THE RELATIONSHIP BETWEEN WORD COMPLEXITY AND COMPUTATIONAL COMPLEXITY IN SUBSHIFTS

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ABSTRACT. We prove several results about the relationship between the word complexity function of a subshift and the set of Turing degrees of points of the subshift, which we call the Turing spectrum. Among other results, we show that a Turing spectrum can be realized via a subshift of linear complexity if and only if it consists of the union of a finite set and a finite number of cones, that a Turing spectrum can be realized via a subshift of exponential complexity (i.e. positive entropy) if and only if it contains a cone, and that every Turing spectrum which either contains degree **0** or is a union of cones is realizable by subshifts with a wide range of 'intermediate' complexity growth rates between linear and exponential.

1. INTRODUCTION

In this work, we study various notions of complexity for subshifts. A subshift is a set X of bi-infinite sequences of symbols from a finite alphabet $\mathcal{A}$ which is invariant under shifts and closed in the product topology; any such X is then a dynamical system when associated with the shift map σ.

Another, equivalent, way to define subshifts is through families of finite forbidden words: any such set induces the subshift of sequences in which no forbidden word appears. Different hypotheses on the set of forbidden words yield classes of subshifts; for instance, subshifts of finite type (SFTs) correspond to finite sets of forbidden words, and effective subshifts correspond to recursively enumerable sets of forbidden words.

There are many ways to measure how "complicated" a subshift is. One of the simplest is the so-called word complexity function $(c_n(X))$, where $c_n(X)$ is the number of n-letter words appearing within points of X. The minimum unbounded growth rate of the complexity function is linear complexity, and the maximum growth rate is exponential growth, i.e. positive topological entropy.

A more recent measurement of complexity comes from computability theory, specifically the notion of Turing degrees. The Turing degree of a sequence is a measure of its computational power [Rog87], and the Turing spectrum [JV13] is the set of all Turing degrees of points in a subshift X. In [JV13], the authors proved several results about the Turing spectra of various subshifts, focusing mostly on so-called multidimensional shifts of finite type. One of the results, however, applied to minimal subshifts in any dimension: namely, such a subshift's spectrum either consists only of **0** (the degree of computable sequences) or must contain the cone

above any of its degrees (this result is recalled in Lemma 2.4). It was later proved that spectra of minimal subshifts may contain several cones [HV17] and then that they actually correspond exactly to the enumeration cones of co-total enumeration degrees [McC18].

It is then natural to wonder about the relationship between these two notions of complexity, especially since substantial connections between entropy and computability are known (see [HM10]).

In this work, we prove several results in this direction. Namely, under various complexity assumptions, we give necessary conditions on the spectrum and sufficient conditions on a set of Turing degrees for there to exist a subshift with the desired complexity which realizes that spectrum. The somewhat surprising summary of these results is that the spectrum is heavily restricted near minimal (i.e. linear) and maximal (i.e. exponential) complexity, and these restrictions relax as the complexity moves further from these extremes.

For subshifts with linear complexity, the following result shows that the spectrum is comprised of a finite number of cones and isolated degrees, with size bounded by the 'slope' (i.e. linear growth rate). (See Section 2 for formal definitions.)

Theorem 3.2. *If a subshift X has strong linear complexity with index t, then $Sp(X) \backslash \{\mathbf{0}\}$ can be written as the union of some number c of cones and m other (isolated) degrees, where $c + m < t$.*

In the opposite direction, the following theorem shows that any spectrum of this type can be realized by a subshift with linear complexity.

Theorem 4.3. *Let S and S' be finite sets of Turing degrees. There exists a subshift X with strong linear complexity whose spectrum is*

$$Sp(X) = \{\mathbf{0}\} \cup S \cup \bigcup_{\mathbf{d}' \in S'} \mathcal{C}_{\mathbf{d}'}.$$

The other extreme for complexity is exponential growth, also called positive (topological) entropy, which also influences the spectrum:

Theorem 3.11. *If a subshift X has positive entropy, then its spectrum contains a cone.*

Again, a result in the opposite direction holds: every such spectrum can be realized by a subshift with positive entropy, which can be taken to be arbitrarily large.

Theorem 4.16. *For any subshift X with entropy h, if $Sp(X)$ contains a cone C, then there exists a subshift P with $Sp(P) = Sp(X)$ and entropy h', for any $h' > h$ such that $\deg_T h' \in C$.*

Our remaining results treat complexities strictly between linear and exponential. First, we show that subshifts with complexities arbitrarily close to linear can realize more spectra than subshifts with linear complexity.

Theorem 4.6. *Let S be a countable set of Turing degrees, and let S' be a finite set of Turing degrees. Then the set of subshifts X with $Sp(X) = \{\mathbf{0}\} \cup S \cup \bigcup_{\mathbf{d}' \in S'} \mathcal{C}_{\mathbf{d}'}$ exhibits arbitrarily slow superlinear complexity.*

By slightly strengthening the assumption on complexity to require that it is "computably close" to linear, we can realize an extremely large class of spectra.

Theorem 4.7. *Let S be a closed nonempty set of Turing degrees. Then the sets of subshifts X with $Sp(X) = S \cup \{\mathbf{0}\}$ or $Sp(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ exhibit arbitrarily computably slow superlinear complexity.*

By a closed set of Turing degrees, we mean here that the set of Turing degrees is realizable by a closed subset of $\{0,1\}^{\mathbb{N}}$. Since subshifts are closed, the Turing spectrum of a subshift must be closed, and so some assumption of this type on S is necessary. Closed sets of Turing degrees are of course a strict subclass of unconstrained sets of Turing degrees: for instance, apart from the set of all degrees and the empty set, a set and its complement cannot both be closed sets of Turing degrees (see [KL10]).

Similar results hold for complexity close to exponential.

Theorem 4.10. *Let S be a closed nonempty set of Turing degrees. Then the set of subshifts with $Sp(X) = S \cup \{\mathbf{0}\}$ exhibits arbitrarily computably fast subexponential complexity.*

Theorem 4.11. *Let S be a closed nonempty set of Turing degrees. Then the set of subshifts with $Sp(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ exhibits arbitrarily fast subexponential complexity.*

Finally, we show that given an intermediate growth rate, i.e. neither linear nor exponential, we can can construct subshifts with complexity close to this growth rate and realising these same classes of spectra.

Theorem 4.12. *Let S be a closed nonempty set of Turing degrees, and let $g : \mathbb{N} \to \mathbb{N}$ be any computable function which is superlinear (i.e. $\frac{g(n)}{n} \to \infty$) and subexponential (i.e. $\frac{\log g(n)}{n} \to 0$). Then there exists a subshift X with $Sp(X) = \{\mathbf{0}\} \cup S$ and a sequence (n_k) so that*

$$\lim_{k \to \infty} \frac{c_{n_k}(X)}{g(n_k)} = 1.$$

Theorem 4.15. *Let S be a closed nonempty set of Turing degrees, and let $g : \mathbb{N} \to \mathbb{N}$ be any computable function which is superquadratic (i.e. $\frac{g(n)}{n^2} \to \infty$) and subexponential (i.e. $\frac{\log g(n)}{n} \to 0$). Then there exists a subshift X with $Sp(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ and a sequence (n_k) so that*

$$\lim_{k \to \infty} \frac{c_{n_k}(X)}{g(n_k)} = 1.$$

We note that all of our realization results are for Turing spectra which either contain the degree $\mathbf{0}$ or are unions of cones. We also show that there exist subshifts with spectrum outside of these two categories, and so our results do not realize all possible spectra for subshifts.

Theorem 5.1. *Given any degree $\mathbf{d_1} \leq \mathbf{0}'$ and any degree $\mathbf{d_2}$, where $\mathbf{0}'$ is the degree of the halting problem, there exists a subshift X such that $Sp(X) = \mathcal{C}_{\mathbf{d_1}} \cup \{\mathbf{d_2}\}$. In particular, $\mathbf{d_2}$ is not in $\mathcal{C}_{\mathbf{d_1}}$ if we take $\mathbf{d_2} < \mathbf{d_1}$ or $\mathbf{d_2}$ and $\mathbf{d_1}$ incomparable.*

We give relevant definitions in Section 2, then restrictions imposed on the Turing spectrum by word complexity in Section 3, give realization results in Section 4 and end with giving an example of spectrum that differs from our realizations in Section 5.

2. Definitions

2.1. Subshifts.
Given a finite set $\mathcal{A}$ (called the **alphabet**), a **subshift** over $\mathcal{A}$ is a subset of $\mathcal{A}^{\mathbb{Z}}$ which is closed in the product topology and invariant under the shift map σ defined by $(\sigma x)(n) = x(n+1)$.

For any $x \in \mathcal{A}^{\mathbb{Z}}$, the **orbit** of x is the set of all its translates $\mathcal{O}(x) = \{\sigma^k(x) \mid k \in \mathbb{Z}\}$.

A **word** is any element of $\mathcal{A}^n$ for some $n \in \mathbb{N}$. For any word w, its **length**, written $|w|$, is the integer n so that $w \in \mathcal{A}^n$. Subshifts can alternately be defined in terms of words; for any set $\mathcal{F}$ of words, one can define the subshift $X(\mathcal{F})$ given by the set of all sequences on $\mathcal{A}$ which do not contain any words of $\mathcal{F}$. A subshift is a **subshift of finite type (SFT)** if it is $X(\mathcal{F})$ for some finite $\mathcal{F}$.

A point $x \in \mathcal{A}^{\mathbb{Z}}$ is **periodic** if there exists $p \neq 0$ so that $x(n) = x(n+p)$ for all $n \in \mathbb{Z}$, and **aperiodic** otherwise. We say that x is **eventually periodic on the left** if there exists $p \neq 0$ and N so that $x(n) = x(n+p)$ for all $n < N$, and **eventually periodic the right** if there exists $p \neq 0$ and N so that $x(n) = x(n+p)$ for all $n > N$.

A point $x \in \mathcal{A}^{\mathbb{Z}}$ is **recurrent** if every word appearing within x appears infinitely many times in x. We note that any aperiodic sequence which is eventually periodic on the left and right (e.g. $^{\omega}(01)2(345)^{\omega}$ is not recurrent, since any portion which contains letters breaking the periodicity on the left and right can appear only finitely many times.

A subshift is **minimal** if it does not properly contain any nonempty subshift.

For a subshift X, the **language** $\mathcal{L}(X)$ is the set of all words appearing within some point in X. This notion can be extended to individual points, i.e. for $x \in X$, $\mathcal{L}(x)$ is the set of all words appearing in x. The set of all words of $\mathcal{L}(X)$ of length n will be denoted $\mathcal{L}_n(X) = \mathcal{L}(X) \cap \mathcal{A}^n$.

2.2. Complexity.
The **complexity function** $c_n(X)$ of a subshift is defined by $c_n(X) = |\mathcal{L}_n(X)|$, i.e. $c_n(X)$ is the number of all words with length n appearing in some $x \in X$.

One of the first results in symbolic dynamics is the Morse-Hedlund theorem, which states that if there exists n for which $c_n(X) \leq n$, then X is finite, and in particular all points in X are periodic. Therefore, all nontrivial X satisfy $c_n(X) > n$ for all n, i.e. have complexity growing at least linearly.

A subshift X has **strong linear complexity (with index t)** if there exists t so that $\limsup c_n(X) - tn < \infty$, and **weak linear complexity (with index t)** if there exists t so that $\liminf c_n(X) - tn < \infty$. We say that X has (weak/strong) linear complexity if it has (weak/strong) linear complexity for some index t.

A collection S of subshifts **exhibits arbitrarily slow super-linear complexity** if, for every increasing unbounded $f : \mathbb{N} \to \mathbb{N}$, there exists $X \in S$ where $c_n(X) < nf(n)$ for sufficiently large n.

A collection S of subshifts **exhibits arbitrarily computably slow super-linear complexity** if, for every computable increasing unbounded $f : \mathbb{N} \to \mathbb{N}$, there exists $X \in S$ where $c_n(X) < nf(n)$ for sufficiently large n.

A collection S of subshifts **exhibits arbitrarily fast subexponential complexity** if, for every increasing unbounded $f : \mathbb{N} \to \mathbb{N}$, there exists $X \in S$ with $\lim \frac{\log c_n(X)}{n} = 0$, but $\frac{\log c_n(X)}{n} > \frac{1}{f(n)}$ for sufficiently large n.

A collection S of subshifts **exhibits arbitrarily computably fast subexponential complexity** if, for every computable increasing unbounded $f : \mathbb{N} \to \mathbb{N}$, there exists $X \in S$ with $\lim \frac{\log c_n(X)}{n} = 0$, but $\frac{\log c_n(X)}{n} > \frac{1}{f(n)}$ for sufficiently large n.

A subshift X has **positive entropy** if $\lim_{n \to \infty} \frac{\log c_n(X)}{n} > 0$; this limit is called the **topological entropy** of X and written $h(X)$. For more information on entropy (including a proof of the existence of the limit), see [Wal82].

2.3. **Ergodic theory.** Here we summarize a few results from ergodic theory which will be used in a couple of later proofs; see [Wal82] for a detailed introduction to the subject. Throughout this section, all measures considered are Borel probability measures on $\mathcal{A}^{\mathbb{Z}}$. Several statements are slightly weaker than the full strength of the theorem in question; we've stated versions suited to our setting.

Theorem 2.1 (Poincaré recurrence theorem). *If X is a subshift and $\mu(X) = 1$, then μ-a.e. $x \in X$ is recurrent.*

Definition 2.1. *A measure μ is **ergodic** if every measurable A for which $A = \sigma A$ satisfies $\mu(A) \in \{0, 1\}$.*

Theorem 2.2 (Birkhoff pointwise ergodic theorem). *If μ is an ergodic measure and w is a word, then for μ-a.e. x,*

$$\lim_{n \to \infty} \frac{|\{0 \leq i < n \ : \ \sigma^i x \in [w]\}|}{n} \to \mu([w]),$$

where $[w]$ is the set of x containing an occurrence of w starting at the 0th coordinate.

Lemma 2.3. *If X is a subshift with positive topological entropy, then there exists an ergodic measure μ where $\mu(X) = 1$ and μ-a.e. $x \in X$ is not periodic.*

Proof. We prove the contrapositive, and assume that all ergodic μ with $\mu(X) = 1$ satisfy $\mu(P) > 0$, where P is the set of periodic points of X. For every ergodic μ, since $P = \sigma P$ and $\mu(P) > 0$, in fact $\mu(P) = 1$. It is easy to see from the definition that any such μ has measure-theoretic entropy $h(\mu) = 0$ (the definition of measure-theoretic entropy is long, and so we do not include it here.) By the so-called ergodic decomposition, every measure μ with $\mu(X) = 1$ can be written as a convex combination of ergodic measures, and since measure-theoretic entropy is linear, $h(\mu) = 0$ for all such μ. Finally, the Variational Principle ([Wal82]) states that the topological entropy $h(X)$ is the supremum of the measure-theoretic entropy $h(\mu)$ over all μ with $\mu(X) = 1$, and so $h(X) = 0$, completing the proof. $\square$

2.4. **Computability.** A set $S \subseteq \{0, 1\}^{\mathbb{N}}$ is called **effectively closed** or a Π_1^0 **class** iff there exists a Turing machine M such that the sequences of S are exactly the ones on which M does not halt. Equivalently, there is a Turing machine that enumerates the prefixes of all sequences not in S.

An effectively closed subshift, or **effective subshift** is thus a subshift $X(\mathcal{F})$ for some recursively enumerable set $\mathcal{F}$ of forbidden words. SFTs are effectively closed, but not all effectively closed subshifts are SFTs.

Define the partial order $\leq_T$ on $\{0, 1\}^{\mathbb{N}}$ as follows: for $x, y \in \{0, 1\}^{\mathbb{N}}$, $x \leq_T y$ if there exists a Turing machine which, given y as an oracle, outputs x. We say that $x \equiv_T y$ whenever $x \leq_T y$ and $y \leq_T x$, and since $\leq_T$ is clearly reflexive and transitive, $\equiv_T$ is an equivalence relation. A **Turing degree** is an equivalence class for this relation. Given a sequence $x \in \{0, 1\}^{\mathbb{N}}$, we denote its Turing degree by

$\deg_T x$. The computable sequences all share the lowest degree, denoted by $\mathbf{0}$. For any degree $\mathbf{d}$, the **cone with base d** is the set $\{\mathbf{e} \mid \mathbf{d} \leq_T \mathbf{e}\}$ of all degrees above $\mathbf{d}$, and is denoted by $\mathcal{C}_d$. We will often write $\mathcal{C}_x$ instead of the more formally correct $\mathcal{C}_{\deg_T x}$.

The **Turing spectrum** of a subshift X, denoted by $\mathrm{Sp}(X)$, is the set $\{\mathbf{d} \mid \exists x \in X, \deg_T(x) = \mathbf{d}\}$ of all Turing degrees of all points of X.

The following lemma describes a simple hypothesis guaranteeing that $\mathrm{Sp}(X)$ contains a cone. It is essentially the same as Theorem 5.3 from [JV13], and we present an abbreviated proof describing why the hypothesis of minimality of X can be replaced by aperiodicity and recurrence of x.

Lemma 2.4 ([JV13]). *For any subshift X and any aperiodic recurrent point $x \in X$, $\mathcal{C}_x \subseteq Sp(X)$.*

Proof. Since x is aperiodic and recurrent, it cannot be both eventually periodic on the left and right. Since reflecting a sequence about the origin does not affect its Turing degree, we may assume without loss of generality that x is not eventually periodic on the right.

We construct two functions $\mathbf{enc} : \{0,1\}^{\mathbb{N}} \to X$ and $\mathbf{dec} : \mathbf{enc}(\{0,1\}^{\mathbb{N}}) \to \{0,1\}^{\mathbb{N}}$ such that $\mathbf{enc}(y)$ is an "encoding" of y derived from x and $\mathbf{dec}$ is a "decoding," with the property that $\mathbf{dec}(\mathbf{enc}(y)) = y$ for all y.

Let $\prec$ be a fixed order on Σ the alphabet of X. We compute $\mathbf{enc}(y)$ in an inductive way:

- We start with $w_0 = x_0$ the letter in the center of x.
- For each $k \geq 0$, the word w_k will be a subword of x. Then, by Lemma 5.2 from [JV13], there exist subwords $w_{k,0}$ and $w_{k,1}$ of x with the following properties: w_k occurs exactly twice within each $w_{k,i}$, the first differing letters a, b following the left/right occurrences of w_k in $w_{k,0}$ satisfy $a \prec b$, and the corresponding letters e, f for $w_{k,1}$ satisfy $f \prec e$. (The proof of Lemma 5.2 was stated for minimal X, but clearly only used the facts that x is recurrent and not eventually periodic on the right.)

  Depending on whether $y_k = 0$ or 1, consider the word w_{k,y_k}, and take the occurrence of this word within x closest to the origin. Let $i \in \mathbb{Z}$ be the position within x of the first letter of the first w within this w_{k,y_k}, and let j be the position within x of b or f, i.e. the rightmost of the first differing letters following w_k within w_{k,y_k}. Finally, define $w_{k+1} = x_{[i-(j-i);j]}$.

As w_k is a subword in the center of w_{k+1}, the sequence $(w_k)_{k \in \mathbb{N}}$ converges, and the sequence to which it converges is $\mathbf{enc}(y) \in X$.

The decoding $\mathbf{dec}$ is now straightforward: for any $z \in \mathbf{enc}(\{0,1\}^{\mathbb{N}})$, we start by looking at the letter in the center, which recovers w_0 from the definition of $\mathbf{enc}$. We then look for its next occurrence to the right, and we then scan the letters following both until we find a difference (this process succeeds by definition of $\mathbf{enc}$.) Once one is found, we check which case it is and recover y_0. This allowed us to recover w_1, and so we now do the same procedure again starting with w_1 and so on.

Clearly $\mathbf{dec}$ is computable, and $\mathbf{enc}$ is computable given x. Therefore, for any $y \in \{0,1\}^{\mathbb{N}}$, $y \leq_T \mathbf{enc}(y) \leq_T \sup(x,y)$. This means that if $y \geq_T x$, then $\deg_T \mathbf{enc}(y) = \deg_T y$. Since $\mathbf{enc}(y) \in X$ and $y \geq_T x$ was arbitrary, this shows that the spectrum of X contains the cone above $\deg_T x$. $\square$

The following corollary is nearly immediate.

Corollary 2.5. *For any subshift X, $Sp(X)$ contains either a cone or 0.*

Proof. By Theorem 2.1, X contains a recurrent point x. If x is periodic, then obviously $\deg_T(x) = \mathbf{0} \in \mathrm{Sp}(X)$. If x is aperiodic, then by Lemma 2.4, $\mathcal{C}_x \subseteq \mathrm{Sp}(X)$. $\square$

2.5. **Sturmian subshifts.** A particular class of subshifts which will be relevant for several of our results are the so-called Sturmian subshifts. For any $\alpha \in (0,1) \cap \mathbb{Q}^c$, the **Sturmian subshift with rotation number** α is defined as follows:

$$S_\alpha := \overline{\{(\lfloor (n+1)\alpha + c \rfloor - \lfloor n\alpha + c \rfloor)_{n \in \mathbb{Z}} \ : \ c \in \mathbb{R}\}}.$$

The relevant properties of Sturmian subshifts for our purposes are the following.

- $c_n(S_\alpha) = n + 1$ for all $n \in \mathbb{N}$.
- Every S_α is minimal.
- For every n and every $w \in \mathcal{L}_n(S_\alpha)$, the number of 1s in w is either $\lfloor n\alpha \rfloor$ or $\lfloor n\alpha \rfloor + 1$.

The third property above implies that $\mathrm{Sp}(S_\alpha) \subset \mathcal{C}_{\deg_T \alpha}$; for any $x \in S_\alpha$, the frequency of 1s in the first n letters of x is within $\frac{1}{n}$ of α, and so $\deg_T(x) \geq_T \deg_T(\alpha)$. Conversely, the point in S_α corresponding to $c = 0$ in the definition above clearly has degree $\deg_T \alpha$, and so by Lemma 2.4, $\mathrm{Sp}(S_\alpha) \supset \mathcal{C}_{\deg_T \alpha}$. We conclude that $\mathrm{Sp}(S_\alpha) = \mathcal{C}_{\deg_T \alpha}$ for any Sturmian subshift S_α.

3. Restrictions imposed by complexity on the (Turing) spectrum

In this section, we prove several results about restrictions on $\mathrm{Sp}(X)$ given knowledge about the growth rate of the word complexity function $c_n(X)$. Somewhat unsurprisingly, the largest restriction is imposed when X has linear complexity, the slowest possible (nontrivial) growth rate.

3.1. **Implications of linear complexity.** When $c_n(X)$ grows as slowly as possible, namely like n plus a constant, $\mathrm{Sp}(X)$ is extremely restricted.

Theorem 3.1. *If X is a subshift and $\limsup c_n(X) - n < \infty$, then $Sp(X)$ is equal to either the singleton $\mathbf{0}$, a single cone, or the union of $\{\mathbf{0}\}$ and a single cone.*

Proof. By [CJJM03], if $\limsup c_n(x) - n < \infty$, then x is either periodic, eventually periodic in both directions but aperiodic (e.g. $\dots 00012323\dots$), or quasi-Sturmian, i.e. the image of a Sturmian sequence under a shift-commuting homeomorphism. In this third case, x is aperiodic and recurrent (and in fact X is minimal).

Consider any X as in the theorem. We claim that X cannot contain two subsystems of the third (i.e. quasi-Sturmian) type. Indeed, unequal minimal systems are disjoint, and so if X contained two quasi-Sturmian systems Y, Z, they would be disjoint. Then for large enough n, $\mathcal{L}_n(Y)$ and $\mathcal{L}_n(Z)$ would be disjoint, implying $c_n(X) \geq 2n$ for large enough n, contradicting the original assumption on X. Clearly, all x of the first two types (periodic or eventually periodic in both directions) yield only points in X with degree $\mathbf{0}$. It therefore remains only to show that a quasi-Sturmian system has spectrum equal to a single cone. This follows immediately from the fact that the spectrum of a Sturmian system is a single cone, and the fact that a shift-commuting homeomorphism is a computable map. $\square$

For subshifts of linear complexity, the spectrum is still heavily constrained by the growth rate of $c_n(X)$.

Theorem 3.2. *If a subshift X has strong linear complexity with index t, then $Sp(X) \setminus \{\mathbf{0}\}$ can be written as the union of some number c of cones and m other (isolated) degrees, where $c + m < t$.*

The proof of Theorem 3.2 will be divided into two parts. In Lemma 3.3, we prove that $Sp(X)$ contains some number $m < t$ of nonzero isolated degrees whose associated cones are not subsets of $Sp(X)$.

Then, we prove that $Sp(X)$ is the union of these isolated degrees and some number $c < t - m$ of cones. Interestingly, most portions of this proof require only the weaker assumption of weak linear complexity.

3.1.1. *Linear complexity and number of isolated degrees.* From Lemma 2.4, we know that isolated degrees in the spectrum of a subshift, degrees that are not part of a cone, can only come from non-recurrent points.

Lemma 3.3. *If a subshift X has weak linear complexity with index t, then there are fewer than t different orbits of non-recurrent points in X.*

As a straightforward corollary, we can bound the number of degrees that are not the base of some cone:

Corollary 3.4. *If a subshift X has weak linear complexity with index t, then there are fewer than t nonzero degrees in $Sp(X)$ whose associated cones are not subsets of $Sp(X)$.*

Proof of Lemma 3.3. Suppose for a contradiction that there are at least t non-recurrent points $x_1, \ldots, x_t \in X$ with distinct orbits. We may then define words $w_1, \ldots, w_t$ s.t. w_i appears in x_i exactly once for each i. Note that we can freely extend any w_i to a larger subword of x_i without losing this property.

We claim that by extending the w_i if necessary, we may impose the following additional property: for all $j \neq k$, either w_j is not a subword of x_k or w_k is not a subword of x_j. Again, we note that once this property holds for a pair (j, k), it is not lost if any w_i is extended to a larger subword of x_i. It then suffices to show that for each fixed pair (j, k) in turn, we may extend w_j and/or w_k to larger words for which the desired property holds.

Fix any $j \neq k$. If w_k is not a subword of x_j, then the desired property already holds. Similarly, if w_k appears at least twice as a subword of x_j, then x_j contains a subword containing w_j and two occurrences of w_k, which itself cannot be a subword of x_k since w_k appears only once in x_k. Then, the desired property could be obtained by extending w_j to this larger word. The only remaining case is that w_k appears exactly once in x_j. We claim that w_k can then be extended to a larger subword of x_k which does not appear in x_j at all. Suppose for a contradiction that this is not the case, i.e. every subword of x_k containing w_k also appears in x_j. Since w_k appears exactly once in each of x_j and x_k, this would mean that for every n, the n letters preceding and following w_k are the same in x_j and x_k, and so that x_j is just a shift of x_k. We know that this is false, and so have a contradiction, implying that w_k can be extended to a longer subword of x_k for which the desired property holds.

By earlier observations, since $j \neq k$ were arbitrary, we may now assume without loss of generality that w_i appears exactly once in x_i for each i, and that for all $j \neq k$,

either w_j is not a subword of x_k or w_k is not a subword of x_j. By shifting the x_i and/or extending the w_i if necessary, we may also assume that there exists N so that $x_i(-N) \ldots x_i(N) = w_i$ for each i. Now, for any $n > 2N$, define the following words in $\mathcal{L}_n(X)$: $v_{i,m} = x_i(m) \ldots x_i(m+n-1)$ for $1 \leq i \leq t$ and $N - n < m \leq -N$. Note that by definition, $v_{i,m}$ is always a subword of x_i containing w_i. We claim that all $v_{i,m}$ are distinct. First, note that for $j \neq k$ and any m, m', $v_{j,m}$ contains w_j and is a subword of x_j, and $v_{k,m'}$ contains w_k and is a subword of x_k. If $v_{j,m} = v_{k,m'}$, then w_j is a subword of x_k and w_k is a subword of x_j, which we know not to be the case. Therefore, $v_{j,m} \neq v_{k,m'}$ whenever $j \neq k$. It remains only to show that $v_{i,m} \neq v_{i,m'}$ whenever $m \neq m'$. To see this, note that each of $v_{i,m}$ and $v_{i,m'}$ contains w_i exactly once, and that w_i appears in each at different locations. This completes the proof that all $v_{i,m}$ are distinct, which implies that $c_n(X) \geq t(n-2N) = tn - 2tN$ for all $n > 2N$. Therefore, $\liminf c_n(X) - tn > -\infty$, contradicting weak linear complexity of X with index t. The assumption that X contains at least t orbits of non-recurrent points was then false, completing the proof. $\qquad\square$

3.1.2. *Linear complexity and number of cones.* Now that we have control over the degrees of non-recurrent points, we focus on showing that the remaining nonzero degrees can be written as a finite union of cones.

In order to prove this, we will need several intermediate steps. The first is to show that there are fewer than t possibilities for the set of subwords of an aperiodic point of X. For $x \in X$, we denote its set of subwords by $\mathcal{L}(x)$, and accordingly define $c_n(x)$ and the notions of strong/weak linear complexity of x.

Lemma 3.5. *If a subshift X has weak linear complexity with index t, and m is the finite number of orbits of non-recurrent points in X guaranteed by Lemma 3.3, then the set*

$$\{\mathcal{L}(x) \mid x \in X \text{ is aperiodic and recurrent}\}$$

has cardinality less than $t - m$.

Proof. Recall from the proof of Lemma 3.3 that there exists N so that for all $n > 2N$, there are at least $m(n - 2N)$ words in $\mathcal{L}_n(X)$ which each contain some word in $\mathcal{L}_{2N+1}(X)$ only once (these were the words w_i in that proof.)

Now, suppose that $x_1, \ldots, x_k$ have different languages and are recurrent and aperiodic. There exists an $N' \geq 2N + 1$ such that $\mathcal{L}_{N'}(x_1), \ldots, \mathcal{L}_{N'}(x_k)$ are all distinct. By recurrence of the x_i, there exists a window M such that every word in $\mathcal{L}_{N'}(x_i)$ appears at least twice in each $v_i := x_i([-M, M])$.

We now claim that for every $n > N'$, there are at least $n - 2M$ words in $\mathcal{L}_n(x_i)$ that contain v_i. To see this, choose any $1 \leq i \leq k$.

- Suppose that every n-letter subword of x_i contains v_i. By the Morse-Hedlund theorem, since x_i is aperiodic, $c_n(x_i) > n$, immediately yielding the desired result.
- Otherwise, there exists an n-letter subword of x_i not containing v_i. Without loss of generality (by shifting if necessary), assume that $x_i(1) \ldots x_i(n)$ does not contain v_i and is immediately preceded or followed by v_i; we treat only the former case, as the latter is trivially similar. Then consider the n-letter subwords of x_i defined by $u_j = x_i(-j) \ldots x_i(n - j - 1)$ for $2M \leq j < n$. Each u_j contains its rightmost occurrence of v_i starting at its $(j - 2M + 1)$th letter, and so all u_j are distinct, verifying the claim.

For each of the x_i, denote by L_i this collection of words. The L_i are all disjoint; for $x \in L_i$, $\mathcal{L}_{N'}(x) = \mathcal{L}_{N'}(x_i)$, since x contains v_i. Therefore, $\mathcal{L}_n(X)$ contains at least $k(n-2M)$ words which contain some v_i and therefore contain every word in $\mathcal{L}_{N'}(X)$ at least twice. Since $N' \geq 2N+1$, by the above, there are at least $m(n-2N)$ words in $\mathcal{L}_n(X)$ which contain some word in $\mathcal{L}_{N'}(X)$ only once. Therefore, for $n > 4M$, $c_n(X) > k(n - 2M) + m(n - 2N)$, and so by weak linear complexity of X with index t, $k < t - m$. $\qquad\square$

We will now show that the set of nonzero degrees of recurrent points of X is precisely the union of the cones above the degrees of the finite set $\{\mathcal{L}(x) \ : \ x \in X\}$.

Lemma 3.6. *If X is a subshift with strong linear complexity, then for every $x \in X$, $\deg_T x \geq_T \deg_T \mathcal{L}(x)$.*

In order to prove this result, we will need the notion of a **right special word** within a sequence x: a word $w \in \mathcal{L}(x)$ is **right special for** x if there exist at least two ways to extend it on the right, i.e. letters $a \neq b$ such that $wa \in \mathcal{L}(x)$ and $wb \in \mathcal{L}(x)$. The degree $d_x(w)$ of w a right special word for x is the number of additional such extensions it has: a right special word with $d+1$ extensions in $\mathcal{L}(x)$ has degree[1] d. It follows that for any n, the sum of the degrees of all right special words of length n is $c_{n+1}(x) - c_n(x)$. It's obvious then that if $c_{n+1}(x) - c_n(x)$ is bounded, then x has strong linear complexity. Surprisingly, the converse is also true, as proved by Cassaigne.

Theorem 3.7 (Cassaigne [Cas95])**.** *If x is a sequence with strong linear complexity, then $c_{n+1}(x) - c_n(x)$ is bounded from above by a constant.*

In fact, the application of Theorem 3.7 is the only place where we require the assumption of strong (rather than weak) linear complexity.

Proof of Lemma 3.6. We now want to prove that from any point $x \in X$ we can compute $\mathcal{L}(x)$. By Theorem 3.7, there exist N, M such that for every $n > N$, $c_{n+1}(x) - c_n(x) \leq M$, and for infinitely many n, $c_{n+1}(x) - c_n(x) = M$. This means that for a given $n > N$, once we find a set S of right special words of length n where $\sum_{w \in S} d_x(w) = M$, then we know we have found all possible right special words for x of length n, and all possible letters which can follow them in x.

Our algorithm is the following: for input n, the algorithm scans larger and larger portions of x, looking for words of the form wa and wb for w of length at least n. It keeps a list of such words w it has found, along with the letters that can follow w, and whenever it begins a new scan of a longer portion of x, it increments the maximum length of w in its search (beginning by only looking for w of length exactly n, then n or $n + 1$, and so on.)

By definition of M, it will eventually find $k > n$ and a set S of right-special words for x of length k with extensions yielding degrees which sum to M, meaning that the algorithm has the complete set of right special words for x of length k and their followers.

We now claim that a further algorithm can recover the entire k-language $\mathcal{L}_k(x)$. In the sequence x, every word that is not right special has exactly one possible follower. This means that every k-letter word appearing in x which is not right

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[1]It should always be clear from context whether the degree in consideration is a degree of a right special word or a Turing degree since Turing degrees of finite words are always **0**.

special forces the following letter, which in turn forces another letter, until one arrives at a right special word in S. Therefore, every word in $\mathcal{L}_k(x)$ is a subword of some word of the form saw, where $s \in S$, $a \in \mathcal{A}$, w is forced by s and a, and saw ends with a word in S. The algorithm can then just search x for occurrences of every possible sa for $s \in S$ and $a \in \mathcal{A}$ and follow them to the right until hitting another word in S; $\mathcal{L}_k(x)$ is then the set of all k-letter words encountered in this process. Finally, $\mathcal{L}_n(x)$ is just the set of subwords of $\mathcal{L}_k(x)$, and so the algorithm has computed $\mathcal{L}_n(x)$ from x; since n was arbitrary, the proof is complete.

$\square$

Lemma 3.8. *For any subshift X and aperiodic recurrent $x \in X$, there exists an aperiodic recurrent $y \in X$ such that $\mathcal{L}(x) = \mathcal{L}(y)$ and $\deg_T y \leq_T \deg_T \mathcal{L}(x)$.*

Proof. We give a simple algorithm to construct y from $\mathcal{L}(x)$. Define w_1 to be the minimal word (in the lexicographic order) in $\mathcal{L}(x)$ which contains all words of $\mathcal{L}_1(x)$ at least twice (such words exist by recurrence of x).

For all $i > 0$, define w_{i+1} to be the minimal word in $\mathcal{L}(x)$ which contains all words of $\mathcal{L}_{|w_i|}(x)$ at least twice and which contains w_i in its center (again using recurrence of x). The sequence (w_i) then converges to a point, which we call y. Since y contains all words in $\mathcal{L}_{|w_i|}(x)$ for all i, and y is a limit of words in $\mathcal{L}(x)$, $\mathcal{L}(y) = \mathcal{L}(x)$. Aperiodicity of y follows immediately from this fact and aperiodicity of x. Every word in $\mathcal{L}(x)$ occurs at least twice in y by construction, and so y is recurrent. Finally, y was computed from $\mathcal{L}(x)$, and by Lemma 3.6, $\deg_T y \geq_T \deg_T \mathcal{L}(y) = \deg_T \mathcal{L}(x)$. Therefore, $\deg_T y = \deg_T \mathcal{L}(x)$.

$\square$

The following is an immediate corollary of Lemma 3.6 and Lemma 3.8.

Corollary 3.9. *If X is a subshift with strong linear complexity, then for every aperiodic recurrent $x \in X$, there exists an aperiodic recurrent $y \in X$ such that $\mathcal{L}(x) = \mathcal{L}(y)$ and $\deg_T y = \deg_T \mathcal{L}(x)$.*

We may now prove Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.3, there exist only finitely many nonzero degrees $\mathbf{d_1}, \ldots, \mathbf{d_m} \in \mathrm{Sp}\,(X)$ for $m < t$ associated to non-recurrent points in X. Then by Lemma 2.4, for every $\mathbf{d} \in \mathrm{Sp}\,(X) \backslash \{\mathbf{0}\}$ not equal to any d_i, $\mathcal{C}_d \subseteq \mathrm{Sp}\,(X)$.

By Lemma 3.5, there exist $x_1, \ldots, x_c$ for $c < t - m$ so that for all aperiodic non-recurrent $x \in X$, $\mathcal{L}(x) = \mathcal{L}(x_i)$ for some i. For every $1 \leq i \leq c$, by Corollary 3.9, there exists aperiodic recurrent y_i with $\mathcal{L}(y_i) = \mathcal{L}(x_i)$ and $\deg_T(y_i) = \deg_T(\mathcal{L}(x_i))$.

Then, by Lemma 2.4, $\mathcal{C}_{y_i} \subset \mathrm{Sp}\,(X)$ for $1 \leq i \leq k$, and so $\{\mathbf{d_1}, \ldots, \mathbf{d_m}\} \cup \bigcup_{i=1}^{c} \mathcal{C}_{y_i} \subset \mathrm{Sp}\,(X)$. Conversely, for every aperiodic recurrent $x \in X$, by Lemma 3.6, $\deg_T x \geq_T \deg_T \mathcal{L}(x)$. There exists i so that $\mathcal{L}(x) = \mathcal{L}(y_i)$, and so in fact $\deg_T x \geq_T \deg_T \mathcal{L}(y_i) = \deg_T y_i$. Therefore, the set of degrees of aperiodic recurrent $x \in X$ is contained in $\bigcup_{i=1}^{c} \mathcal{C}_{y_i}$, meaning that $\mathrm{Sp}\,(X) \backslash \{\mathbf{0}\} \subset \{\mathbf{d_1}, \ldots, \mathbf{d_m}\} \cup \bigcup_{i=1}^{c} \mathcal{C}_{y_i} \subset \mathrm{Sp}\,(X)$, and so that $\mathrm{Sp}\,(X) \backslash \{\mathbf{0}\} = \{\mathbf{d_1}, \ldots, \mathbf{d_m}\} \cup \bigcup_{i=1}^{c} \mathcal{C}_{y_i} \subset \mathrm{Sp}\,(X)$, completing the proof.

$\square$

3.1.3. *Minimal subshifts with linear complexity have spectrum equal to a cone.* The preceding results also have strong consequences for minimal subshifts with linear complexity; the spectrum of any such shift is a single cone, whose base is the degree of its language.

Theorem 3.10. *If an infinite minimal subshift X has strong linear complexity, then $Sp(X) = \mathcal{C}_{\deg_T \mathcal{L}(X)}$.*

Proof. Consider an infinite minimal subshift X with strong linear complexity. For every $x \in X$, $\mathcal{L}(x) = \mathcal{L}(X)$ (otherwise, the orbit closure of x would be a nonempty proper subsystem of X, contradicting minimality), and x is aperiodic and recurrent (otherwise, the orbit closure of x would contain a point not containing all words in $\mathcal{L}(X)$). Therefore, by Corollary 3.9, there exists a point $y \in X$ with $\deg_T y = \deg_T \mathcal{L}(X)$. Also, for all $x \in X$, Lemma 3.6 implies that $\deg_T x \geq_T \deg_T \mathcal{L}(X) = \deg_T \mathcal{L}(y)$, and so $Sp(X) \subset \mathcal{C}_{\deg_T \mathcal{L}(X)}$. Finally, by Lemma 2.4, $Sp(X) \supset \mathcal{C}_{\deg_T \mathcal{L}(X)}$, completing the proof. $\square$

3.2. **Implications of exponential complexity.** In the opposite direction, if X has exponentially growing complexity function (i.e. X has positive topological entropy), this also restricts $Sp(X)$.

Theorem 3.11. *If a subshift X has positive entropy, then its spectrum contains a cone.*

Proof. Suppose that $h^{top}(X) > 0$. Then by Lemma 2.3, there exists a measure μ giving zero measure to the set of periodic points of X. By Theorem 2.1, μ-a.e. $x \in X$ is recurrent, and so there exists an aperiodic recurrent point $x \in X$. Then, by Lemma 2.4, $Sp(X) \supset \mathcal{C}_x$. $\square$

4. Realizing (Turing) spectrums for various growth rates of complexity functions

In this section, we will present examples of subshifts with various complexity restrictions which exhibit different sets of Turing degrees as spectra. In several cases, our results come close to showing that the restrictions from Section 3 are tight (i.e. cannot be relaxed). The main obstacle is that every spectrum we can realize is either a union of cones or contains $\mathbf{0}$.

4.1. **Examples with strong linear complexity.**

Theorem 4.1. *Let $S = \{\mathbf{d_1}, \ldots, \mathbf{d_t}\}$ be a finite set of Turing degrees. Then there exists a subshift X having strong linear complexity whose spectrum is*

$$\bigcup_{i=1}^{t} \mathcal{C}_{\mathbf{d_i}}.$$

Proof. Given a real $\alpha \in \{0,1\}^{\mathbb{N}}$, the Sturmian subshift S_α corresponding to the rotation of the circle by α has complexity function $c_n(S_\alpha) = n+1$ and its spectrum is $\mathcal{C}_\alpha$.

Now take $\alpha_1, \ldots, \alpha_t$ sequences of degrees $\mathbf{d_1}, \ldots, \mathbf{d_t}$. Then $X = \bigcup_{i=1}^{t} X_{\alpha_i}$ is a subshift and has strong linear complexity (with index t), and its spectrum is exactly $\bigcup_{i=0}^{t} \mathcal{C}_{\mathbf{d_i}}$. $\square$

Theorem 4.2. *Let $S = \{\mathbf{d_1}, \ldots, \mathbf{d_t}\}$ be a finite set of Turing degrees. Then there exists a subshift X with strong linear complexity whose spectrum is*

$$Sp(X) = S \cup \{\mathbf{0}\}.$$

Furthermore, if S can be realized by a Π_1^0 class, then X is effective.

Proof. Take $s_1, \ldots, s_t$ sequences of $\{1, 2\}^{\mathbb{N}}$ with degrees $\mathbf{d_1}, \ldots, \mathbf{d_t}$ respectively. Choose a strictly increasing computable function $(m_i)_{i \in \mathbb{N}}$. We define a subshift $X \subset \{0, 1, 2\}^{\mathbb{Z}}$ consisting of the closure of the union of the orbits of the following sequences $\{x_i\}_{i=1}^t$:

$$x_i := \quad {}^\omega 0.s_i(1)0^{m_1}s_i(2)0^{m_2}s_i(3)0^{m_3}s_i(4)0^{m_4}s_i(5)0^{m_5}s_i(6)0^{m_6}\cdots$$

Each x_i is just s_i with growing numbers of 0 interspersed between its letters, and so since (m_i) is computable, $\deg_T(x_i) = \deg_T(s_i)$ for $1 \le i \le t$. The only points in X other than the orbits of the x_i are those obtained by limits of shifts of various x_i. It is not hard to check that the only such sequences are in the orbit of either $y_1 = {}^\omega 0.10^\omega$, $y_2 = {}^\omega 0.20^\omega$, or $y_3 = {}^\omega 0.0^\omega$. These all trivially have Turing degree $\mathbf{0}$, and so $\mathrm{Sp}(X) = S \cup \{\mathbf{0}\}$.

We must now show that X has strong linear complexity by bounding $c_n(X)$ from above. Subwords of y_1, y_2, and y_3 have at most one symbol not equal to 0, and there are $2n + 1$ n-letter words of this type. We now need to find the contribution to $c_n(X)$ from subwords of some x_i. Fix i, and consider any w a subword of x_i.

- if w contains fewer than two symbols from $\{1, 2\}$, then w has already been counted as a subword of y_1, y_2, or y_3.
- if w contains at least two $\{1, 2\}$ symbols, then w must occur within $0^{n-1}s_1^i 0^{m_1}s_2^i 0^{m_2}\ldots s_k^i 0^{m_k}$, where k is the unique integer for which $m_{k-1} < n \le m_k$. The number of such n-letter subwords of x_i is $\sum_{i=1}^{k}(1 + m_i)$. If we choose, for instance, $m_i = 2^i - 1$, then this is always strictly less than $4n$.

So, by combining these counts over all $1 \le i \le t$, we see that $c_n(X) < (2n+1) + 4tn$, and therefore X clearly has strong linear complexity. If S is a Π_1^0 class, then the set of forbidden words can be recursively enumerated, and so X is effectively closed. $\square$

Theorem 4.3. *Let S and S' be finite sets of Turing degrees. There exists a subshift X with strong linear complexity whose spectrum is*

$$Sp(X) = \{\mathbf{0}\} \cup S \cup \bigcup_{\mathbf{d'} \in S'} \mathcal{C}_{\mathbf{d'}}.$$

Proof. Simply take the union of the subshifts guaranteed by Theorems 4.1 and 4.2. $\square$

4.2. Examples with arbitrarily slow superlinear complexity.

Theorem 4.4. *Let $S = \{\mathbf{d_0}, \mathbf{d_1}, \ldots\}$ be a countable set of Turing degrees. Then the set of subshifts X with $Sp(X) = \bigcup_{k \ge 0} \mathcal{C}_{\mathbf{d_k}}$ exhibits arbitrarily slow superlinear complexity.*

Proof. Choose any increasing unbounded $f : \mathbb{N} \to \mathbb{N}$. Define (m_i) an strictly increasing (not necessarily computable) sequence of natural numbers satisfying $m_{f(n)/2} > n$ for all sufficiently large n; this is possible since f is unbounded.

Choose a sequence $(\alpha_k)_{k \ge 0}$ where α_k has degree $\mathbf{d_k}$ for $k \ge 0$, and where α_k approaches the limit α_0 quickly enough to guarantee $\mathcal{L}_{m_k}(S_{\alpha_k}) = \mathcal{L}_{m_k}(S_{\alpha_0})$ for $k \in \mathbb{N}$ (recall that S_α is the Sturmian shift with rotation number α); this is possible since the set of reals with any fixed Turing degree is obviously dense. Define $X =$

$\bigcup_{k=0}^{\infty} S_{\alpha_k}$; X is closed (and therefore a subshift) since $\alpha_k \to \alpha_0$. Since $\mathrm{Sp}\,(S_{\alpha_k}) = \mathcal{C}_{\mathbf{d_k}}$ for all n, $\mathrm{Sp}\,(X) = \bigcup_{k \geq 0} \mathcal{C}_{\mathbf{d_k}}$, as desired.

It remains to bound $c_n(X)$ from above. Choose any n, and define k so that $m_{k-1} \leq n < m_k$. Since $m_{f(n)/2} > n$ for sufficiently large n, $k \leq f(n)/2$ for all such n. For any $i \geq k$, note that since $\mathcal{L}_{m_i}(S_{\alpha_i}) = \mathcal{L}_{m_i}(S_{\alpha_0})$, $\mathcal{L}_n(S_{\alpha_i}) = \mathcal{L}_n(S_{\alpha_0})$. Therefore, $c_n(X) \leq k(n+1)$, which is bounded from above by $(n+1)(f(n)/2) \leq nf(n)$ for sufficiently large n, completing the proof.

$\square$

Theorem 4.5. *Let S be a countable set of Turing degrees. Then the set of subshifts X with $Sp\,(X) = S \cup \{\mathbf{0}\}$ exhibits arbitrarily slow superlinear complexity.*

Proof. Let $S = \{\mathbf{d_1}, \mathbf{d_2}, \ldots\}$ be a countable set of Turing degrees, and choose $\{0,1\}$-sequences $(s_i)_{i \in \mathbb{N}}$ with $\deg_T(s_i) = \mathbf{d_i}$. Fix any increasing unbounded $f : \mathbb{N} \to \mathbb{N}$. Define (m_i) an strictly increasing (not necessarily computable) sequence of natural numbers satisfying $m_{(f(n)-6)/4} > n$ for all sufficiently large n; this is possible since f is unbounded.

Now, for each i, define a sequence

$$x_i := {}^{\omega}0.10^{2^1-1}10^{2^2-1}10^{2^3-1}\ldots 10^{2^{m_i}-1}10^{2^{m_i+1}+s_i(1)}10^{2^{m_i+2}+s_i(2)}10^{2^{m_i+3}+s_i(3)}\ldots$$

The runs of 0s in x_i are just consecutive powers of 2 until the gap of length 2^{m_i}, after which the runs of 0s switch to increasing powers of 2 which encode the letters of s_i. It should be clear from this that $\deg_T(x_i) = \deg_T(s_i) = \mathbf{d_i}$. Define a subshift X as the closure of the union of the orbits of all x_i. All $x \in X$ are then either some shift of an x_i, or a limit of shifts of various x_i. It is easy to see that the only such points, apart from the x_i themselves, are in the orbit of either

$y_1 = {}^{\omega}0.10^{2^1-1}10^{2^2-1}10^{2^3-1}\ldots$, $y_2 = {}^{\omega}0.10^{\omega}$, or $y_3 = {}^{\omega}0.0^{\omega}$. Clearly all of these have Turing degree $\mathbf{0}$, and so $\mathrm{Sp}\,(X) = S \cup \{\mathbf{0}\}$.

It remains to bound $c_n(X)$ from above. We first note that for every n, the number of combined n-letter subwords of y_1, y_2, and y_3 is less than or equal to $6n+1$ by exactly the same argument as used in Theorem 4.2. Now, choose any n, and take the unique $k \in \mathbb{N}$ for which $m_{k-1} < n \leq m_k$. Recall that for sufficiently large n, $m_{(f(n)-6)/4} > n$, and so for such n, $k \leq (f(n)-6)/4$. Then, note that for $i \geq k$, every n-letter subword of x_i appears also as a subword of y_1. Therefore, we need only count the n-letter subwords within x_i for $i < k$. Each x_i is a sequence whose gaps of 0 have lengths which are increasing powers of 2, and so by a similar argument as in Theorem 4.2, each x_i has less than or equal to $4k$ subwords of length n. Combining the above yields

$$c_n(X) \leq (4k+6)n,$$

which is less than or equal to $nf(n)$ for large enough n, completing the proof.

$\square$

Theorem 4.6. *Let S be a countable set of Turing degrees, and let S' be a finite set of Turing degrees. Then the set of subshifts X with $Sp\,(X) = \{\mathbf{0}\} \cup S \cup \bigcup_{\mathbf{d'} \in S'} \mathcal{C}_{\mathbf{d'}}$ exhibits arbitrarily slow superlinear complexity.*

Proof. Just take the unions of subshifts guaranteed by Theorems 4.1 and 4.5. $\square$

4.3. **Examples with arbitrarily computably slow superlinear complexity.**

Theorem 4.7. *Let S be a closed nonempty set of Turing degrees. Then the sets of subshifts X with $Sp(X) = S \cup \{\mathbf{0}\}$ or $Sp(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ exhibit arbitrarily computably slow superlinear complexity.*

The proof is composed of the following two lemmas.

Lemma 4.8. *Let S be an arbitrary nonempty closed set of Turing degrees, then the set of subshifts X with $Sp(X) = S \cup \{\mathbf{0}\}$ exhibits arbitrarily computably slow superlinear complexity.*

Proof. Let S be an arbitrary closed nonempty set of Turing degrees, and choose a closed set of $\{1, 2\}$-sequences $(s_{\mathbf{d}})_{\mathbf{d} \in S}$ with $\deg_T(s_{\mathbf{d}}) = \mathbf{d}$. Fix any increasing unbounded computable $f : \mathbb{N} \to \mathbb{N}$, and choose a strictly increasing computable function $(m_i)_{i \in \mathbb{N}}$ so that $m_{(\log_2 f(n))/6} > n$ for all n; this is possible since f is unbounded and computable. We define a subshift $X \subset \{0, 1, 2\}^{\mathbb{Z}}$ consisting of the closure of the union of the orbits of the following sequences $(x_{\mathbf{d}})_{\mathbf{d} \in S}$:

$$x_{\mathbf{d}} := \quad {}^{\omega}0.s_{\mathbf{d}}(1)0^{m_1}s_{\mathbf{d}}(2)0^{m_2}s_{\mathbf{d}}(3)0^{m_3}s_{\mathbf{d}}(4)0^{m_4}\cdots$$

Since (m_i) is computable, it should be clear that $\deg_T(x_{\mathbf{d}}) = \deg_T(s_{\mathbf{d}}) = \mathbf{d}$ for all $\mathbf{d} \in S$. The only $x \in X$ other than the orbits of the $x_{\mathbf{d}}$ are those obtained by limits of shifts of various $x_{\mathbf{d}}$. Since $(s_{\mathbf{d}})_{\mathbf{d} \in S}$ is closed, the only such points outside of the $x_{\mathbf{d}}$ themselves are in the orbit of either $y_1 = {}^{\omega}0.10^{\omega}$, $y_2 = {}^{\omega}0.20^{\omega}$, or $y_3 = {}^{\omega}0.0^{\omega}$. These all trivially have Turing degree $\mathbf{0}$, and so $Sp(X) = S \cup \{\mathbf{0}\}$.

We must now bound $c_n(X)$ from above. We recall from the proof of Theorem 4.2 that any particular $x_{\mathbf{d}}$ has complexity less than or equal to $6n$; let's for instance fix x to be the sequence induced by $s_{\mathbf{d}} = 1^{\omega}$. Now, consider any n, and define k so that $m_{k-1} < n \leq m_k$. Recall that for sufficiently large n, $m_{(\log_2 f(n))/6} > n$, and so for such n, $k \leq (\log_2 f(n))/6$. Clearly any n-letter word in $\mathcal{L}(X)$ has less than or equal to k symbols in $\{1, 2\}$, and is obtained by taking some subword of x and changing the 1s to a combination of 1s and 2s, which can be done in less than or equal to 2^k ways. Therefore, $c_n(X) \leq 2^k(6n)$, which is less than $nf(n)$ for sufficiently large n, completing the proof.

$\square$

Lemma 4.9. *Let S be an arbitrary closed nonempty set of Turing degrees, then the set of subshifts X with $Sp(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ exhibits arbitrarily computably slow superlinear complexity.*

Proof. This proof is based on a construction of Miller [Mil12]. Let S be an arbitrary non-empty closed set of Turing degrees and $f : \mathbb{N} \to \mathbb{N}$ be any increasing computable unbounded function. Choose $g : \mathbb{N} \to \mathbb{N}$ a computable function satisfying $g(n) > 2$ and $2^{2n+5} \leq f(g(n))$ for all n. Take $(s_{\mathbf{d}})_{\mathbf{d} \in S}$ to be a closed set of $\{0, 1\}$ sequences with $s_{\mathbf{d}}$ of degree $\mathbf{d}$.

Now, for each finite word $\sigma \in \{0, 1\}^*$, we inductively define two words a_{σ} and b_{σ} as follows: (here ϵ represents the empty word)

$$a_\epsilon = 1 \qquad\qquad\qquad\qquad b_\epsilon = 0$$

$$a_{\sigma 0} = b_\sigma \underbrace{a_\sigma \cdots a_\sigma}_{g(|\sigma|)-1} \qquad\qquad b_{\sigma 0} = a_\sigma \underbrace{b_\sigma \cdots b_\sigma}_{g(|\sigma|)}$$

$$a_{\sigma 1} = a_\sigma \underbrace{b_\sigma \cdots b_\sigma}_{g(|\sigma|)-1} \qquad\qquad b_{\sigma 1} = b_\sigma \underbrace{a_\sigma \cdots a_\sigma}_{g(|\sigma|)}$$

Note that a_σ is always a prefix of b_σ, and so for every σ, all subwords of a_σ appear in b_σ. Now let $\mathcal{F}_S$ be the set of words that are not subwords of any word of the form b_σ for σ a prefix of some $s_\mathbf{d}$. Now let $X = X(\mathcal{F}_S) \subset \{0,1\}^\mathbb{N}$ be the subshift defined by the set $\mathcal{F}_S$ of forbidden words.

We first show that $\mathrm{Sp}\,(X)$ is $\bigcup_{\mathbf{d} \in S} \mathcal{C}_\mathbf{d}$. By definition of $\mathcal{F}_S$, for every $x \in X$ and $n \in \mathbb{N}$, x is a concatenation of a_σ and b_σ for some σ of length n, and σ is the prefix of some $s_\mathbf{d}$. By the recursive definition of the words a_σ and b_σ and the fact that $(s_\mathbf{d})_{\mathbf{d} \in S}$ is closed, any fixed x is induced in this way by prefixes of a single $s_\mathbf{d}$. Furthermore, this $s_\mathbf{d}$ can be computed from x by just counting the number of successive occurences of a_σ and b_σ. Start with $\sigma = \epsilon$. Then, if the sequence contains two consecutive a_ϵ, then the first bit of $s_\mathbf{d}$ is 0, and otherwise it is 1. From this information, we can deduce subsequent bits of $s_\mathbf{d}$ in a similar fashion. Therefore, each sequence of X has degree above some degree of S, and so $\mathrm{Sp}\,(X) \subseteq \bigcup_{\mathbf{d} \in S} \mathcal{C}_\mathbf{d}$.

For the opposite containment, we fix any $s_\mathbf{d}$, and define σ_n to be its prefix of length n for every $n \in \mathbb{N}$. The sequence b_{σ_n} approaches a limit $x_\mathbf{d}$, which is in X by definition. Clearly $\deg_T(x_\mathbf{d}) = \mathbf{d}$, and $x_\mathbf{d}$ is recurrent. Therefore, by Lemma 2.4, $\mathrm{Sp}\,(X)$ contains $\mathcal{C}_\mathbf{d}$. Since $\mathbf{d} \in S$ was arbitrary, the reverse containment is proved, and so $\mathrm{Sp}\,(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_\mathbf{d}$.

It remains to bound $c_n(X)$ from above. We first note that for every σ of length k, every word a_σ, b_σ has length between $\prod_{i=0}^{k-1} g(i)$ and $\prod_{i=0}^{k-1}(g(i)+1) \leq 2^k \prod_{i=0}^{k-1} g(i)$. For convenience, we denote $h(k) = \prod_{i=0}^{k-1} g(i)$ for all k. Now, for any n, there exist k and $1 \leq j < g(k)$ so that $jh(k) \leq n < (j+1)h(k)$. For every point of x, there exists σ of length k so that x is a concatenation of a_σ and b_σ, and there are either at least $g(k) - 1$ copies of a_σ between any two b_σ or vice versa. Therefore, every word in $\mathcal{L}_n(X)$ is either a subword of $(a_\sigma)^{j+1}b_\sigma(a_\sigma)^j$ or $(b_\sigma)^{j+1}a_\sigma(b_\sigma)^j$ (recall that $j < g(k)$, $n < (j+1)h(k)$, and all a_σ, b_σ have length at least $h(k)$.) For fixed σ, the number of such words is clearly bounded from above by the sum of the lengths of these words, which is less than or equal to $(4j + 4)2^k h(k)$. Since there are 2^k possible σ, we see that $c_n(X) \leq (4j + 4)2^{2k} h(k) \leq 2^{2k+3} n \leq f(g(k-1))n$ and $f(g(k-1))n \leq nf(n)$ since $g(k-1) \leq h(k) \leq n$, completing the proof. $\qquad\square$

4.4. Examples with arbitrarily computably fast subexponential complexity.

Theorem 4.10. *Let S be a closed nonempty set of Turing degrees. Then the set of subshifts with $\mathrm{Sp}\,(X) = S \cup \{\mathbf{0}\}$ exhibits arbitrarily computably fast subexponential complexity.*

Proof. Let S be an arbitrary nonempty closed set of Turing degrees, and choose a closed set of $\{2,3\}$-sequences $(s_\mathbf{d})_{\mathbf{d} \in S}$ with $\deg_T(s_\mathbf{d}) = \mathbf{d}$. Fix any increasing unbounded computable $f : \mathbb{N} \to \mathbb{N}$, and choose a strictly increasing computable function $(m_i)_{i \in \mathbb{N}}$ so that $m_{f(n)} > n$ for sufficiently large n; this is possible since f

is unbounded and computable. For each i, define $S_i \subset \{0,1\}^{m_i}$ to be the set of all $\{0,1\}$ words of length m_i which do not contain 1s separated by a distance of less than i. (For instance, if $m_2 = 3$, then $S_2 = \{000, 001, 010, 100, 101\}$.) For each i, define w_i to be the concatenation of all words in S_i in lexicographic order, followed by a second copy of 0^{m_i}. (In the above example, w_2 would be 000001010100101000.) Since (m_i) is computable, the sequence (w_i) is also computable.

We define a subshift $X \subset \{0,1,2,3\}^{\mathbb{Z}}$ consisting of the closure of the union of the orbits of the following sequences $(x_{\mathbf{d}})_{\mathbf{d} \in S}$:

$$x_{\mathbf{d}} := \quad {}^{\omega}0.s_{\mathbf{d}}(1)w_1 s_{\mathbf{d}}(2)w_2 s_{\mathbf{d}}(3)w_3 s_{\mathbf{d}}(4)w_4 \cdots$$

Since (w_i) is computable, $\deg_T(x_{\mathbf{d}}) = \deg_T(s_{\mathbf{d}}) = \mathbf{d}$ for all $\mathbf{d} \in S$. The only $x \in X$ other than the orbits of the $x_{\mathbf{d}}$ are those obtained by limits of shifts of various $x_{\mathbf{d}}$. The only such points are either shifts of some $x_{\mathbf{d}}$ by closedness of $(s_{\mathbf{d}})_{\mathbf{d} \in S}$ or in the orbit of either $y_1 = {}^{\omega}0.10^{\omega}$, $y_2 = {}^{\omega}0.20^{\omega}$, or $y_3 = {}^{\omega}0.30^{\omega}$. These all trivially have Turing degree $\mathbf{0}$, and so $\mathrm{Sp}(X) = S \cup \{\mathbf{0}\}$.

If $h(X)$ were positive, then by Lemma 2.3, there would exist an ergodic μ with the property that μ-a.e. $x \in X$ is aperiodic. However, the letters 1, 2, and 3 have frequency zero in every point of X, and so applying Theorem 2.2 with $w = 0$ shows that $\mu([0]) = 1$. By σ-invariance of μ, we see that $\mu(\{0^{\omega}\}) = 1$, i.e. μ-a.e. $x \in X$ is periodic, a contradiction. Therefore, our original assumption was false and $h(X) = 0$.

Finally, we must bound $c_n(X)$ from below. For any n, choose k such that $n_{k-1} < n \leq n_k$. Recall that for sufficiently large n, $m_{f(n)} > n$, and so for such n, $k \leq f(n)$. Every word of length n which does not contain 1s separated by distance less than k is a subword of w_k, and so is in $\mathcal{L}(X)$. There are at least $2^{\lceil n/k \rceil}$ such words; for instance, any word which contains 0 at all indexes except multiples of k has this property. Therefore,

$$c_n(X) \geq 2^{\lceil n/k \rceil} \implies \frac{\log c_n(X)}{n} \geq \frac{1}{k},$$

which is greater than $\frac{1}{f(n)}$ for all sufficiently large n, completing the proof. $\qquad \square$

4.5. Examples with arbitrarily fast subexponential complexity.

Theorem 4.11. *Let S be a closed nonempty set of Turing degrees. Then the set of subshifts with $\mathrm{Sp}(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ exhibits arbitrarily fast subexponential complexity.*

Proof. Consider any such set S, a subshift Y (with alphabet $\mathcal{A}$) as in Lemma 4.9 with $\mathrm{Sp}(Y) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ and subexponential complexity, and any increasing unbounded $f : \mathbb{N} \to \mathbb{N}$. Choose a sequence (n_k) so that $n_{k+1} < kf(k)$ for every k, and for which the sequence $(n_{k+1} - n_k)$ approaches infinity; this is possible since $f(k)$ is increasing and unbounded (for instance, we could define $n_k = \lfloor kf(k) \rfloor$).

Now, define a subshift Z with alphabet $\{0,1\}$ as follows. Define the set C of $\{0,1\}$-sequences which are 0 at all locations not indexed by any n_k (and which can be 0 or 1 at all (n_k)-indexed locations). Then, take $Z = \overline{\bigcup_{c \in C} \mathcal{O}(c)}$, and note that trivially the sequence 0^{ω} is in Z. The same argument as was used in the proof of Theorem 4.10 shows that $h(Z) = 0$, i.e. that Z has subexponential complexity.

Now, just define $X = Y \times Z$, the subshift on $\mathcal{A} \times \{0,1\}$ of sequences projecting to a point of Y along the first coordinate and to a point of Z along the second. For every $y \in Y$, the point $y \times \{0^{\omega}\}$ is in X, and clearly has the same degree as y.

Therefore, $\bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}} = \mathrm{Sp}\,(Y) \subseteq \mathrm{Sp}\,(X)$. For the reverse containment, note that all points of X project to points of Y along the first coordinate, and so for every $x \in X$, there exists $y \in Y$ s.t. $\deg_T x \geq_T \deg_T y$. Since $\deg_T y \in \mathrm{Sp}\,(Y) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$, by definition of cone, $\deg_T x \in \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ as well. Since $x \in X$ was arbitrary, the reverse containment is shown, and so $\mathrm{Sp}\,(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$.

It remains only to bound the complexity of X. By definition, $\mathcal{L}_n(X) = \mathcal{L}_n(Y) \times \mathcal{L}_n(Z)$, and since both Y and Z have subexponential complexity, this means that X does as well. For a lower bound, choose any $n \geq n_1$, and choose k s.t. $n_k \leq n < n_{k+1}$. Then every $w \in \{0,1\}^n$ which has 0 at all locations not equal to some $n_1, \ldots, n_k$ is in $\mathcal{L}_n(Z)$, and so $c_n(Z) \geq 2^k$. Clearly then $c_n(X) \geq 2^k$ as well, and so

$$\frac{\log c_n(X)}{n} \geq \frac{\log 2^k}{n} \geq \frac{k}{n_{k+1}} > \frac{1}{f(n_k)} \geq \frac{1}{f(n)}.$$

Since f was arbitrary, this completes the proof. $\qquad\square$

4.6. Examples with intermediate complexity.
We begin by showing that subshifts can get close to any "intermediate" computable complexity function with any Turing spectrum containing $\mathbf{0}$.

Theorem 4.12. *Let S be a closed nonempty set of Turing degrees, and let $g : \mathbb{N} \to \mathbb{N}$ be any computable function which is superlinear (i.e. $\frac{g(n)}{n} \to \infty$) and subexponential (i.e. $\frac{\log g(n)}{n} \to 0$). Then there exists a subshift X with $\mathrm{Sp}\,(X) = \{\mathbf{0}\} \cup S$ and a sequence (n_k) so that*

$$\lim_{k \to \infty} \frac{c_{n_k}(X)}{g(n_k)} = 1.$$

We need the following very simple lemma.

Lemma 4.13. *For every alphabet $\mathcal{A}$, $n \in \mathbb{N}$, and $w \in \mathcal{A}^*$, denote by N the number of n-letter subwords of w. Then, for every $1 \leq k \leq N$, there exists a prefix p of w containing exactly k different subwords of length n.*

Proof. If we define $f : [n, |w|] \to \mathbb{N}$ by taking $f(i)$ to be the number of different n-letter subwords of $w(1) \ldots w(i)$, the following facts are clear: $f(n) = 1$, $f(|w|) = N$, and $f(i+1) - f(i) \in \{0, 1\}$ for all i. It follows immediately that for every $1 \leq k \leq N$, there exists m so that $f(m) = k$; taking $p = w(1) \ldots w(m)$ then satisfies the conditions of the lemma. $\qquad\square$

Proof of Theorem 4.12. Let S be an arbitrary nonempty closed set of Turing degrees, and choose a closed set $(s_{\mathbf{d}})_{\mathbf{d} \in S}$ of $\{4,5\}$-sequences with $\deg_T(s_{\mathbf{d}}) = \mathbf{d}$. Then, inductively defining a computable sequence $(n_k)_{k \in \mathbb{N}}$ so that $\frac{\log g(n_{k+1})}{n_{k+1}} < \frac{1}{n_k}$ and $n_k | n_{k+1}$ for all k (here we use the fact that g is computable and subexponential). By passing to a subsequence if necessary, we may also assume that $n_k > 2^k \sum_{i=1}^{k-1} n_i(2 + 2^i)$ for all k. Since g is subexponential, we may also assume that $g(n_k) < 2^{n_k}$ for all k. Define w_1 by concatenating all words in $\{1,2\}^{n_1}$ in lexicographic order; then, since $g(n_1) < 2^{n_1}$, we can use Lemma 4.13 to construct $p_1 \in \{1,2\}^*$ containing exactly $g(n_1)$ words of length n_1. For $k \geq 1$, define w_{k+1} by concatenating all words (again in lexicographic order) of the form $0^{n_k-1}\ell_1 0^{n_k-1}\ell_2 \ldots 0^{n_k-1}\ell_{n_{k+1}/n_k}$ for $\ell_i \in \{1,2\}$; clearly then w_{k+1} contains at least $2^{n_{k+1}/n_k}$ words of length n_{k+1}. By definition, $g(n_{k+1}) < 2^{n_{k+1}/n_k}$, and so we can

use Lemma 4.13 to construct a prefix p_{k+1} of w_{k+1} which contains exactly $g(n_{k+1})$ words of length n_{k+1}. Note that since g and (n_k) were chosen computable, and since the procedure from the proof of Lemma 4.13 is clearly algorithmic, the sequence (p_k) is computable as well. For future reference, we note that $|w_k| = n_k 2^{n_k/n_{k-1}} < n_k 2^{n_k}$ for all k, and so $|p_k| < n_k 2^{n_k}$ as well. Then, for $\mathbf{d} \in S$, define

$$x_{\mathbf{d}} := \quad {}^\omega 0 . p_1 (0^{n_1} s_{\mathbf{d}}(1) 0^{n_1 - 1}) p_2 (0^{n_2} s_{\mathbf{d}}(2) 0^{n_2 - 1}) p_3 (0^{n_3} s_{\mathbf{d}}(3) 0^{n_3 - 1}) p_4 \cdots$$

We define the subshift $X \subset \{0, 1, 2, 3, 4\}^{\mathbb{Z}}$ to be the closure of the union of the orbits of $(x_{\mathbf{d}})_{\mathbf{d} \in S}$. Since the sequence (p_k) is computable, $\deg_T(x_{\mathbf{d}}) = \deg_T(s_{\mathbf{d}}) = \mathbf{d}$ for all $\mathbf{d} \in S$. The only $x \in X$ other than the orbits of the $x_{\mathbf{d}}$ are those obtained by limits of shifts of various $x_{\mathbf{d}}$. Since $(s_{\mathbf{d}})_{\mathbf{d} \in S}$ is closed, the only such points begin with ${}^\omega 0$ and end with 0^ω, and so all trivially have Turing degree $\mathbf{0}$. Therefore, $\mathrm{Sp}\,(X) = S \cup \{\mathbf{0}\}$. It remains only to prove the desired estimates on $c_{n_k}(X)$.

Clearly, $c_{n_k}(X) \geq g(n_k)$ for every k, since p_k contains $g(n_k)$ subwords of length n_k. For an upper bound, we consider an arbitrary n_k-letter subword w of some $x^{\mathbf{d}}$, and break into several cases. If the final letter of w is to the right of the 0^{n_k} immediately following p_k, then w has at most one non-0 symbol; there are at most $1 + 4n_k$ options for such words. If the final letter of w is inside the 0^{n_k} immediately following p_k, then the location of that final letter determines w, and so there are at most n_k options for w in that case. If the initial letter of w is to the left of p_k, then as long as w is not the word 0^{n_k} (which was already counted in a previous case), it is determined by the location of its rightmost letter and the values of $s^{\mathbf{d}_i}$ for $1 \leq i \leq k$, giving not more than $2^k \sum_{i=1}^{k-1} |p_i| + 2n_i < 2^k \sum_{i=1}^{k-1} n_i(2 + 2^{n_i})$ possibilities. Combining all of this yields

$$c_{n_k}(X) \leq g(n_k) + 5n_k + 1 + 2^k \sum_{i=1}^{k-1} n_i(2 + 2^{n_i}) \leq g(n_k) + 6n_k.$$

So, $g(n_k) \leq c_{n_k}(X) < g(n_k) + 6n_k$, which implies that $\lim_{k \to \infty} \frac{c_{n_k}(X)}{g(n_k)} = 1$ since g was assumed to be superlinear.

$\square$

We now prove that every union of cones can be realized as a Turing spectrum for a slightly smaller range of intermediate complexities, and begin by proving that cones can be realized by such subshifts.

Lemma 4.14. *Let $\mathbf{d}$ be any Turing degree, and let $g : \mathbb{N} \to \mathbb{N}$ be any computable function which is superquadratic (i.e. $\frac{g(n)}{n^2} \to \infty$) and subexponential (i.e. $\frac{\log g(n)}{n} \to 0$). Then there exists a subshift X with $\mathrm{Sp}\,(X) = \mathcal{C}_{\mathbf{d}}$ and a sequence (n_k) so that*

$$\lim_{k \to \infty} \frac{c_{n_k}(X)}{g(n_k)} = 1.$$

Proof. Consider any such $\mathbf{d}$ and g. Choose any α for which $\deg_T \alpha = \mathbf{d}$, and define the Sturmian subshift S_α; recall that $\mathrm{Sp}\,(S_\alpha) = \mathcal{C}_{\mathbf{d}}$.

Since g is superquadratic, computable, and subexponential, the function $h(n) := \lfloor g(n)/n \rfloor$ is superlinear, computable, and subexponential. Therefore, by Theorem 4.12, there exists a subshift Y with $\mathrm{Sp}\,(Y) = \{\mathbf{0}\}$ and a sequence n_k for which $\lim_{k \to \infty} \frac{c_{n_k}(Y)}{h(n_k)} = 1$. Now, simply define $X = Y \times S_\alpha$. Clearly $\mathcal{L}_{n_k}(X) =$

$\mathcal{L}_{n_k}(Y) \times \mathcal{L}_{n_k}(S_\alpha)$, and so $c_{n_k}(X) = c_{n_k}(Y)c_{n_k}(S_\alpha) = (n_k + 1)c_{n_k}(Y)$. However, this clearly implies that

$$\lim_{k\to\infty} \frac{c_{n_k}(X)}{g(n_k)} = \lim_{k\to\infty} \frac{c_{n_k}(X)/n_k}{g(n_k)/n_k} = \lim_{k\to\infty} \frac{c_{n_k}(Y)}{h(n_k)} = 1.$$

Finally, we note that every point of X projects to a point of S_α along the second coordinate, and so since $\text{Sp}\,(S_\alpha) = \mathcal{C}_{\mathbf{d}}$, $\text{Sp}\,(X) \subseteq \mathcal{C}_{\mathbf{d}}$. For the reverse containment, note that Y contains a computable point y, and so for every $s \in S_\alpha$, $x = y \times s \in X$, and $\deg_T x = \deg_T s$. Therefore, $\text{Sp}\,(X) = \mathcal{C}_{\mathbf{d}}$, completing the proof. $\square$

Theorem 4.15. *Let S be a closed nonempty set of Turing degrees, and let $g :$ $\mathbb{N} \to \mathbb{N}$ be any computable function which is superquadratic (i.e. $\frac{g(n)}{n^2} \to \infty$) and subexponential (i.e. $\frac{\log g(n)}{n} \to 0$). Then there exists a subshift X with $\text{Sp}\,(X) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$ and a sequence (n_k) so that*

$$\lim_{k\to\infty} \frac{c_{n_k}(X)}{g(n_k)} = 1.$$

Proof. Consider any such S, g, and an arbitrary $\mathbf{d} \in S$. By Lemma 4.14, there exists a subshift Y with $\lim_{k\to\infty} \frac{c_{n_k}(Y)}{g(n_k)} = 1$ and $\text{Sp}\,(Y) = \mathcal{C}_{\mathbf{d}}$. Since g is computable and superlinear, by Lemma 4.9 there exists a subshift Z with $\lim_{n\to\infty} \frac{c_n(Z)}{g(n)} = 0$ and $\text{Sp}\,(Z) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$. Then, we simply define $X = Y \cup Z$. It's immediate that $\text{Sp}\,(X) = \text{Sp}\,(Y) \cup \text{Sp}\,(Z) = \bigcup_{\mathbf{d} \in S} \mathcal{C}_{\mathbf{d}}$, and

$$\lim_{k\to\infty} \frac{c_{n_k}(X)}{g(n_k)} = \lim_{k\to\infty} \frac{c_{n_k}(Y)}{g(n_k)} + \lim_{k\to\infty} \frac{c_{n_k}(Z)}{g(n_k)} = 1.$$

$\square$

4.7. **Examples with exponential complexity/positive entropy.** Though it is still not known exactly which sets of Turing degrees can be the spectrum of a subshift, we can show that there are no restrictions beyond those of Theorem 3.11 on the spectra of positive entropy subshifts.

Theorem 4.16. *For any subshift X with entropy h, if $\text{Sp}\,(X)$ contains a cone C, then there exists a subshift P with $\text{Sp}\,(P) = \text{Sp}\,(X)$ and entropy h', for any $h' > h$ such that $\deg_T h' \in C$.*

Proof. Consider X as in the theorem, and $h' > h$ with $\deg_T h' = \mathbf{d}$ and suppose that $\mathcal{C}_{\mathbf{d}} \subset \text{Sp}\,(X)$. There exists an integer k such that $h'/k \in (0,1)$, choose $\alpha = h'/k$, clearly $\deg_T(\alpha) = \mathbf{d}$. Define the Sturmian subshift S_α with rotation number α, and recall that $\text{Sp}\,(S_\alpha) = \mathcal{C}_{\deg_T \alpha} = \mathcal{C}_{\mathbf{d}}$. Define $Y \subset S_\alpha \times \{a,b\}^{\mathbb{Z}}$ to be the subshift with alphabet $\{(0,a),(1,a),(1,b)\}$ of sequences which project to some point of S along the first coordinate (the lack of $(0,b)$ in the alphabet imposes the additional constraint that only a can appear with 0).

Let us compute the entropy of Y by bounding $c_n(Y)$. In every word of length n appearing in Y, the number of 1s is either $\lfloor n\alpha \rfloor$ or $\lfloor n\alpha \rfloor + 1$, and for each 1 there are two choices for the second component and for each 0 only one choice. Therefore,

$$(n+1)2^{\lfloor n\alpha \rfloor} \le c_n(Y) \le (n+1)2^{\lfloor n\alpha \rfloor + 1}, \text{ and}$$

$$\frac{\log\left((n+1)2^{\lfloor n\alpha \rfloor}\right)}{n} \le \frac{\log c_n(Y)}{n} \le \frac{\log\left((n+1)2^{\lfloor n\alpha \rfloor + 1}\right)}{n},$$

and by taking logs, dividing by n, and letting $n \to \infty$, we see that $h(Y) = \alpha$.

We now claim that $\mathrm{Sp}(Y) = \mathcal{C}_{\mathbf{d}}$. Clearly, for all $y \in Y$, one can compute a point $s \in S_\alpha$ by projecting to the first coordinate, and so $\deg_T y \geq_T \deg_T s \geq_T \mathbf{d}$. Therefore, $\mathrm{Sp}(Y) \subseteq \mathcal{C}_{\mathbf{d}}$. For the opposite containment, take any $\mathbf{d}' \geq \mathbf{d}$. We can choose $z \in \{a, b\}^{\mathbb{Z}}$ with $\deg_T(z) = \mathbf{d}'$ and $s \in S_\alpha$ with $\deg_T(s) = \mathbf{d}'$, since $\mathrm{Sp}(S_\alpha) = \mathcal{C}_{\mathbf{d}}$. Define $y \in Y$ as the point with s occupying the first coordinate, and z occupying the locations in the second coordinate where s has 1s. Then $\deg_T(y) = \sup(\deg_T(s), \deg_T(z)) = \mathbf{d}'$. Therefore, $\mathcal{C}_{\mathbf{d}} \subseteq \mathrm{Sp}(Y)$, and so indeed $\mathrm{Sp}(Y) = \mathcal{C}_{\mathbf{d}}$.

Now, we consider the k-fold Cartesian product Y^k of Y, i.e. the subshift consisting of ordered k-tuples of points of Y (viewed as a sequence over the alphabet $\{(0, a), (1, a), (1, b)\}^k$.) Clearly $h(Y^k) = k\alpha = h'$. Since points of Y^k come from k-tuples of points of Y, it's easy to see that

$$\mathrm{Sp}\left(Y^k\right) = \{\sup(\deg_T(y_1), \dots \deg_T(y_k)) \ : \ y_1, \dots, y_k \in Y\}.$$

Since $\mathrm{Sp}(Y) = \mathcal{C}_{\mathbf{d}}$, we see that $\mathrm{Sp}\left(Y^k\right) = \mathcal{C}_{\mathbf{d}}$ as well. Finally, define $P = X \sqcup Y^k$. By the preceding, $\mathrm{Sp}(P) = \mathrm{Sp}(X) \cup \mathrm{Sp}(Y) = \mathrm{Sp}(X) \cup \mathcal{C}_{\mathbf{d}} = \mathrm{Sp}(X)$. Finally, since P is a disjoint union of X and Y^k, its entropy is $h(P) = \max\left(h(X), h(Y^k)\right) = h'$. $\square$

5. AN EXAMPLE OF SPECTRUM WITH AN ISOLATED DEGREE

All our previous realizations of spectra are of either unions of cones or an arbitrary set of Turing degrees containing $\mathbf{0}$. We now construct an example of subshift whose spectrum is not a union of cones and does not contain $\mathbf{0}$.

Theorem 5.1. *Given any degree $\mathbf{d_1} \leq \mathbf{0}'$ and any degree $\mathbf{d_2}$, where $\mathbf{0}'$ is the degree of the halting problem, there exists a subshift X such that $\mathrm{Sp}(X) = \mathcal{C}_{\mathbf{d_1}} \cup \{\mathbf{d_2}\}$. In particular, $\mathbf{d_2}$ is not in $\mathcal{C}_{\mathbf{d_1}}$ if we take $\mathbf{d_2} < \mathbf{d_1}$ or $\mathbf{d_2}$ and $\mathbf{d_1}$ incomparable.*

Proof. Let $\alpha \in (0, 1)$ be of degree $\mathbf{d_1}$, by Schoenfield's limit lemma (see for example Cooper [Coo04, Proposition 10.5.10]), there exists a computable sequence of rationals $(r_n)_{n \in \mathbb{N}}$ such that $r_n \to \alpha$. Let $f(n)$ be a strictly increasing function of degree $\mathbf{d_2}$.

Let us now consider the following sequence:

$$z := \quad \cdots 2w_6 2w_4 2w_2 2w_1 2w_3 2w_5 2 \cdots$$

where each $w_i \in \{0, 1\}^{f(i)}$ and where $w_i(k) = \lfloor (k+1)r_i \rfloor - \lfloor kr_i \rfloor$ for $k \in \{1, \dots, f(i)\}$. We claim that the degree of z is $\mathbf{d_2}$: since $(r_n)_{n \in \mathbb{N}}$ is computable, using $\mathbf{d_2}$ as an oracle, it is easy to compute z, while $\mathbf{d_2}$ can be computed from z just by computing the distances between subsequent 2s.

Now let X be the orbit closure of z. We claim that $X = S_\alpha \cup \{\sigma^i(z) \mid i \in \mathbb{Z}\}$.

It is clear that each word w of $\mathcal{L}(S_\alpha)$ is in X: since $r_n \to \alpha$, there exists some k such that w is a subword of w_k. So $S_\alpha \subseteq X$.

Conversely take a word w appearing in some w_k, w appears in a sequence of X which is not a translate of z iff there is an infinite increasing sequence $(s_i)_{i \in \mathbb{N}}$ such that w is a subword of w_{s_i}. This is true iff $w \in \mathcal{L}(S_\alpha)$. Therefore $X \subseteq S_\alpha \cup \{\sigma^i(z) \mid i \in \mathbb{Z}\}$.

Now it is clear that $\mathrm{Sp}(X) = \mathcal{C}_{\mathbf{d_1}} \cup \{\mathbf{d_2}\}$. $\square$

Note that the proof can be generalized to use the assumption that $\mathbf{d_1} \leq \mathbf{a'}$ and $\mathbf{d_2} \geq \mathbf{a}$ for any degree $\mathbf{a}$; the only change is that the sequence (r_n) will be computable using a, a sequence of degree $\mathbf{a}$, as an oracle rather than computable.

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