Low complexity subshifts have discrete spectrum

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Abstract We prove results about subshifts with linear (word) complexity, meaning that $\limsup \frac{p(n)}{n} < \infty$, where for every n, $p(n)$ is the number of n-letter words appearing in sequences in the subshift. Denoting this limsup by C, we show that when $C < \frac{4}{3}$, the subshift has discrete spectrum, i.e. is measurably isomorphic to a rotation of a compact abelian group with Haar measure. We also give an example with $C = \frac{3}{2}$ which has a weak mixing measure. This partially answers an open question of Ferenczi, who asked whether $C = \frac{5}{3}$ was the minimum possible among such subshifts; our results show that the infimum in fact lies in $[\frac{4}{3}, \frac{3}{2}]$. All results are consequences of a general S-adic/substitutive structure proved when $C < \frac{4}{3}$.

Introduction

The main objects of study in symbolic dynamics are **subshifts**, which are dynamical systems defined by a finite alphabet $\mathcal{A}$, a closed shift-invariant set of sequences $X \subset \mathcal{A}^{\mathbb{Z}}$, and the left-shift map σ. We sometimes speak of subshifts as measure-theoretic dynamical systems by associating a measure μ; in this case μ is always assumed to be a Borel probability measure invariant under σ. One of the most basic ways to measure the 'size' of a subshift X is the **word complexity function** $p(n)$, which measures the number of finite words of length n which appear within points of X. In addition to being intimately connected with the fundamental notion of topological entropy (the entropy $h(X)$ is just the exponential growth rate of $p(n)$ when $p(n)$ grows exponentially), many recent works prove that slow growth of $p(n)$ forces various strong structural properties of X.

The well-known Morse-Hedlund theorem implies that if X is infinite, then $p(n) \geq n + 1$ for all n. There are subshifts which achieve this minimal value (i.e. $p(n) = n + 1$ for all n), which are called **Sturmian** subshifts. We do not give a full treatment here, but briefly say that Sturmian subshifts are defined by symbolic codings of orbits for irrational circle rotations, and in fact are measure-theoretically isomorphic to these rotations (associated with Lebesgue measure).

Slightly above the minimum possible complexity is the property of linear complexity, meaning that $\limsup p(n)/n = C < \infty$. This implies a great deal about X; a full list is beyond this work, but we list a few such results here. In the following, X is **transitive** when there exists $x \in X$ whose **orbit** $\{\sigma^n x\}$ is dense in X, and **minimal** when every $x \in X$ has dense orbit.

1. If X is transitive, then the number of ergodic measures on X is bounded from above by $\lfloor C \rfloor$. If $C < 3$, then in fact there is only one σ-invariant measure on X, in which case X is said to be **uniquely ergodic**. ([Bos92], [DOP22])
2. For all X, the number of nonatomic generic measures on X is bounded from above by $\lfloor C \rfloor$ ([CK19])
3. If X is minimal, then the automorphism group of X is virtually $\mathbb{Z}$ (in particular, there are at most $\lfloor C \rfloor$ cosets once one mods out by the shift action) ([CK15], [DDMP16])
4. If X is minimal, then X has finite topological rank ([DDMP21])
5. X cannot have any nontrivial strongly mixing measure ([Fer96])

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6. If X is transitive and $C < 1.5$, then X is minimal ([OP19])

(In fact, the weaker condition $\liminf p(n)/n < \infty$ is sufficient for some of the structure above, but as our results don't involve this quantity, we don't comment on it further here.) The final item above is one of surprisingly few results proved about subshifts with C close to 1, and understanding more about the structure of such shifts was a main motivation of this work. In a sense, we show that for C sufficiently close to 1, a subshift must have structure more and more similar to the Sturmian subshifts, which achieve minimal possible complexity. Recall that Sturmian subshifts are measure-theoretically isomorphic to a (compact abelian) group rotation; this property is called **discrete spectrum**. In fact this property is equivalent to $L^2(X)$ being spanned by the measurable eigenfunctions of σ (i.e. f for which $f(\sigma x) = \lambda f(x)$ for some λ). When X has no eigenfunctions at all, it is said to be **weak mixing**, which is in a sense an opposite property to discrete spectrum.

Ferenczi ([Fer96]) proved that the property of **strong mixing** (which means that $\mu(A \cap \sigma^{-n}B) \to \mu(A)\mu(B)$ for all measurable A, B) cannot hold for any nontrivial measure on a linear complexity subshift. He also gave an example of X with a strongly mixing measure and $p(n)$ quadratic and asked whether this complexity was the lowest possible. This was proved not to be the case in [Cre22a] and [CPR22], which provided examples first on the order of $n \log n$, and then below any possible superlinear growth rate, establishing linear complexity as the 'threshold' for existence of such a measure. In a different work, Ferenczi ([Fer95]) examined the same question for weakly mixing measures, where it is known that linear complexity can occur via the well-known Chacon subshift. He there gave an example of X with a nontrivial weakly mixing measure and $C = 5/3$, and again asked whether this was minimal. This was shown not to be the case in [Cre22b], where examples were given of C arbitrarily close to (but above) $3/2$.

Our main results are the following.

Theorem 1. If X is an infinite transitive subshift with $\limsup \frac{p(q)}{q} < \frac{4}{3}$, then X is uniquely ergodic with unique measure which has discrete spectrum.

Theorem 2. There exists an infinite transitive subshift X which is uniquely ergodic, has unique measure which is weak mixing, and for which $\limsup \frac{p(q)}{q} = \frac{3}{2}$.

In [Cre22b], it was also suggested that perhaps a subshift X having a nontrivial weakly mixing measure forces $\limsup \frac{p(q)}{q} > \frac{3}{2}$; Theorem 2 answers this negatively. In fact, the examples from Theorem 2 satisfy $\lim p(q) - 1.5q = -\infty$, in contrast to Theorem C from [Cre22b], which showed that for rank-one subshifts, even total ergodicity implies $\limsup p(q) - 1.5q = \infty$. The examples also satisfy $\liminf \frac{p(q)}{q} = 1$ and for any $f(q) \to \infty$, there exist examples such that $p(q) < q + f(q)$ infinitely often.

The proof of Theorem 1 depends on proving a substitutive structure for subshifts with $C < \frac{4}{3}$. In fact, for any $C < 2$, Corollary 5.28 from [PS22] already implies that X can be generated by a sequence of substitutions τ_k on the alphabet $\{0, 1\}$; this is known as having **alphabet rank two**. Similar results from [DDMP21] prove that even $\liminf p(n)/n < \infty$ implies finite alphabet rank. However, in general it is not so easy to prove dynamical properties of a subshift purely from such a structure; the key of our arguments is that when C is closer to 1, these substitutions come from a very restricted class.

Specifically, our Proposition 2.1 shows that any such subshift is induced by a sequence of substitutions of the form $\tau_{m_k,n_k} : 0 \mapsto 0^{m_k-1}1, 1 \mapsto 0^{n_k-1}1$ where $n \leq 2m$ for $m > 1$ and $n \leq 3$ for $m = 1$. This is related to the well-known Pisot conjecture for subshifts, which states that a subshift generated by iterating a single substitution τ should have discrete spectrum if the associated matrix (in which the (a, b) entry is the number of occurrences of b in $\tau(a)$) has largest eigenvalue which is a Pisot number (i.e. a complex number with modulus greater than 1 all of whose conjugates have modulus less than 1).

The Pisot conjecture has been proved in some settings, including when $|\mathcal{A}| = 2$ ([HS03]) and whenever the so-called balanced pair algorithm terminates ([SS02]). Our proof of Theorem 1 is in fact based on this algorithm.

In our case, the substitutive structure comes from a sequence of substitutions and not a single one; this is sometimes called the S-adic Pisot conjecture, based on the often-used term 'S-adic' (among other references, see [DLR13]) to refer to sequences obtained by a sequence of substitutions on a fixed alphabet. This is much more difficult. The strongest result is due to [BST19], which is too long to state formally here, but which proves discrete spectrum in a fairly general S-adic setting. They do require, however, that the sequence of substitutions (τ_n) be recurrent, meaning that for every k, there exists L so that $\tau_i = \tau_{i+L}$ for $1 \leq i \leq k$.

We cannot enforce any such condition on our substitutions, as it's quite possible to have low complexity for τ_{m_k,n_k} all distinct (for instance, consider Sturmian subshifts, which can be generated by an infinite sequence of distinct substitutions if the digits of its continued fraction expansion are distinct). Nevertheless, due to the extremely simple form of τ_{m_k,n_k} (in which both 0 and 1 are mapped to words of the form 0^i1), we are able to prove discrete spectrum.

We note that indeed our substitutive structure is in some sense Pisot; the associated matrix for $\tau_{m,n}$ is $\left(\begin{smallmatrix} m-1 & 1 \\ n-1 & 1 \end{smallmatrix}\right)$, whose eigenvalues are $\frac{\sqrt{m^2+4(n-m)}\pm m}{2}$. This matrix is Pisot when $m < n \leq 2m$. Our Proposition 2.1 implies $m < n \leq 2m$, with the possible exception $m = 1, n = 3$. Though this substitution is not Pisot, Proposition 2.1 implies that when it occurs, the previous substitution has $n = m + 1$, and the composition of those substitutions has matrix $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} m-1 & 1 \\ m & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} m & 1 \\ 3m-2 & 3 \end{smallmatrix}\right)$, which is always Pisot.

One of course should not expect that simply assuming each τ_i to be Pisot should guarantee discrete spectrum; informally, if the second eigenvalues have moduli each less than 1 but which converge to 1 quickly, then the 'average behavior' will be that of a non-Pisot number. This is essentially the construction of our example from Theorem 2, which not only does not have discrete spectrum, but is weak mixing (i.e. has no eigenvalue at all).

1 Definitions and preliminaries

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}$; the **full shift** is the set $\mathcal{A}^{\mathbb{Z}}$ associated with the product topology. We use σ to denote the left shift homeomorphism on $\mathcal{A}^{\mathbb{Z}}$. A **subshift** is a closed σ-invariant subset $X \subset \mathcal{A}^{\mathbb{Z}}$. The **orbit** of $x \in X$ is the set $\{\sigma^n x\}_{n\in\mathbb{Z}}$. A subshift X is **transitive** when it is the closure of the orbit of a single sequence x, and **minimal** when it is the closure of the orbit of every $x \in X$. For a minimal subshift X, in a slight abuse of notation, we sometimes refer to X as the orbit closure of a one-sided sequence $y \in \mathcal{A}^{\mathbb{N}}$; this simply means that X is the orbit closure of a two-sided sequence $x \in X$ containing y.

A **word** is any element of $\mathcal{A}^n$ for some $n \in \mathbb{N}$, referred to as its **length** and denoted by $|w|$. We denote $\mathcal{A}^* = \bigcup_{n\geq 1} \mathcal{A}^n$. We represent the concatenation of words $w_1, w_2, \ldots, w_n$ by $w_1 w_2 \ldots w_n$.

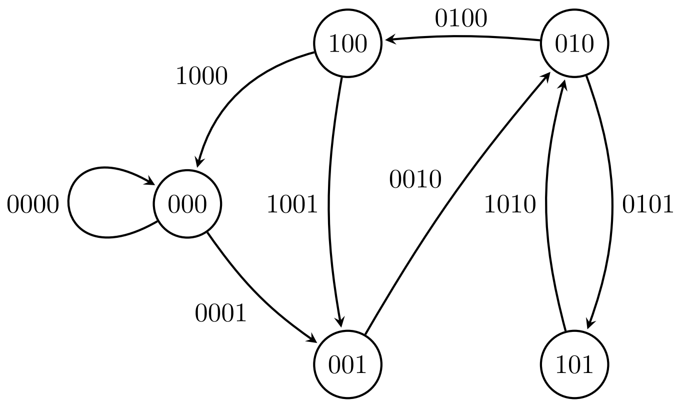
The **language** of a subshift X on $\mathcal{A}$, denoted $L(X)$, is the set of all finite words appearing as subwords of points in X. For any $n \in \mathbb{N}$, we denote $L_n(X) = L(X) \cap \mathcal{A}^n$, the set of n-letter words in $L(X)$. For a subshift X, the **word complexity function** of X is defined by $p(n) := |L_n(X)|$. For a subshift X and word $w \in L(X)$ we denote by $[w]$ the clopen subset in X consisting of all $x \in X$ such that $x_0 \ldots x_{|w|-1} = w$.

One way to generate subshifts is via substitutions. A **substitution** (sometimes called a morphism) is a map $\tau : \mathcal{A} \to \mathcal{B}^*$ for finite alphabets $\mathcal{A}$ and $\mathcal{B}$. Substitutions can be composed when viewed as homomorphisms on the monoid of words under composition, i.e. if $\tau : \mathcal{A} \to \mathcal{B}^*$ and $\rho : \mathcal{B} \to \mathcal{C}^*$, then $\rho\circ\tau : \mathcal{A} \to \mathcal{C}^*$ can be defined by $(\rho\circ\tau)(a) = \rho(b_1)\rho(b_2)\ldots\rho(b_k)$, where $\tau(a) = b_1 \ldots b_k$. When a sequence of substitutions $\tau_k : \mathcal{A} \to \mathcal{A}^*$ shares the same alphabet, and when there exists $a \in \mathcal{A}$ for which $\tau_k(a)$ begins with a for all k, clearly $(\tau_1 \circ \cdots \circ \tau_k)(a)$ is a prefix of $(\tau_1 \circ \cdots \circ \tau_{k+1})(a)$ for all k. In this situation one may then speak of the (right-infinite) limit of $(\tau_1 \circ \cdots \circ \tau_k)(a)$.

For any subshift X, there is a convenient way to represent the n-language and possible transitions between words in points of X by a directed graph called the Rauzy graph.

Definition 1.1. For a subshift X and $n \in \mathbb{N}$, the *nth Rauzy graph of* X is the directed graph $G_{X,n}$ with vertex set $L_n(X)$, and directed edges from $w_1 \ldots w_n$ to $w_2 \ldots w_{n+1}$ for all $w_1 \ldots w_{n+1} \in L_{n+1}(X)$.

Example 1.2. If X is the golden mean subshift consisting of bi-infinite sequences on $\{0, 1\}$ without consecutive 1s, and $n = 3$, then $G_{X,3}$ is the following directed graph:



There is a natural association from bi-infinite paths on the Rauzy graph to sequences in $\mathcal{A}^{\mathbb{Z}}$; a sequence of vertices (v_k) corresponds to the sequence $x \in A^{\mathbb{Z}}$ defined by $x(k) \dots x(k+n-1) = v_k$ for all k. The main usage of the Rauzy graph is that every point of X corresponds to a bi-infinite path in the Rauzy graph. However, the opposite is not necessarily true; if X has restrictions/forbidden words of length greater than $n+1$, then there may be paths in the Rauzy graph whose associated sequences are not in X. However, when X has low word complexity function, the set of paths in the Rauzy graph is sufficiently restrictive to give us useful information about (but not necessarily a complete description of) X.

We note that when X is transitive, $G_{X,n}$ is strongly connected for all n, i.e. there is a path between any two vertices. Rauzy graphs are particularly useful for working with so-called left/right special words in $L(X)$.

Definition 1.3. A word $w \in L(X)$ is **left-special** (resp. **right-special**) if there exist $a \neq b \in \mathcal{A}$ so that $aw, bw \in L(X)$ (resp. $wa, wb \in L(X)$). A word is **bi-special** if it is both left- and right-special.

For a given n, the left- and right-special words in $L_n(X)$ correspond to vertices of $G_{X,n}$ with multiple incoming/outgoing edges respectively. When $G_{X,n}$ has relatively few such vertices, large portions of bi-infinite paths are 'forced' in the sense that when such a path visits a vertex which is not right-special, there is only one choice for the following edge. Note that if X contains no right-special words of some length n, then any edge of $G_{X,n}$ forces all subsequent edges, meaning that $G_{X,n}$ has only finitely many bi-infinite paths and X is finite. Therefore every infinite subshift X has right-special words of every length, and a similar argument shows that it has left-special words of every length as well.

A particularly simple case that we deal with repeatedly is when $p(n+1) - p(n) = 1$; this means that $G_{X,n}$ has exactly one more edge than the number of vertices, which means that it has a single vertex r with two outgoing edges and a single vertex ℓ with two incoming edges (ℓ and r may be the same vertex), which correspond to the unique right- and left-special words in $L_n(X)$. It's not hard to show that when X is transitive and $p(n+1) - p(n) = 1$, the structure of the Rauzy graph $G_{X,n}$ must be a (possibly empty) path from ℓ to r and two edge-disjoint paths from r to ℓ.

We will frequently make use of the following standard lemma for estimating word complexity.

Lemma 1.4. *Let X be a subshift on alphabet $\mathcal{A}$, for all n let $RS_n(X)$ denote the set of right-special words of length n in the language of X, and for all right-special w, let $F(w)$ denote the set of letters which can follow w, i.e. $\{a : wa \in L(X)\}$. Then, for all $q > r$,*

$$p(q) = p(r) + \sum_{i=r}^{q-1} \sum_{w \in RS_i(X)} (|F(w)| - 1).$$

Proof. Consider the map $f : L_{r+1}(X) \to L_r(X)$ obtained by removing the final letter, i.e. $f(wa) = w$. It's clear that f is surjective and that $|f^{-1}(w)| = 1$ for w which is not right-special and $|f^{-1}(w)| =$

$|F(w)|$ for $w \in RS_r(X)$. The result for $q = r + 1$ follows immediately, and the general case follows by induction. □

The following corollary is immediate.

Corollary 1.5. *If X is an infinite subshift and $T \subset \mathbb{N}$ denotes the set of lengths n for which $|RS_n(X)| > 1$, then for all $q > r$,*

$$p(q) \geq p(r) + (q - r) + |T \cap \{r, \dots, q - 1\}|.$$

If $|RS_i(X)| \leq 2$ for all $m \leq i < n$ and $|F(w)| = 2$ for all right-special w with lengths in $[r, q)$, then the inequality above is an equality.

2 Structure of subshifts with $C < 4/3$

As mentioned above, our results rely on a substitutive/S-adic structure for subshifts with sufficiently low complexity. The substitutions in question all have the same form. Namely, for all positive integers $m < n$, define the substitution

$$\tau_{m,n} : \begin{cases} 0 \mapsto 0^{m-1}1 \\ 1 \mapsto 0^{n-1}1. \end{cases}$$

When $m_1, \dots, m_k$ and $n_1, \dots, n_k$ are understood, we use the shorthand notation

$$\rho_k = \tau_{m_1,n_1} \circ \cdots \circ \tau_{m_k,n_k}.$$

Proposition 2.1. *If X is an infinite transitive subshift with $\limsup \frac{p(q)}{q} < \frac{4}{3}$, then there exists a substitution $\pi : \{0, 1\} \to \mathcal{A}^*$ where $\pi(0), \pi(1)$ begin with different letters and $|\pi(0)| < |\pi(1)| < 2|\pi(0)|$ and sequences $(m_k), (n_k)$ satisfying $0 < m_k < n_k$ so that X is the orbit closure of*

$$x^{(m_k),(n_k)} = \lim_k (\pi \circ \tau_{m_1,n_1} \circ \cdots \circ \tau_{m_k,n_k})(0) = \lim_k \pi(\rho_k(0)).$$

In addition,

- *$n_k \leq 2m_k$ whenever $m_k > 1$;*
- *$n_k < 1.9m_k$ whenever $m_k > 4$;*
- *$n_k \leq 3$ whenever $m_k = 1$;*
- *if $m_{k+1} = 1, n_{k+1} = 3$ then $n_k = m_k + 1$; and*
- *every right-special word of length at least $|s(\pi(0))^{m_1-1}|$, where s is the maximal common suffix of $(\pi(0))^\infty$ and $(\pi(0))^\infty \pi(1)$, is a suffix of a concatenation of $\pi(0)$ and $\pi(1)$.*

Definition 2.2. A word v is a **root** of w if $|v| \leq |w|$ and w is a suffix of the left-infinite word v^∞. The **minimal root** of w is the shortest v which is a root of w.

Every word w has a unique minimal root since it is a root of itself (and all roots of w are suffixes of w).

Lemma 2.3 ([Cre22a] Lemma 5.7)**.** *If w and v are words with $|v| \leq |w|$ such that wv has w as a suffix then v is a root of w.*

Lemma 2.4 ([Cre22a] Lemma 5.8)**.** *If $uv = vu$ then u and v are multiples of the same word, i.e. there exists a word v_0 and integers $t, s > 0$ such that $u = v_0^t$ and $v = v_0^s$.*

Lemma 2.5. *Let u and v be words with $|v| < |u|$. Let s be the maximal common suffix of v^∞ and $v^\infty u$. If $|s| \geq |vu|$ then u and v are multiples of the same word.*

Proof. If $|s| \geq |vu|$ then s has vu as a suffix. Since v is a root of s, v is a root of u so $u = u'v^t$ for some $t \geq 1$ and suffix u' of v. Then s has $u'v^tv$ as a suffix since that is a suffix of v^∞ and $|s| \geq |u'v^tv|$. Then

uv is a suffix of s so $uv = vu$ as they are both suffixes of s and have the same length so Lemma 2.4 gives the claim. □

Lemma 2.6. *Let v and u be words with $|v| < |u|$ which are not multiples of the same word and where v is a suffix of u. Let s be the maximal common suffix of v^∞ and $v^\infty u$ (which must be finite by Lemma 2.5). Then s is a suffix of any left-infinite concatenation of u and v.*

Proof. By Lemma 2.5, $|s| < |vu|$ so we need only verify that s is a suffix of uv^q for $q \geq 1$ and of uu. Since v is a suffix of u, uu has vu as a suffix hence has s as a suffix. If $|s| \geq |u|$ then v is a root of u so $u = u'v^t$ and $uv^q = u'v^tv^q$ is a suffix of v^∞ so s is a suffix of uv^q. If $|s| < |u|$ then $u = u_0s'v^t$ for some (possibly empty) suffix s' of v and $t \geq 1$ (as $s = s'v^t$ has v as a root and $|s| \geq |v|$ as v is a suffix of u). Then $uv^q = u_0s'v^{t+q}$ has $s = s'v^t$ as a suffix. □

Lemma 2.7. *Let v and u be words and s be the maximal common suffix of v^∞ and $v^\infty u$. Let y and z be suffixes of some (possibly distinct) concatenations of u and v, both of length at least $|s|$. Then for any word w, the maximal common suffix of yvw and zuw is sw.*

Proof. Since y is a suffix of a concatenation of u and v, so is yv. Then yv has sv as a suffix by Lemma 2.6. Likewise zu has su as a suffix. As s is a suffix of v^∞, then so is yv. Likewise, zu is a suffix of $v^\infty u$. Therefore the maximal common suffix of yv and zu is s (as they are both at least as long as s). □

Lemma 2.8. *If $p(q+1) - p(q) = 1$ then there exists a bi-special word which has length in $[q, q+p(q)]$, has exactly two successors, and is the unique right-special word of its length and also the unique left-special word of its length.*

Proof. Let w be the unique right-special word of length q (which must have exactly two successors) and y be the unique left-special word and write z for the label of the path from y to w in the Rauzy graph. Then $|z| \leq p(|w|)$. The word yz is left-special and right-special and $|yz| = |y| + |z| \leq q + p(q)$.

If x is a word of the same length as yz which is right-special then x must have w as a suffix. Then $x = x_0w$ and $|x_0| = |z|$. Since there is only one path in the Rauzy graph ending at w of length $|z|$ (due to y being the unique left-special word), we have that $x = yz$. □

Lemma 2.9. *Let X be an infinite transitive subshift with $p(q) \leq \frac{4}{3}q$ for all sufficiently large q. Then there exist words a and b which begin with different letters with $|a| < |b| < 2|a|$ and $p(q) < \frac{4}{3}q$ for all $q \geq |a|$ and where a is a root of b such that every $x \in X$ can be written in exactly one way as a concatenation of a and b. If we define s to be the maximal common suffix of a^∞ and $a^\infty b$, there exists $t \geq 0$ so sa^t is the unique right-special and left-special word of its length.*

Proof. There exist infinitely many q such that $p(q+1) - p(q) = 1$ by Corollary 1.5. By Lemma 2.8, there exists a bi-special word w with $|w|$ arbitrarily large which is the unique left-special and right-special word of its length and which has exactly two successors. We may assume $p(q) \leq \frac{4}{3}q$ for all $q \geq |w|$. We note that by [OP19], X is infinite and minimal.

Let u and v be the shortest two return words for w (meaning wu and wv both have w as a suffix) which will be the labels of the two paths from w to itself in the Rauzy graph $G_{X,|w|}$ for words of length $|w|$, with v being the shorter of the two. All bi-infinite words in X can be written in exactly one way as a concatenation of v and u, as every such word must be the label of a path in the Rauzy graph (which visits the vertex w infinitely many times by minimality of X), and the only two such paths have labels v and u.

Since $|u| + |v| \leq p(|w|) + 1 \leq \frac{4}{3}|w| + 1$, we have $2|v| \leq \frac{4}{3}|w| + 1$ so $|v| \leq \frac{2}{3}|w| + \frac{1}{2}$. This is less than $|w|$ (since $|w| > 1$), and so v is a root of w by Lemma 2.3. Note that v cannot be a proper multiple of any word since if $v = v_0^t$ then wv_0 has w as a suffix so v_0 is a root of w making v_0 a return word for w which is shorter than v.

Observe that if $|w| < 3|v|$ then $|u| \leq \frac{4}{3}|w| + 1 - |v| < \frac{4}{3}|w| - \frac{1}{3}|w| + 1$ so u is a suffix of w making v a root of u. We write $u = u^\star v^s$ for some proper suffix $u^\star$ of v (which cannot be empty as u and v start with different letters) and define $a = v$ and $b = u^\star v$. Then as before, every bi-infinite word in X can be written uniquely as a concatenation of $v = a$ and $u = ba^{s-1}$, hence the same is true of a and b (since $a = v$). Clearly a is a root of b, and $|a| < |b| < 2|a|$ as $0 < |u^\star| < |a|$.

So assume from here on that $|w| \geq 3|v|$.

Suppose now that for every suffix w_0 of w with $|v| \leq |w_0| < 2|v|$, we have $p(|w_0| + 1) - p(|w_0|) \geq 2$. Then, by Corollary 1.5, $p(2|v|) = p(2|v|) - p(|v|) + p(|v|) \geq 2(2|v| - |v|) + |v| + 1 = 3|v| + 1$ so $\frac{p(2|v|)}{2|v|} > \frac{3}{2}$, contradicting our hypothesis.

Therefore there exists w_0 a suffix of w with $|v| \leq |w_0| < 2|v|$ which is the unique right-special word of its length and it has exactly two successors.

Since w_0 is a suffix of w, v is a root of w_0. As there must also be a unique left-special word of the same length as w_0, w_0 extends to a bi-special word w_{00} which is the unique left-special and right-special word of its length and which has exactly two successors (Lemma 2.8). Now $|w_{00}| \leq |w_0| + |v|$ since the path from the left-special to the right-special vertex in the Rauzy graph for words of length $|w_0|$ must be no longer than v (as $w_0 v$ must have w_0 as a suffix). Then $|w_{00}| < 2|v| + |v| = 3|v| \leq |w|$ so w_{00} is a proper suffix, and prefix, of w.

Let v_0 and u_0 be the shortest return words for w_{00} with v_0 beginning with the same letter as v (and u_0 beginning with a different letter). Then all bi-infinite words in X are concatenations of u_0 and v_0. Since v is a return word for w_{00}, v must be a concatenation of u_0 and v_0 which means that v_0 must be a prefix of v by virtue of sharing a common first letter. Likewise u_0 must be a prefix of u.

Since v is a suffix of w, then vv_0 has v as a suffix so v_0 is a root of v by Lemma 2.3. Write $v = v' v_0^t$ for some $t \geq 1$ and v' a proper suffix of v_0. Then $v_0 = v'' v'$ so v has $v' v_0 = v' v'' v'$ as a prefix. But v_0 is also a prefix of v so both $v' v''$ and $v'' v'$ are prefixes of v. Therefore they are equal so by Lemma 2.4 both are multiples of the same word. But then v is a multiple of that word and it cannot be a proper multiple of any word so either v' or v'' is empty and so $v_0 = v$.

If $|u_0| \leq |v|$ then u_0 is a root of w_{00} hence of v. Write $v = v^\star u_0^s$ for some proper suffix $v^\star$ of u_0 (which cannot be empty as v begins with a different letter than u) and $s \geq 1$. Taking $a = u_0$ and $b = v^\star u_0$, then every bi-infinite word in X is a concatenation of $u_0 = a$ and $v = ba^{s-1}$. Clearly a is a root of b and $|a| < |b| < 2|a|$.

So we are left with $|u_0| > |v|$. Here $|u_0| \leq p(|w_{00}|) + 1 - |v| < \frac{4}{3}|w_{00}| + 1 - \frac{1}{3}|w_{00}|$ as $|w_{00}| < 3|v|$. Therefore $|u_0| \leq |w_{00}|$. So u_0 is a suffix of w hence v is a root of u_0. Writing $u_0 = u^\star v^s$ for some proper suffix $u^\star$ of v and $s \geq 1$ then taking $a = v$ and $b = u^\star v$, just as before we have that every bi-infinite word in X is a unique concatenation of $v = a$ and $u_0 = ba^{s-1}$, hence of a and b. As before, clearly a is a root of b and $|a| < |b| < 2|a|$.

In all cases, one of a, b is a prefix of u and the other is a prefix of v. Since u and v begin with different letters, a and b begin with different letters. It remains to verify the claim about the maximal common suffix s and that a may be taken arbitrarily long.

In the case when a is a root of w (and w_{00} was not introduced), set $w_{00} = w$ and $t = 0$. Then in all cases, a is a root of w_{00} as a is either v or u_0 so w_{00} is a suffix of a^∞. In all cases, ba^t is the other return word for w_{00} for some $t \geq 0$. Then $w_{00} a^\ell b a^t$ has w_{00} as a suffix for all $\ell \geq 0$ so w_{00} is a suffix of $a^\infty b a^t$. Since w_{00} is left-special and a and ba^t are its two return words, the maximal common suffix of a^∞ and $a^\infty b a^t$ must be no longer than w_{00}. Therefore $w_{00} = sa^t$ where s is the maximal common suffix of a^∞ and $a^\infty ba$.

Let $\{w_\ell\}$ be a sequence of such bi-special words with $|w_\ell|$ increasing to ∞ and let $\{a_\ell\}$ and $\{v_\ell\}$ be the corresponding a and v above. Since either $a = v$ or $a = u_0$, and in both cases it is a root of w_{00}, a_ℓ is a root of v_ℓ.

Since w_ℓ is the unique right-special word of its length, it is a suffix of $w_{\ell+1}$ and therefore v_ℓ is a suffix of $v_{\ell+1}$. If $|v_\ell|$ were bounded then there would exist L such that $v_\ell = v_L$ for $\ell \geq L$ but then v_L would be a root of w_ℓ for $\ell \geq L$ so $v_L^\infty \in X$, a contradiction. So $|v_\ell| \to \infty$. Likewise, since a_ℓ is a root of v_ℓ, if

$|a_\ell|$ were bounded then for some L we would have $a_L^\infty \in X$. Therefore $|a_\ell| \to \infty$ so we may take a and b such that for all $q \geq |a|$, we have $p(q) < \frac{4}{3}q$. □

The following lemma is our main tool to recursively demonstrate the structure from Proposition 2.1. The key is control over the lengths of the suffixes from Lemmas 2.5 and 2.6.

Lemma 2.10. *Let X be an infinite transitive subshift with $\frac{p(q)}{q} < \frac{4}{3}$ for $q > N$. Let u and v be words with $N < |v| < |u|$ such that v is a suffix of u and v is not a prefix of u. Let s be the maximal common suffix of v^∞ and $v^\infty u$ and let p be the maximal common prefix of u and v.*

Assume that $|p| + |s| < |u| + |v|$ and $|p| + |s| < 3|v|$ and that every bi-infinite word in X can be written as a concatenation of u and v. Then there exist $0 < m < n$ such that every concatenation of u and v which represents a point in X has only v^{m-1} and v^{n-1} appearing between nearest occurrences of u and satisfying:

- *$n \leq 2m$ whenever $m > 1$;*
- *$n < 1.9m$ whenever $m > 4$;*
- *$n \leq 3$ whenever $m = 1$*

and the words $sv^{n-2}p$ and $sv^{m-1}uv^{m-1}p$ are right-special.

Proof. For brevity, whenever we refer to a 'concatenation' in the following, it is a concatenation of u, v which represents a point of X or a subword of such a point. We again note that by [OP19], X is infinite and minimal, and so no concatenation can contain infinitely many consecutive v. Similarly, if there was only a single number of v which may occur between nearest occurrences of u, then X would be finite, contradicting our assumptions. So there are at least two different numbers of v which can occur between nearest occurrences of u.

Suppose for a contradiction that uv^xu and uv^yu and uv^zu all appear in some concatenations and that $x < y < z$. We may assume that x is the minimal value such that uv^xu appears in a concatenation. Since uv^xu and uv^yu are necessarily preceded by v^x (due to x being minimal), then v^xuv^xu and v^xuv^xv both appear in concatenations (as $y > x$). By Lemma 2.6 (as v is not a prefix of u, they cannot be multiples of the same word), s is a suffix of every left-infinite concatenation. This means that v^xuv^xu and v^xuv^xv are both preceded by s in the bi-infinite concatenations they respectively appear in, and so sv^xuv^x can be followed by either u or v, meaning that sv^xuv^xp is right-special (since the letters appearing after p in u and v are distinct by maximality of p).

Likewise, v^xuv^yu and v^xuv^yv appear in some concatenations (due to $z > y$) so sv^xuv^yp is also right-special. By Lemma 2.7, the maximal common suffix of sv^xuv^xp and sv^xuv^yp is sv^xp. Therefore there are at least two right-special words of length ℓ for $|sv^xp| < \ell \leq |sv^xuv^xp|$ (namely, the unequal suffixes of sv^xuv^xp and sv^xuv^yp of length ℓ). Then, since $|p| + |s| < |v| + |u| < 2|u|$, by Corollary 1.5

$$\frac{p(|sv^xuv^xp|)}{|sv^xuv^xp|} \geq 1 + \frac{|sv^xuv^xp| - |sv^xp|}{|sv^xuv^xp|} = 1 + \frac{x|v| + |u|}{|p| + |s| + 2x|v| + |u|} > 1 + \frac{x|v| + |u|}{2|u| + 2x|v| + |u|}.$$

The final expression is increasing for $x \geq 0$, hence is at least $\frac{4}{3}$ (its value at $x = 0$), contradicting our hypothesis that $p(q)/q < \frac{4}{3}$ for $q > N$. Therefore such $x < y < z$ cannot exist so there are only two distinct values x and y. Writing $x = m - 1$ and $y = n - 1$ then shows that v^{m-1} and v^{n-1} are the only words appearing between occurrences of u in a concatenation.

By similar reasoning as above, we observe that $sv^{m-1}uv^{m-1}p$ is right-special and that $sv^{n-2}p$ is also right-special since $sv^{n-1}u$ appears in a concatenation and it has $sv^{n-2}v$ as a prefix and $sv^{n-2}u$ as a suffix. Again by similar reasoning as above, their maximal common suffix is $sv^{m-1}p$.

Suppose $|sv^{m-1}uv^{m-1}p| \leq |sv^{n-2}p|$. Then there are at least two right-special words of length ℓ for $|sv^{m-1}p| < \ell \leq |sv^{m-1}uv^{m-1}p|$ so, by Corollary 1.5 and the fact that $|p| + |s| < |u| + |v| < 2|u|$,

$$\frac{p(|sv^{m-1}uv^{m-1}p|)}{|sv^{m-1}uv^{m-1}p|} \geq 1 + \frac{(m-1)|v| + |u|}{|p| + |s| + 2(m-1)|v| + |u|} > 1 + \frac{(m-1)|v| + |u|}{2(m-1)|v| + 3|u|} \geq \frac{4}{3}$$

which contradicts our hypothesis.

So instead $|sv^{n-2}p| < |sv^{m-1}uv^{m-1}p|$. Then there are at least two right-special words of length ℓ for $|sv^{m-1}p| < \ell \leq |sv^{n-2}p|$ so, by Corollary 1.5 and the fact that $|p| + |s| < 3|v|$,

$$\frac{p(|sv^{n-2}p|)}{|sv^{n-2}p|} \geq 1 + \frac{(n-m-1)|v|}{|p|+|s|+(n-2)|v|} > 1 + \frac{(n-m-1)|v|}{3|v|+(n-2)|v|} = 1 + \frac{n-m-1}{n+1}.$$

Consider first when $m = 1$. If $n \geq 4$ then $\frac{n-m-1}{n+1} = \frac{n-2}{n+1} \geq \frac{2}{5} > \frac{1}{3}$ which contradicts our hypothesis.

Now consider when $m > 1$. If $n \geq 2m+1$ then $\frac{n-m-1}{n+1} \geq \frac{2m+1-m-1}{2m+1+1} = \frac{m}{2m+2} \geq \frac{2}{2(2)+2} = \frac{1}{3}$ contradicting our hypothesis. So $n \leq 2m$ when $m > 1$.

Finally, consider when $m \geq 5$. Suppose $n \geq 1.9m$. Then

$$\frac{n-m-1}{n+1} \geq \frac{1.9m-m-1}{1.9m+1} = \frac{0.9m-1}{1.9m+1} \geq \frac{4.5-1}{9.5+1} = \frac{1}{3}$$

contradicting our hypothesis. So $n < 1.9m$ whenever $m > 4$. □

Proof of Proposition 2.1. We prove by induction that such sequences exist, using the notation $v_k := \pi(\rho_{k-1}(0))$ and $u_k := \pi(\rho_{k-1}(1))$.

By [OP19], X is minimal. Write s_k for the maximal common suffix of v_k^∞ and $v_k^\infty u_k$ and p_k for the maximal common prefix of v_k and u_k.

Our inductive hypotheses are the following:

- all $x \in X$ can be written as concatenations of u_k and v_k;
- v_k is a suffix of u_k and is not a prefix of u_k;
- $|p_k| + |s_k| < \min(|v_k| + |u_k|, 3|v_k|)$;
- $v_k = (\pi \circ \tau_{m_1,n_1} \circ \cdots \circ \tau_{m_{k-1},n_{k-1}})(0) = \pi(\rho_{k-1}(0))$ and $u_k = (\pi \circ \tau_{m_1,n_1} \circ \cdots \circ \tau_{m_{k-1},n_{k-1}})(1) = \pi(\rho_{k-1}(1))$.

Since $\limsup \frac{p(q)}{q} < \frac{4}{3}$, eventually $p(q) < \frac{4}{3}q$. Lemma 2.9 gives v_1 and u_1 with v_1 a suffix of u_1 and $|v_1| < |u_1| < 2|v_1|$ which start with different letters such that every infinite word is a concatenation of u_1 and v_1. By Lemma 2.5, $|s_1| < |v_1u_1| < 3|v_1|$. As u_1 and v_1 begin with different letters, p_1 is empty. Therefore the base case is established by setting $\pi(0) = v_1$ and $\pi(1) = u_1$. Lemma 2.9 ensures that $p(q) < \frac{4}{3}q$ for all $q \geq |\pi(0)|$.

Given v_k and u_k, by Lemma 2.10 there exist $0 < m_k < n_k$ such that every infinite word is a concatenation of $v_{k+1} = v_k^{m_k-1}u_k$ and $u_{k+1} = v_k^{n_k-1}u_k$. Observe that $u_{k+1} = v_k^{n_k-1}u_k = (\pi(\rho_{k-1}(0)))^{n_k-1}\pi(\rho_{k-1}(1)) = \pi(\rho_{k-1}(0^{n_k-1}1)) = \pi(\rho_{k-1}(\tau_{m_k,n_k}(1))) = \pi(\rho_k(1))$ and similarly $v_{k+1} = \pi(\rho_k(0))$.

Clearly v_{k+1} is a suffix of u_{k+1}. If v_{k+1} were a prefix of u_{k+1} then u_k would be a prefix of $v_k^{n_k-m_k}u_k$ but that would make v_k a prefix of u_k. So v_{k+1} is not a prefix of u_{k+1}, and $p_{k+1} = v_k^{m_k-1}p_k$.

By definition, s_{k+1} is the maximal common suffix of v_{k+1}^∞ and $v_{k+1}^\infty u_{k+1}$. We can rewrite these as $y = \ldots u_k v_k^{m_k-1}u_k$ and $z = \ldots v_k v_k^{m_k-1}u_k$. These share a suffix of $v_k^{m_k-1}u_k$, so we must just find the maximal common suffix of the portions with this removed, i.e. $y' = \ldots u_k$, a concatenation ending with u_k, and $z' = \ldots v_k$, a concatenation ending with v_k. But y' then agrees with $v_k^\infty u_k$ on a suffix of length $|u_k|+|s_k| > |s_k|$ by Lemma 2.6 and z' agrees with v_k^∞ on a suffix of length $|v_k|+|s_k| > |s_k|$ by Lemma 2.6, meaning that y' and z' have maximal common suffix s_k. Therefore, $s_{k+1} = s_k v_k^{m_k-1}u_k = s_k v_{k+1}$. Then,

$$|p_{k+1}| + |s_{k+1}| = |p_k| + |s_k| + 2(m_k-1)|v_k| + |u_k| < (2m_k-1)|v_k| + 2|u_k| = 2|v_{k+1}| + |v_k|$$

and since $|v_{k+1}| + |v_k| \leq |u_{k+1}|$ and $|v_k| < |v_{k+1}|$, the inductive hypotheses are verified.

Lemma 2.10 gives that $n_k \leq 2m_k$ when $m_k > 1$ and $n_k \leq 1.9m_k$ when $m_k > 4$ and that $n_k \leq 3$ when $m_k = 1$.

Suppose that $m_k = 1$ and $n_k = 3$ and $n_{k-1} \geq m_{k-1} + 2$. By Lemma 2.10, the words $s_k v_k p_k$ and

$s_{k-1}v_{k-1}^{n_{k-1}-2}p_k$ and $s_k u_k p_k$ are right-special. By Lemma 2.7, the maximal common suffix of $s_k v_k p_k$ and $s_k u_k p_k$ is $s_k p_k$. Using Lemma 2.6 and that $p_k = v_{k-1}^{m_{k-1}-1}p_{k-1}$, both $s_k v_k p_k$ and $s_k u_k p_k$ have $s_{k-1}u_{k-1}v_{k-1}^{m_{k-1}-1}p_{k-1}$ as a suffix. By Lemma 2.7, the maximal common suffix of either of them and $s_{k-1}v_{k-1}^{n_{k-1}-2}p_{k-1}$ is then $s_{k-1}v_{k-1}^{m_{k-1}-1}p_{k-1}$. Therefore there are least $|s_k v_k p_k| + |s_k v_k p_k| - |s_k p_k| + |s_{k-1}v_{k-1}^{n_{k-1}-1}p_{k-1}| - |s_{k-1}v_{k-1}^{m_{k-1}-1}p_{k-1}|$ right-special words of length at most $|s_k v_k p_k|$.

Since $p_k = v_{k-1}^{m_k-1}p_{k-1}$, $s_k = s_{k-1}v_k$ and $|p_{k-1}| + |s_{k-1}| < 3|v_{k-1}|$,

$$|p_k| + |s_k| = (m_{k-1}-1)|v_{k-1}| + |v_k| + |p_{k-1}| + |s_{k-1}| < |v_k| + (m_{k-1}+2)|v_{k-1}| = 2|v_k| - |u_k| + 3|v_{k-1}|.$$

Therefore, since $n_{k-1} \geq m_{k-1}+2$,

$$\frac{p(|s_k v_k^{n_k-2}p_k|)}{|s_k v_k^{n_k-2}p_k|} \geq 1 + \frac{|v_k| + (n_{k-1}-m_{k-1}-1)|v_{k-1}|}{|v_k| + |p_k| + |s_k|} > 1 + \frac{|v_k| + |v_{k-1}|}{3|v_k| - |u_{k-1}| + 3|v_{k-1}|} > 1 + \frac{1}{3}$$

contradicting our hypothesis. So if $m_k = 1$ and $n_k = 3$ then $n_{k-1} = m_{k-1}+1$.

Since $s_1 v_1^t$ is the unique right-special and unique left-special word of its length for some $t \geq 0$ (Lemma 2.9) and $u_1 v_1^t$ and v_1 are the two return words for $s_1 v_1^t$, we have that $t \leq m_1 - 1$ as u_1 is always followed by v_1^t. Since $s_1 v_1^t$ is left-special, $u_1 v^t s_1 v^t$ must appear meaning that $t = m_1 - 1$. Therefore any right-special word of length at least $|s_1 v_1^{m_1-1}|$ must have $s_1 v_1^{m_1-1}$ as a suffix. As the return words for $s_1 v_1$ are v_1 and $u_1 v_1^{m_1-1}$, then every right-special word of at least that length is a suffix of a concatenation of u_1 and v_1.

Finally, since v_k is in the language for all k, there exists a two-sided sequence containing $x^{(m_k),(n_k)} = \lim v_k$. Then since X is minimal, X is the orbit closure of $x^{(m_k),(n_k)}$. □

Remark 2.11. In future arguments, for any subshift X satisfying the structure of Proposition 2.1, we use the notation of the proof, i.e. $u_k = \pi(\rho_{k-1}(1))$, $v_k = \pi(\rho_{k-1}(0))$, p_k is the maximal prefix of v_k and u_k, and s_k is the maximal suffix of v_k^∞ and $v_k^\infty u_k$. In addition, as shown in the proof of Proposition 2.1, the sequence (p_k) satisfies the recursion $p_{k+1} = v_k^{m_k-1}p_k = v_k p_{k+1}$, the sequence (s_k) satisfies the recursion $s_{k+1} = s_k v_{k+1}$, and $|p_k| + |s_k| < \min(|u_k| + |v_k|, 3|v_k|)$ for all k.

Remark 2.12. By induction on k, each substitution $\pi \circ \rho_k$ is uniquely decomposable, in the sense that each $x \in X$ can be decomposed uniquely into words $(\pi \circ \rho_k)(a)$ for $a \in \{0,1\}$. For $k = 0$, this follows from Lemma 2.9 since $\pi(0) = v_1$ and $\pi(1) = u_1$ were constructed using that lemma. If $\pi \circ \rho_k$ is uniquely decomposable, then every x is representable uniquely as a concatenation of $(\pi \circ \rho_k)(0)$ and $(\pi \circ \rho_k)(1)$, and then the same must be true of $(\pi \circ \rho_{k+1})(0) = (\pi \circ \rho_k)(0)^{m_{k+1}-1}(\pi \circ \rho_k)(1)$ and $(\pi \circ \rho_{k+1})(1) = (\pi \circ \rho_k)(0)^{n_{k+1}-1}(\pi \circ \rho_k)(1)$ (since each of these contains $(\pi \circ \rho_k)(1)$ exactly once.)

3 Subshifts with $C < 4/3$ have discrete spectrum

Theorem 1. *If X is an infinite transitive subshift with $\limsup \frac{p(q)}{q} < \frac{4}{3}$, then X is uniquely ergodic with unique measure which has discrete spectrum.*

Our proof relies on first proving exponential decay of some quantities, which will later be used to verify discrete spectrum via so-called mean almost periodicity.

Proposition 3.1. *Let X be the orbit closure of $x^{(m_k),(n_k)}$ where $(m_k),(n_k)$ satisfy the conclusions of Proposition 2.1. Then there exist ϵ_k which converge to 0 exponentially so that for every k,*

$$\frac{(n_{k+1}+1)|\pi(0)| \prod_{i=1}^{k}(n_i - m_i)}{|(\pi \circ \rho_k)(0)|} < \epsilon_k.$$

Proof. We first set some preliminary notation. Define $a_1 = 1$ and $a_k = n_{k-1} - m_{k-1}$ and $b_k = m_k$ for $k > 0$. Note that by Proposition 2.1, all b_k and a_k are positive; $a_{k+1} \leq b_k$ whenever $b_k > 1$; $a_{k+1} < 0.9b_k$

whenever $b_k > 4$; and $a_{k+1} \leq 2$ whenever $b_k = 1$. We also define $d_k = |(\pi \circ \rho_k)(0)|$, and note that (d_k) satisfies the recursion

$$d_{k+1} = b_{k+1}d_k + a_{k+1}d_{k-1} \tag{1}$$

where $d_{-1} = |\pi(1)| - |\pi(0)|$ and $d_0 = |\pi(0)|$.

For ease of notation, define

$$\beta_j = \frac{a_{j+1}d_{j-1}}{d_j}$$

and observe that, by (1),

$$\beta_{j+1} = \frac{a_{j+2}d_j}{d_{j+1}} = \frac{a_{j+2}}{b_{j+1} + a_{j+1}\frac{d_{j-1}}{d_j}} = \frac{a_{j+2}}{b_{j+1} + \beta_j}.$$

Note that $\beta_0 = \frac{a_1 d_{-1}}{d_0} = \frac{d_{-1}}{|\pi(0)|}$. Then

$$\frac{|\pi(0)|a_1 \cdots a_{k+1}}{d_k} = \frac{|\pi(0)|}{d_{-1}} \prod_{j=0}^{k} \frac{a_{j+1}d_{j-1}}{d_j} = \frac{|\pi(0)|}{d_{-1}} \beta_0 \prod_{j=1}^{k} \beta_j = \prod_{j=1}^{k} \beta_j. \tag{2}$$

Claim. $0 < \beta_j < 2$ for all $j \geq 0$.

Proof. Since $a_{j+1} \leq b_j + 1$ for all j, $\beta_j \leq \frac{b_j+1}{b_j+\beta} < 1 + \frac{1}{b_j} \leq 2$. □

Claim. If $a_{j+1} \leq b_j$ then $\beta_j < 1$.

Proof. Since $\beta_{j-1} > 0$, $\beta_j = \frac{a_{j+1}}{b_j+\beta_{j-1}} < \frac{a_{j+1}}{b_j} \leq 1$. □

Claim. If $a_{j+1} = b_j + 1$ then at least one of $\beta_j < 1$ or $\beta_j\beta_{j-1} \leq 1$.

Proof. When $a_{j+1} = 2$ and $b_j = 1$, by Proposition 2.1, $\tau_{1,3}$ cannot occur for consecutive values so we have $a_j \leq b_j$ so $\beta_{j-1} \leq 1$. Since $\beta_j = \frac{2}{1+\beta_{j-1}} \geq 1$, we have $\beta_j\beta_{j-1} = 2 - \beta_j \leq 1$. □

This implies $\prod_{j=1}^{k} \beta_j \leq \beta_1 \leq 2$.

By the assumptions on (m_k) and (n_k), we see that $a_{k+1} \leq b_k$ when $b_k > 1$ and $a_{k+1} \leq 2$ when $b_k = 1$ and $a_{k+1} < 0.9b_k$ when $b_k > 4$. We now break into several cases.

Case 1: If $b_j > 4$ then $\beta_j < 0.9$.

Proof. If $b_j > 4$ then, as $d_j > b_j d_{j-1}$ by (1), $\beta_j = \frac{a_{j+1}d_{j-1}}{d_j} < 0.9$. □

Case 2: If $a_{j+1} \leq b_j \leq 4$ and $b_{j-1} \leq 4$ then $\beta_j < 0.96$.

Proof. If $a_{j+1} \leq b_j \leq 4$ and $b_{j-1} \leq 4$ then by (1),

$$d_j = b_j d_{j-1} + a_j d_{j-2} \leq b_j d_{j-1} + (b_{j-1}+1)d_{j-2} < b_j d_{j-1} + d_{j-1} + d_{j-2} \leq (b_j+2)d_{j-1} \leq 6d_{j-1}.$$

Then, again by (1), using that $a_{j+1} \leq b_j$,

$$\frac{d_j}{d_{j-1}} = b_j + \frac{a_j d_{j-2}}{d_{j-1}} > b_j + \frac{1}{6} \geq a_{j+1} + \frac{1}{6}.$$

Therefore, since $a_{j+1} \leq b_j \leq 4$,

$$\beta_j = \frac{a_{j+1}d_{j-1}}{d_j} < \frac{a_{j+1}}{a_{j+1} + (1/6)} = \left(1 + \frac{1}{6a_{j+1}}\right)^{-1} < \left(1 + \frac{1}{24}\right)^{-1} = 0.96.$$ □

Case 3: If $a_{j+1} \leq b_j \leq 4$ and $b_{j-1} > 4$ then at least one of $\beta_j < 0.96$ or $\beta_j\beta_{j-1} < 0.5$ holds.

Proof. Consider when $a_{j+1} \leq b_j \leq 4$ and $b_{j-1} > 4$ so $\beta_{j-1} < 0.96$. Suppose $\beta_j > \frac{8}{9}$. Then

$$\frac{8}{9} < \frac{a_{j+1}}{b_j + \beta_{j-1}} \leq \frac{b_j}{b_j + \beta_{j-1}} \leq \frac{4}{4 + \beta_{j-1}}$$

so $8 + 2\beta_{j-1} < 9$ so $\beta_{j-1} < \frac{1}{2}$. Then $\beta_j\beta_{j-1} < \beta_{j-1} < 0.5$ since $a_{j+1} \leq b_j$ implies $\beta_j < 1$. So at least one of $\beta_j \leq \frac{8}{9} < 0.96$ or $\beta_j\beta_{j-1} < 0.5$ must hold. □

Any j where $a_{j+1} \leq b_j$ is covered by Case 1 if $b_j > 4$ and Case 2 or 3 if $b_j \leq 4$. The only remaining case is then $a_{j+1} > b_j$, which happens only if $a_{j+1} = 2$ and $b_j = 1$.

Case 4: If $a_{j+1} = 2$ and $b_j = 1$ then at least one of $\beta_j\beta_{j-1} < \frac{48}{49}$ or $\beta_j\beta_{j-1}\beta_{j-2} < 0.52$ holds.

Proof. Consider any such j. By Proposition 2.1, $\tau_{1,3}$ cannot occur consecutively so $a_j \leq b_{j-1}$, and so $j-1$ is in one of Cases 1-3. If $\beta_{j-1} < 0.96$, then

$$\beta_j\beta_{j-1} = \frac{a_{j+1}}{b_j + \beta_{j-1}}\beta_{j-1} = \frac{2\beta_{j-1}}{1 + \beta_j - 1} = 1 + \frac{\beta_{j-1} - 1}{\beta_{j-1} + 1} < 1 + \frac{0.96 - 1}{0.96 + 1} = \frac{48}{49}.$$

If $\beta_{j-1} \geq 0.96$, then $j-1$ must be in Case 3 and $\beta_{j-1}\beta_{j-2} < 0.5$. Then

$$\beta_j\beta_{j-1}\beta_{j-2} = \frac{2}{1 + \beta_{j-1}}\beta_{j-1}\beta_{j-2} < \frac{1}{1 + \beta_{j-1}} \leq \frac{1}{1 + 0.96} < 0.52.$$ □

Claim. For all $k \geq 1$,

$$\prod_{j=1}^{k} \beta_j < 2\Big(\frac{48}{49}\Big)^{k/2}.$$

Proof. All $j > 2$ are in one of the cases above, and so at least one of the following hold: $\beta_j < 0.96$, $\beta_j\beta_{j-1} < \frac{48}{49}$, or $\beta_j\beta_{j-1}\beta_{j-2} < 0.52$. For every k, we can group the product $\prod_{j=1}^{k} \beta_j$ into products of one, two, or three consecutive terms bounded from above in this way, with the possible exception of β_1 or $\beta_1\beta_2$. As $0.96 < \sqrt{\frac{48}{49}}$ and $0.52 < (\frac{48}{49})^{3/2}$, and since $\beta_1\beta_2 < 1$ whenever $\beta_1 > 1$, this yields

$$\prod_{j=1}^{k} \beta_j < \beta_1\Big(\frac{48}{49}\Big)^{k/2} < 2\Big(\frac{48}{49}\Big)^{k/2}.$$ □

Since $n_{k+1} \leq 2m_{k+1} + 1 = 2b_{k+1} + 1$, we have $\frac{(n_{k+1}+1)d_k}{d_{k+1}} \leq \frac{(2b_{k+1}+2)d_k}{b_{k+1}d_k} = 2 + \frac{2}{b_{k+1}} \leq 4$, and so

$$\frac{n_{k+1}|\pi(0)|\prod_{i=1}^{k}(n_i - m_i)}{d_{k+1}} = \frac{n_{k+1}d_k}{d_{k+1}}\frac{|\pi(0)|a_1 \cdots a_{k+1}}{d_k} \leq 4\prod_{j=1}^{k} \beta_j < 8\left(\frac{48}{49}\right)^{k/2}.$$

Defining $\epsilon_k := 8(\frac{48}{49})^{k/2}$ completes the proof. □

Proof of Theorem 1. Our technique for verifying discrete spectrum of X is by using mean almost periodicity, which requires a definition. The **upper density** of $A \subset \mathbb{N}$, denoted $\overline{d}(A)$, is $\limsup \frac{|A \cap \{1,\ldots,n\}|}{n}$. It's easy to check that upper density is subadditive, i.e. $\overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B)$ for every A, B.

A subshift X is **mean almost periodic** if for all $\epsilon > 0$ and all $x \in X$, there exists a syndetic set S so that for all $s \in S$, x and $\sigma^s x$ differ on a set of locations with upper density less than ϵ. It is well-known that mean almost periodicity implies discrete spectrum; see for instance Theorem 2.8 of [LS09].

By Proposition 2.1, X is the orbit closure of

$$x^{(m_k),(n_k)} = \lim_{k\to\infty} (\pi \circ \tau_{m_1,n_1} \circ \tau_{m_2,n_2} \circ \cdots \tau_{m_k,n_k})(0) = \lim_{k\to\infty} (\pi \circ \rho_k)(0)$$

for some $\pi : \{0,1\} \to \mathcal{A}^*$ where $\pi(0), \pi(1)$ begin with different letters and $|\pi(0)| < |\pi(1)| < 2|\pi(0)|$ and some sequences $(m_k), (n_k)$ satisfying $0 < m_k < n_k \leq 2m_k$ or $(m_k, n_k) = (1,3)$.

We again use the notations $a_{k+1} = n_k - m_k$ and $d_k = |(\pi \circ \rho_k)(0)|$ as in the proof of Proposition 3.1.

For any $k > 0$ and $p \in \mathbb{N}$, define the words

$$y_{0,k,p} = ((\pi \circ \rho_k)(0))^p (\pi \circ \rho_k)(1),\ z_{0,k,p} = (\pi \circ \rho_k)(1)((\pi \circ \rho_k)(0))^p,$$
$$y_{1,k,p} = ((\pi \circ \rho_k)(1))^p (\pi \circ \rho_k)(0),\ z_{1,k,p} = (\pi \circ \rho_k)(0)((\pi \circ \rho_k)(1))^p.$$

We will prove the following by induction:

$$y_{i,k,p},\ z_{i,k,p} \text{ differ on fewer than } 2|\pi(1)|pa_1 \dots a_{k+1} \text{ locations } (i \in \{0,1\}). \tag{3}$$

The base case $k = 0$ trivially holds, since the lengths of $y_{0,0,p}, z_{0,0,p}, y_{1,0,p}, z_{1,0,p}$ are less than $2p|\pi(1)|$.

Assume now that (3) holds for some $k - 1$ (and all p).

Consider first the case when $n_k \leq 2m_k$.

Then by definition of τ_{m_k,n_k}, if we write $u = (\pi \circ \rho_{k-1})(1)$, $v = (\pi \circ \rho_{k-1})(0)$, $m = m_k$, and $n = n_k$, then $y_{0,k,p} = (v^{m-1}u)^p v^{n-1}u$ and $z_{0,k,p} = v^{n-1}u(v^{m-1}u)^p$.

Since v is a suffix of u, write $u = u'v$. Then, using that $m < n \leq 2m$,

$$\begin{aligned} y_{0,k,p} &= (v^{m-1}u)^p v^{n-1}u = (v^{m-1}u'v)^p v^{m-1}v^{n-m}u = v^{m-1}(u'v^m)^p v^{n-m}u \\ &= v^{m-1}(u'v^{n-m}v^{2m-n})^p v^{n-m}u, \\ z_{0,k,p} &= v^{n-1}u(v^{m-1}u)^p = v^{m-1}v^{n-m}(u'v^m)^p u = v^{m-1}v^{n-m}(u'v^{2m-n}v^{n-m})^p u \\ &= v^{m-1}(v^{n-m}u'v^{2m-n})^p v^{n-m}u. \end{aligned}$$

Since $|u'v^{n-m}| = |v^{n-m}u'|$, this means $y_{0,k,p}$ and $z_{0,k,p}$ differ at a number of locations equal to p times the number of locations where $u'v^{n-m}$ and $v^{n-m}u'$ differ. Clearly $u'v^{n-m}$ and $v^{n-m}u'$ differ on the same number of locations as $u'v^{n-m}v = uv^{n-m}$ and $v^{n-m}u'v = v^{n-m}u$ differ. Since $uv^{n-m} = z_{0,k-1,n-m}$ and $v^{n-m}u = y_{0,k-1,n-m}$, the inductive hypothesis gives that they differ on fewer than $2|\pi(1)|(n-m)a_1 \cdots a_k$ locations. Then $y_{0,k,p}$ and $z_{0,k,p}$ differ on fewer than $2|\pi(1)|p(n-m)a_1 \cdots a_k$ locations. Since $a_{k+1} = n - m$, this proves the claim. Similarly,

$$\begin{aligned} y_{1,k,p} &= (v^{n-1}u)^p v^{m-1}u = v^{n-1}(u'v^n)^{p-1}u'v^m u \\ &= v^{m-1}v^{n-m}(u'v^m v^{n-m})^{p-1}u'v^m u = v^{m-1}(v^{n-m}u'v^m)^p u, \\ z_{1,k,p} &= v^{m-1}u(v^{n-1}u)^p = v^{m-1}(u'v^n)^p u = v^{m-1}(u'v^{n-m}v^m)^p u. \end{aligned}$$

so $y_{1,k,p}$ and $z_{1,k,p}$ differ on fewer than $2|\pi(1)|pa_1 \cdots a_{k+1}$ locations.

Consider now the case when $n_k = 3$ and $m_k = 1$. Here

$$(\pi \circ \rho_k)(0) = (\pi \circ \rho_{k-1})(1), (\pi \circ \rho_k)(1) = ((\pi \circ \rho_{k-1})(0))^2 (\pi \circ \rho_{k-1})(1)$$

By Proposition 2.1, $n_{k-1} = m_{k-1} + 1$ so we have $(\pi \circ \rho_{k-1})(1) = (\pi \circ \rho_{k-2})(0)(\pi \circ \rho_{k-1})(0)$.

First consider when $m_{k-1} > 1$. Here $(\pi \circ \rho_{k-2})(0)$ is a prefix of $(\pi \circ \rho_{k-1})(0)$ so there are words $g = (\pi \circ \rho_{k-2})(0)$ and h such that $(\pi \circ \rho_{k-1})(0) = gh$ and $(\pi \circ \rho_{k-1})(1) = ggh$. Then $(\pi \circ \rho_k)(0) = ggh$ and $(\pi \circ \rho_k)(1) = (gh)^2 ggh$ so

$$\begin{aligned} y_{0,k,p} &= (ggh)^p (ghghggh) = ggh(ggh)^{p-1} ghghggh \\ z_{0,k,p} &= (ghghggh)(ggh)^p = ghg(hgg)^{p-1} hgghggh \end{aligned}$$

which differ on two pairs of gh and hg and on $p - 1$ pairs of ggh and hgg.

Our inductive hypothesis does apply directly to gh and hg, however gh and hg differ on the same number of letters as $ggh = (\pi \circ \rho_{k-2})(0)((\pi \circ \rho_{k-2})(0))^{m_{k-1}-1}(\pi \circ \rho_{k-2})(1)$ and $ghg = ((\pi \circ \rho_{k-2})(0))^{m_{k-1}-1}(\pi \circ \rho_{k-2})(1)(\pi \circ \rho_{k-2})(0)$. Those words differ on the same number of letters as $(\pi \circ \rho_{k-2})(0)(\pi \circ \rho_{k-2})(1)$

and $(\pi \circ \rho_{k-2})(1)(\pi \circ \rho_{k-2})(0)$, and by hypothesis they differ on fewer than $2|\pi(1)|a_1 \cdots a_{k-1}$ locations.

Similarly, $gggh = ((\pi \circ \rho_{k-2})(0))^{m_{k-1}+1}(\pi \circ \rho_{k-2})(1)$ and $ghgg = ((\pi \circ \rho_{k-2})(0))^{m_{k-1}-1}(\pi \circ \rho_{k-2})(1)((\pi \circ \rho_{k-2})(0))^2$ differ on the same number of letters as $(\pi \circ \rho_{k-2})(1)((\pi \circ \rho_{k-2})(0))^2$ and $((\pi \circ \rho_{k-2})(0))^2(\pi \circ \rho_{k-2})(1)$ which by hypothesis is fewer than $2|\pi(1)|2a_1 \cdots a_{k-1}$ locations.

Therefore $y_{0,k,p}$ and $z_{0,k,p}$ differ on fewer than $2 \cdot 2|\pi(1)|a_1 \cdots a_{k-1} + 2(p-1)2|\pi(1)|a_1 \cdots a_{k-1}$ locations. Since $a_k = 1$ and $a_{k+1} = 2$, they differ on fewer than $2|\pi(1)|pa_1 \cdots a_{k+1}$ locations. Similarly,

$$y_{1,k,p} = (ghghggh)^p ggh = ghg(hgghghg)^{p-1}hgghggh$$
$$z_{1,k,p} = ggh(ghghggh)^p = ggh(ghghggh)^{p-1}ghghggh$$

differ on two pairs of gh and hg and on $p-1$ pairs of $hgghghg$ and $ghghggh$. As $hgghghg$ and $ghghggh$ differ on two pairs of gh and hg, the total number of differences is $2p$ times the number of differences between gh and hg. Since gh and hg differ on fewer than $2|\pi(1)|a_1 \cdots a_{k-1}$ locations, and since $a_k = 1$ and $a_{k+1} = 2$, $y_{1,k,p}$ and $z_{1,k,p}$ differ on fewer than $2|\pi(1)|pa_1 \cdots a_{k+1}$ locations.

Now consider when $m_{k-1} = 1$. Here $(\pi \circ \rho_{k-1})(0) = (\pi \circ \rho_{k-2})(1)$ so $(\pi \circ \rho_{k-2})(0)$ is a suffix of $(\pi \circ \rho_{k-1})(1)$. So there are words $g = (\pi \circ \rho_{k-2})(0)$ and h such that $(\pi \circ \rho_{k-1})(0) = hg$. Then $(\pi \circ \rho_k)(0) = ghg$ and $(\pi \circ \rho_k)(1) = (hg)^2 ghg$ so

$$y_{0,k,p} = (ghg)^p hghgghg = gh(ggh)^{p-1}ghghgghg$$
$$z_{0,k,p} = hghgghg(ghg)^p = hg(hgg)^{p-1}hgghgghg$$

which differ on two pairs of gh and hg and on $p-1$ pairs of ggh and hgg. Since $gghg = (\pi \circ \rho_{k-2})(0)^2(\pi \circ \rho_{k-1})(0) = (\pi \circ \rho_{k-2})(0)^2(\pi \circ \rho_{k-2})(1)$ and $hggg = (\pi \circ \rho_{k-2})(1)((\pi \circ \rho_{k-2})(0))^2$, by hypothesis they differ on fewer than $2|\pi(1)|2a_1 \cdots a_{k-1}$ locations. Then, as above, $y_{0,k,p}$ and $z_{0,k,p}$ differ on fewer than $2|\pi(1)|pa_1 \cdots a_{k+1}$ locations. Similarly,

$$y_{1,k,p} = (hghgghg)^p ghg = hg(hgghghg)^{p-1}hgghgghg$$
$$z_{1,k,p} = ghg(hghgghg)^p = gh(ghghggh)^{p-1}ghghgghg$$

differ on $2p$ pairs of gh and hg so $y_{1,k,p}$ and $z_{1,k,p}$ differ on fewer than $2|\pi(1)|pa_1 \cdots a_{k+1}$ locations.

We will now prove that X is mean almost periodic. Fix any k, and as before, define $u = (\pi \circ \rho_{k-1})(1)$, $v = (\pi \circ \rho_{k-1})(0)$, $m = m_k$, and $n = n_k$. Choose any $y \in X$; by minimality of X, y can be written as a bi-infinite concatenation of the words $(\pi \circ \rho_k)(0) = v^{m-1}u$ and $(\pi \circ \rho_k)(1) = v^{n-1}u$. We may assume without loss of generality that y contains $v^{m-1}u$ starting at the origin, since any syndetic set S as in the definition of mean almost periodicity for y also works for any shift of y. Since $d_k = |v^{m-1}u|$, let us write

$$y = \ldots .v^{m-1}uv^{i_1-1}uv^{i_2-1}u \ldots$$
$$\sigma^{d_k} y = \ldots .v^{i_1-1}uv^{i_2-1}u \ldots$$

where each i_k is either m or n. We can rewrite as

$$y = \ldots .v^{m-1}(uv^{i_1-m})v^{m-1}(uv^{i_2-m})v^{m-1} \ldots$$
$$\sigma^{d_k} y = \ldots .v^{m-1}(v^{i_1-m}u)v^{m-1}(v^{i_2-m}u)v^{m-1} \ldots$$

The words inside parentheses are unequal exactly when $i_j = n$, in which case they are the pair uv^{n-m}, $v^{n-m}u$. Since the lengths of uv^{n-m} and $v^{n-m}u$ are the same, this means that the only differences in y and $\sigma^{d_k} y$ occur within pairs uv^{n-m}, $v^{n-m}u$. By (3), the number of differences in any such pair is bounded from above by $2|\pi(1)|(n-m)a_1 \ldots a_k = 2|\pi(1)|a_1 \ldots a_{k+1}$. When y is partitioned into its level-$(k+1)$ words $(\pi \circ \rho_{k+1})(0)$ and $(\pi \circ \rho_{k+1})(1)$ (and σ^{d_k} is partitioned at the same locations), each partitioned segment contains exactly one such pair uv^{n-m}, $v^{n-m}u$. Since each such segment has length at least $|(\pi \circ \rho_{k+1})(0)| = d_{k+1}$,

$$\overline{d}\left(\{t \ : \ y(t) \neq (\sigma^{d_k} y)(t)\}\right) \leq \frac{2|\pi(1)|a_1 \ldots a_{k+1}}{d_{k+1}}.$$

For ease of notation, we define $D_q = \{t \ : \ y(t) \neq y(t+q)\}$ for every q; by the above,

$$\overline{d}(D_{d_k}) \leq \frac{2|\pi(1)|a_1 \dots a_k a_{k+1}}{d_{k+1}}. \tag{4}$$

Now, fix any k and consider the set

$$S_k := \left\{ \sum_{i=k}^{r} p_i d_i \ : \ r > k, 0 \leq p_i \leq n_{i+1} + 1 \right\}.$$

We claim that S_k is syndetic. To see this, note that $n_{i+1}d_i > d_{i+1}$ for all i since $d_{i+1} = m_{i+1}d_i + (n_i - m_i)d_{i-1} \leq m_{i+1}d_i + (m_i+1)d_{i-1} < m_{i+1}d_i + d_i + d_{i-1} \leq (m_{i+1}+2)d_i \leq (n_{i+1}+1)d_i$, and so a simple greedy algorithm shows that for all $M \in \mathbb{N}$, there exists $s \in S_k$ with $M \leq s < M + d_k$.

Finally, choose any $s = \sum_{i=k}^{r} p_i d_i \in S_k$. For any $\ell_1, \ell_2 \in \mathbb{N}$, $D_{\ell_1+\ell_2} \subset D_{\ell_1} \cup (D_{\ell_2} - \ell_1)$ since $t \in D_{\ell_1+\ell_2}$ implies at least one of $y(t) \neq y(t+\ell_1)$ or $y(t+\ell_1) \neq y(t+\ell_1+\ell_2)$, and so $\overline{d}(D_{\ell_1+\ell_2}) \leq \overline{d}(D_{\ell_1}) + \overline{d}(D_{\ell_2})$. Using this repeatedly implies

$$\overline{d}(D_s) = \overline{d}\left(D_{\sum_{i=k}^{r} p_i d_i}\right) \leq \sum_{i=k}^{r} p_i \overline{d}(D_{d_i}) \leq \sum_{i=k}^{r} \frac{2|\pi(1)|n_{i+1}a_1 \dots a_{i+1}}{d_{i+1}}.$$

Proposition 3.1 implies that $\frac{(n_{i+1}+1)|\pi(0)|a_1 \cdots a_{i+1}}{d_{i+1}} < \epsilon_i$ for a sequence ϵ_i which is exponentially decaying. Then $\overline{d}(D_s) < \sum_{i=k}^{r} \frac{2|\pi(1)|}{|\pi(0)|}\epsilon_i$. Since (ϵ_i) is summable, the right-hand side becomes arbitrarily small as $k \to \infty$, and so X is mean almost periodic, and therefore has discrete spectrum. □

Remark 3.2. We remark that in fact this proof yields an explicit formula for an eigenvalue of X. Namely, define a sequence (c_k) by $c_{-1} = 1$, $c_0 = 0$, and the same recursion $c_{k+1} = b_{k+1}c_k + a_{k+1}c_{k-1}$. Basic continued fraction theory implies that $\frac{c_k}{d_k}$ approaches a limit α, and that for all k,

$$\left|\frac{c_k}{d_k} - \alpha\right| < \left|\frac{c_k}{d_k} - \frac{c_{k+1}}{d_{k+1}}\right| = \frac{|\pi(0)|a_1 \dots a_{k+1}}{d_k d_{k+1}} = \frac{|\pi(0)| \prod_{i=1}^{k}(n_i - m_i)}{d_k d_{k+1}}.$$

Therefore, the distance from $d_k\alpha$ to the nearest integer is less than $\frac{|\pi(0)| \prod_{i=1}^{k}(n_i - m_i)}{d_{k+1}}$, which decays exponentially by Proposition 3.1. If we define $\lambda = e^{2\pi i \alpha}$, then $\lambda^{d_k} = \lambda^{|(\pi\circ\rho_k)(0)|}$ approaches 1 with exponential rate. By definition, $|(\pi \circ \rho_k)(1)| = d_k + (n_k - m_k)d_{k-1}$. The distance from $(n_k - m_k)d_{k-1}\alpha$ to the nearest integer is less than $\frac{n_k|\pi(0)| \prod_{i=1}^{k-1}(n_i - m_i)}{d_k}$, which again decays exponentially by Proposition 3.1. Therefore, $\lambda^{|(\pi\circ\rho_k)(1)|}$ approaches 1 with exponential rate as well.

From this, an essentially identical argument to that of Host from [Hos86] (see also p. 170-171 from [Que10]) shows that λ is an eigenvalue (in fact a continuous one). (His argument was for a single substitution τ, but the construction can be done virtually without change with τ^k replaced by $\pi \circ \rho_k$.)

We can even represent α (and therefore λ) in terms of generalized continued fractions. If we defined an alternate sequence (e_k) by the same recursion with $e_{-1} = 0$ and $e_0 = 1$, then $\frac{c_k}{e_k}$ is just the kth convergent to the generalized continued fraction

$$\beta = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = \cfrac{1}{m_1 + \cfrac{n_1 - m_1}{m_2 + \cfrac{n_2 - m_2}{m_3 + \ddots}}}.$$

In particular, $\frac{c_k}{e_k} \to \beta$. Since $c_{-1} = 1, c_0 = 0, e_{-1} = 0, e_1 = 1$ and c_k, d_k and e_k are all defined by the same (linear) recursion, $d_k = d_{-1}c_k + d_0 e_k$ for all k. Then, as $d_{-1} = |\pi(1)| - |\pi(0)|$ and $d_0 = |\pi(0)|$,

$$\alpha = \lim \frac{c_k}{d_k} = \lim \Big(d_{-1} + d_0\Big(\frac{e_k}{c_k} - 1\Big)\Big)^{-1} = (d_{-1} + d_0(\beta^{-1} - 1))^{-1} = \frac{\beta}{|\pi(1)|\beta + |\pi(0)|(1-\beta)}$$

Therefore, the eigenvalue λ can be written as $\exp\left(2\pi i\left(\frac{\beta}{|\pi(1)|\beta+|\pi(0)|(1-\beta)}\right)\right)$.

4 A weak mixing subshift with $C = 3/2$

Theorem 2. *There exists an infinite transitive subshift X which is uniquely ergodic, has unique measure which is weak mixing, and for which* $\limsup \frac{p(q)}{q} = \frac{3}{2}$.

The complexity estimates in Theorem 2 will follow from a general formula for word complexity of subshifts with the structure from Proposition 2.1, which may be of independent interest.

Proposition 4.1. *Let X be the orbit closure of $x^{(m_k),(n_k)}$ for π and (τ_{m_k,n_k}) satisfying the conclusions of Proposition 2.1. Then there exists a constant K such that for $k \geq 2$,*

$$p(q) = \begin{cases} q + \sum_{j=2}^{k}(n_j - m_j - 1)|v_j| + K & \text{if } |s_k v_k^{n_k-2} p_k| \leq q \leq |s_{k+1} v_{k+1}^{m_{k+1}-1} p_{k+1}| \\ 2q - |s_k v_k^{m_k-1} p_k| + \sum_{j=2}^{k-1}(n_j - m_j - 1)|v_j| + K & \text{if } |s_k v_k^{m_k-1} p_k| \leq q \leq |s_k v_k^{n_k-2} p_k|. \end{cases}$$

Proof. We claim first that the words $p_\infty := \lim s_k p_k = \lim s_1 v_2 \cdots v_k v_k^{m_k-1} v_{k-1}^{m_{k-1}-1} \cdots v_1^{m_1-1}$ and $s_k v_k^{n_k-2} p_k$ for $k > 0$ are right-special.

Since $v_{k+1} = v_k^{m_k-1} u_k$ and $u_{k+1} = v_k^{n_k-1} u_k$, $p_{k+1} = v_k^{m_k-1} p_k$. By induction then $p_{k+1} = v_k^{m_k-1} \cdots v_1^{m_1-1}$ as p_1 is empty. By Lemma 2.6, $s_k p_k$ is a suffix of $s_{k+1} p_{k+1} = s_k v_k^{m_k-1} u_k v_k^{m_k-1} p_k$. As $|s_{k+1}| > |s_k|$, this shows p_∞ exists and is left-infinite.

By definition of p_k as the maximal common prefix, $p_k \pi(0)$ and $p_k \pi(1)$ are both in the language since each of u_k and v_k must have one of them as a prefix and they cannot have the same one. So p_k is right-special for each k (as $\pi(0)$ and $\pi(1)$ begin with different letters) hence p_∞ is right-special. That $s_k v_k^{n_k-2} p_k$ is right-special follows from Lemma 2.10.

Next we claim that every right-special word is a suffix of p_∞ or of $s_k v_k^{n_k-2} p_k$ for some $k > 0$.

Since every right-special word of length at least $|s_1 v_1^{m_1-1}|$ is a suffix of a concatenation of u_1 and v_1, any right-special word with $s_2 p_2 = s_1 v_1^{m_1-1} u_1 v_1^{m_1-1}$ as a suffix is of the form $x u_1 v_1^{m_1-1}$ where x is a suffix of a concatenation of u_1 and v_1. If x were not a suffix of a concatenation of v_2 and u_2 then $u_1 v_1^r u_1$ for $r \neq m_1 - 1, n_1 - 1$ must appear somewhere in x but this is impossible by definition of τ_{m_1,n_1}. So every right-special word with $s_2 p_2$ as a suffix is of the form $x p_2$ where x is a suffix of a concatenation of v_2 and u_2.

Assume that any word with $s_k p_k$ as a suffix is necessarily of the form $x p_k$ where x is a concatenation of u_k and v_k. Let w be a word which has $s_{k+1} p_{k+1}$ as a suffix. Since $s_{k+1} p_{k+1} = s_k v_{k+1} v_k^{m_k-1} p_k$ which has $s_k p_k$ as a suffix, $w = x v_{k+1} v_k^{m_k-1} p_k$ where x is a suffix of a concatenation of u_k and v_k. If x were not a suffix of a concatenation of u_{k+1} and v_{k+1} then somewhere in $x v_{k+}$ there must appear $u_k v_k^r u_k$ for $r \neq n_k - 1, m_k - 1$ or v_k^t for $t > n_k - 1$. But this is impossible by definition of τ_{m_k,n_k}. By induction, then for all k, any word with suffix $s_k p_k$ is of the form $x p_k$ where x is a suffix of a concatenation of u_k and v_k.

Since v_k is a suffix of u_k for $k > 1$, write $u_k = u'_k v_k^{\ell_k}$ for $\ell_k \geq 1$ maximal. Note that s_k has $v_k^{\ell_k}$ as a suffix.

Let w be a right-special word with $|w| \geq |s_1 p_1|$. Take $k \geq 1$ maximal so that w has $s_k p_k$ as a suffix. By the above, $w = x p_k$ is where x is a suffix of a concatenation of u_k and v_k in any left-infinite word. Choose $t \geq 0$ maximal so that $v_k^t p_k$ is a suffix of w.

Suppose w is not a suffix of $s_k v_k^{t-\ell_k} p_k$. Then $u_k v_k^{t-\ell_k} p_k$ must be right-special since all letters to the left of s_k are forced to come from u_k by maximality of ℓ_k. As the p_k must appear as a prefix of both v_k and u_k, then $u_k v_k^{t-\ell_k} u_k$ and $u_k v_k^{t-\ell_k} v_k$ are in the language so $t - \ell_k = m_k - 1$. But then w has $v_k^{m_k-1} p_k = p_{k+1}$ as a suffix, contradicting the maximality of k.

So w is a suffix of $s_k v_k^{t-\ell_k} p_k$. Suppose $t - \ell_k \geq n_k - 1$. Then w has $v_k^{n_k-1+\ell_k} p_k$ as a suffix. As w is right-special, this requires $v_k^{n_k-1+\ell_k} v_k$ be in the language. But that is only possible if u_k has $v_k^{\ell_k+1}$ as a suffix, contradicting the maximality of ℓ_k.

So $t - \ell_k \leq n_k - 2$. Then w, being a suffix of $s_k v_k^{t-\ell_k} p_k$, is a suffix of $s_k v_k^{n_k-2} p_k$.

Finally, we establish the complexity function is as claimed. Since p_∞ has $s_{k+1}p_{k+1} = s_k v_{k+1} v_k^{m_k-1} p_k$ as a suffix, by Lemma 2.6, it has $s_k u_k v_k^{m_k-1} p_k$ as a suffix. By Lemma 2.7, the maximal common suffix of p_∞ and $s_k v_k^{n_k-2} p_k$ is then $s_k v_k^{m_k-1} p_k$. Likewise the maximal common suffix of $s_k v_k^{n_k-2} p_k$ and $s_{k'} v_{k'}^{n_{k'}-2} p_{k'}$ for $k' > k$ is $s_k v_k^{m_k-1} p_k$ as v_{k+1} has u_k as a suffix. Therefore each $s_k v_k^{n_k-2} p_k$ provides $(n_k - 2 - (m_k - 1))|v_k|$ right-special words (with lengths in $(|s_k v_k^{m_k-1} p_k|, |s_k v_k^{n_k-2} p_k|])$ which are not suffixes of p_∞. Set $K = p(|s_2 p_2|) - |s_2 p_2|$ and the claim follows. □

Proof of Theorem 2. Define any increasing (n_k), (m_k) so that $n_k = 2m_k$ for all k, $m_1 = 1$, and the sum $\sum_k (n_k)^{-1} < \infty$. Then define π to be the identity, define τ_{m_k,n_k}, $\rho_k, a_k, b_k, c_k, d_k$ as in the proof of Proposition 3.1, and note that $a_{k+1} = n_k - m_k = m_k = b_k$ for all k and $\sum_k (b_k)^{-1} < \infty$. Just as before, $d_k = |\rho_k(0)|$ for all k, and we wish to impose the additional condition that d_k is prime for every $k > 1$. This is easily achieved via induction. First, $d_0 = d_1 = 1$, so $d_2 = b_2 d_1 + a_2 d_0 = m_2 + 1$, which can clearly be chosen prime. Then, assume that d_k is prime, and recall that $d_{k+1} = b_{k+1} d_k + a_{k+1} d_{k-1}$. Both $a_{k+1} = b_k$ and d_{k-1} are positive and less than the prime d_k (since $d_k = b_k d_{k-1} + a_k d_{k-2}$ and d_{k-2} is positive for $k > 1$), meaning that d_k and $a_{k+1} d_{k-1}$ are positive and coprime. Then by Dirichlet's theorem, there exist infinitely many choices of b_{k+1} so that d_{k+1} is prime. As long as the sequence (b_k) is chosen large enough at each step, we will maintain the condition $\sum_k (b_k)^{-1} < \infty$.

Let X be the orbit closure of $x^{(m_k),(n_k)}$. X is minimal by construction so by [Bos92], X is uniquely ergodic with unique measure μ.

Suppose for a contradiction that X is not weak mixing, and so there is an eigenvalue $\lambda \neq 1$ with measurable eigenfunction f. The following is somewhat classical and so we elide some details; see [Hos86].

One can define Rokhlin towers by $B_k = [\rho_k(0)]$, $h_k = |\rho_k(0)|$, and $T_k = \bigcup_{j=0}^{h_k-1} \sigma^j B_k$; since $m_k, n_k \to \infty$, $\mu(T_k) \to 1$. By Remark 2.12, X is uniquely decomposable so the levels of the towers are disjoint. Then, for each k, define

$$f_k(x) = \sum_{j=0}^{h_k-1} \frac{1}{\mu(B_k)} \Big(\int_{\sigma^j B_k} f \; d\mu \Big) \mathbb{1}_{\sigma^j B_k}(x)$$

i.e., $f_k(x) = (\mu(B_k))^{-1} (\int_{\sigma^j B_k} f \; d\mu)$ for $x \in \sigma^j B_k$ and $f_k(x) = 0$ for $x \notin T_k$.

By the Lebesgue Differentiation Theorem, as $\mu(T_k) \to 1$ and $\mu(\sigma^j B_k) \to 0$, f_k converge almost everywhere to f.

Observe that $\sigma^{d_k} = \sigma^{|\rho_k(0)|}$ takes every occurrence of $\rho_k(0)$ to an occurrence of $\rho_k(0)$ except for those which are immediately prior to an occurrence of $\rho_k(1)$ in some $\rho_{k+1}(0)$ or $\rho_{k+1}(1)$. Then for all $t > 0$, $\sigma^{d_{k+t}}$ takes all occurrences of $\rho_k(0)$ appearing in a $\rho_{k+t}(0)$ to an occurrence of $\rho_k(0)$ except possibly for those appearing in a $\rho_{k+t}(0)$ immediately prior to a $\rho_{k+t}(1)$.

Let $\{i_k\}$ be any sequence such that $0 < i_k < 0.5(m_{k+1} - 1)$. Then as above, for all $t > 0$, $\sigma^{i_{k+t} d_{k+t}}$ takes all occurrences of $\rho_k(0)$ in a $\rho_{k+t}(0)$ to an occurrence of $\rho_k(0)$ except possibly for those appearing in a $\rho_{k+t}(0)$ less than i_{k+t} occurrences before a $\rho_{k+t}(1)$ in some $\rho_{k+t+1}(0)$ or $\rho_{k+t+1}(1)$. We also note that since $n_{k+t+1} = 2m_{k+t+1}$, at least one-third of the $\rho_k(0)$ appearing in any $x \in X$ are part of some $\rho_{k+t}(0)$. Therefore,

$$\mu(\sigma^{i_{k+t} d_{k+t}} [\rho_k(0)] \cap [\rho_k(0)]) \geq \frac{m_{k+t} - 1 - i_{k+t}}{m_{k+t} - 1} \left(\frac{1}{3} \mu([\rho_k(0)]) \right)$$

so, since $i_{k+t} < 0.5(m_{k+t} - 1)$,

$$\mu(\sigma^{i_{k+t} d_{k+t}} (\sigma^j B_k) \cap (\sigma^j B_k)) > \frac{1}{6} \mu(\sigma^j B_k).$$

Then $f_k(\sigma^{i_{k+t} d_{k+t}} x) = f_k(x)$ for a set of measure at least $\frac{1}{6}\mu(T_k)$. Since $f_k \to f$ almost everywhere and $\mu(T_k) \to 1$, there is then a positive measure set such that for any sufficiently small $\epsilon > 0$ and almost every x in the set, there exists k so that for all t, $|f(\sigma^{i_{k+t} d_{k+t}} x) - f(x)| < \epsilon$. Therefore $\lambda^{i_k d_k} \to 1$.

We will prove that this is impossible. Define $r \in (0,1)$ by $\lambda = e^{2\pi i r}$; then $\langle i_k d_k r\rangle \to 0$ whenever $0 < i_k < 0.5(m_{k+1} - 1)$, which implies that for large enough k (say $k \geq k_0$), $\langle d_k r\rangle < 0.05(m_{k+1} - 1)^{-1}$. Clearly r cannot be rational, since all d_k are 1 or prime. Since $5n_{k+1} = 10m_{k+1} < 20(m_{k+1} - 1)$, for $k \geq k_0$, $\langle d_k r\rangle < 0.2(n_{k+1})^{-1}$. This implies that for all $k \geq k_0$, there exists $c'_k \in \mathbb{Z}$, so that $\left|r - \frac{c'_k}{d_k}\right| < 0.2(d_k n_{k+1})^{-1} < 0.2(d_{k+1})^{-1}$. (Recall that $d_{k+1} = b_{k+1}d_k + a_{k+1}d_{k-1} < 2b_{k+1}d_k = n_{k+1}d_k$.) We will prove the following: for all $k > k_0$,

$$c'_{k+1} = b_{k+1}c'_k + a_{k+1}c'_{k-1}. \tag{5}$$

Assume that $k > k_0$, and denote the right-hand side of (5) by c''_{k+1}. Then,

$$\left|r - \frac{c'_k}{d_k}\right| < 0.2(d_{k+1})^{-1} \text{ and } \left|r - \frac{c'_{k-1}}{d_{k-1}}\right| < 0.2(d_k)^{-1}, \tag{6}$$

and so

$$\left|d_{k+1}r - c'_k\frac{d_{k+1}}{d_k}\right| < 0.2. \tag{7}$$

We can simplify

$$\left|c'_k\frac{d_{k+1}}{d_k} - c''_{k+1}\right| = \left|c'_k\left(b_{k+1} + \frac{a_{k+1}d_{k-1}}{d_k}\right) - b_{k+1}c'_k - a_{k+1}c'_{k-1}\right| = \left|c'_k a_{k+1}\frac{d_{k-1}}{d_k} - a_{k+1}c'_{k-1}\right|. \tag{8}$$

By the second inequality in (6),

$$\left|a_{k+1}d_{k-1}r - a_{k+1}c'_{k-1}\right| < \frac{0.2a_{k+1}d_{k-1}}{d_k} = \frac{0.2b_k d_{k-1}}{d_k} < 0.2. \tag{9}$$

Similarly, by the first inequality in (6),

$$\left|a_{k+1}d_{k-1}r - c'_k a_{k+1}\frac{d_{k-1}}{d_k}\right| < \frac{0.2a_{k+1}d_{k-1}}{d_{k+1}} = \frac{0.2b_k d_{k-1}}{d_{k+1}} < 0.2. \tag{10}$$

Therefore, by the triangle inequality and (8)-(10),

$$\left|c'_k\frac{d_{k+1}}{d_k} - c''_{k+1}\right| < 0.4.$$

Combining with (7) via the triangle inequality yields

$$\left|d_{k+1}r - c''_{k+1}\right| < 0.6. \tag{11}$$

Recall that by definition,

$$\left|r - \frac{c'_{k+1}}{d_{k+1}}\right| < 0.2(d_{k+2})^{-1}, \text{ and so } \left|d_{k+1}r - c'_{k+1}\right| < 0.2\frac{d_{k+1}}{d_{k+2}} < 0.2. \tag{12}$$

Finally, (11) and (12) imply that $c'_{k+1} = c''_{k+1}$ (since they are both integers), completing the proof that (5) holds for $k > k_0$.

Since r is irrational and $\frac{c'_k}{d_k} \to r$, we may also assume without loss of generality (by increasing k_0) that $\frac{c'_{k_0}}{d_{k_0}} \neq \frac{c'_{1+k_0}}{d_{1+k_0}}$. Then, it is easily proved by induction that for all $k > k_0$,

$$\left|\frac{c'_k}{d_k} - \frac{c'_{k+1}}{d_{k+1}}\right| = |c'_{1+k_0}d_{k_0} - c'_{k_0}d_{1+k_0}|\frac{a_{1+k_0}\dots a_{k+1}}{d_k d_{k+1}}.$$

We abbreviate $Q = |c'_{1+k_0}d_{k_0} - c'_{k_0}d_{1+k_0}|$, and note that $Q \neq 0$ by the assumption that $\frac{c'_{k_0}}{d_{k_0}} \neq \frac{c_{1+k_0}}{d_{1+k_0}}$. We can now bound the distance from above using that $a_{j+1}d_{j-1} \leq d_j$:

$$\left|\frac{c'_k}{d_k} - \frac{c'_{k+1}}{d_{k+1}}\right| = \frac{Qa_{1+k_0}\dots a_{k+1}}{d_k d_{k+1}} = \frac{Q}{d_{k_0-1}d_{k+1}} \prod_{j=k_0}^{k} \frac{a_{j+1}d_{j-1}}{d_j} > \frac{Q}{d_{k_0-1}d_{k+1}} \prod_{j=k_0}^{\infty} \frac{a_{j+1}d_{j-1}}{d_j}. \tag{13}$$

Note that

$$\frac{d_j}{a_{j+1}d_{j-1}} = \frac{b_j}{a_{j+1}} + \frac{a_j d_{j-2}}{a_{j+1}d_{j-1}} \leq \frac{b_j}{a_{j+1}} + \frac{b_{j-1}d_{j-2}}{a_{j+1}d_{j-1}} < \frac{b_j+1}{a_{j+1}} < \frac{b_j}{a_{j+1}-1} = \frac{b_j}{b_j-1} = \left(1-b_j^{-1}\right)^{-1}.$$

Therefore, the product $\prod_{j=k_0}^{\infty} \frac{a_{j+1}d_{j-1}}{d_j}$ is greater than $\prod_{j=k_0}^{\infty}\left(1-\frac{1}{b_j}\right)$, which converges to a positive limit L by the assumption that $\sum b_k^{-1} < \infty$. Combining with (13) yields that there exists a positive constant $K = \frac{QL}{d_{k_0-1}}$ so that for all $k > k_0$,

$$\left|\frac{c'_k}{d_k} - \frac{c'_{k+1}}{d_{k+1}}\right| > \frac{K}{d_{k+1}}. \tag{14}$$

However, recall that $|r - \frac{c'_k}{d_k}| < 0.2(d_k n_{k+1})^{-1}$ meaning $|rd_{k+1}n_{k+1} - c'_k n_{k+1}| < 0.2$ so $c'_k n_{k+1}$ is the closest integer to $rd_{k+1}n_{k+1}$. Since $\langle 0.25 n_{k+1}d_k r\rangle \to 0$, this implies there exists $k_1 > k_0$ such that $|rd_{k+1}n_{k+1} - c'_k n_{k+1}| < 0.5K$. Then $|r - \frac{c'_k}{d_k}| < 0.5K(n_{k+1}d_k)^{-1}$. Since $d_{k+1} < n_{k+1}d_k$, then $|r - \frac{c'_k}{d_k}| < 0.5K(d_{k+1})^{-1}$. Then for $k > k_1$,

$$\left|r - \frac{c'_k}{d_k}\right| < 0.5K(d_{k+1})^{-1} \text{ and } \left|r - \frac{c'_{k+1}}{d_{k+1}}\right| < 0.5K(d_{k+2})^{-1} < 0.5K(d_{k+1})^{-1},$$

which contradicts (14) by the triangle inequality. Therefore, our original assumption is false and X is weak mixing.

It remains only to show that the complexity function satisfies the claimed bounds. Since $|p_1| = 0$ and by Remark 2.11, $p_{k+1} = v_k^{m_k-1}p_k$, we have $|p_k| = \sum_{j=1}^{k-1}(m_j-1)|v_j|$ and therefore, since $n_j - m_j = m_j$,

$$\sum_{j=1}^{k}(n_j - m_j - 1)|v_j| = \sum_{j=1}^{k}(m_j-1)|v_j| = (m_k-1)|v_k| + |p_k|.$$

By Proposition 4.1, then

$$p(|s_k v_k^{2m_k-2}p_k|) = |s_k v_k^{2(m_k-1)}p_k| + (m_k-1)|v_k| + |p_k| + K = 1.5|s_k v_k^{2m_k-2}p_k| - 0.5(|s_k| - |p_k|) + K.$$

Since $|p_k| + |s_k| < 3|v_k|$ and $m_k \to \infty$, $\lim \frac{p(|s_k v_k^{2m_k-2}p_k|)}{|s_k v_k^{2m_k-2}p_k|} = 1.5$. Proposition 4.1 implies that the limsup of $\frac{p(q)}{q}$ is achieved along some subsequence of $|s_k v_k^{n_k-2}p_k|$, so $\limsup \frac{p(q)}{q} = 1.5$. □

Remark 4.2. The examples in Theorem 2 also satisfy $p(q) - 1.5q \to -\infty$ and $\liminf \frac{p(q)}{q} = 1$. For any $f(q) \to \infty$, such a subshift exists which also satisfies $p(q) < q + f(q)$ infinitely often.

Proof. By Remark 2.11, $s_{k+1} = s_k v_{k+1}$, so we have $|s_k| - |p_k| \leq |s_k| - |v_k| = |s_{k-1}| \to \infty$ so $p(q) - 1.5q \to -\infty$. By Proposition 4.1,

$$p(|v_k^{m_k-1}p_k|) = |v_k^{m_k-1}p_k| + \sum_{j=1}^{k-1}(n_j - m_j - 1)|v_j| + K = |v_k^{m_k-1}p_k| + |p_k| + K$$

and $|p_k| < 3|v_k|$ so since $m_k \to \infty$, $\liminf \frac{p(q)}{q} = 1$. Now let $f(q) \to \infty$ be arbitrary. For all k, if v_k and p_k are given, we can choose $b_k = m_k$ large enough so that $f((m_k-1)|v_k| + |p_k|) > |p_k| + K$, which implies that $p(|v_k^{m_k-1}p_k|) < |v_k^{m_k-1}p_k| + f(|v_k^{m_k-1}p_k|)$. □

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