UBIQUITY OF ENTROPIES OF INTERMEDIATE FACTORS

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Abstract. We consider topological dynamical systems (X, T), where X is a compact metrizable space and T denotes an action of a countable amenable group G on X by homeomorphisms. For two such systems (X, T) and (Y, S) and a factor map $\pi : X \to Y$, an intermediate factor is a topological dynamical system (Z, R) for which π can be written as a composition of factor maps $\psi : X \to Z$ and $\varphi : Z \to Y$. In this paper we show that for any countable amenable group G, for any G-subshifts (X, T) and (Y, S), and for any factor map $\pi : X \to Y$, the set of entropies of intermediate subshift factors is dense in the interval $[h(Y, S), h(X, T)]$. As corollaries, we also prove that the set of entropies of intermediate zero-dimensional factors is equal to the interval $[h(Y, S), h(X, T)]$, and even when (X, T) is a zero-dimensional G-system, the set of entropies of its zero-dimensional factors is equal to the interval $[0, h(X, T)]$. Our proofs rely on a generalized Marker Lemma that may be of independent interest.

1. Introduction

In this work, we continue a line of research, initiated by Shub and Weiss [12], concerning the following seemingly basic question: given a topological dynamical system, what can be said about the topological entropies of its factors? For the purposes of this paper, we consider a topological dynamical system to be a pair (X, T), where X is a compact metrizable space and T is an action of a countable amenable group G on X by homeomorphisms. Additionally, a factor of such a system (X, T) is another system (Y, S) for which there exists a continuous surjection $\pi : X \to Y$ that commutes with the actions of S and T. Some general results about the entropies of factors are available [8, 9, 12], but the question above has not been completely resolved. For instance, it is

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still not known whether every system with positive, finite entropy must have a nontrivial factor with strictly smaller entropy.

Under certain hypotheses, there are some existing results about the entropies of factors. For instance, Lindenstrauss proved [8] that if (X, T) is a finite-dimensional $\mathbb{Z}$-system, then the set of entropies of factors of (X, T) is the entire interval $[0, h(X, T)]$. Furthermore, Lindenstrauss provided examples showing that this is not necessarily the case if X has infinite dimension.

Lindenstrauss also proved what we call a 'relative version' of his result, which concerns the entropies of intermediate factors. Given a countable amenable group G, G-systems (X, T) and (Y, S), and a factor map $\pi : X \to Y$, an intermediate factor is a G-system (Z, R) for which π can be written as a composition of factor maps $\psi : X \to Z$ and $\varphi : Z \to Y$. With this definition, Lindenstrauss showed that if π is a factor map from a finite dimensional $\mathbb{Z}$-system (X, T) onto (Y, S), then the set of entropies of intermediate factors is the entire interval $[h(Y, S), h(X, T)]$ [8]. (The relative version is of course more general than the original result concerning entropies of factors of a single system, since the original result may be obtained by taking (Y, S) to be the trivial factor in the relative version.) Using the notion of mean dimension [10], Lindenstrauss also generalized these results to extensions of minimal $\mathbb{Z}$-systems with zero mean dimension [9].

Here we examine the question in the setting of subshifts, or symbolic dynamical systems. Of course subshifts are zero-dimensional, and therefore previous results about zero-dimensional systems can be applied to subshifts. However, we are interested in the finer question of what can be said about the entropies of intermediate factors, where the intermediate systems must come from a restricted class (such as subshifts or zero-dimensional systems). In this direction, for $\mathbb{Z}$-subshifts, it was shown in [12] that any system with positive entropy must have nontrivial factors of strictly smaller entropy (and in fact that they can be taken arbitrarily close to $h(X, T)$). In addition, in [4], in the case where (X, T) is a sofic $\mathbb{Z}$-subshift, it was shown that the set of entropies of subshift factors is dense in the interval $[0, h(X, T)]$, which is in some sense the most that can be hoped for, since a subshift has only countably many subshift factors by the Curtis-Lyndon-Hedlund theorem. A corresponding relative result was also established when (X, T) and (Y, S) are both sofic $\mathbb{Z}$-subshifts.

In this work, we show that the previously mentioned result of [4] holds even when (X, T) is an arbitrary (not necessarily sofic) subshift. Furthermore, we generalize these results from $\mathbb{Z}$-subshifts to G-subshifts, where G is an arbitrary countable amenable group. Note that the computation/realization of entropies is much more difficult in the setting of general countable amenable groups. For instance, realization of arbitrary entropies for G-subshifts has only recently been addressed for general classes of countable amenable groups [3, 13]. In order to establish our main results at this level of generality, we first prove a generalized Marker Lemma for countable groups.

Let us now state our main results. When referring to subshifts, we omit explicit mention of the action (since it is always the shift action). In the following two results, G denotes any countable amenable group, X and Y represent G-subshifts, $\mathcal{H}_{sub}(X)$ denotes the set of numbers $r \in \mathbb{R}$ such that there exists a G-subshift Z so that X factors onto Z and $h(Z) = r$, and $\mathcal{H}^{\pi}_{sub}(X, Y)$ denotes the set of numbers $r \in \mathbb{R}$ such that there exists a G-subshift Z with $h(Z) = r$ and factor maps $\varphi : X \to Z$ and $\psi : Z \to Y$ such that $\pi = \psi \circ \varphi$.

Theorem 1.1. *Let G be a countable amenable group, and let X be a G-subshift. Then $\mathcal{H}_{sub}(X)$ is dense in the interval $[0, h(X)]$.*

The main technique in the proof of Theorem 1.1 is a general version of the classical Marker Lemma of [1], which may be of independent interest; see Section 3. We also establish the following relative version.

Theorem 1.2. *Let G be a countable amenable group. Let X and Y be G-subshifts with a factor map $\pi : X \to Y$. Then $\mathcal{H}^{\pi}_{sub}(X, Y)$ is dense in the interval $[h(Y), h(X)]$.*

From these results, we are actually able to establish corollaries about general zero-dimensional factors of subshifts as well. In the following two results, X and Y represent G-subshifts and $\mathcal{H}_0(X)$ denotes the set of numbers $r \in \mathbb{R}$ such that there exists a zero-dimensional G-system (Z, R) so that X factors onto (Z, R) and $h(Z, R) = r$.

Theorem 1.3. *Let G be a countable amenable group, and let X be a G-subshift. Then $\mathcal{H}_0(X) = [0, h(X)]$.*

Theorem 1.3 is actually proved as a corollary of a relative version. Let $\mathcal{H}^{\pi}_0(X, Y)$ be the set of numbers $r \in \mathbb{R}$ such that there exists a zero-dimensional G-system (Z, R) with $h(Z, R) = r$ and factor maps $\varphi : X \to Z$ and $\psi : Z \to Y$ such that $\pi = \psi \circ \varphi$.

Theorem 1.4. *Let G be a countable amenable group. Let X and Y be G-subshifts with a factor map $\pi : X \to Y$. Then $\mathcal{H}^{\pi}_0(X, Y) = [h(Y), h(X)]$.*

Finally, we can prove a corollary about entropies of factors of general zero-dimensional G-systems. In the following result, G is a countable amenable group, (X, T) is a zero-dimensional G-system, and $\mathcal{H}_0(X, T)$ is the set of numbers $r \in \mathbb{R}$ such that there exists a zero-dimensional G-system (Z, R) such that (X, T) factors onto (Z, R) and $h(Z, R) = r$.

Theorem 1.5. *Let G be a countable amenable group, and let (X, T) be a zero-dimensional G-system. Then $\mathcal{H}_0(X, T) = [0, h(X, T)]$.*

We note that this result provides more information about the aforementioned result of [8] when (X, T) is zero-dimensional: in this case, not only do we know that every entropy in $[0, h(X, T)]$ can be achieved by factors, but that those factors can always be chosen to be zero-dimensional.

The rest of the paper is organized as follows. Section 2 contains necessary background and notation, Sections 3 and 4 contain some preliminary results required for later proofs, and Sections 5, 6, 7, and 8 contain the proofs of Theorems 1.1-1.5.

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2. Background and notation

2.1. Countable amenable groups.
Let G be a countable amenable group. For sets $A, K \subset G$, we let $AK = \{ak : a \in A, k \in K\}$ and $A \,\Delta\, K = (A \setminus K) \cup (K \setminus A)$. A sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of G is called a Følner sequence if for each finite set $K \subset G$, we have

$$\lim_n \frac{|KF_n \,\Delta\, F_n|}{|F_n|} = 0.$$

The existence of a Følner sequence is equivalent to the amenability of the group G. Let $\{F_n\}_{n=1}^{\infty}$ be a Følner sequence, which we fix here and use throughout the paper.

Definition 2.1. Suppose U and K are non-empty finite subsets of G and $\delta > 0$. We say that U is (K, δ)-*invariant* if

$$\frac{|KU \triangle U|}{|U|} < \delta.$$

Note that the definition of Følner sequence yields that for any finite set $K \subset G$ and any $\delta > 0$, the set F_n is (K, δ)-invariant for all large enough n.

Definition 2.2. Suppose $\{A_1, \ldots, A_r\}$ is a collection of finite sets of G. We say that the collection is *δ-disjoint* if there exist subsets $\{B_1, \ldots, B_r\}$ such that

(1) $B_i \subset A_i$ for all i,
(2) for $i \neq j$, we have $B_i \cap B_j = \varnothing$, and
(3) $|B_i|/|A_i| > 1 - \delta$ for all i.

Definition 2.3. Suppose $\{A_1, \ldots, A_r\}$ is a collection of finite sets of G, and let $B \subset G$. We say that the collection *α-covers* B if

$$\big|B \cap \big(\bigcup_i A_i\big)\big| \geq \alpha |B|.$$

Definition 2.4. ([11]) For $\delta > 0$, a collection of finite sets $\{T_1, \ldots, T_N\}$ is said to *δ-quasitile* a group G (or to be a *set of δ-quasitiles for* G) if $\{e\} \in T_1 \subset \cdots \subset T_N$ and for any finite set $D \subset G$, there are finite sets $C_i \subset G$, for $1 \leq i \leq N$, such that

(1) for fixed i, the collection $\{T_i c : c \in C_i\}$ is δ-disjoint
(2) for $i \neq j$, $T_i C_i \cap T_j C_j = \varnothing$, and
(3) the collection $\{T_1 C_1, \ldots, T_N C_N\}$ $(1-\delta)$-covers D.

The fundamental result of Ornstein and Weiss [11, Theorem 6] states that for any $\delta > 0$, there exists N such that for any finite set K and any $\delta' > 0$, there exist finite sets $\{T_1, \ldots, T_N\}$ that are (K, δ')-invariant and that δ-quasi-tile G. We remark that by making $\delta' > 0$ small enough and K large enough, one may force $|T_1| = \min_i |T_i|$ to be arbitrarily large.

For a collection of quasitiles $\{T_1, \ldots, T_N\}$, we refer to any collection $\{C_1, \ldots, C_N\}$ satisfying (1)-(3) above as a collection of *center sets* corresponding to D. If $\{C_1, \ldots, C_N\}$ is a collection of center sets for D, then let $C_i' = \{c \in C_i : T_i c \cap D \neq \varnothing\}$, and observe that the collection $\{C_1', \ldots, C_N'\}$ is again a collection of center sets for D. Thus, for a collection of quasitiles $\{T_1, \ldots, T_N\}$ and finite $D \subset G$, there exists a collection of center sets $\{C_1, \ldots, C_N\}$, and we may assume without loss of generality that for all $c \in C_i$, we have $T_i c \cap D \neq \varnothing$.

Lemma 2.5. *Let $\delta \in (0,1)$. Suppose that $\{S_1, \ldots, S_N\}$ is a collection of finite sets that δ-quasitiles G. Let $m = |S_1| = \min\{|S_1|, \ldots, |S_N|\}$, and $S = \bigcup_i S_i = S_N$. Then for all large enough n, there exists $C = C(n) \subset G$ such that $\{Sc : c \in C\}$ $(1-\delta)$-covers F_n and*

$$|C| \leq \frac{(1+\delta)|F_n|}{(1-\delta)m}.$$

Proof. Since $\{F_n\}_{n=1}^{\infty}$ is a Følner sequence, for all large enough n, we have

$$\frac{|SS^{-1}F_n \triangle F_n|}{|F_n|} < \delta. \tag{2.1}$$

Fix n satisfying this condition. Since $\{S_1, \dots, S_N\}$ is a collection of δ-quasitiles, there exists a collection of center sets corresponding to the set F_n with the additional property that for each $c \in C_i$, we have $S_i c \cap F_n \neq \varnothing$.

Let $C = \bigcup_i C_i$. Then

$$A = \bigcup_i S_i C_i \subset SC,$$

and since A $(1-\delta)$-covers F_n, we see that SC $(1-\delta)$-covers F_n.

Now let us estimate $|C|$. Let $C_i = \{c_1^i, \dots, c_{M_i}^i\}$, and let $A_k^i = S_i c_k^i$, for $k = 1, \dots, M_i$. Then the collection $\{A_k^i : 1 \leq i \leq N, 1 \leq k \leq M_i\}$ is δ-disjoint (by the quasitile properties (1) and (2)). Then there exist sets $B_k^i \subset A_k^i$ such that $|B_k^i| \geq |A_k^i|(1-\delta)$ and if $B_k^i \cap B_\ell^j \neq \varnothing$, then $i = j$ and $k = \ell$. Then we have

$$\begin{aligned}
\left| \bigcup_{i=1}^{N} \bigcup_{k=1}^{M_i} B_k^i \right| &= \sum_{i=1}^{N} \sum_{k=1}^{M_i} |B_k^i| \\
&\geq \sum_{i=1}^{N} \sum_{k=1}^{M_i} (1-\delta)|A_k^i| \\
&= (1-\delta) \sum_{i=1}^{N} \sum_{k=1}^{M_i} |S_i| \\
&= (1-\delta) \sum_{i=1}^{N} |S_i| \cdot |C_i| \\
&\geq (1-\delta) m \sum_{i=1}^{N} |C_i| \\
&\geq (1-\delta) m |C|.
\end{aligned}$$

Fix $1 \leq i \leq N$ and $1 \leq k \leq M_i$. Recall that $A_k^i \cap F_n = S_i c_k^i \cap F_n \neq \varnothing$. Let $g \in S_i c_k^i \cap F_n$. Then there exists $s \in S_i$ such that $g = s c_k^i$, which implies that $c_k^i = s^{-1} g \in S^{-1} F_n$, and hence $S c_k^i \subset SS^{-1} F_n$. Then we see that

$$\begin{aligned}\left|\bigcup_{i=1}^{N}\bigcup_{k=1}^{M_i} B_k^i\right| &\leq \left|\bigcup_{i=1}^{N}\bigcup_{k=1}^{M_i} A_k^i\right| \\ &\leq |SS^{-1}F_n| \\ &\leq (1+\delta)|F_n|,\end{aligned}$$

where we have used (2.1). Combining the above inequalities gives

$$(1-\delta)m|C| \leq \left|\bigcup_{i=1}^{N}\bigcup_{k=1}^{M_i} B_k^i\right| \leq (1+\delta)|F_n|,$$

from which we conclude that

$$|C| \leq \frac{(1+\delta)|F_n|}{(1-\delta)m},$$

as desired. □

2.2. **Topological and symbolic dynamics.** We present some basic definitions from topological and symbolic dynamics; for a more thorough introduction to symbolic dynamics, see [7].

Definition 2.6. Let G be a countable amenable group. A *G-system* is a pair (X, T), where X is a compact metrizable space and $T = (T^g)_{g\in G}$ is an action of G on X by homeomorphisms. Additionally, a *zero-dimensional G-system* is a G-system (X, T) such that X is zero-dimensional (i.e., it has a topological base consisting of clopen sets).

Definition 2.7. A *factor map* from a G-system (X, S) to a G-system (Y, T) is a surjective function $\varphi : X \to Y$ that is continuous and commutes with the actions of S and T. A *conjugacy* is a factor map that is also injective. When $\varphi : X \to Y$ is a factor map, we refer to Y as a factor of X.

When G is a countable amenable group, one may associate to each G-system (X, T) an extended real number called the *topological entropy* of (X, T). We let $h(X)$ denote the topological entropy of (X, T) when the action is understood, and we note that it serves as an important conjugacy invariant in the study of topological dynamical systems. We will not need a general definition of topological entropy here, but will define it formally only for the class of G-subshifts (below) and then will extend to some zero-dimensional G-systems via inverse limits; see Section 2.3. For more information about the general entropy theory of countable amenable groups, see [5].

Definition 2.8. For any finite alphabet $\mathcal{A}$ and countable amenable group G, the *full G-shift* over $\mathcal{A}$ is the set $\mathcal{A}^G$, which is viewed as a compact topological space with the (discrete) product topology.

Definition 2.9. A *pattern* over $\mathcal{A}$ is any function w from a finite set $S \subset G$ to $\mathcal{A}$, in which case S is called the *shape* of w.

Definition 2.10. The *shift action* $\sigma = (\sigma_g)_{g \in G}$ is the action of G on $\mathcal{A}^G$ by automorphisms defined by $(\sigma_g x)_h = x_{gh}$ for all $g, h \in G$.

Definition 2.11. A *G-subshift* is a closed subset of a full shift $\mathcal{A}^G$ that is invariant under σ.

Any subshift X is a compact space with the induced topology from $\mathcal{A}^G$, and so (X, σ) is a topological G-dynamical system. In fact, this topology on X is also generated by the ultra-metric given by

$$d(x, y) = 2^{-\min\{n \,:\, x_{g_n} \neq y_{g_n}\}},$$

where $G = \{g_n\}_{n=0}^{\infty}$ is an enumeration of G.

Definition 2.12. For any G-subshift X and finite $S \subset G$, the *S-language* of X, denoted by $\mathcal{L}_S(X)$, is the set of all patterns w with shape S which appear as subpatterns of points of X, i.e., for which there exists $x \in X$ and $g \in G$ with $x(gS) = w$.

Definition 2.13. For any G-subshift X and $w \in \mathcal{L}_S(X)$, the *cylinder set* $[w]$ is the set of all $x \in X$ with $x(S) = w$.

All factor maps between subshifts have a simple combinatorial description.

Definition 2.14. Given alphabets $\mathcal{A}$ and $\mathcal{B}$, a finite set $S \subset G$, and a function $f : \mathcal{A}^S \to \mathcal{B}$, the *sliding block code* induced by S and f is the map $\varphi : \mathcal{A}^G \to \mathcal{B}^G$ defined by

$$(\varphi(x))_g = f(x(gS))$$

for all $x \in \mathcal{A}^G$ and $g \in G$. A 1-*block code* is a sliding block code when $S = \{e\}$.

If X and Y are subshifts, then for every factor map $\pi : X \to Y$, there is a sliding block code φ such that $\pi = \varphi|_X$; this is the classical Curtis-Lyndon-Hedlund theorem when $G = \mathbb{Z}$, and the proof is essentially the same for general G. Even more is true: up to conjugacy, every factor map can be written as a 1-block code, i.e., if $\varphi : X \to Y$ is a factor map, then there exists a conjugacy ψ on X and a 1-block code ρ on $\psi(X)$ so that $\varphi = \rho \circ \psi$.

Definition 2.15. The *topological entropy* of a G-subshift X is

$$h(X) := \lim_{n\to\infty} \frac{1}{|F_n|} \log |\mathcal{L}_{F_n}(X)|,$$

where $\{F_n\}_{n=1}^{\infty}$ is a Følner sequence for G.

This limit is consistent with the general entropy theory for countable amenable groups, and in particular it always exists and is independent of the Følner sequence chosen (see [5] for a proof).

It is well-known that for countable amenable G, factor maps cannot increase topological entropy, i.e., if φ is a factor map from X to Y, then $h(X) \geq h(Y)$. This observation motivates the main question of this paper, namely whether a large set of entropies is achieved for intermediate systems 'between X and Y.'

2.3. **Inverse limits.**

Definition 2.16. Suppose $\{Z_n\}_{n=0}^{\infty}$ is a sequence of G-subshifts, and for each $n \geq 1$, we have a factor map $\pi_n : Z_n \to Z_{n-1}$. The inverse limit $Z = \varprojlim Z_n$ is the set

$$Z = \left\{ (z_0, z_1, z_2, \dots) \in \prod_{n=0}^{\infty} Z_n : \forall n \geq 1, \pi_n(z_n) = z_{n-1} \right\}.$$

Since each Z_n is compact, the product of these systems is compact by Tychonoff's theorem. In fact, if d_n is the usual ultra-metric on Z_n for each $n \geq 0$, then we can metrize the product topology on Z with the metric

$$d(z, z') = \sum_{n=0}^{\infty} \frac{d_n(z_n, z'_n)}{2^n}.$$

Note that since each Z_n is zero-dimensional, Z is zero-dimensional as well. Additionally, there is a natural action of G on Z, given by $S^g(z) = (\sigma_g(z_0), \sigma_g(z_1), \dots)$, which is continuous. We require the following lemma about entropy of inverse limits; for a proof in the case $G = \mathbb{Z}$, see [8, Lemma 4.9]. The straight-forward adaptation of this proof for general countable amenable groups is left to the reader.

Lemma 2.17. *For any inverse limit $Z = \varprojlim Z_n$, we have*

$$h(Z) = \lim_{n} h(Z_n).$$

Finally, we will need the following simple lemma, which shows that factor maps can be carried to the inverse limit; the proof is standard and left to the reader.

Lemma 2.18. *Suppose that X is a G-subshift with a surjective factor map $\varphi_0 : X \to Z_0$ and that for each $n \geq 1$, there exist surjective factor maps $\varphi_n : X \to Z_n$ and $\pi_n : Z_n \to Z_{n-1}$ such that $\varphi_{n-1} = \pi_n \circ \varphi_n$. Define the map $\varphi : X \to Z$ by the rule $\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots)$. Then φ is a surjective factor map.*

2.4. **Periodic patterns.**

Definition 2.19. For any finite alphabet $\mathcal{A}$, countable amenable G, $S, T \subset G$ with S finite, $k \in \mathbb{N}$, and $w \in \mathcal{A}^T$, we say that *w has k periods from S* if there exists a collection $\{s_1, \dots, s_k\}$ of k distinct elements of S such that $w_g = w_{gs_i}$ whenever g and gs_i are both in T. In this case, we may refer to $\{s_1, \dots, s_k\}$ as a *period set* for w.

Lemma 2.20. *Let $k \in \mathbb{N}$, and let $S \subset G$ be a finite set with $|S| \geq k$. Let $T \subset G$ be any set such that*

$$k \log_{|\mathcal{A}|} |S| < \frac{|T|}{2k}$$

and for each $s \in S$,

$$|T \triangle Ts| < \frac{|T|}{2k^2}.$$

Then

$$\Big|\Big\{w \in \mathcal{A}^T : w \text{ has } k \text{ periods from } S\Big\}\Big| \leq |\mathcal{A}|^{2|T|/k}.$$

Proof. Let $k \in \mathbb{N}$, and let $S \subset G$ be a finite set with $|S| \geq k$. Let T be as above. To simplify the notation in this proof, let $\delta > 0$ be such that $\delta < k^{-1}/2$ and such that

$$k \log_{|\mathcal{A}|} |S| < \delta |T|,$$

and for each $s \in S$,

$$|T \triangle Ts| < \frac{\delta |T|}{k}.$$

Now let $P \subset S$ such that $|P| = k$. First, define a finite undirected graph with vertex set $V = T$ and edge set $E \subset T \times T$, where $(g, h) \in E$ if there exists $p \in P$ such that $h = gp$ or $g = hp$. Let $C \subset T$ be the set of vertices corresponding to an arbitrary connected component of (V, E) such that $|C| < k$ (which might not exist). Let $g \in C$. If $p \in P$ and $h = gp^{-1} \in T$, then $g = hp$, which means $(g, h) \in E$, and then $gp^{-1} = h \in C$. Since $|C| < k = |gP^{-1}|$, there exists $p \in P$ such that $gp^{-1} \notin T$, i.e., $g \in T \setminus Tp$. We now conclude that for any connected component C of (V, E) with $|C| < k$, we have

$$C \subset \bigcup_{p \in P} (T \setminus Tp).$$

For $g \in T$, let $C(g)$ denote the connected component of (V, E) containing g. Then the above containment and our second hypothesis on T combine to give

$$\left|\{g \in T : |C(g)| < k\}\right| \leq \sum_{p \in P} |T \setminus Tp| < \frac{\delta |T|}{k} |P| = \delta |T|.$$

Let N_ℓ be the number of connected components of (V, E) with cardinality ℓ. We have

$$\begin{aligned} |T| &\geq \left|\{g \in T : |C(g)| \geq k\}\right| \\ &= \sum_{\ell=k}^{|T|} N_\ell \cdot \ell \\ &\geq k \sum_{\ell=k} N_\ell, \end{aligned}$$

and therefore the number of connected components of cardinality at least k is bounded above by $|T|/k$.

Now we turn to counting patterns in $\mathcal{A}^T$ with period set P. Notice that if $w \in \mathcal{A}^T$ has P as a period set, then w must be constant on any connected component of the graph (V, E). Then the estimates in the previous paragraph yield that

$$\left|\{w \in \mathcal{A}^T : P \text{ is a period set for } w\}\right| \leq |\mathcal{A}|^{|T|/k} \cdot |\mathcal{A}|^{\delta |T|} = |\mathcal{A}|^{(k^{-1}+\delta)|T|}.$$

Finally, let us estimate the number of patterns in $\mathcal{A}^T$ with k periods from S. Using the previous estimates and our first hypothesis on T, we have

$$\begin{aligned} &\left|\{w \in \mathcal{A}^T : w \text{ has } k \text{ periods from } S\}\right| \\ &\qquad \leq \sum_{\substack{P \subset S \\ |P| = k}} \left|\{w \in \mathcal{A}^T : P \text{ is a period set for } w\}\right| \\ &\qquad \leq |\mathcal{A}|^{(k^{-1}+\delta)|T|} \cdot \left|\{P \subset S : |P| = k\}\right| \\ &\qquad \leq |\mathcal{A}|^{(k^{-1}+\delta)|T|} \cdot |S|^k \\ &\qquad \leq |\mathcal{A}|^{(k^{-1}+\delta)|T|} \cdot |\mathcal{A}|^{\delta |T|} \\ &\qquad = |\mathcal{A}|^{(k^{-1}+2\delta)|T|}. \end{aligned}$$

Since $\delta < k^{-1}/2$, the proof of the lemma is complete. $\square$

2.5. **Other preliminaries.** We denote the usual binary entropy function by $H : [0,1] \to \mathbb{R}$, where $H(x) = -x\log(x) - (1-x)\log(1-x)$, with the convention that $0 \cdot \log 0 = 0$. The following two facts are elementary and presented without proof.

Lemma 2.21. *For any n and $\alpha < 1/2$, we have*

$$\sum_{k=0}^{\lfloor \alpha n \rfloor} \binom{n}{k} \leq 2^{H(\alpha)n}.$$

Lemma 2.22. *If A and B are finite sets and $\phi : A \to B$ satisfies $|\phi^{-1}(b)| \leq M$ for all $b \in B$, then $|A| \leq M|B|$.*

3. Marker Lemma

Here we will prove a Marker Lemma for countable groups which generalizes the classical Krieger Marker Lemma ([1], [6]). First, we require a definition.

Definition 3.1. Let $\mathcal{F}$ be a finite collection of sets. For $k \in \mathbb{N}$, we say that $\mathcal{F}$ is *k-fold disjoint* if for all collections $\{F_1, \dots, F_k\}$ of k distinct elements of $\mathcal{F}$, we have

$$\bigcap_{\ell=1}^{k} F_\ell = \varnothing.$$

Now we state our general Marker Lemma. Note that it does not require amenability of the group G.

Lemma 3.2. *Let G be a countable group, and let X be a G-subshift. Let $S \subset G$ and $T \subset G$ be finite, and let $1 \leq k \leq |S|$. Then there exists a clopen set $F \subset X$ such that*

(1) *the collection $\{\sigma_s(F) : s \in S\}$ is $(k+1)$-fold disjoint, and*
(2) *if*

$$x \notin \bigcup_{s \in S^{-1}S} \sigma_s(F),$$

then $x(T)$ has k periods from $S^{-1}S$.

The classical Krieger Marker Lemma corresponds to the case $G = \mathbb{Z}$, $k = 1$, $S = [0,n]$, and $T = [-n,n]$ for some $n \in \mathbb{N}$. Our use for arbitrary k can be thought of as allowing shifts of the marker set to have weaker disjointness properties in exchange for stronger periodicity properties away from it.

Proof. Let $\mathcal{N} = \{w_1, \dots, w_r\}$ be an enumeration of the patterns w in $\mathcal{A}^T$ such that w *does not* have k periods from $S^{-1}S$. Inductively define the following sets. Let $A_1 = [w_1]$. For $i = 1, \dots, r-1$, let

$$A_{i+1} = [w_{i+1}] \setminus \left(\bigcup_{j=1}^{i} \bigcup_{s \in S^{-1}S} \sigma_s(A_j) \right).$$

Let $F = \bigcup_{i=1}^{r} A_i$. Note that F is clopen.

To establish (1), suppose for contradiction that there exists $P = \{p_1, \dots, p_{k+1}\} \subset S$ with $|P| = k+1$ and there exists $x \in X$ with

$$x \in \bigcap_{p \in P} \sigma_p(F).$$

Then for each $p \in P$, we have that $\sigma_{p^{-1}}(x) \in F = \bigcup_i A_i$. For each $p \in P$, choose $i(p) \in \{1, \dots, r\}$ such that $\sigma_{p^{-1}}(x) \in A_{i(p)}$. We claim that there exists $w \in \mathcal{N}$ such that $\sigma_{p^{-1}}(x) \in [w]$ for all $p \in P$. To see this, suppose for contradiction that there exist $p, q \in P$ such that $i(p) \neq i(q)$. Assume without loss of generality that $i(p) < i(q)$ (otherwise reverse the roles of p and q). Then

$$\sigma_{q^{-1}}(x) \in A_{i(q)} \cap \left(\sigma_{q^{-1}p}(A_{i(p)})\right),$$

which gives a contradiction, since $q^{-1}p \in S^{-1}S$ and $A_{i(q)}$ is defined to be disjoint from $\sigma_s(A_{i(p)})$ for all $s \in S^{-1}S$. Hence, there exists $w \in \mathcal{N}$ such that $\sigma_{p^{-1}}(x) \in [w]$ for all $p \in P$. Therefore

$$\sigma_{p_1^{-1}}(x) \in [w] \cap \sigma_{p_1^{-1}p_2}[w] \cap \dots \cap \sigma_{p_1^{-1}p_{k+1}}[w].$$

Since $|P| = k+1$, the non-emptiness of the intersection in the previous display gives that the set $\{p_1^{-1}p_2, \dots, p_1^{-1}p_{k+1}\}$ is a set of k periods from $S^{-1}S$ for w. However, this contradicts the fact that $w \in \mathcal{N}$. Thus we have established (1).

Now let $x \in X$ such that $x(T)$ *does not* have k periods from $S^{-1}S$. Then $x(T) = w_i$ for some $i = 1, \dots, r$. If $i = 1$, then $x \in F$. If $i > 1$, then either $x \in A_i \subset F$ or else there exists $j < i$ and $s \in S^{-1}S$ such that $x \in \sigma_s(A_j)$. In all cases, we obtain that

$$x \in \bigcup_{s \in S^{-1}S} \sigma_s(F).$$

Taking the contrapositive, we conclude that if

$$x \notin \bigcup_{s \in S^{-1}S} \sigma_s(F),$$

then $x(T)$ has k periods from $S^{-1}S$. This establishes (2) and finishes the proof. $\square$

4. Density

In this section, we define some basic notions of (upper) density for subsets of countable amenable G (in terms of our previously chosen Følner sequence F_n). This will be used to quantify a way in which visits to the marker set from Lemma 3.2 are rare when k is taken much smaller than $|S|$.

Definition 4.1. Let X be a G-subshift, and let $F \subset X$. For a finite set $E \subset G$ and $x \in X$, let

$$N_E(x, F) = \big|\{g \in E : \sigma_g(x) \in F\}\big|.$$

Then let

$$D_n(F) = \sup_{x \in X} \frac{N_{F_n}(x, F)}{|F_n|},$$

and $\overline{D}(F) = \limsup_n D_n(F)$.

Definition 4.2. Let X be a G-subshift. Let $S \subset G$ be finite, and let $k \in \mathbb{N}$. We say that $F \subset X$ is (S, k)-*disjoint* if $\{\sigma_s(F) : s \in S\}$ is k-fold disjoint.

Lemma 4.3. *Let X be a G-subshift. Let $\{S_1, \ldots, S_N\}$ be a collection that δ-quasitiles G with $m = \min_i |S_i|$ and $S = \bigcup_i S_i = S_N$, and let $k \geq 1$. If $F \subset X$ is (S^{-1}, k)-disjoint, then*

$$\overline{D}(F) \leq \frac{k(1+\delta)}{(1-\delta)m} + \delta.$$

Proof. By Lemma 2.5, for all large enough n, there exists $C = C(S, n, \delta)$ such that $\{Sc : c \in C\}$ $(1-\delta)$-covers F_n and

$$|C| \leq \frac{(1+\delta)|F_n|}{(1-\delta)m}.$$

Also, since F is (S^{-1}, k)-disjoint, for any $x \in X$, we have

$$\begin{aligned} N_S(x, F) &= |\{g \in S : \sigma_g(x) \in F\}| \\ &= |\{g \in S : x \in \sigma_{g^{-1}}(F)\}| \\ &\leq k. \end{aligned}$$

Then for any $x \in X$, the previous two displays and the fact that $\{Sc : c \in C\}$ $(1-\delta)$-covers F_n gives that

$$\begin{aligned} N_{F_n}(x,F) &\le N_{SC}(x,F) + N_{F_n \setminus SC}(x,F) \\ &\le \sum_{c \in C} N_{Sc}(x,F) + |F_n \setminus SC| \\ &\le \sum_{c \in C} N_S(\sigma_c(x),F) + \delta|F_n| \\ &\le k|C| + \delta|F_n| \\ &\le \frac{k(1+\delta)|F_n|}{(1-\delta)m} + \delta|F_n|. \end{aligned}$$

After dividing by $|F_n|$, taking the supremum over $x \in X$, and letting n tend to infinity, we obtain the desired estimate. □

Now we show that if a factor map only changes a small percentage of symbols, then the entropy drop across the factor map cannot be large.

Definition 4.4. Let $\pi : X \to Y$ be a factor map between G-subshifts. For a finite set $E \subset G$, $y \in Y$, and $x \in \pi^{-1}(y)$, we define

$$N_E(x,y) = |\{g \in E : x_g \neq y_g\}|.$$

Then define

$$D_n(y) = \sup_{x \in \pi^{-1}(y)} \frac{N_{F_n}(x,y)}{|F_n|},$$

and $\overline{D}(\pi) = \limsup_n \sup_{y \in Y} D_n(y)$.

Lemma 4.5. *Let $\pi : X \to Y$ be a factor map between G-subshifts on alphabet $\mathcal{A}$. Suppose that $\overline{D}(\pi) < 1/2$. Then $h(X) - h(Y) \le H(\overline{D}(\pi)) + \overline{D}(\pi)\log|\mathcal{A}|$.*

Proof. Let $\gamma = \overline{D}(\pi)$, and let $\epsilon > 0$ be such that $\gamma + \epsilon < 1/2$. Choose n large enough so that for all $y \in Y$ and $x \in \pi^{-1}(y)$, we have

$$\frac{N_{F_n}(x,y)}{|F_n|} < \gamma + \epsilon.$$

For $w \in \mathcal{L}_{F_n}(X)$, choose $x \in X$ such that $x(F_n) \in [w]$. Let $y = \pi(x)$, and let $u = y(F_n)$. Additionally, let $K_w = \{g \in F_n : w_g \neq u_g\}$. Note that

$$|K_w| = N_{F_n}(x,y) \le (\gamma+\epsilon)|F_n|.$$

Now consider the map that sends $w \in \mathcal{L}_{F_n}(X)$ to $(u, K_w, w(K_w))$, which is injective. Then

$$|\mathcal{L}_{F_n}(X)| \le |\mathcal{L}_{F_n}(Y)| \cdot 2^{H(\gamma+\epsilon)|F_n|} \cdot |\mathcal{A}|^{(\gamma+\epsilon)|F_n|},$$

where we have used Lemma 2.21. Taking logarithm, dividing by $|F_n|$, and letting n tend to infinity gives

$$h(X) \leq h(Y) + H(\gamma + \epsilon) + (\gamma + \epsilon) \log |\mathcal{A}|.$$

Since ϵ may be taken arbitrarily small, we obtain the desired result. $\square$

5. Proof of Theorem 1.1

Let X be a G-subshift with alphabet $\mathcal{A}$. We assume that $|\mathcal{A}| \geq 2$, since when $|\mathcal{A}| = 1$, $h(X) = 0$ and the theorem trivially holds. Let $\epsilon \in (0, 1/2)$. We will show that the set of entropies of subshift factors of X is ϵ-dense in the interval $[0, h(X)]$. Choose $k \geq 1$ such that $4 \log(|\mathcal{A}|)/k < \epsilon$. Choose $\delta \in (0, 1)$ and $m_0 \geq 1$ such that

$$H\left(\frac{(k+1)(1+\delta)}{(1-\delta)m_0} + \delta\right) + \left(\frac{(k+1)(1+\delta)}{(1-\delta)m_0} + \delta\right) \log |\mathcal{A}| < \epsilon/2. \tag{5.1}$$

Choose a collection $\{S_1, \dots, S_M\}$ that δ-quasitiles G such that $\min_i |S_i| \geq m_0$ and $S = \bigcup_i S_i$ has cardinality at least k. Choose $\eta \in (0, 1)$ such that

$$\log(|\mathcal{A}|)(2k^{-1}(1-\eta)^{-1} + \eta) < \epsilon/2. \tag{5.2}$$

Choose a collection $\{T_1, \dots, T_N\}$ that η-quasitiles G such that $2k^2 \log_{|\mathcal{A}|} |SS^{-1}| \leq |T_1| = \min_i |T_i|$ and for all $s \in SS^{-1}$ and $i = 1, \dots, N$, we have

$$|T_i \triangle T_i s| < \frac{|T_i|}{2k^2}.$$

Let $T = \bigcup_i T_i$.

Now apply the Marker Lemma (Lemma 3.2) with parameters k, S^{-1}, and T. We get a clopen set $F \subset X$ such that F is $(S^{-1}, k+1)$-disjoint and if $x \in X$ satisfies

$$x \notin \bigcup_{s \in SS^{-1}} \sigma_s(F),$$

then $x(T)$ has k periods in SS^{-1}. By Lemma 4.3, we obtain that

$$\overline{D}(F) \leq \frac{(k+1)(1+\delta)}{(1-\delta)m_0} + \delta. \tag{5.3}$$

Before defining our factor maps, we require a few more definitions. Let $G = \{g_k\}_{k=0}^{\infty}$ be an enumeration of G, with the convention that $g_0 = e$. Let $G_m = \{g_0, \dots, g_m\}$. We suppose that a and b are symbols that are *not* contained in $\mathcal{A}$, and we let $\mathcal{B} = \mathcal{A} \cup \{a, b\}$.

Now we define our factor maps. First, let $\varphi_0 : X \to \mathcal{B}^G$ be defined by the rule

$$\varphi_0(x)_g = \begin{cases} a, & \sigma_g(x) \in F \\ x_g, & \text{otherwise.} \end{cases}$$

Since F is clopen, φ_0 is a sliding block code. Let $X_0 = \varphi_0(X)$. Inductively, suppose that $\varphi_0, \dots, \varphi_m$ and $X_0, \dots, X_m$ have been defined. Define $\varphi_{m+1} : X_m \to \mathcal{B}^G$ by the rule

$$\varphi_{m+1}(x)_g = \begin{cases} b, & \text{if } x_g \neq a \text{ and } x_{g_{m+1}^{-1}g} = a \\ x_g, & \text{otherwise.} \end{cases}$$

It is clear that φ_{m+1} is a sliding block code. Let $X_{m+1} = \varphi_{m+1}(X_m)$. This concludes our definition of the factor maps $\{\varphi_m\}_{m=0}^{\infty}$ and the subshifts $\{X_m\}_{m=0}^{\infty}$. The remainder of the proof will be devoted to showing that the set of entropies of the subshifts X_m is ϵ-dense in the interval $[0, h(X)]$ by verifying that the 'entropy gaps' $h(X) - h(X_0)$ and $h(X_m) - h(X_{m+1})$ are smaller than ϵ and that the entropy $h(X_m)$ is 'eventually small,' i.e. $h(X_m) < \epsilon$ for sufficiently large m.

Both claims will be proved by appealing to properties of F guaranteed by the Marker Lemma. The former will follow from the fact that visits to F have small density, meaning that the changes made via each φ_m have small density. The latter will follow from the fact that portions of points of X which are not near visits to F are highly periodic, and since letters at locations near visits to F are changed to a for large m, such X_m will have small entropy by Lemma 2.20.

5.1. **Small entropy gaps.**

Claim 5.1. $\overline{D}(\varphi_0) \leq \overline{D}(F)$.

(In fact, this inequality is an equality, but we will not need that fact.)

Proof. Let $\epsilon_1 > 0$. Choose n large enough so that $D_n(F) \leq \overline{D}(F) + \epsilon_1$. Let $y \in X_0$ and $x \in \varphi_0^{-1}(y)$. Then

$$\begin{aligned} N_{F_n}(x, y) &= |\{g \in F_n : x_g \neq y_g\}| \\ &= |\{g \in F_n : \sigma_g(x) \in F\}| \\ &= N_{F_n}(x, F) \\ &\leq (\overline{D}(F) + \epsilon_1)|F_n| \end{aligned}$$

Dividing by $|F_n|$, taking supremum over $y \in Y$ and $x \in \varphi_0^{-1}(y)$, and letting n tend to infinity yields

$$\overline{D}(\varphi_0) \leq \overline{D}(F) + \epsilon_1.$$

Since ϵ_1 may be taken arbitrarily small, we obtain that $\overline{D}(\varphi_0) \leq \overline{D}(F)$. $\square$

Claim 5.2. $h(X) - h(X_0) < \epsilon$.

Proof. Note that $\overline{D}(F) < 1/2$ (by (5.1) and (5.3)). Then by Lemma 4.5 and Claim 5.1, we see that

$$\begin{aligned} h(X) - h(X_0) &\leq H(\overline{D}(\varphi_0)) + \overline{D}(\varphi_0) \log |\mathcal{A}| \\ &\leq H(\overline{D}(F)) + \overline{D}(F) \log |\mathcal{A}|. \end{aligned}$$

By (5.1) and (5.3), we conclude that $h(X) - h(X_0) < \epsilon$. $\square$

Claim 5.3. For all $m \geq 0$, we have $\overline{D}(\varphi_{m+1}) \leq \overline{D}(F)$.

Proof. Let $\epsilon_1 > 0$. Choose n large enough so that $D_n(F) \leq \overline{D}(F) + \epsilon_1$ and $|(g_{m+1}^{-1} F_n) \setminus F_n| \leq \epsilon_1 |F_n|$. Let $y \in X_{m+1}$ and $x \in \varphi_{m+1}^{-1}(y)$. Let $z \in X$ be such that $x = \varphi_m \circ \ldots \varphi_0(z)$. Then

$$\begin{aligned} N_{F_n}(x, y) &= |\{g \in F_n : x_g \neq y_g\}| \\ &\leq |\{g \in F_n : x_g \neq a \text{ and } x_{g_{m+1}^{-1} g} = a| \\ &\leq |\{g \in F_n : x_{g_{m+1}^{-1} g} = a\}| \\ &= N_{g_{m+1}^{-1} F_n}(z, F) \\ &\leq N_{F_n}(z, F) + N_{(g_{m+1}^{-1} F_n) \setminus F_n}(z, F) \\ &\leq (\overline{D}(F) + \epsilon_1)|F_n| + |(g_{m+1}^{-1} F_n) \setminus F_n| \\ &\leq (\overline{D}(F) + \epsilon_1)|F_n| + \epsilon_1 |F_n|. \end{aligned}$$

Dividing by $|F_n|$, taking the supremum over $y \in X_{m+1}$ and $x \in \varphi_{m+1}^{-1}(y)$, and letting n tend to infinity yields

$$\overline{D}(\varphi_{m+1}) \leq \overline{D}(F) + 2\epsilon_1.$$

Since ϵ_1 may be taken arbitrarily small, we obtain that $\overline{D}(\varphi_{m+1}) \leq \overline{D}(F)$. $\square$

Claim 5.4. For all $m \geq 0$, we have $h(X_{m+1}) - h(X_m) < \epsilon$.

Proof. Recall that $\overline{D}(F) < 1/2$. Then by Lemma 4.5 and the previous claim, we see that

$$\begin{aligned} h(X_{m+1}) - h(X_m) &\leq H(\overline{D}(\varphi_{m+1})) + \overline{D}(\varphi_{m+1}) \log |\mathcal{A}| \\ &\leq H(\overline{D}(F)) + \overline{D}(F) \log |\mathcal{A}|. \end{aligned}$$

By (5.1) and (5.3), we conclude that $h(X_{m+1}) - h(X_m) < \epsilon$. $\square$

5.2. **Eventually small entropy.** Note that by our choice of the quasitiles $T_1, \dots, T_N$, we may apply Lemma 2.20 with parameters k, SS^{-1}, and T_i, obtaining that for each i, we have

$$|\{v \in \mathcal{A}^{T_i} : v \text{ has } k \text{ periods from } SS^{-1}\}| \leq |\mathcal{A}|^{2|T_i|/k}. \tag{5.4}$$

Lemma 5.5. *For large enough m, we have $h(X_m) < \epsilon$.*

Proof. Choose m large enough that $TSS^{-1} \subset G_m$. Let $\delta_1 \in (\overline{D}(F), 1/2)$ and $\delta_2 > 0$ be arbitrary. Choose n large enough that $D_n(F) \leq \delta_1$ and

$$\max\left(\frac{|F_n \triangle G_m^{-1} F_n|}{|F_n|}, \frac{|F_n \triangle TT^{-1} F_n|}{|F_n|}\right) < \delta_2.$$

Since $\{T_1, \dots, T_N\}$ is a set of η-quasitiles, there exists a collection $\{C_1, \dots, C_N\}$ of center sets corresponding to F_n with the additional property that if $c \in C_i$, then $T_i c \cap F_n \neq \varnothing$.

Let $w \in \mathcal{L}_{F_n}(X_m)$. Choose $y \in X_m$ such that $y(F_n) = w$, and choose $x \in X$ such that $y = \varphi_m \circ \dots \varphi_0(x)$. Let $J_w = \{g \in G_m^{-1} F_n : y_g = a\}$. Note that for $g \in F_n$, we have that $w_g = a$ if and only if $g \in J_w$, and $w_g = b$ if and only if $g \in (G_m J_w) \setminus J_w$. Furthermore,

$$\begin{aligned} |J_w| &= |J_w \cap F_n| + |J_w \setminus F_n| \\ &\leq N_{F_n}(x, F) + |(G_m^{-1} F_n) \setminus F_n| \\ &\leq \delta_1 |F_n| + \delta_2 |F_n|, \end{aligned} \tag{5.5}$$

where we have used our choice of n in the last estimate.

Now for each i, let C_i^w be the set of $c \in C_i$ such that $(F_n \cap T_i c) \setminus (G_m J_w) \neq \varnothing$. Note that C_i^w is completely determined by J_w (along with the already chosen F_n, T_i, and G_m).

Claim 5.6. If $c \in C_i^w$, then $x(T_i c)$ has k periods from SS^{-1}.

Proof. To begin, suppose that $c \in C_i$, $g \in F_n \cap T_i c$, and

$$\sigma_c(x) \in \bigcup_{s \in SS^{-1}} \sigma_s(F).$$

Then there exists $s \in SS^{-1}$ such that $\sigma_{sc}(x) \in F$. Hence $y_{sc} = a$. Let $g' = sc$. Then $c = s^{-1} g'$. Now let $t \in T_i$ be such that $tc = g$. Then $g' = st^{-1} g \in SS^{-1} T^{-1} F_n \subset G_m^{-1} F_n$, by our choice of m. Therefore $g' \in J_w$, and then $g = ts^{-1} g' \in TSS^{-1} J_w \subset G_m J_w$, again using our choice of m. We conclude that if $c \in C_i$ and

$$\sigma_c(x) \in \bigcup_{s \in SS^{-1}} \sigma_s(F),$$

then $F_n \cap T_i c \subset G_m J_w$. By the contrapositive, if $c \in C_i$ and $(F_n \cap T_i c) \setminus (G_m J_w) \neq \varnothing$, then

$$\sigma_c(x) \notin \bigcup_{s \in SS^{-1}} \sigma_s(F),$$

which gives that $x(Tc)$ has k periods from SS^{-1} by our choice of F. Thus, we have shown that if $c \in C_i^w$, then $x(Tc)$ has k periods from SS^{-1}, and therefore so does $x(T_i c)$ (since $T_i \subset T$). $\square$

For a finite set E, let $\mathcal{P}(E)$ denote the power set of E. Now consider the map $\phi : \mathcal{L}_{F_n}(X_m) \to \mathcal{P}(G_m^{-1}F_n)$ defined by $w \mapsto J_w$.

Claim 5.7. For each $J \subset \mathcal{P}(G_m^{-1}F_n)$, we have

$$|\phi^{-1}(J)| \leq |\mathcal{A}|^{\eta|F_n|} \cdot |\mathcal{A}|^{(2/k)\sum_i |T_i| \cdot |C_i|}.$$

Proof. Let $J \in \mathcal{P}(G_m^{-1}F_n)$. Define $C_i' = \{c \in C_i : (F_n \cap T_i c) \setminus (G_m J) \neq \varnothing\}$. Now let $w \in \phi^{-1}(J)$, i.e., $J_w = J$, and let $x \in X$ be such that $x(F_n) = w$. Note that since $J_w = J$, we also have $C_i^w = C_i'$ for each i.

Let $g \in F_n$. For $g \in G_m J$, we have that $w_g = a$ whenever $g \in J$ and $w_g = b$ whenever $g \notin J$. Now suppose $g \in T_i c \setminus (G_m J)$ for some $c \in C_i$. Then $w_g = x_g$. Also, we have $c \in C_i'$, and by Claim 5.6, $x(T_i c)$ has k periods from SS^{-1}. Hence $w \in \phi^{-1}(J)$ is uniquely determined by a tuple of the form

$$\left(\big(x(T_1 c)\big)_{c \in C_1'}, \ \ldots, \ \big(x(T_N c)\big)_{c \in C_N'}, \ w(F_n \setminus (\bigcup_i T_i C_i)) \right)$$

where each $x(T_i c)$ has k periods from SS^{-1}. Thus, we have

$$\begin{aligned} |\phi^{-1}(J)| &\leq \prod_{i=1}^{N} |\{v \in \mathcal{A}^{T_i} : v \text{ has } k \text{ periods from } SS^{-1}\}|^{|C_i'|} \\ &\qquad \cdot |\mathcal{A}|^{|F_n \setminus (\bigcup_i T_i C_i)|} \\ &\leq \prod_{i=1}^{N} |\{v \in \mathcal{A}^{T_i} : v \text{ has } k \text{ periods from } SS^{-1}\}|^{|C_i|} \\ &\qquad \cdot |\mathcal{A}|^{|F_n \setminus (\bigcup_i T_i C_i)|}. \end{aligned}$$

Then by Lemma 2.20 and the fact that $\{T_1 C_1, \ldots, T_N C_N\}$ $(1-\eta)$-covers $|F_n|$, we obtain the desired inequality

$$|\phi^{-1}(J)| \leq |\mathcal{A}|^{\eta|F_n|} \cdot |\mathcal{A}|^{(2/k)\sum_i |T_i| \cdot |C_i|}.$$

$\square$

Finally, using (5.5) in combination with Lemma 2.21 and Claim 5.7 yields the following estimate on the cardinality of $|\mathcal{L}_{F_n}(X_m)|$:

$$|\mathcal{L}_{F_n}(X_m)| \leq 2^{H(\delta_1+\delta_2)|F_n|}|\mathcal{A}|^{\eta|F_n|} \cdot |\mathcal{A}|^{(2/k)\sum_i |T_i|\cdot|C_i|}.$$

Using the η-disjointness of $T_1, \ldots, T_N$, we see that for each i, we have

$$\begin{aligned} |T_i C_i| &= \left|\bigcup_{c\in C_i} T_i c\right| \\ &\geq \sum_{c\in C_i}(1-\eta)|T_i c| \\ &\geq (1-\eta)|T_i|\cdot|C_i|. \end{aligned}$$

Recall that by our choice of centers, if $c \in C_i$, then $T_i c \cap F_n \neq \varnothing$. Let $g \in T_i c \cap F_n$. Then $g = tc$ for some $t \in T_i$, and so $c = t^{-1}g \in T^{-1}F_n$. Hence $T_i C_i \subset TT^{-1}F_n$. Combining the previous displayed formula with the quasi-invariance of F_n with respect to TT^{-1} and δ_2 (by choice of n), we obtain

$$\begin{aligned} \sum_i |T_i|\cdot|C_i| &\leq \frac{1}{1-\eta}\sum_i |T_i C_i| \\ &\leq \frac{1}{1-\eta}\left|\bigcup_i T_i C_i\right| \\ &\leq \frac{1}{1-\eta}|TT^{-1}F_n| \\ &\leq \frac{1+\delta_2}{1-\eta}|F_n|. \end{aligned}$$

Finally, putting together all of the above estimates, we get

$$\begin{aligned} |\mathcal{L}_{F_n}(X_m)| &\leq 2^{H(\delta_1+\delta_2)|F_n|}|\mathcal{A}|^{(2/k)\sum_i |T_i|\cdot|C_i|}|\mathcal{A}|^{\eta|F_n|} \\ &\leq 2^{H(\delta_1+\delta_2)|F_n|}|\mathcal{A}|^{(2/k)\frac{1+\delta_2}{1-\eta}|F_n|}|\mathcal{A}|^{\eta|F_n|}. \end{aligned}$$

Taking logarithms, dividing by $|F_n|$, and letting n tend to infinity, we get

$$h(X_m) \leq H(\delta_1+\delta_2) + \left(\frac{2(1+\delta_2)}{k(1-\eta)} + \eta\right)\log|\mathcal{A}|.$$

Since $\delta_2 > 0$ was arbitrary, and since $\delta_1 \in (\overline{D}(F), 1/2)$ was arbitrary, we see that

$$h(X_m) \leq H(\overline{D}(F)) + \left(\frac{2}{k(1-\eta)} + \eta\right)\log|\mathcal{A}|.$$

By (5.1), (5.2), and (5.3), we conclude that $h(X_m) < \epsilon$. This finishes the proof of Lemma 5.5. $\square$

This concludes the proof of the ϵ-density of the sequence $\{h(X_m)\}_{m=0}^{\infty}$ within $[0, h(X)]$. Since $\epsilon > 0$ was arbitrary, we conclude that $\mathcal{H}_{sub}(X)$ is dense in $[0, h(X)]$, which finishes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

This proof is quite similar to that of Theorem 1.1. The main difference is that rather than completely removing information from locations in points of X by replacing letters by a and b (as in the previous proof), the factor maps in this proof apply the block code defining $\pi : X \to Y$ at those locations.

Let X have alphabet $\mathcal{A}_X$, and let Y have alphabet $\mathcal{A}_Y$. Suppose without loss of generality that the factor map $\pi : X \to Y$ results from a 1-block code $\pi_0 : \mathcal{A}_X \to \mathcal{A}_Y$. We assume without loss of generality that $|\mathcal{A}_X| \geq 2$. Let $\epsilon \in (0, 1/2)$. We will show that $\mathcal{H}_{sub}(X, Y)$ is ϵ-dense in $[h(Y), h(X)]$.

Choose the following parameters as in the proof of Theorem 1.1: k, δ, m_0, S, η, T, and F. Also, choose the same enumeration $G = \{g_k\}_{k=0}^{\infty}$ as before, and define the same sequence of maps $\{\varphi_m\}_{m=0}^{\infty}$ and G-subshifts $\{X_m\}_{m=0}^{\infty}$.

Now define the alphabets $\mathcal{B}_Y = \mathcal{A}_Y \times \{a, b\}$ and $\mathcal{B} = \mathcal{A}_X \cup \mathcal{B}_Y$. We suppose that a and b are chosen so that $\mathcal{A}_X \cap \mathcal{B}_Y = \varnothing$. Define $\varphi_0 : X \to \mathcal{B}^G$ by the rule

$$\psi_0(x)_g = \begin{cases} (\pi_0(x_g), a), & \text{if } \sigma_g(x) \in F \\ x_g, & \text{otherwise.} \end{cases}$$

Let $Z_0 = \psi_0(X)$. Inductively, let $m \geq 0$, and suppose we have defined $Z_0, \ldots, Z_m$ and $\psi_0, \ldots, \psi_m$. Define $\psi_{m+1} : Z_m \to \mathcal{B}^G$ by the rule

$$\psi_{m+1}(z)_g = \begin{cases} (\pi_0(z_g), b), & \text{if } z_g \in \mathcal{A}_X \text{ and } z_{g_{m+1}^{-1} g} \in \mathcal{A}_Y \times \{a\} \\ z_g, & \text{otherwise.} \end{cases}$$

Let $Z_{m+1} = \psi_{m+1}(Z_m)$.

Let us now define some auxiliary maps. First, for $m \geq 0$, let $\overline{\varphi}_m = \varphi_m \circ \cdots \circ \varphi_0$ and $\overline{\psi}_m = \psi_m \circ \cdots \circ \psi_0$. Note that $\overline{\varphi}_m : X \to X_m$ and $\overline{\psi}_m : X \to Z_m$ are factor maps. Next, let $p_Y : \mathcal{A}_Y \times \{a, b\} \to \mathcal{A}_Y$ be the projection $p_Y(u, v) = u$. For any $m \geq 0$, we define the map $P_m^Y : Z_m \to (\mathcal{A}_Y)^G$ by the rule

$$P_m^Y(z)_g = \begin{cases} p_Y(z_g), & \text{if } y_g \in \mathcal{A}_Y \times \{a, b\} \\ \pi_0(z_g), & \text{otherwise.} \end{cases}$$

Note that $\pi = P_m^Y \circ \overline{\psi}_m$, and since π is a factor map, P_m^Y must be a factor map onto Y as well. Also, let $p_X : \mathcal{A}_Y \times \{a, b\}$ be the projection $p_X(u, v) = v$. For $m \geq 0$, let $P_m^X : Z_m \to (\mathcal{A}_X \cup \{a, b\})^G$ be defined by the rule

$$P_m^X(z)_g = \begin{cases} p_X(z_g), & \text{if } z_g \in \mathcal{A}_Y \times \{a, b\} \\ z_g, & \text{otherwise.} \end{cases}$$

Observe that for any $m \geq 0$, we have $\overline{\varphi}_m = P_m^X \circ \overline{\psi}_m$, and since φ_m is a factor map, we see that P_m^X is a factor map from Z_m onto X_m.

Claim 6.1. $h(X) - h(Z_0) < \epsilon$.

Proof. Use Lemma 4.5 as in the proof of Claim 5.2. □

Claim 6.2. For any $m \geq 0$, we have $h(Z_m) - h(Z_{m+1}) < \epsilon$.

Proof. Use Lemma 4.5 as in the proof of Claim 5.4. □

Claim 6.3. For any $m \geq 0$, we have $h(Z_m) \leq h(Y) + h(X_m)$.

Proof. Note that P_m^Y and P_m^X are 1-block codes, and hence they are well-defined maps on the language of Z_m. Given $w \in \mathcal{L}_{F_n}(Z_m)$, let $u = P_m^X(w)$ and $v = P_m^Y(w)$. The map $w \mapsto (u, v)$ is injective: i) if $u_g = a$, then $w_g = (v_g, a)$, ii) if $u_g = b$, then $w_g = (v_g, b)$, and iii) if $u_g \in \mathcal{A}_X$, then $w_g = u_g$. Hence

$$|\mathcal{L}_{F_n}(Y_m)| \leq |\mathcal{L}_{F_n}(X_m)| \cdot |\mathcal{L}_{F_n}(Y)|.$$

Taking logarithms, dividing by $|F_n|$, and taking the limit as n tends to infinity, we conclude that $h(Z_m) \leq h(Y) + h(X_m)$, as desired. □

By Lemma 5.5, we see that for all large m, we have $h(X_m) < \epsilon$. Combining this fact with Claims 6.1, 6.2, and 6.3, we see that $\mathcal{H}_{sub}(X, Y)$ is ϵ-dense in $[h(Y), h(X)]$. Since ϵ was arbitrary, this concludes the proof of Theorem 1.2.

7. Proof of Theorem 1.4

To achieve arbitrary entropies $r \in [h(Y), h(X)]$, we construct zero-dimensional factors as inverse limits of intermediate subshift factors using Lemma 2.18 and Theorem 1.2.

The case $r = h(Y)$ is trivial, and so we let $r \in (h(Y), h(X)]$. Let $Z_0 = Y$, $\varphi_0 = \pi$, and ψ_0 be the identity map on Y. By Theorem 1.2, there exists a subshift Z_1 and factor maps $\varphi_1 : X \to Z_1$ and $\pi_1 : Z_1 \to Z_0$ such that $\pi = \pi_1 \circ \psi_1$ and $h(Z_1) \in (r - 1, r)$. Now suppose we have defined Z_n, $\varphi_n : X \to Z_n$, and $\pi_n : Z_n \to Z_{n-1}$. By Theorem 1.2, there exists a subshift Z_{n+1} and factor maps $\varphi_{n+1} : X \to Z_{n+1}$ and $\pi_{n+1} : Z_{n+1} \to Z_n$ such that $\pi_n = \pi_{n+1} \circ \varphi_{n+1}$ and $h(Z_{n+1}) \in (r - \frac{1}{n+1}, r)$.

Let $Z = \varprojlim Z_n$, and let $\varphi : X \to Z$ be the natural factor map (as in Lemma 2.18). By Lemma 2.17 and our choice of $h(Z_n)$ for each n, we have that $h(Z) = \lim_n h(Z_n) = r$.

8. Proof of Theorem 1.5

Consider any zero-dimensional system X. By definition, there are finite clopen partitions $\mathcal{P}_n$ of X with elements whose diameters approach 0. For each n, we can associate a factor map f_n to the full shift $(\mathcal{P}_n)^G$ via the usual symbolic coding, i.e., $(f_n(x))(g)$ is the element of $\mathcal{P}_n$ in which $T_g x$ lies. Define $X_n = f_n(X)$; then $h(X_n)$ is equal to $h(X, \mathcal{P}_n)$, the entropy of (X, T) with respect to the partition $\mathcal{P}_n$ (see [14] for a full definition). Since $\mathrm{diam}(P_n) \to 0$, Theorem 7.6 from [14] (proved there for $G = \mathbb{Z}$, but the general proof is similar) implies that the entropies $h(X_n) = h(X, \mathcal{P}_n)$ increase to $h(X)$.

The case $r = h(X)$ is trivial, and so we choose arbitrary $r \in [0, h(X))$. Since $h(X_n)$ increases to $h(X)$, there exists n so that $h(X_n) > r$. By Theorem 1.3, there exists a factor map ψ so that $\psi(X_n)$ is zero-dimensional and $h(\psi(X_n)) = r$. This means that $h((\psi \circ f_n)X) = r$, and since $r \in [0, h(X))$ was arbitrary, the proof is complete.

Remark 8.1. A natural question is whether the relative version of Theorem 1.5 holds, i.e. whether given a factor map between zero-dimensional systems, all entropies in the interval between them are achieved by entropies of intermediate zero-dimensional factors. We considered this question, but if such a relative theorem holds, it cannot be proved via our methods.

Here we explain why our methods cannot be used to obtain a relative zero-dimensional result. For $G = \mathbb{Z}$, there exist zero-dimensional systems Y such that all symbolic extensions of Y have entropy bounded away from Y (these systems have positive residual entropy, as in [2]). Specifically, consider Y to be the zero-dimensional system from Example 8.10 in [2]; it was shown that $h(Y) = \log 2$, but any subshift Z factoring onto Y has $h(Z) \geq 2 \log 2$.

This means that if we were to consider the projection factor map $\varphi : Y \times Y \to Y$, there are in fact no intermediate subshift factors with entropy in $[h(Y), h(Y \times Y)]$. Therefore, we could not possibly construct intermediate zero-dimensional factors with entropies in $(h(Y), h(Y \times Y))$ as inverse limits of subshifts.

Of course, the relative version of our zero-dimensional result could still be true, but a proof would require completely different techniques.

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