qr}$ and $\pi(1) = 0^{qr}1^r$. Let $m_k = n_k = 1$ for all k. Then it's an immediate implication of Lemma 3.16 that $p(q+1) - p(q)$ is eventually 1, and so $\limsup p(q)/q = 1$.

Also, $|v_0|a_0 \cdots a_k = qr$ and $\gcd(|v_{k+1}|, |v_k|) = \gcd(|v_k| + |v_{k-1}|, |v_k|) = \gcd(|v_{k-1}|, |v_k|)$ for all k. Then, since $\gcd(|v_1|, |v_0|) = \gcd(qr + qr + r, qr) = r$, $\gcd(|v_{k+1}|, |v_k|) = r$ for all k. Therefore, X defined as above has the claimed properties by Theorem 6.4. □

Finally, we address Examples 1.2-1.4 from the introduction. Example 1.2 is fairly straightforward; it is determined by substitutions with $|u_0| = |v_0| = 1$, $m_k = 3$ and $n_k = 5$. Therefore, by (2), $a_0 = 1$, all other $a_k = 2$, and all $b_k = 3$. The verification that $\limsup p(q)/q < 3/2$ follows from Corollary 3.17 and Remark 3.14. Namely, by Corollary 3.17, the limsup of $p(q)/q$ is achieved along the sequence $q_k = |s_k v_k^{n_k-2}p_k| = |s_k| + 3|v_k| + |p_k|$, which equals $\sum_{i=0}^{k-1} 3|v_i| + 4|v_k|$ by Remark 3.14. The value of $p(q_k)$ is equal to $q_k + \sum_{i=0}^{k}(n_k - m_k - 1)|v_i| = \sum_{i=0}^{k-1} 4|v_i| + 5|v_k| + C$ for some constant C. Finally, we note that by the Perron-Frobenius theorem, the lengths $|v_i|$ grow exponentially with base the Perron eigenvalue of the incidence matrix $\left(\begin{smallmatrix} 2 & 4 \\ 1 & 1 \end{smallmatrix}\right)$, which is $\kappa = \frac{3+\sqrt{17}}{2}$. Therefore, $\frac{\sum_{i=0}^{k-1}|v_i|}{|v_k|} \to \frac{1}{\kappa - 1} = \frac{\sqrt{17}-1}{8}$, and so

$$p(q_k)/q_k = \frac{\sum_{i=0}^{k-1} 4|v_i| + 5|v_k| + C}{\sum_{i=0}^{k-1} 3|v_i| + 4|v_k|} \to \frac{4(\sqrt{17}-1)/8 + 5}{3(\sqrt{17}-1)/8 + 4} = \frac{105 + \sqrt{17}}{86} \approx 1.2689 < 3/2.$$

It remains to verify that the MEF is a rotation of $\mathcal{M}_2 = (\mathbb{R} \times \mathbb{Q}_2)/\mathbb{Z}[1/2]$. As noted above, $a_0 = 1$ and all a_k for $k > 0$ equal 2. Also, $|v_0| = 1$, and it is easily checked by induction that all $|v_k|$ are odd; since $\gcd(|v_{k+1}|, |v_k|)$ divides $|v_0|a_0 \ldots a_k = 2^k$ by Lemma 5.4, all $\gcd(|v_{k+1}|, |v_k|) = 1$. Therefore, in the language of Theorem 6.4, $\mathcal{O}_X$ is trivial and $\mathcal{M}_X$ is $\mathcal{M}_2$.

Since the computations are significantly more unpleasant, we omit details of Example 1.3, except to note the following differences from Example 1.2. First, the limsup of $p(q)/q$ is now achieved along the sequence $q_k = |s_k v_k^{r_k-1}u_k v_k^{r_k-1}p_k| = |s_k u_k p_k|$. Second, now every a_k for $k > 0$ is equal to $2^{r_k}(n_k - m_k) = 4$, and $\gcd(|v_k|, |v_{k+1}|) = 2^k$, which implies by Theorem 6.4 that $\mathcal{M}_X = \mathcal{M}_2$ and $\mathcal{O}_X$ is the binary odometer.

For Example 1.4, we cannot solve exactly for $\limsup p(q)/q$ since we do not have a closed form for m_k and n_k. However, we note that by Corollary 3.17, increasing m_k while keeping $n_k - m_k$ and r_k constant can only decrease this limsup; since (m_k, n_k) is always either $(3, 5)$ or $(5, 7)$, this limsup is then clearly less than or equal to $\frac{105+\sqrt{17}}{86}$ from Example 1.2. As in Example 1.2, $a_0 = 1$ and all other $a_k = 2$. It is easily checked by induction and the definition of the ρ_k that for all k, $|v_k|$ is divisible by 2^k, but not by 2^{k+1}. Therefore, $\gcd(|v_k|, |v_{k+1}|) = 2^k = |v_0|a_0 \cdots a_k$, and so by Theorem 6.4, $\mathcal{M}_X$ is $\mathbb{R}/\mathbb{Z} = S^1$ and $\mathcal{O}_X$ is the binary odometer.

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