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# Perturbations of Multidimensional Shifts of Finite Type

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*Abstract.* In this paper, we study perturbations of multidimensional shifts of finite type. Specifically, for any $\mathbb{Z}^d$ shift of finite type $X$ with $d > 1$ and any finite pattern $w$ in the language of $X$, we denote by $X_w$ the set of elements of $X$ not containing $w$. For strongly irreducible $X$ and patterns $w$ with shape a $d$-dimensional cube, we obtain upper and lower bounds on $h^{top}(X) - h^{top}(X_w)$ dependent on the size of $w$. This extends a result of Lind for $d = 1$. We also apply our methods to an undecidability question in $\mathbb{Z}^d$ symbolic dynamics.

## 1. *Introduction*

Among the main objects of study in symbolic dynamics are the $\mathbb{Z}^d$ shifts of finite type (or SFTs). Informally, a $\mathbb{Z}^d$ shift of finite type is defined by specifying a finite alphabet $A$ and a finite set of finite "forbidden patterns" $\mathcal{F}$ made up of letters from $A$, and then defining $X_\mathcal{F}$ to be the set of configurations in $A^{\mathbb{Z}^d}$ in which no pattern from $\mathcal{F}$ appears. The simplest nondegenerate example of an SFT is the $\mathbb{Z}^d$ golden mean shift $G$, where $A = \{0, 1\}$ and the forbidden patterns are any adjacent pair of 1s. Then for $d = 1$, $G$ is the set of all biinfinite strings of 0s and 1s in which 1s never appear consecutively. For $d = 2$, $G$ is the set of all ways of assigning 0 or 1 to all points in $\mathbb{Z}^2$ so that there is no adjacent pair of 1s horizontally or vertically.

For any $\mathbb{Z}^d$ SFT $X$, its topological entropy $h^{top}(X)$ measures the exponential growth rate of the number of patterns which appear in points of $X$. For any $\mathbb{Z}^d$ SFT $X$, and any pattern $w$ which appears in some point of $X$, define a new SFT $X_w$ by adding $w$ to the list of forbidden patterns. Then clearly $X_w \subseteq X$, and so the topological entropy of $X_w$ is at most that of $X$. We are interested in estimating the drop $h^{top}(X) - h^{top}(X_w)$ in topological entropy, and how this quantity behaves as the extra forbidden pattern $w$ becomes large.

As is often the case in symbolic dynamics, to obtain any general result, some sort of mixing condition must be imposed on $X$. We use the term "mixing condition" here to describe a wide range of properties which can be possessed by a SFT, all of which deal with the question: given points $x, y \in X$ and finite subsets $S$ and $T$ of $\mathbb{Z}^d$, does there exist a point $z \in X$ such that $z|_S = x|_S$ and $z|_T = y|_T$? There is a fairly large hierarchy of mixing conditions for $\mathbb{Z}^d$ SFTs, but the only ones which we will concern ourselves with in this paper, in order from weakest to strongest, are irreducibility, topological mixing, and strong irreducibility. (All of these are defined in Section 2.) For a more detailed examination of this hierarchy of mixing conditions, see [**2**].

For $d = 1$, the following theorem of Lind shows that quite strong statements may be made about $h^{top}(X) - h^{top}(X_w)$ even under our weakest mixing condition of irreducibility.

THEOREM 1.1. ([**7**], p. 360, Theorem 3) *For any irreducible $\mathbb{Z}$ SFT $X$ on an alphabet $A$ with positive topological entropy $h^{top}(X)$, there exist constants $C_X$, $D_X > 0$ and $N_X \in \mathbb{N}$ such that for any $n > N_X$ and any pattern $w \in A^{[1,n]}$ which appears as a subpattern of some point of $X$,*

$$\frac{C_X}{e^{h^{top}(X)n}} < h^{top}(X) - h^{top}(X_w) < \frac{D_X}{e^{h^{top}(X)n}}.$$

The main result of this paper is a version of Theorem 1.1 for $d > 1$.

THEOREM 1.2. *For any $d > 1$ and any strongly irreducible $\mathbb{Z}^d$ SFT $X$ on an alphabet $A$ with $|X| > 1$, there exist constants $C_X, D_X > 0$, $A_X, B_X$, and $N_X \in \mathbb{N}$ such that for any $n > N_X$ and any pattern $w \in A^{[1,n]^d}$ which appears as a subpattern of some point of $X$,*

$$\frac{C_X}{e^{h^{top}(X)(n+A_X)^d}} < h^{top}(X) - h^{top}(X_w) < \frac{D_X}{e^{h^{top}(X)(n+B_X)^d}}.$$

Note that our hypothesis is the much stronger mixing condition of strong irreducibility, and the bounds on $h^{top}(X) - h^{top}(X_w)$ do not differ by a multiplicative constant as they do in Theorem 1.1. (It is not necessary to assume that $h^{top}(X) > 0$; Lemma 4.11 shows that any strongly irreducible subshift containing at least two points has positive topological entropy.) In Section 3, we will demonstrate why such sacrifices must be made in any meaningful multidimensional extension of Theorem 1.1. This discussion will have the dual purposes of motivating our result and illustrating some of the complexities that arise in symbolic dynamics when $d > 1$.

We now briefly outline the content of this paper. In Section 2, we give some necessary definitions and terminology.

In Section 3, we discuss some of the difficulties inherent in extending Theorem 1.1 to multiple dimensions, and attempt to justify Theorem 1.2 as the "correct" such extension.

In Section 4, we prove some preliminary results about generic patterns for measures on subshifts that will be integral in the proof of Theorem 1.2.

Section 5 contains the proof of the upper bound of Theorem 1.2.

Section 6 is devoted to a somewhat complicated self-contained result about replacements of patterns which is necessary for the proof in Section 5.

Section 7 contains the proof of the lower bound portion of Theorem 1.2.

Section 8 is about the construction of a pattern $w_{o,d}$ whose existence is a necessary ingredient in the proof in Section 7.

Section 9 presents an application of our results to an undecidability question for SFTs in higher dimensions.

For a thorough introduction to higher-dimensional symbolic dynamics, see [**8**].

## 2. *Definitions and terminology*

We begin with some basic definitions and terminology. In this paper, an **alphabet** will always be a finite set with at least two elements.

*Definition 2.1.* The $\mathbb{Z}^d$ **full shift** on $A$ is the set $A^{\mathbb{Z}^d}$. For any full shift $A^{\mathbb{Z}^d}$, we define the $\mathbb{Z}^d$**-shift action** $\{\sigma_{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^d}$ on $A^{\mathbb{Z}^d}$ as follows: for any $\vec{v} \in \mathbb{Z}^d$ and $x \in A^{\mathbb{Z}^d}$, $(\sigma_{\vec{v}}(x))(\vec{u}) = x(\vec{v} + \vec{u})$ for all $\vec{u} \in \mathbb{Z}^d$.

*Definition 2.2.* A $\mathbb{Z}^d$ **subshift** on an alphabet $A$ is a set $X \subseteq A^{\mathbb{Z}^d}$ with the following two properties:

(i) $X$ is shift-invariant, meaning that for any $x \in X$ and $\vec{v} \in \mathbb{Z}^d$, $\sigma_{\vec{v}}(x) \in X$.
(ii) $X$ is closed in the product topology on $A^{\mathbb{Z}^d}$.

When the value of $d$ is clear, we will sometimes omit the $\mathbb{Z}^d$ and just use the term subshift.

A **pattern** $u$ on the alphabet $A$ is any mapping from a non-empty subset $S$ of $\mathbb{Z}^d$ to $A$, where $S$ is called the **shape** of $u$. For patterns $u$ and $u'$, where $u$ has shape $S$, we say that $u$ is a **subpattern** of $u'$ if there exists $\vec{p} \in \mathbb{Z}^d$ such that $u(\vec{q}) = u'(\vec{q} + \vec{p})$ for all $\vec{q} \in S$. We use the term **occurrence** to refer to a specific instance of a pattern $u$ as a subpattern within a larger pattern $u'$. For any pattern $u$ with shape $S$ and any $T \subseteq S$, denote by $u|_T$ the restriction of $u$ to $T$, i.e. the subpattern of $u$ occupying $T$.

*Definition 2.3.* The **language** of a subshift $X$, denoted by $L(X)$, is the set of patterns which appear as subpatterns of elements of $X$. The set of patterns with a particular shape $S$ which are in the language of $X$ is denoted by $L_S(X)$.

For any pattern $u$ with shape $S$ in a subshift $X$, denote by $[u]$ the set $\{x \in X \ : \ x|_S = u\}$, called the **cylinder set** of $u$. Although it is a slight abuse of notation, in certain situations we will also refer by $[u]$ to the set of all patterns $w$ such that $w|_S = u$; for instance we use $L_T(X) \cap [u]$ to refer to the set of patterns $w$ with shape $T$ such that $w|_S = u$.

*Definition 2.4.* A $\mathbb{Z}^d$ **shift of finite type** (or SFT) $X$ is defined by specifying a finite collection of finite patterns on $A$ (call this collection $\mathcal{F}$), and then defining $X = (A^{\mathbb{Z}^d})_{\mathcal{F}}$ to be the set of elements of $A^{\mathbb{Z}^d}$ which do not contain any member of $\mathcal{F}$ as subpatterns. For any fixed $X$, the **type** of $X$ is the minimum nonnegative integer $t$ such that for some $\mathcal{F}$ consisting entirely of patterns with shape $[1,t]^d$, $X = (A^{\mathbb{Z}^d})_{\mathcal{F}}$. A shift of type two is called a **Markov shift**.

It is not hard to see that any $\mathbb{Z}^d$ shift of finite type is a $\mathbb{Z}^d$ subshift.

We will often make use of the $l_\infty$ metric on $\mathbb{R}^d$: for $\vec{p}, \vec{q} \in \mathbb{R}^d$, $\|\vec{p} - \vec{q}\|_\infty = \max_{1 \leq i \leq d} |p_i - q_i|$. This metric on $\mathbb{R}^d$ induces a notion of "distance" between subsets of $\mathbb{Z}^d$: for $S, T \subset \mathbb{Z}^d$, we say $\rho(S,T) = \min_{\vec{s} \in S, \vec{t} \in T} \|\vec{s} - \vec{t}\|_\infty$.

*Definition 2.5.* A $\mathbb{Z}^d$ subshift $X$ is **topologically mixing** if for any $x, y \in X$ and any finite $S, T \subset \mathbb{Z}^d$, there exists $R$ such that for any $\vec{p} \in \mathbb{Z}^d$ with the property that $\rho(S, T + \vec{p}) > R$, there exists $z \in X$ such that $z|_S = x|_S$ and $z|_{T+\vec{p}} = y|_{T+\vec{p}}$.

*Definition 2.6.* A $\mathbb{Z}^d$ subshift $X$ is **strongly irreducible** if there exists $R$ such that for any $x, y \in X$ and any finite $S, T \subset \mathbb{Z}^d$, and for any $\vec{p} \in \mathbb{Z}^d$ with the property that $\rho(S, T + \vec{p}) > R$, there exists $z \in X$ such that $z|_S = x|_S$ and $z|_{T+\vec{p}} = y|_{T+\vec{p}}$. We call the minimum such $R$ the **uniform filling length** of $X$.

For $d = 1$, an SFT $X$ is **irreducible** if it contains a dense forward orbit, i.e. if there exists $x \in X$ so that $\{\sigma_n x\}_{n \in \mathbb{Z}^+}$ is dense in $X$. There are several natural extensions of this definition when $d > 1$, but all of them are extremely weak conditions (much weaker than topological mixing), and so we do not present them here.

*Definition 2.7.* Two $\mathbb{Z}^d$ subshifts $X$ and $Y$ are **topologically conjugate** (denoted $X \cong Y$) if there exists a bijective map from $X$ to $Y$ which is continuous in the product topology and which commutes with the $\mathbb{Z}^d$-shift action. Such a map is called a **topological conjugacy**.

We denote by $P_{j_1,j_2,\ldots,j_d}$ the $j_1 \times j_2 \times \ldots \times j_d$ rectangular prism $\prod_{i=1}^d \{1, 2, \ldots, j_i\}$ in $\mathbb{Z}^d$, and use the notation $\Gamma_j$ for the cube $P_{j,j,\ldots,j}$. We call $j_1, \ldots, j_d$ the **sizes** of $P_{j_1,\ldots,j_d}$, and call $j$ the **size** of $\Gamma_j$.

*Definition 2.8.* The **topological entropy** of a $\mathbb{Z}^d$ subshift $X$, denoted by $h^{top}(X)$, is defined by

$$h^{top}(X) = \lim_{j_1,j_2,\ldots,j_d \to \infty} \frac{\ln |L_{P_{j_1,j_2,\ldots,j_d}}(X)|}{j_1 j_2 \cdots j_d}.$$

Topological entropy is an invariant under topological conjugacy, i.e. if $X \cong Y$, then $h^{top}(X) = h^{top}(Y)$. In fact, it is one of the most useful and important invariants associated to a subshift.

The following notations and definitions will be needed in some of the more technical details of our proofs.

For each $1 \leq i \leq d$, we denote by $\vec{e_i}$ the $i$th element of the standard basis for $\mathbb{Z}^d$, and for $\vec{p}, \vec{q} \in \mathbb{R}^d$, the **distance in the $\vec{\mathbf{e_i}}$-direction** between $\vec{p}$ and $\vec{q}$ is defined to be $|p_i - q_i|$. We denote by $\vec{\mathbf{1}} \in \mathbb{Z}^d$ the vector whose entries are all ones.

For any $S, S', T \subseteq \mathbb{Z}^d$ satisfying $S, S' \supseteq T$, we say that patterns $u$ with shape $S$ and $u'$ with shape $S'$ **agree on** $T$ if $u|_T = u'|_T$, i.e. $u$ and $u'$ have the same letters on $T$.

The **boundary of thickness** $k$ of a subset $S$ of $\mathbb{Z}^d$, which is denoted by $S^{(k)}$, is the set of $\vec{p} \in S$ for which there exists $\vec{q} \in \mathbb{Z}^d \diagdown S$ with $\|\vec{p} - \vec{q}\|_\infty \leq k$. Whenever we refer to just the **boundary** of a shape $S$, we mean the boundary of thickness one.

For any $S \subseteq \mathbb{Z}^d$, a set $T \subseteq \mathbb{Z}^d$ is called a **copy** of $S$ if $T = S + \vec{v}$ for some $\vec{v} \in \mathbb{Z}^d$. This $\vec{v}$ is then called the **difference** of $S$ and $T$, denoted by $T - S$. For any sets $A \subset S$ and $B \subset T$, we say that $A$ in $S$ **corresponds to** $B$ in $T$ if $A = B + (S - T)$.

A pattern $v$ with shape $S$ **forces** a pattern $w$ with shape $T$ in a subshift $X$ if for every $x \in X$, $x|_S = v \Rightarrow x|_T = w$.

A pattern $w$ with shape $S$ is **periodic** with respect to $\vec{v} \in \mathbb{Z}^d$ if $S \cap (S - \vec{v}) \neq \varnothing$ and $w(\vec{u}) = w(\vec{u} + \vec{v})$ for all $\vec{u} \in S \cap (S - \vec{v})$. We also say that $\vec{v}$ is a **period** of $w$. A pattern is **aperiodic** if it has no periods.

## 3. *Motivation and examples*

The question of what a "natural" multidimensional version of Theorem 1.1 should look like is not easily answered. First, let's examine the conclusion. One might think that in Theorem 1.2, we should have just replaced the $e^{h^{top}(X)n}$ in Theorem 1.1 by $e^{h^{top}(X)n^d}$. However, we can quickly see that this is a bit too much to hope for.

*Example 3.1.* For any $d > 1$, define $Y$ to be the full shift $\{0,1\}^{\mathbb{Z}^d}$. For every $j \in \mathbb{N}$, define the higher block code $f_j : Y \to (A^{\Gamma_j})^{\mathbb{Z}^d}$ by $(f_j(x))(\vec{p}) = x|_{\vec{p} + \Gamma_j - \vec{\mathbf{1}}}$ for every $x \in Y$ and $\vec{p} \in \mathbb{Z}^d$. In other words, for every $\vec{p} \in \mathbb{Z}^d$, $(f_j(x))(\vec{p})$ is defined to be the subpattern of $x$ which lies in $\Gamma_j + \vec{p} - \vec{\mathbf{1}}$, or a copy of $\Gamma_j$ whose least corner in the usual lexicographic order on $\mathbb{Z}^d$ is $\vec{p}$. Higher block codes are topological conjugacies (see [**9**] for a proof in one dimension, which is easily extendable to $\mathbb{Z}^d$), so $Y$ is topologically conjugate to $f_j(Y)$ for every $j \in \mathbb{N}$. We can also think of $f_j$ as a bijection between $L_{\Gamma_n}(Y)$ and $L_{\Gamma_{n-j+1}}(f_j(Y))$ for all $n \geq j$. This means that for any $n \geq j$ and any $w \in L_{\Gamma_n}(Y)$, $Y_w$ and $(f_j(Y))_{f_j(w)}$ are topologically conjugate as well, and therefore have the same topological entropy.

Let us suppose that a $\mathbb{Z}^d$ version of Theorem 1.1 could be made by simply changing the $n$ in the exponent to $n^d$. Then for any $d > 1$ and any strongly irreducible $\mathbb{Z}^d$ SFT $X = (A^{\mathbb{Z}^d})_{\mathcal{F}}$, there exist constants $C_X, D_X > 0$, and $N_X \in \mathbb{N}$ such that for any $n > N_X$ and any pattern $w \in A^{[1,n]^d}$ which appears as a subpattern of some point of $X$, $\frac{C_X}{e^{h^{top}(X)n^d}} < h^{top}(X) - h^{top}(X_w) < \frac{D_X}{e^{h^{top}(X)n^d}}$. In particular, for any pattern $w \in L_{\Gamma_n}(Y)$ with $n > N_Y$,

$$\frac{C_Y}{e^{h^{top}(Y)n^d}} < h^{top}(Y) - h^{top}(Y_w) < \frac{D_Y}{e^{h^{top}(Y)n^d}}.$$

This means that for any $j$, if in addition $n > N_Y + j - 1$, then

$$\frac{C_Y}{e^{h^{top}(Y)n^d}} < h^{top}(f_j(Y)) - h^{top}((f_j(Y))_{f_j(w)}) < \frac{D_Y}{e^{h^{top}(Y)n^d}}. \tag{1}$$

Since $f_j(w) \in L_{\Gamma_{n-j+1}}(f_j(Y))$, if $n > N_{f_j(Y)}$ then

$$\frac{C_{f_j(Y)}}{e^{h^{top}(f_j(Y))(n-j+1)^d}} < h^{top}(f_j(Y)) - h^{top}((f_j(Y))_{f_j(w)}) < \frac{D_{f_j(Y)}}{e^{h^{top}(f_j(Y))(n-j+1)^d}}. \tag{2}$$

However, since $d > 1$ and $h^{top}(f_j(Y)) = h^{top}(Y)$, $e^{h^{top}(f_j(Y))n^d}$ grows much more quickly than $e^{h^{top}(Y)(n-j+1)^d}$ as $n \to \infty$. Therefore (1) and (2) contradict each other. Example 3 then shows that $A_X$ and $B_X$ cannot be omitted in the statement of Theorem 1.2. Since $j$ could be arbitrarily large in Example 3, it must be the case that $A_X$ and $B_X$ depend on $X$ rather than being absolute constants. If $X$ has uniform filling length $R$, then the values we actually achieve in the proof of Theorem 1.2 are $A_X = 44R+71$ and $B_X = -2R$. We note that these constants $A_X$ and $B_X$ could be thought of as being present in Theorem 1.1 as well, but hidden inside the constants $C_X$ and $D_X$.

We now address the hypotheses of Theorem 1.2. There are examples which show that for $d > 1$, topological mixing is not sufficient to get meaningful bounds on $h^{top}(X) - h^{top}(X_w)$. Regarding the upper bound, there is a topologically mixing $\mathbb{Z}^2$ SFT $X_*$ defined in [**6**] such that $h^{top}(X_*) > 0$, and for which there exist square patterns $c$ of arbitrarily large size such that $h^{top}((X_*)_c) = 0$. ($c$ is defined on p. 29 of [**6**]. In the language used there, for every $n$, every level-$n$ rectangle contains the $2n \times 2n$ pattern $c$, and the set of points containing no level-$n$ rectangle is called the set of exceptional points, which is shown to have zero entropy.) For the lower bound, there is a topologically mixing $\mathbb{Z}^2$ SFT $X^{(N)}_{MS}$ defined in [**2**] with positive topological entropy whose alphabet contains a letter $g$ so that $h^{top}(X^{(N)}_{MS}) = h^{top}((X^{(N)}_{MS})_g)$. This clearly implies that $h^{top}(X^{(N)}_{MS}) = h^{top}((X^{(N)}_{MS})_w)$ for any pattern $w$ containing $g$ as well. These two examples show that for $d > 1$, no meaningful upper and lower bounds on $h^{top}(X) - h^{top}(X_w)$ dependent only on the size of $w$ can exist for all topologically mixing $\mathbb{Z}^d$ SFTs. To obtain a conclusion of the strength we want, a stronger mixing condition is then necessary. The hypothesis we use is that of strong irreducibility.

The difference in the notions of strong irreducibility and topological mixing is that for strong irreducibility, the distance $R$ between shapes $S$ and $T$ necessary for interpolation between patterns in $L_S(X)$ and $L_T(X)$ is independent of $S$ and $T$, whereas in topological mixing, this distance depends on $S$ and $T$. When $d = 1$, these two notions coincide. (To see this, note that for a $\mathbb{Z}$ Markov shift, any distance sufficient for interpolating between any two letters of the alphabet is sufficient to interpolate between any two patterns.) However, strong irreducibility is a strictly stronger notion in more than one dimension: for any shape $S$ and $\vec{v} \in \mathbb{Z}^d$ with $\rho(\{\vec{v}\}, S) > R$ and any $a \in A$, by the definition of strong irreducibility there is some $y \in X$ with $y|_S = w$ and $y(\vec{v}) = a$; yet there exist topologically mixing SFTs for $d > 1$ such that for any $k$, some patterns force letters a distance $k$ away, including the checkerboard island shift of Quas and Sahin ([**11**]) and the aforementioned $X_*$ shift of Hochman ([**6**]). Any such system is then an example of a topologically mixing but not strongly irreducible SFT.

Our example $f_j(Y)$ used above to show the necessity of the constants $A_X$ and $B_X$ in Theorem 1.2 did not actually show that $A_X$ and $B_X$ are not the same. The following example will show that in fact $A_X$ and $B_X$ must be distinct, even if $X$ is assumed to be a strongly irreducible $\mathbb{Z}^d$ SFT.

*Example 3.2.* For any $d > 1$ and $j > 1$, define the $\mathbb{Z}^d$ SFT $Z_j$ on the alphabet $\{0, 1\}$ with the set of forbidden patterns $\mathcal{F}_j = \{w \in \{0, 1\}^{\Gamma_j} \ : \ w$ has at least two 1s$\}$. In other words, $Z_j$ consists of all infinite patterns of 0s and 1s on $\mathbb{Z}^d$ such that any two 1s are a $\| \ \|_\infty$-distance of at least $j$ from each other. $Z_j$ is clearly an SFT. It is also strongly irreducible with $R = j - 1$: for any pair of patterns in $L(Z_j)$ which are a distance of at least $j$ from each other, one can make a point of $X_j$ by filling the rest of $\mathbb{Z}^2$ with 0s.

We claim that in the language of Theorem 1.2, it must be the case that $A_{Z_j} \neq B_{Z_j}$, and in fact that $A_{Z_j} - B_{Z_j} \geq 2j - 2 = 2R$. To prove this, we first make a definition: for any $n > 0$, we define the pattern $a_n$ with shape $\Gamma_{nj+1}$ by $a_n(\vec{p}) = 1$ if and only if all coordinates of $\vec{p}$ are equal to 1 (mod $j$). Clearly, $a_n \in L(Z_j)$. Let us also for any $n > 0$ define the pattern $b_n$ with shape $\Gamma_{(n+2)j-1}$ where $b_n$ has $a_n$ as a subpattern occupying its central copy of $\Gamma_{nj+1}$, and has 0s on all of $\Gamma^{(j-1)}_{(n+2)j-1}$. By the definition of $Z_j$, any occurrence of $a_n$ in an element of $Z_j$ forces an occurrence of $b_n$ containing it. Therefore, $(Z_j)_{a_n} = (Z_j)_{b_n}$ for any $n$. However, the size of the shape of $b_n$ is $2j - 2$ bigger than that of $a_n$ for any $n$, and so for any $A_{Z_j}$ and $B_{Z_j}$ which satisfy Theorem 1.2, it must be the case that $A_{Z_j} - B_{Z_j} \geq 2j - 2 = 2R$ for any $j$. This shows that $A_X - B_X$ must be at least linear in $R$, and so the value of $46R + 71$ for $A_X - B_X$ attained in Theorem 1.2 can be improved by at most additive and multiplicative constants.

In addition to these necessary changes in the statement of a multidimensional version of Theorem 1.1, changes must be made to the proof as well. Lind's proof of Theorem 1.1 relied strongly on techniques from linear algebra. In particular, he uses the fact that the topological entropy of a $\mathbb{Z}$ SFT is the logarithm of the Perron eigenvalue of its (integer-valued) transition matrix. However, there is no known procedure for using transition matrices to explicitly compute $h^{top}(X)$ when $d > 1$. Even if the notion is extendable in some way, the situation cannot be nearly as simple as in the $\mathbb{Z}$ case, since there exists a $\mathbb{Z}^2$ SFT $X$ where $h^{top}(X)$ is not the logarithm of an algebraic number! ([**5**]) Therefore, to prove a multidimensional version of Theorem 1.1, we must use a different approach. Our methods are similar in places to those used in [**12**], where the effect on entropy of removing patterns from SFTs is also examined.

## 4. *Some measure-theoretic preliminaries*

The proof of Theorem 1.2 rests on some general measure-theoretic facts about $\mathbb{Z}^d$ subshifts, generic patterns, and measures of maximal entropy, which we will prove in this section. For our purposes, a measure on a subshift $X$ will always be a shift-invariant Borel probability measure.

*Definition 4.1.* The **measure-theoretic entropy** of a subshift $X$ with respect to a measure $\mu$ on $X$, which is denoted by $h_\mu(X)$, is defined by

$$h_\mu(X) = \lim_{j_1, j_2, \ldots, j_d \to \infty} \frac{1}{j_1 j_2 \cdots j_d} \sum_{w \in L_{P_{j_1,\ldots,j_d}}(X)} f(\mu([w])),$$

where $[w]$ is the cylinder set $\{x \in X \ : \ x|_{P_{j_1,\ldots,j_d}} = w\}$, $f(x) := -x \ln x$ for $x > 0$, and $f(0) = 0$.

*Definition 4.2.* A measure $\mu$ on a subshift $X$ is called a **measure of maximal entropy** if $h_\mu(X) = h^{top}(X)$.

Such measures are said to have maximal entropy because of the following Variational Principle.

THEOREM 4.3. *For any $\mathbb{Z}^d$ subshift $X$, $h^{top}(X) = \sup h_\mu(X)$. This supremum is achieved for some $\mu$.*

See [**10**] for a proof. It is well-known that in fact any $\mathbb{Z}^d$ subshift has an ergodic measure of maximal entropy. The following proposition gives a useful property for measures of maximal entropy on SFTs.

PROPOSITION 4.4. ([**3**], p. 281, Proposition 1.20) *For any $d$, any $\mathbb{Z}^d$ shift $X$ of finite type $t$, any $\mu$ a measure of maximal entropy for $X$, and any shape $U \subseteq \mathbb{Z}^d$, the conditional distribution of $\mu$ on $U$ given any fixed pattern $w \in L_{(U^c)^{(t)}}(X)$ is uniform over all patterns $x \in L_U(X)$ such that the pattern $y$ with shape $U \cup (U^c)^{(t)}$ defined by $y|_U = x$ and $y|_{(U^c)^{(t)}} = w$ is in $L(X)$.*

Proposition 4.4 says that any pair of patterns with the same shape $U \cup (U^c)^{(t)}$ which agree on $(U^c)^{(t)}$ have the same measure for any measure of maximal entropy. By taking $U = S \setminus S^{(t)}$ for any shape $S$, it also implies that any two patterns with shape $S$ which agree on $S^{(t)}$ have the same measure for any measure of maximal entropy. It is then clear that two patterns with shape $S$ which agree on a set containing $S^{(t)}$ have the same measure for any measure of maximal entropy as well, which gives the following corollary:

COROLLARY 4.5. *For any $d$, any $\mathbb{Z}^d$ shift $X$ of finite type $t$, any measure $\mu$ of maximal entropy on $X$, any shapes $S$ and $T$ satisfying $S^{(t)} \subseteq T \subseteq S$, and any pattern $w \in L_S(X)$, $\mu([w]) = \frac{\mu([w|_T])}{|L_S(X) \cap [w|_T]|}$.*

LEMMA 4.6. *For any $d$, any strongly irreducible $\mathbb{Z}^d$ shift $X$ of finite type $t$ with uniform filling length $R$, any measure $\mu$ of maximal entropy on $X$, any shape $S$, and any $u \in L_S(X)$,*

$$\frac{1}{\left|L_{\left(S \cup (S^c)^{(t+R)}\right) \setminus \left(S \cup (S^c)^{(t+R)}\right)^{(t)}}(X)\right|} \leq \mu([u]) \leq \frac{1}{|L_{S \setminus S^{(R)}}(X)|}.$$

*Proof.* Fix $d$, $X$, $\mu$, $S$, and $u \in L_S(X)$. We begin by bounding $\mu([u])$ from below. Choose any pattern $v \in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)$. We claim that $\rho(S,(S \cup (S^c)^{(t+R)})^{(t)}) > R$. Consider any $\vec{p} \in S$ and $\vec{q} \in (S \cup (S^c)^{(t+R)})^{(t)}$. By definition, there exists $\vec{r} \notin S \cup (S^c)^{t+R}$ such that $\|\vec{q} - \vec{r}\|_\infty \leq t$. Also by definition, $\|\vec{p} - \vec{r}\|_\infty > R + t$. Therefore, by the triangle inequality, $\|\vec{p} - \vec{q}\|_\infty > R$, and since $\vec{p} \in S$ and $\vec{q} \in (S \cup (S^c)^{(t+R)})^{(t)}$ were arbitrary, $\rho(S,(S \cup (S^c)^{(t+R)})^{(t)}) > R$ as claimed. This means that by strong irreducibility, there exists a pattern $w_v \in L_{S\cup(S^c)^{(t+R)}}(X)$ such that $w_v|_S = u$ and $w_v|_{(S\cup(S^c)^{(t+R)})^{(t)}} = v$. By Corollary 4.5, $\mu([w_v]) = \frac{\mu([v])}{|L_{S\cup(S^c)^{(t+R)}}(X)\cap[v]|}$. Clearly the number of patterns in $L(X)$ with shape $S\cup(S^c)^{(t+R)}$ which equal $v$ on $(S\cup(S^c)^{(t+R)})^{(t)}$ is less than or equal to the number of patterns in $L(X)$ with shape $\big(S \cup (S^c)^{(t+R)}\big)\setminus\big(S \cup (S^c)^{(t+R)}\big)^{(t)}$, and so

$$\mu([w_v]) \geq \frac{\mu([v])}{\left|L_{\left(S\cup(S^c)^{(t+R)}\right)\setminus\left(S\cup(S^c)^{(t+R)}\right)^{(t)}}(X)\right|}. \tag{3}$$

If we sum (3) over all possible choices for $v$, then we get

$$\sum_{v\in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)} \mu([w_v]) \geq \frac{\sum_{v\in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)} \mu([v])}{\left|L_{\left(S\cup(S^c)^{(t+R)}\right)\setminus\left(S\cup(S^c)^{(t+R)}\right)^{(t)}}(X)\right|}. \tag{4}$$

Note that all $w_v$ are distinct, and for all $w_v$, $w_v|_S = u$. Therefore,

$$\bigcup_{v\in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)} [w_v] \subseteq [u],$$

and so $\sum_{v\in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)} \mu([w_v]) \leq \mu([u])$. Since $\bigcup_{v\in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)}[v] = X$, $\sum_{v\in L_{(S\cup(S^c)^{(t+R)})^{(t)}}(X)} \mu([v]) = 1$. By combining these facts with (4), we see that

$$\mu([u]) \geq \frac{1}{\left|L_{\left(S\cup(S^c)^{(t+R)}\right)\setminus\left(S\cup(S^c)^{(t+R)}\right)^{(t)}}(X)\right|}.$$

We now bound $\mu([u])$ from above. Choose any pattern $v \in L_{S\cup(S^c)^{(t)}}(X)$ with $v|_S = u$. We wish to use Corollary 4.5 with $(S^c)^{(t)}$ as our $T$. To do so, we need to know that $(S\cup(S^c)^{(t)})^{(t)} \subseteq (S^c)^{(t)} \subseteq S\cup(S^c)^{(t)}$. The second containment is trivial. To prove the first, suppose that $\vec{p} \in (S\cup(S^c)^{(t)})^{(t)}$. By definition, there then exists $\vec{q} \notin S \cup (S^c)^{(t)}$ such that $\|\vec{p} - \vec{q}\|_\infty \leq t$. Since $\vec{q} \notin S \cup (S^c)^{(t)}$, $\vec{q} \in S^c$. Therefore, since $\|\vec{p} - \vec{q}\|_\infty \leq t$, $\vec{p} \in (S^c)^{(t)}$ by definition. We now apply Corollary 4.5:

$$\mu([v]) = \frac{\mu([v((S^c)^{(t)})])}{|L_{S\cup(S^c)^{(t)}}(X) \cap [v((S^c)^{(t)})]|} \tag{5}$$

We claim that $\rho(S\setminus S^{(R)},(S^c)^{(t)}) > R$; choose any $\vec{p} \in S\setminus S^{(R)}$ and $\vec{q} \in (S^c)^{(t)}$. Clearly $\vec{q} \in S^c$. If $\|\vec{p} - \vec{q}\|_\infty \leq R$, then $\vec{p} \in S^{(R)}$, a contradiction. Thus, $\|\vec{p} - \vec{q}\|_\infty > R$, and so $\rho(S\setminus S^{(R)},(S^c)^{(t)}) > R$. This implies that for any $v \in L_{S\cup(S^c)^{(t)}}(X)$ with $v|_S = u$, and for any $v' \in L_{S\setminus S^{(R)}}(X)$, there exists

$y \in L_{S \cup (S^c)^{(t)}}(X)$ with $y|_{(S^c)^{(t)}} = v|_{(S^c)^{(t)}}$ and $y|_{S \setminus S^{(R)}} = v'$. Therefore, $|L_{S \cup (S^c)^{(t)}}(X) \cap [v|_{(S^c)^{(t)}}]| \geq |L_{S \setminus S^{(R)}}(X)|$. By using this fact and summing (5) over all possible $v$, we get

$$\sum_{v \in L_{S \cup (S^c)^{(t)}}(X) \cap [u]} \mu([v]) \leq \frac{\sum_{v \in L_{S \cup (S^c)^{(t)}}(X) \cap [u]} \mu([v|_{(S^c)^{(t)}}])}{|L_{S \setminus S^{(R)}}(X)|}. \tag{6}$$

Note that since all $v$ are distinct, $\bigcup_{v \in L_{S \cup (S^c)^{(t)}}(X) \cap [u]} [v] = [u]$, and so $\sum_{v \in L_{S \cup (S^c)^{(t)}}(X) \cap [u]} \mu([v]) = \mu([u])$. Also note that since all $v$ are distinct, but all have $v|_S = u$, it must be the case that all $v|_{(S^c)^{(t)}}$ are distinct, and so $\sum_{v \in L_{S \cup (S^c)^{(t)}}(X) \cap [u]} \mu([v((S^c)^{(t)})]) \leq 1$. Combining these facts with (6), we see that

$$\mu([u]) \leq \frac{1}{|L_{S \setminus S^{(R)}}(X)|},$$

which, along with the lower bound on $\mu([u])$ already achieved, completes the proof. □

To avoid confusion, from now on $\widetilde{\mu}$ will be used to denote a measure of maximal entropy on a subshift $X$, and $\mu$ to denote a measure which may or may not be of maximal entropy.

*Definition 4.7.* For any $d$, any $\mathbb{Z}^d$ subshift $X$, any ergodic measure $\mu$ on $X$, any finite set of patterns $u_1 \in L_{S_1}(X)$, $u_2 \in L_{S_2}(X), \ldots, u_j \in L_{S_j}(X)$, any $k \in \mathbb{N}$ and any $\epsilon > 0$, define $A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)$ to be the set of patterns in $L_{\Gamma_k}(X)$ which have between $k^d(\mu([u_i]) - \epsilon)$ and $k^d(\mu([u_i]) + \epsilon)$ occurrences of $u_i$ for all $1 \leq i \leq j$.

LEMMA 4.8. *For any $d$, any $\mathbb{Z}^d$ subshift $X$, any ergodic measure $\mu$ on $X$, any finite set of patterns $u_1 \in L_{S_1}(X)$, $u_2 \in L_{S_2}(X), \ldots, u_j \in L_{S_j}(X)$, any $k \in \mathbb{N}$ and any $\epsilon > 0$, $\liminf_{k \to \infty} \frac{\ln |A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)|}{k^d} \geq h_\mu(X)$.*

*Proof.* For brevity, we will use the shorthand notation $\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X))$ for the measure of the union of all cylinder sets corresponding to patterns in $A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)$. Fix any $\epsilon > 0$. Firstly, we notice that it is a simple consequence of the pointwise ergodic theorem for $\mathbb{Z}^d$ actions ([**13**]) that $\lim_{k \to \infty} \mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)) = 1$.

We now write out the formula for the measure-theoretic entropy of $X$:

$$h_\mu(X) = \lim_{k \to \infty} \frac{1}{k^d} \sum_{u \in L_{\Gamma_n}(X)} f(\mu([u])),$$

where again $f(x) = -x \ln x$ for $x > 0$ and $f(0) = 0$. Partition $L_{\Gamma_k}(X)$ into the two pieces $A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)$ and $A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)^c$:

$$h_\mu(X) = \lim_{k \to \infty} \Big( \frac{1}{k^d} \sum_{u \in A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)} f(\mu([u])) + \frac{1}{k^d} \sum_{u' \in A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)^c} f(\mu([u'])) \Big).$$

To estimate each summand, we use the easily checkable fact that for any set of nonnegative reals $\alpha_1, \alpha_2, \ldots, \alpha_{j'}$ whose sum is $\beta$, the maximum value of $\sum_{i=1}^{j'} f(\alpha_i)$ occurs when all terms are equal, and is $\beta \ln \frac{j'}{\beta}$. Using this, we see that the above sum is bounded from above by

$$\frac{1}{k^d}\left(\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)) \ln \frac{|A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)|}{\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X))}\right)$$
$$+\frac{1}{k^d}\left(\big(1-\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X))\big) \ln \frac{|L_{\Gamma_k}(X)|}{1-\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X))}\right).$$

Since $\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X))$ approaches 1 and $\frac{\ln |L_{\Gamma_k}(X)|}{k^d}$ approaches $h^{top}(X) < \infty$ as $k \to \infty$, the second term approaches 0 as $k \to \infty$. By replacing $\mu(A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X))$ by 1 in the limit in the first term, we see that

$$h_\mu(X) \leq \liminf_{k\to\infty} \frac{1}{k^d} \ln |A_{k,\epsilon,\mu,u_1,\ldots,u_j}(X)|,$$

which was exactly what needed to be shown.

□

*Definition 4.9.* For any $d$, any strongly irreducible $\mathbb{Z}^d$ shift $X$ of finite type $t$ with uniform filling length $R$, any $\epsilon > 0$, and any positive integers $k$ and $M$, define $A_{k,\epsilon,M}(X)$ to be the set of patterns $u$ in $L_{\Gamma_k}(X)$ such that for every $S_i \subseteq \Gamma_M$ and every $u_i \in L_{S_i}(X)$, $u$ has between $k^d\Big(\frac{1}{\left|L_{(S_i \cup (S_i^c)^{(t+R)}) \setminus (S_i \cup (S_i^c)^{(t+R)})^{(t)}}(X)\right|} - \epsilon\Big)$ and $k^d\big(\frac{1}{|L_{S_i \setminus S_i^{(R)}}(X)|} + \epsilon\big)$ occurrences of $u_i \in L_{S_i}(X)$.

Corollary 4.10. *For any $d$, any strongly irreducible $\mathbb{Z}^d$ shift $X$ of finite type $t$ with uniform filling length $R$, any $\epsilon > 0$, and any positive integer $M$,*

$$\lim_{k\to\infty} \frac{\ln |A_{k,\epsilon,M}(X)|}{k^d} = h^{top}(X).$$

*Proof.* For any fixed $\widetilde{\mu}$ an ergodic measure of maximal entropy on $X$, Lemmas 4.6 and 4.8 show that

$$\liminf_{k\to\infty} \frac{\ln |A_{k,\epsilon,M}(X)|}{k^d} \geq h_{\widetilde{\mu}}(X) = h^{top}(X).$$

And by the definition of topological entropy,

$$\limsup_{k\to\infty} \frac{\ln |A_{k,\epsilon,M}(X)|}{k^d} \leq \limsup_{k\to\infty} \frac{\ln |L_{\Gamma_k}(X)|}{k^d} = h^{top}(X).$$

□

Finally, we will eventually need the following lemma about strongly irreducible systems, whose proof is standard.

Lemma 4.11. *For any $d$ and any strongly irreducible $\mathbb{Z}^d$ subshift $X$ with uniform filling length $R$ and for any rectangular prism $P_{j_1,j_2,\ldots,j_d}$, $e^{h^{top}(X)j_1 j_2 \cdots j_d} \leq |L_{P_{j_1,j_2,\ldots,j_d}}(X)| \leq e^{h^{top}(X)(j_1+R)(j_2+R)\cdots(j_d+R)}$.*

Note that in particular, Lemma 4.11 implies that any strongly irreducible subshift consisting of at least two points has positive topological entropy.

## 5. *The proof of the upper bound in Theorem 1.2*

We begin with a sketch of the proof of the upper bound of Theorem 1.2. For any $d > 1$, consider a strongly irreducible $\mathbb{Z}^d$ shift $X$ of finite type $t$ with uniform filling length $R$ containing more than one point, an integer $n$, and $w \in L_{\Gamma_n}(X)$. For any integer $k$ much larger than $n$, we would like to define a map from $L_{\Gamma_{k+C}}(X)$ to $L_{\Gamma_k}(X_w)$ ($C$ is some constant independent of $k$) which replaces occurrences of $w$ with a pattern $\widetilde{w}$ agreeing with $w$ on the boundary of thickness $t$. (By the definition of the type $t$ of $X$, this agreement ensures that when one performs such replacements on a pattern in $X$, the resulting pattern is still in $X$.) We then wish to estimate the sizes of preimages under this map to get an inequality relating $|L_{\Gamma_{k+C}}(X)|$ and $|L_{\Gamma_k}(X_w)|$, which upon taking logarithms and letting $k \to \infty$ would give the desired bound. We can make certain helpful assumptions about the patterns in which these replacements take place (such as the approximate number of occurrences of $w$) by using the results of Section 4.

There is a subtlety though; a replacement of $w$ by $\widetilde{w}$ in a pattern $v$ can create a new occurrence of $w$, i.e. an occurrence of $w$ which was not present before the replacement, but is afterwards. We would like to choose $\widetilde{w}$ so that such occurrences of $w$ cannot be created, but this is impossible for some choices of $w$. We prove a slightly weaker fact that is still sufficient for the purposes of proving the upper bound in Theorem 1.2: for any large enough $n$ and $w \in L_{\Gamma_n}(X)$, there exist a subpattern $w' \in L_{\Gamma_m}(X)$ with $m$ not much smaller than $n$ and $w'' \neq w'$ with the same shape $\Gamma_m$ such that $w'$ and $w''$ agree on $\Gamma_m^{(t)}$ and such that a replacement of $w'$ by $w''$ can never create a new occurrence of $w'$. This weaker fact, plus a few technical details, is Theorem 6.1, whose statement and proof we defer to Section 6.

We can then define a map $\phi_k : L_{\Gamma_{k+C}}(X) \to L_{\Gamma_k}(X_w)$ by replacing occurrences of $w'$ by $w''$. (Such replacements can be used to get rid of all occurrences of $w$ since $w'$ is a subpattern of $w$.) We then wish to get an upper bound on the size of preimages $\phi_k^{-1}(u)$ for all $u \in L_{\Gamma_k}(X_w)$ in order to get an upper bound on $\frac{|L_{\Gamma_{k+C}}(X)|}{|L_{\Gamma_k}(X_w)|}$, which yields the desired upper bound on $h^{top}(X) - h^{top}(X_w)$. However, we cannot get a good enough upper bound of this sort for all $u \in L_{\Gamma_k}(X_w)$. Instead, we consider the restriction of $\phi_k$ to $A_{k+C,\epsilon,3n}(X)$ (Definition 4.9) for small $\epsilon$, and then use bounds on $\phi_k^{-1}(u) \cap A_{k+C,\epsilon,3n}(X)$ for $u \in L_{\Gamma_k}(X_w)$ to get an upper bound on $\frac{|A_{k+C,\epsilon,3n}(X)|}{|L_{\Gamma_k}(X_w)|}$. Then taking the logarithm, dividing by $k$ and letting $k$ approach infinity gives an upper bound on $h^{top}(X) - h^{top}(X_w)$ by Corollary 4.10, which will approach the upper bound from Theorem 1.2 as $\epsilon \to 0$.

*Remark 5.1.* For the remainder of the paper, we will frequently talk about a cube of size $m$ being "central" in another cube of size $n > m$. Obviously if $m$ and $n$ have different parities, this is impossible. If this is the case, then the size of the inner "central" cube is decreased by one. We will not comment on this subtlety when it comes up in our proofs, as it quickly becomes unwieldy, and none of the bounds on sizes of cubes that we work with are tight enough for these slight changes by 1 to pose a problem.

We now begin the proof of the upper bound of Theorem 1.2. By Theorem 6.1 in Section 6, there exists $N_1$ so that for any $n > N_1$ and $w \in L_{\Gamma_n}(X)$, there exist $m > n - n^{1-\frac{1}{4d}}$, $w'$ a subpattern of $w$ with shape $\Gamma_m$, and $w'' \in L_{\Gamma_m}(X)$ agreeing with $w'$ on $\Gamma_m^{(t)}$ such that replacing $w'$ by $w''$ cannot create new occurrences of $w'$. If we define $\ell = \ell(n) = \left\lceil \left( \frac{d \ln n}{h^{top}(X)} \right)^{\frac{1}{d}} \right\rceil + 1$, then $w'$ and $w''$ actually agree outside some copy $U$ of $\Gamma_{2R+\ell}$ which is contained in the central copy of $\Gamma_{3(2R+\ell)\lceil n^{1-\frac{1}{3d}} \rceil}$ in $\Gamma_m$. Also, $w''|_U$ contains a subpattern $a \in L_{\Gamma_\ell}(X)$ which is not a subpattern of $w$. We from now on will only consider $n$ larger than $N_1$, and for any such $n$ and $w \in L_{\Gamma_n}(X)$, we use $m$, $w'$, $w''$, $U$, $a$, and $\ell$ to denote the objects listed above whose existence is guaranteed by Theorem 6.1. Denote by $\widetilde{w} \in L_{\Gamma_n}(X)$ the pattern obtained by replacing $w'$ by $w''$ in $w$.

*Definition 5.2.* For any $n > N_1$ and $w \in L_{\Gamma_n}(X)$, two occurrences of the pattern $w'$ which occur in copies $S$ and $S'$ of $\Gamma_m$ are said to be in **undesirable position** if $\frac{m}{2} - 2n^{1-\frac{1}{4d}} \leq \|S - S'\|_\infty < n$.

We point out that there are fewer than $2^d n^d$ patterns (up to translation) which are made up of a pair of $w'$ in undesirable position. This is because for a given occurrence of $w'$, in order for another occurrence to be in undesirable position with the first, it must be contained in a copy of $\Gamma_{2n+m-2}$ concentric with the first, and there are fewer than $2^d n^d$ distinct copies of $\Gamma_m$ inside this copy of $\Gamma_{2n+m}$. Call these patterns $u_1, u_2, \ldots, u_b$, where $b < 2^d n^d$, and call $S_i$ the shape of $u_i$ for any $1 \leq i \leq b$. $S_i$ is a union of two copies of $\Gamma_m$ for all $i$.

Lemma 5.3. *There exists $N_2 > N_1$ so that for any $n > N_2$ and $w \in L_{\Gamma_n}(X)$, if a pattern $v$ has no occurrences of any $u_i$, then replacing an occurrence of $w$ in $v$ by $\widetilde{w}$ cannot yield a new occurrence of $w$ in $v$.*

*Proof.* For a contradiction, assume that for some $w$ a particular replacement of $w$ by $\widetilde{w}$ in $v$ could create a new occurrence of $w$, i.e. that there exist $v \in L(X)$ and $S$ and $S'$ copies of $\Gamma_n$ such that $v|_S = w$, $v|_{S'} \neq w$, and if we denote by $v'$ the pattern obtained by replacing $v|_S$ by $\widetilde{w}$, then $v'|_{S'} = w$.

We refer now to Figure 1. (All figures in this paper are drawn with $d = 2$, but all constructions and descriptions are carried out in full generality.) In Figure 1, $T \subset S$ is a copy of $\Gamma_m$ such that $v|_T = w'$, $U$ is the copy of $\Gamma_{2R+\ell}$ within $S$ outside which $v$ and $v'$ agree, $T'$ is the copy of $\Gamma_m$ in $S'$ corresponding to $T$ in $S$, and $A$ is the copy of $\Gamma_{3(2R+\ell)\left\lceil n^{1-\frac{1}{3d}} \right\rceil}$ central in $S$ in which $U$ must lie.

Since $v|_S = w$ does not contain $a$ and $v'|_U$ contains $a$ as a subpattern, $S'$ cannot contain all of $U$. However, since $v|_{S'} \neq w = v'|_{S'}$, and $v$ and $v'$ agree outside $U$, $S'$

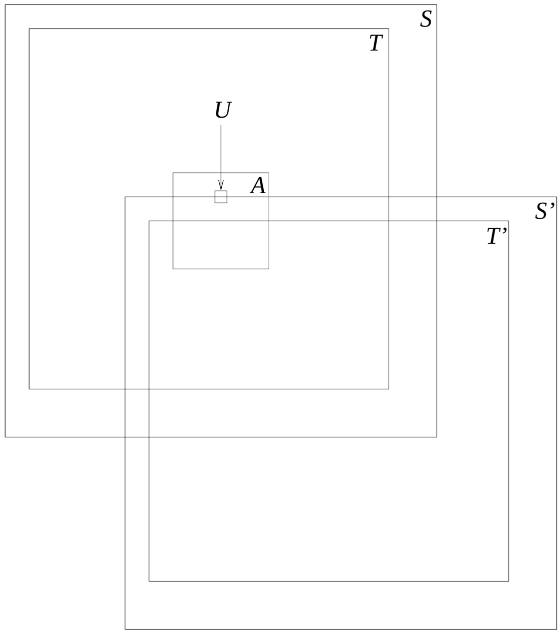


FIGURE 1. Intersecting occurrences of $w$

must have nonempty intersection with $U$. Since the replacement of $w$ by $\widetilde{w}$ cannot possibly create a new occurrence of $w'$, $T'$ must have already been filled with $w'$ before the replacement occurred, i.e. $v|_{T'} = w'$. Since $v|_S = w$, clearly $v|_T = w'$ as well.

We claim that $v|_T$ and $v|_{T'}$ were in undesirable position. Since $U \subseteq A$, and since the size of $A$ is $3(2R+\ell)\lceil n^{1-\frac{1}{3d}} \rceil$, the distance in the $\vec{e_i}$-direction between the center of $U$ and the center of $A$ is less than $\frac{3}{2}(2R+\ell)n^{1-\frac{1}{3d}}$ for each $1 \leq i \leq d$. $A$ is central in $S$, so the center of $A$ is the same as the center of $S$. Since $T$ is a subcube of $S$ whose size is at most $n^{1-\frac{1}{4d}}$ shorter, the distance in the $\vec{e_i}$-direction between the center of $S$ and the center of $T$ is less than $\frac{1}{2}n^{1-\frac{1}{4d}}$ for each $1 \leq i \leq d$. For the same reason, the distance in the $\vec{e_i}$-direction between the center of $S'$ and the center of $T'$ is less than $\frac{1}{2}n^{1-\frac{1}{4d}}$ for each $1 \leq i \leq d$. Finally, since $U$ intersects the boundary of $S'$, and since $U$ has size $2R+\ell$, there exists $1 \leq i \leq d$ for which the distance in the $\vec{e_i}$-direction between the center of $S'$ and the center of $U$ is between $\frac{m}{2} - (R + \frac{\ell}{2})$ and $\frac{m}{2} + (R + \frac{\ell}{2})$. Putting all of these facts together, we see that there exists $1 \leq i \leq d$ so that the distance in the $\vec{e_i}$-direction between the center of $T$ and the center of $T'$ is between $\frac{m}{2} - \frac{3}{2}(2R+\ell)n^{1-\frac{1}{3d}} - n^{1-\frac{1}{4d}} - (R+\frac{\ell}{2})$ and $\frac{m}{2} + \frac{3}{2}(2R+\ell)n^{1-\frac{1}{3d}} + n^{1-\frac{1}{4d}} + (R+\frac{\ell}{2})$. Clearly there exists $N_2 > N_1$ so that for $n > N_2$, $\frac{3}{2}(2R+\ell)n^{1-\frac{1}{3d}} + (R+\frac{\ell}{2}) < n^{1-\frac{1}{4d}}$, which would imply that this distance would be between $\frac{m}{2} - 2n^{1-\frac{1}{4d}}$ and $n$, which would imply that $v|_T$ and $v|_{T'}$ were indeed filled with occurrences of $w'$ in undesirable position, meaning that

$v|_{T\cup T'} = u_i$ for some $i$, a contradiction. Therefore, there exists $N_2 > N_1$ so that for $n > N_2$, our original assumption was wrong, and so we are done.

□

For any $n > N_2$, $w \in L_{\Gamma_n}(X)$, and $k > m$, we define $\phi_k$ as a composition of three maps: $\alpha_k : L_{\Gamma_{k+4m}}(X) \to L_{\Gamma_{k+4m}}(X)$, $\beta_k : \alpha_k(L_{\Gamma_{k+4m}}(X)) \to L_{\Gamma_{k+4m}}(X)$, and $\gamma_k : (\beta_k \circ \alpha_k)(L_{\Gamma_{k+4m}}(X)) \to L_{\Gamma_k}(X_w)$.

Given any $u \in L_{\Gamma_{k+4m}}(X)$, $\alpha_k(u)$ is defined by finding each occurrence of any of the $u_i$ within $u$, and in each one, replacing the first lexicographically of the two occurrences of $w'$ which make up $u_i$ by $w''$. In order to make this operation well-defined, we must specify an order for these replacements to be done in. First perform the operation described above on the lexicographically first occurrence of $u_1$, then on the new lexicographically first occurrence of $u_1$, and continue until no $u_1$ remain. Then perform the same procedure for $u_2$, $u_3$, etc. Since replacing $w'$ by $w''$ can never create a new occurrence of $w'$, the resulting pattern, which we define as $\alpha_k(u)$, will contain no $u_i$, i.e. will have no pair of occurrences of $w'$ in undesirable position.

For any $\alpha_k(u) \in \alpha_k(\Gamma_{k+4m}(X))$, we define $\beta_k(\alpha_k(u))$ by beginning with $\alpha_k(u)$, replacing the first occurrence of $w$ lexicographically by $\widetilde{w}$, then replacing the new first occurrence of $w$ lexicographically by $\widetilde{w}$, and repeating this procedure. Since $\alpha_k(u)$ contained no $u_i$, and since the replacements being made cannot create new $w'$, after each of the replacements of $w$ by $\widetilde{w}$, the resulting pattern will still contain no $u_i$. Therefore, by Lemma 5.3, none of these replacements can create new occurrences of $w$, and therefore this procedure will terminate in a pattern with no occurrences of $w$, which we define to be $(\beta_k \circ \alpha_k)(u)$.

The definition of $\gamma_k$ is much simpler: for any $(\beta_k \circ \alpha_k)(u) \in (\beta_k \circ \alpha_k)(L_{\Gamma_{k+4m}}(X))$, $\gamma_k((\beta_k \circ \alpha_k)(u))$ is the subpattern of $(\beta_k \circ \alpha_k)(u)$ occupying its central copy of $\Gamma_k$. For any $u \in L_{\Gamma_{k+4m}}(X)$, we define $\phi_k(u) = (\gamma_k \circ \beta_k \circ \alpha_k)(u)$.

Lemma 5.4. *For any $n > N_2$, $w \in L_{\Gamma_n}(X)$, $k > m$, and $u \in L_{\Gamma_{k+4m}}(X)$, $\phi_k(u) \in L_{\Gamma_k}(X_w)$.*

*Proof.* Fix any $n > N_2$, $w \in L_{\Gamma_n}(X)$, $k > m$, and $u \in L_{\Gamma_{k+4m}}(X)$. We will inductively define a sequence of patterns $d_j$, where $d_j \in L_{\Gamma_{k+4jm}}(X)$ and each $d_j$ has no occurrences of $w$. Define $d_1 = (\beta_k \circ \alpha_k)(u)$. For any $d_j$, $j > 0$, define $d_{j+1}$ as follows: since $d_j \in L(X)$, it can be extended to an infinite configuration in $X$. Use this fact to create a pattern $d'_j \in L_{\Gamma_{k+4(j+1)m}}(X)$ which has $d_j$ as the pattern occupying its central copy of $\Gamma_{k+4jm}$. We then define $d_{j+1} = (\beta_{k+4jm} \circ \alpha_{k+4jm})(d'_j)$. For the same reasons as above, $d_{j+1}$ will then contain no occurrences of $w$. Since $d_{j+1}$ was obtained from $d'_j$ by making a series of replacements of $w'$ by $w''$, and since $w'$ and $w''$ agree on their boundary of thickness $t$, $d_{j+1}$ and $d'_j$ themselves agree on their boundary of thickness $t$. Therefore, $d_{j+1} \in L(X)$, completing the inductive step and allowing us to define $d_j$ for all $j$. We also note that for any $j$, since $d_j$ contained no occurrences of $w$ or $u_i$ for any $1 \le i \le b$, any occurrences of these patterns in $d'_j$ must have nonempty overlap with the boundary of thickness

$2m$ of $\Gamma_{k+4(j+1)m}$. Therefore, any occurrences of these patterns in $d'_j$ is contained entirely within its boundary of thickness $4m$, and so $d_{j+1}$ and $d_j$ must agree on their respective central copies of $\Gamma_{k+4(j-1)m}$, since no replacements could have affected that portion. This means that we may define the limit of the $d_j$, call it $x$. Then $x$ has no occurrences of $w$ and is therefore in $X_w$. Since all $d_j$ agree on their central copies of $\Gamma_k$, and since $\phi_k(u)$ occupies the central copy of $\Gamma_k$ in $d_1$, $\phi_k(u)$ is a subpattern of $x$, and so $\phi_k(u) \in L_{\Gamma_k}(X_w)$ as claimed.

□

By Lemma 5.4, for $n > N_2$, $\phi_k$ is indeed a function from $L_{\Gamma_{k+4m}}(X)$ to $L_{\Gamma_k}(X_w)$. For any $\epsilon > 0$, we will consider the restriction of $\phi_k$ to $A_{k+4m,\epsilon,3n}(X)$. (Definition 4.9) The purpose of this is to bound from above the number of occurrences of $w$ and any $u_i$ in the patterns $v$ that we apply $\phi_k$ to.

LEMMA 5.5. *There exists $N_3 > N_2$ so that for any $n > N_3$, any $w \in L_{\Gamma_n}(X)$, and any $1 \leq i \leq b$, every element of $A_{k+4m,\epsilon,3n}(X)$ has fewer than $(k+4m)^d(e^{-h^{top}(X)(n-2R)^d} + \epsilon)$ occurrences of $w$ and fewer than $(k+4m)^d(e^{-1.4h^{top}(X)n^d} + \epsilon)$ occurrences of $u_i$.*

*Proof.* Fix any $n > N_2$, $w \in L_{\Gamma_n}(X)$, and $v \in A_{k+4m,\epsilon,3n}(X)$. Since the shape of $w$ is $\Gamma_n \subseteq \Gamma_{3n}$, $v$ has fewer than $(k+4m)^d(\frac{1}{|L_{\Gamma_n \setminus \Gamma_n^{(R)}}(X)|} + \epsilon) = (k+4m)^d(\frac{1}{|L_{\Gamma_{n-2R}}(X)|} + \epsilon)$ occurrences of $w$. By Lemma 4.11, $|L_{\Gamma_{n-2R}}(X)| \geq e^{h^{top}(X)(n-2R)^d}$, and so $v$ has fewer than $(k+4m)^d(e^{-h^{top}(X)(n-2R)^d} + \epsilon)$ occurrences of $w$.

Since each $u_i$ has shape $S_i \subseteq \Gamma_{3n}$, $v$ has fewer than $(k+4m)^d(\frac{1}{|L_{S_i \setminus S_i^{(R)}}(X)|} + \epsilon)$ occurrences of $u_i$ for each $1 \leq i \leq b$. We wish to bound $|L_{S_i \setminus S_i^{(R)}}(X)|$ from below. First, recall that $S_i$ is the union of two copies of $\Gamma_m$, so $S_i \setminus S_i^{(R)}$ is the union of two copies of $\Gamma_{m-2R}$ concentric with the original copies of $\Gamma_m$. By the definition of undesirable position, there exists $1 \leq i \leq d$ for which the distance in the $\vec{e_i}$-direction between the centers of these two $\Gamma_{m-2R}$ is not less than $\frac{m}{2} - 2n^{1-\frac{1}{4d}}$.

In Figure 2, we denote the direction in question by $\vec{e_i}$, the two copies of $\Gamma_{m-2R}$ by $S$ and $S'$, and the distance in the $\vec{e_i}$-direction between the centers of $S$ and $S'$ by $c$. We then define $T$ to be a rectangular prism which is a subset of $S$ a distance of $R+1$ away from $T'$. The sizes of $T$ are $m-2R$ in every direction but $\vec{e_i}$, and $c - R \geq \frac{m}{2} - 2n^{1-\frac{1}{4d}} - R$ in the $\vec{e_i}$-direction.

Clearly, for large enough $n$, this dimension $c-R$ of $T$ is greater than $\frac{m}{2} - 3n^{1-\frac{1}{4d}}$. Since $T$ and $S'$ are a distance of $R+1$ away from each other and are subpatterns of $S_i \setminus S_i^{(R)}$, $|L_{S_i \setminus S_i^{(R)}}(X)| \geq |L_T(X)||L_{S'}(X)|$, which is at least $\exp\big(h^{top}(X)[(m-2R)^{d-1}(\frac{m}{2} - 3n^{1-\frac{1}{4d}}) + (m-2R)^d]\big)$ by Lemma 4.11. $R$, $t$, and $n^{1-\frac{1}{4d}}$ are all small in relation to $m$ for large enough $n$, and $\frac{m}{n} \to 1$ as $n \to \infty$, so for large $n$, this bound is greater than $e^{h^{top}(X)1.4n^d}$. Therefore, there exists $N_3 > N_2$ so that for $n > N_3$, $v$ has fewer than $(k+4m)^d(e^{-1.4h^{top}(X)n^d} + \epsilon)$ occurrences of any $u_i$ for $1 \leq i \leq b$.

□

For large $n$, we will now prove upper bounds on $|\alpha_k^{-1}(v) \cap A_{k+4m,\epsilon,3n}(X)|$ for

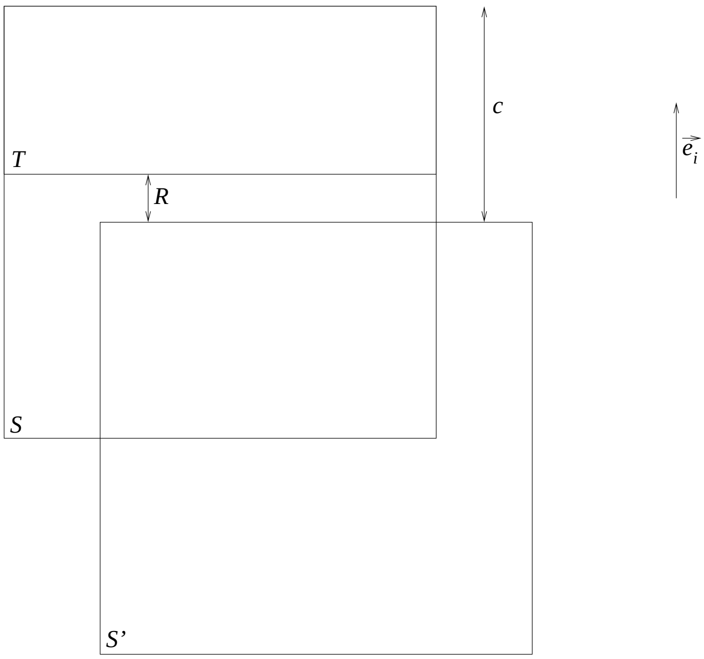


FIGURE 2. $S_i \diagdown S_i^{(R)}$

any $v \in \alpha_k(L_{\Gamma_{k+4m}}(X))$, on $|\beta_k^{-1}(v') \cap \alpha_k(A_{k+4m,\epsilon,3n}(X))|$ for any $v' \in L_{\Gamma_{k+4m}}(X)$, and on $|\gamma_k^{-1}(v'')|$ for any $v'' \in L_{\Gamma_k}(X_w)$.

LEMMA 5.6. *There exists $N_4 > N_3$ so that for any $n > N_4$ and $w \in L_{\Gamma_n}(X)$, there exist $\epsilon_n$ and $k_n$ for which, given any $\epsilon \in (0, \epsilon_n)$, any $k > k_n$, and any $v \in A_{k+4m,\epsilon,3n}(X)$, $|\alpha_k^{-1}(v) \cap A_{k+4m,\epsilon,3n}(X)| \leq \exp((k+4m)^d e^{-h^{top}(X)(n-2R)^d})$.*

*Proof.* Consider any $n > N_3$, any $w \in L_{\Gamma_n}(X)$, any $v \in A_{k+4m,\epsilon,3n}(X)$, and any $u \in \alpha_k^{-1}(v) \cap A_{k+4m,\epsilon,3n}(X)$. Since $b < 2^d n^d$, Lemma 5.5 implies that the total number of letters in $u$ which are part of an occurrence of any $u_i$ is at most $(2m^d)(2^d n^d)(k+4m)^d(e^{-1.4h^{top}(X)n^d} + \epsilon)$. There exists $N_{3.5} > N_3$ so that for $n > N_{3.5}$, $(2m^d)(2^d n^d)(k+4m)^d(e^{-1.4h^{top}(X)n^d}+\epsilon) < (k+4m)^d(e^{-1.3h^{top}(X)n^d}+2^{d+1}n^{2d}\epsilon)$. So, for any $n > N_{3.5}$, $u$ differs from $v$ on less than $(k+4m)^d(e^{-1.3h^{top}(X)n^d} + 2^{d+1}n^{2d}\epsilon)$ letters. The number of $u$ with this property is at most

$$|A|^{(k+4m)^d(\exp(-1.3h^{top}(X)n^d)+2^{d+1}n^{2d}\epsilon)} \sum_{i=0}^{\lfloor (k+4m)^d(\exp(-1.3h^{top}(X)n^d)+2^{d+1}n^{2d}\epsilon)\rfloor} \binom{(k+4m)^d}{i}. \quad (7)$$

For any $n$, we define $\epsilon_n$ so small that $2^{d+1}n^{2d}\epsilon_n < e^{-1.3h^{top}(X)n^d}$. Then, for large enough $n$ (say $n > N_{3.75} > N_{3.5}$) and $\epsilon \in (0, \epsilon_n)$, $e^{-1.3h^{top}(X)n^d} + 2^{d+1}n^{2d}\epsilon < \frac{1}{2}$, which implies that (7) is less than

$$|A|^{(k+4m)^d(2\exp(-1.3h^{top}(X)n^d))}(k+4m)^d \cdot \binom{(k+4m)^d}{\lfloor (k+4m)^d(2e^{-1.3h^{top}(X)n^d})\rfloor}. \quad (8)$$

By Stirling's formula, $\lim_{n\to\infty}\frac{1}{n}\ln\binom{n}{n\alpha} = -\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha)$, which is less than $-2\alpha\ln\alpha$ for any $\alpha < \frac{1}{2}$. Therefore, for any $n$, we may choose $k_n^{(1)}$ so that for any $k > k_n^{(1)}$, $\frac{1}{(k+4m)^d}\ln\binom{(k+4m)^d}{\lfloor (k+4m)^d(2e^{-1.3h^{top}(X)n^d})\rfloor} < -2(2e^{-1.3h^{top}(X)n^d})\ln(2e^{-1.3h^{top}(X)n^d}) < 5.2h^{top}(X)n^d e^{-1.3h^{top}(X)n^d}$. We also define $k_n^{(2)}$ so that for any $k > k_n^{(2)}$, $\frac{\ln((k+4m)^d)}{(k+4m)^d} < e^{-1.3h^{top}(X)n^d}$. Then, take $k_n = \max(k_n^{(1)}, k_n^{(2)})$. For any $k > k_n$, (8) is less than

$$\exp((k+4m)^d e^{-1.3h^{top}(X)n^d}(2\ln|A| + 1 + 5.2h^{top}(X)n^d)), \quad (9)$$

and clearly there exists $N_4 > N_{3.75}$ so that for any $n > N_4$, (9) is less than $\exp((k+4m)^d e^{-h^{top}(X)(n-2R)^d})$.

□

To prove the upper bound on $\beta_k$-preimages, we first need a lemma about $\beta_k$.

Lemma 5.7. *For any $n > N_2$, $w \in L_{\Gamma_n}(X)$, and $u \in \alpha_k(L_{\Gamma_{k+4m}}(X))$, all copies of $\Gamma_n$ where replacements occur in changing $u$ to $\beta_k(u)$ are disjoint.*

*Proof.* Consider any $n > N_2$, $w \in L_{\Gamma_n}(X)$, and $u \in \alpha_k(L_{\Gamma_{k+4m}}(X))$. Consider any pair of occurrences of $w$ in $u$ with nonempty intersection, say at $S$ and $S'$ copies of $\Gamma_n$. Clearly $\|S - S'\|_\infty < n$. Then these occurrences of $w$ contain occurrences of $w'$ as subpatterns, say at $T$ and $T'$ copies of $\Gamma_m$ with $T \subset S$ and $T' \subset S'$. Then $T - T' = S - S'$, so $\|T - T'\|_\infty < n$. Since $u$ is in the image of $\alpha_k$, $u$ has no occurrences of any $u_i$, and so $u|_T$ and $u|_{T'}$ are not in undesirable position. Also define $U$ to be the copy of $\Gamma_{2R+\ell}$ in $T$ which would be changed to change $u|_T$ to $w''$, and $U'$ to be the corresponding copy of $\Gamma_{2R+\ell}$ in $T'$. Since $\|T-T'\|_\infty < n$ and $u|_T$ and $u|_{T'}$ are not in undesirable position, $\|T-T'\|_\infty < \frac{m}{2} - 2n^{1-\frac{1}{4d}}$. Since $n > N_2$, $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil < 2n^{1-\frac{1}{4d}}$, and so $T$ contains the central copy of $\Gamma_{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}$ in $S$, implying that $T$ contains $U'$. Similarly, $T'$ contains $U$. Therefore, if $u|_T$ is replaced by $w''$ sometime during the replacements defining $\beta_k$, then $u|_{S'}$ will be changed to something other than $w$, and if $u|_{T'}$ is replaced by $w''$ at some point in these replacements, then $u|_S$ will be changed to something other than $w$. In other words, in any replacement involved in changing $u$ to $\beta_k(u)$, and any copy $S$ of $\Gamma_n$, if $u|_S$ is changed from $w$ to $\widetilde{w}$, then all occurrences of $w$ with nonempty intersection with $S$ are also changed. Since new occurrences of $w$ cannot be created during these replacements, we have shown that the copies of $\Gamma_n$ where replacements occur in the application of $\beta_k$ must be disjoint.

□

Lemma 5.8. *There exists $N_5 > N_4$ so that for any $n > N_5$, $w \in L_{\Gamma_n}(X)$, $k > m$, $\epsilon > 0$, and $v' \in L_{\Gamma_{k+4m}}(X)$, $|\beta_k^{-1}(v') \cap \alpha_k(A_{k+4m,\epsilon,3n}(X))| \leq 2^{(k+4m)^d(19^d\exp(-h^{top}(X)(n-2R)^d)+43^d n^{2d}\epsilon)}$.*

*Proof.* We first show that there exists $N_{4.5} > N_4$ so that for any $n > N_{4.5}$ and any $w \in L_{\Gamma_n}(X)$, any period $\vec{v}$ of $w''$ satisfies $\|\vec{v}\|_\infty > \frac{m}{2} - \frac{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}{2}$. Define $N_{4.5} > N_4$ so that $\frac{m}{2} - \frac{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}{2} > \frac{n}{3}$ and $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil < \frac{n}{6}$ for all $n > N_{4.5}$.

Recall that for any $n > N_1$ and $w \in L_{\Gamma_n}(X)$, $w''$ contains $a$ as a subpattern somewhere in its central copy of $\Gamma_{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}$, which we denote by $A$. Consider any $\vec{v}$ with $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil \leq \|\vec{v}\|_\infty \leq \frac{m}{2} - \frac{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}{2}$. Since $A$ is a central subcube of $\Gamma_m$ of size less than $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil$, $A + \vec{v} \subseteq \Gamma_m$ as well. If $\vec{v}$ is a period of $w''$, then $w''|_A = w''|_{A+\vec{v}}$, and so $w''|_{A+\vec{v}}$ contains $a$. However, since $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil \leq \|\vec{v}\|_\infty$, $A+\vec{v}$ is disjoint from $A$, and therefore $w''|_{A+\vec{v}} = w'|_{A+\vec{v}}$. Since $w'$ is a subpattern of $w$, $w'$ contains no occurrences of $a$, and we have a contradiction. If $\|\vec{v}\|_\infty < 3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil$, then we first note that if $n > N_{4.5}$, $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil < \frac{n}{6}$ and $\frac{m}{2} - \frac{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}{2} > \frac{n}{3}$. This means that there exists a multiple $r\vec{v}$ of $\vec{v}$ so that $3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil \leq \|r\vec{v}\|_\infty \leq \frac{m}{2} - \frac{3(2R+\ell)\lceil n^{1-\frac{1}{3d}}\rceil}{2}$. Then $r\vec{v}$ is a period of $w''$ as well, and the previous argument again gives a contradiction. Therefore, for $n > N_{4.5}$, any period $\vec{v}$ of $w''$ satisfies $\|\vec{v}\|_\infty > \frac{n}{3}$.

Each of the replacements involved in $\alpha_k$ and $\beta_k$ involves changing letters only on a copy of $\Gamma_m$, and so could possibly create new occurrences of $\widetilde{w}$ only inside a copy of $\Gamma_{m+2n}$ concentric with this copy of $\Gamma_m$. Since $w''$ is a subpattern of $\widetilde{w}$, any period $\vec{v}$ of $\widetilde{w}$ must also satisfy $\|\vec{v}\|_\infty > \frac{n}{3}$. Therefore, any replacement of $w'$ by $w''$ in the replacements defining $\alpha_k$ and $\beta_k$ results in fewer than $6^d$ newly created occurrences of $\widetilde{w}$. We also point out that each of these replacements results in fewer than $(m+n)^d$ new occurrences of $w$ as well. (This is because all such occurrences of $w$ are contained in the aforementioned copy of $\Gamma_{m+2n}$.)

Fix any $u \in A_{k+4m,\epsilon,3n}(X)$. Since $n > N_3$, Lemma 5.5 implies that the total number of occurrences of any $u_i$ in $u$ is smaller than $2^d n^d (k+4m)^d (e^{-1.4h^{top}(X)n^d} + \epsilon)$, and so the total number of replacements made in the application of $\alpha_k$ to $u$ is also less than $2^d n^d (k+4m)^d (e^{-1.4h^{top}(X)n^d} + \epsilon)$. Since $u$ itself had at most $(k+4m)^d (e^{-h^{top}(X)(n-2R)^d} + \epsilon)$ occurrences of $w$ by definition of $A_{k+4m,\epsilon,3n}(X)$, and since each replacement in the application of $\alpha_k$ to $u$ creates less than $(m+n)^d$ new occurrences of $w$, $\alpha_k(u)$ has less than $(k+4m)^d(e^{-h^{top}(X)(n-2R)^d} + \epsilon) + (m+n)^d 2^d n^d (k+4m)^d (e^{-1.4h^{top}(X)n^d} + \epsilon)$ occurrences of $w$. There exists $N_{4.7} > N_{4.5}$ so that for $n > N_{4.7}$, this is less than $(k+4m)^d(2e^{-h^{top}(X)(n-2R)^d} + 5^d n^{2d}\epsilon)$. This means that for any such $n$, the number of replacements involved in changing $\alpha_k(u)$ to $(\beta_k \circ \alpha_k)(u)$ is less than $(k+4m)^d(2e^{-h^{top}(X)(n-2R)^d} + 5^d n^{2d}\epsilon)$ as well. We from now on assume that $n > N_{4.7}$.

We now wish to bound from above the number of occurrences of $\widetilde{w}$ in $(\beta_k \circ \alpha_k)(u)$. $u$ itself contains at most $(k+4m)^d(e^{-h^{top}(X)(n-2R)^d} + \epsilon)$ occurrences of $\widetilde{w}$ by definition of $A_{k+4m,\epsilon,3n}(X)$. The total number of replacements made in the application of $\alpha_k$ to $u$ is less than $2^d n^d (k+4m)^d (e^{-1.4h^{top}(X)n^d} + \epsilon)$, and there exists $N_5 > N_{4.7}$ so that for any $n > N_5$, this quantity is less than $(k+4m)^d(e^{-h^{top}(X)(n-2R)^d} + 2^d n^d \epsilon)$. Each of these replacements can create at most

$6^d$ new occurrences of $\widetilde{w}$, and so at most $(k+4m)^d(6^d e^{-h^{top}(X)(n-2R)^d} + 12^d n^d \epsilon)$ occurrences of $\widetilde{w}$ are added during these replacements. We showed above that there are fewer than $(k+4m)^d(2e^{-h^{top}(X)(n-2R)^d} + 5^d n^{2d}\epsilon)$ replacements involved in changing $\alpha_k(u)$ to $(\beta_k \circ \alpha_k)(u)$. Since each of these replacements can create at most $6^d$ new occurrences of $\widetilde{w}$, the number of occurrences of $\widetilde{w}$ created during these replacements is less than $(k+4m)^d(2 \cdot 6^d e^{-h^{top}(X)(n-2R)^d} + 30^d n^{2d}\epsilon)$. Collecting these facts, for $n > N_5$ the total number of occurrences of $\widetilde{w}$ created in the process of changing $u$ to $(\beta_k \circ \alpha_k)(u)$ is less than $(k+4m)^d((3 \cdot 6^d + 1)e^{-h^{top}(X)(n-2R)^d} + (1 + 12^d n^d + 30^d n^{2d})\epsilon)$. This means that $(\beta_k \circ \alpha_k)(u)$ contains less than $(k+4m)^d(19^d e^{-h^{top}(X)(n-2R)^d} + 43^d n^{2d}\epsilon)$ occurrences of $\widetilde{w}$.

By Lemma 5.7, the set of copies of $\Gamma_n$ on which replacements are performed in the application of $\beta_k$ to $\alpha_k(u)$ is pairwise disjoint. From this, it is clear that for any $u \in A_{k+4m,\epsilon,3n}(X)$ and any $S$ a copy of $\Gamma_n$ which is the location of a replacement involved in changing $\alpha_k(u)$ to $(\beta_k \circ \alpha_k)(u)$, $(\beta_k \circ \alpha_k)(u)|_S = \widetilde{w}$. In other words, once $w$ is changed to $\widetilde{w}$ during the application of $\beta_k$ to $\alpha_k(u)$, that occurrence of $\widetilde{w}$ will not be changed during future replacements. This means that knowing $(\beta_k \circ \alpha_k)(u)$, along with knowing the locations of all copies of $\Gamma_n$ where replacements occur in changing $\alpha_k(u)$ to $(\beta_k \circ \alpha_k)(u)$, is enough to uniquely determine $\alpha_k(u)$. Therefore, for any $n > N_5$ and $v' \in L_{\Gamma_{k+4m}}(X)$, $|\beta_k^{-1}(v') \cap \alpha_k(A_{k+4m,\epsilon,3n}(X))|$ is less than or equal to the total number of subsets of the locations of occurrences of $\widetilde{w}$ in $v'$ which could have been the locations of replacements in the application of $\beta_k$, which is less than or equal to $2^{(k+4m)^d(19^d \exp(-h^{top}(X)(n-2R)^d) + 43^d n^{2d}\epsilon)}$.

□

Lemma 5.9. *For any $k > m$, $\epsilon > 0$, and $v'' \in L_{\Gamma_k}(X_w)$, $|\gamma_k^{-1}(v'')| \leq |A|^{(k+4m)^d - k^d}$.*

*Proof.* Fix $v'' \in L_{\Gamma_k}(X_w)$. Any $u \in \gamma_k^{-1}(v'')$ must have its central copy of $\Gamma_k$ filled with $v''$. This means that $|\gamma_k^{-1}(v'')| \leq |A|^{(k+4m)^d - k^d}$.

□

*Proof of the upper bound of Theorem 1.2.* Choose any $d > 1$, $X$ a strongly irreducible $\mathbb{Z}^d$ SFT with uniform filling length $R$ containing more than one point, $n > N_5$, $w \in L_{\Gamma_n}(X)$, $k > k_n$, and $\epsilon < \epsilon_n$. For any $v \in L_{\Gamma_k}(X_w)$, by Lemmas 5.6, 5.8, and 5.9,

$$|\phi_k^{-1}(v) \cap A_{k+4m,\epsilon,3n}(X)| \leq \exp((k+4m)^d e^{-h^{top}(X)(n-2R)^d}) \cdot 2^{(k+4m)^d(19^d \exp(-h^{top}(X)(n-2R)^d) + 43^d n^{2d}\epsilon)} \cdot |A|^{(k+4m)^d - k^d},$$

which is less than $\exp((k+4m)^d(Ce^{-h^{top}(X)(n-2R)^d} + D\epsilon))$ for large enough $k$ and for constants $C$ independent of $n$ and $k$ and $D$ independent of $k$. This implies that

$$|L_{\Gamma_k}(X_w)| \geq \frac{|A_{k+4m,\epsilon,3n}(X)|}{\exp((k+4m)^d(Ce^{-h^{top}(X)(n-2R)^d} + D\epsilon))}$$

for sufficiently large $k$. We then take natural logarithms of both sides, divide by $(k+4m)^d$, and let $k \to \infty$ to get

$$\lim_{k\to\infty} \frac{k^d}{(k+4m)^d} \frac{\ln |L_{\Gamma_k}(X_w)|}{k^d} \geq \lim_{k\to\infty} \frac{\ln |A_{k+4m,\epsilon,3n}(X)|}{(k+4m)^d} - Ce^{-h^{top}(X)(n-2R)^d} - D\epsilon,$$

and by using the definition of topological entropy and Corollary 4.10, we are left with

$$h^{top}(X_w) \geq h^{top}(X) - Ce^{-h^{top}(X)(n-2R)^d} - D\epsilon.$$

We may let $\epsilon$ approach zero, which leaves

$$h^{top}(X) - h^{top}(X_w) \leq \frac{C}{e^{h^{top}(X)(n-2R)^d}}.$$

□

## 6. *A replacement theorem*

The goal of this section is to prove the following theorem, which was integral in Section 5.

THEOREM 6.1. *For any $d > 1$ and any strongly irreducible $\mathbb{Z}^d$ shift of finite type $X$ with uniform filling length $R$ containing more than one point, there exists $N_1$ such that for any $w \in L_{\Gamma_n}(X)$ with $n > N_1$, there exist $m > n - n^{1-\frac{1}{4d}}$, $a \in L_{\Gamma_\ell}(X)$ (where $\ell = \ell(n) = \left\lceil \left(\frac{d \ln n}{h^{top}(X)}\right)^{\frac{1}{d}} \right\rceil + 1$), and $w', w'' \in L_{\Gamma_m}(X)$ with the following properties:*

*(i) $w'$ is a subpattern of $w$*
*(ii) $a$ is not a subpattern of $w$*
*(iii) There is a copy $U$ of $\Gamma_\ell$ contained in the central copy of $\Gamma_{3(2R+\ell)\lceil n^{1-\frac{1}{3d}} \rceil}$ in $\Gamma_m$ such that $w''|_U = a$*
*(iv) $w'$ and $w''$ agree outside the copy of $\Gamma_{2R+\ell}$ in which $U$ is central, and in particular agree on $\Gamma_m^{(t)}$*
*(v) Replacing $w'$ by $w''$ in any pattern $v$ can never create a new occurrence of $w'$.*

Fix $d > 1$, any strongly irreducible $\mathbb{Z}^d$ shift of finite type $X$ with uniform filling length $R$ containing more than one point, any $n$, and any $w \in L_{\Gamma_n}(X)$. By Lemma 4.11, $h^{top}(X) > 0$. Define $\ell = \ell(n) = \left\lceil \left(\frac{d \ln n}{h^{top}(X)}\right)^{\frac{1}{d}} \right\rceil + 1$. Clearly, there exists $N_{0.5}$ so that if $n > N_{0.5}$, then $\ell < n$. We from now on consider only $n > N_{0.5}$. Then, $|L_{\Gamma_\ell}(X)|$ is, by Lemma 4.11, at least $e^{h^{top}(X)\ell^d} > e^{d \ln n} = n^d$. This is greater than the number of patterns with shape $\Gamma_\ell$ which occur as subpatterns of $w$, and so therefore there exists a pattern in $L_{\Gamma_\ell}(X)$ which is not a subpattern of $w$. Fix such a pattern, and call it $a$.

The general idea of the proof of Theorem 6.1 will be to first check whether or not one can just take $w' = w$. If so, then the proof is complete. If not, we will pass to a subpattern of $w$, and check whether it is possible to take $w'$ to be this subpattern. Again, if so, then we are done. If not, then we again pass to a subpattern. Lemma 6.5 below will show that as long as this continues, each of these patterns is periodic on a much larger proportion than the previous, which will imply that this procedure must terminate within a finite number ($2^d$) of steps.

Fix any $j \in [\frac{n}{2}, n]$ and subpattern $u \in L_{\Gamma_j}(X)$ of $w$. We define $k = k(j) = 3(2R + \ell)\lceil j^{1-\frac{1}{3d}} \rceil$. It is instructive to note that when $n$ is large, $k$ will always be much smaller than $j$. Choose $N_{0.75} > 5t$ so that for $n > N_{0.75}$ and any $j \in [\frac{n}{2}, n]$, $j > 5k$, and from now on assume that $n > N_{0.75}$. Take a copy of $\Gamma_k$ (call it $A$) central in $\Gamma_j$, and partition it into $\left(3\lceil j^{(1-\frac{1}{3d})} \rceil\right)^d$ disjoint copies of $\Gamma_{2R+\ell}$. Consider only the interior copies of $\Gamma_{2R+\ell}$ in this partition, i.e. the ones which are disjoint from the boundary of $A$. (There are more than $2^d j^{d-\frac{1}{3}}$ such cubes.) For every one of these cubes $U$, use strong irreducibility of $X$ to define a **standard replacement** associated to $U$ to be any pattern $u'$ which agrees with $u$ on $\Gamma_j \setminus U$, and such that the pattern $a$ occupies the central copy of $\Gamma_\ell$ in $u'|_U$. To summarize: $a$ occupies a copy of $\Gamma_\ell$, which is contained in $U$ a copy of $\Gamma_{2R+\ell}$, which is contained in $A$ a copy of $\Gamma_k$, which is contained in a copy of $\Gamma_j$ occupied by a pattern $u$, which is contained in $\Gamma_n$, the shape of $w$. Note that since $j > 5k$ and $j \geq \frac{n}{2} > 2.5t$, $j > k + 2t$, meaning that any standard replacement of $u$ agrees with $u$ on $\Gamma_j^{(t)}$.

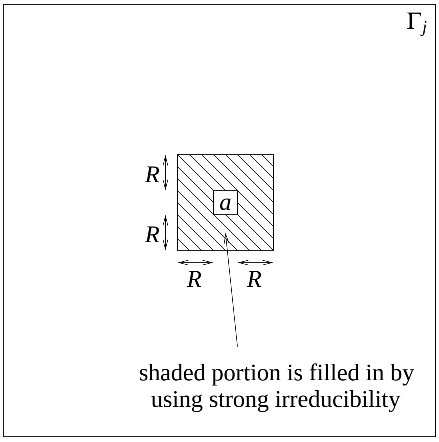


FIGURE 3. A standard replacement of $u$

*Definition 6.2.* A subpattern $u$ of $w$ has **Property A** if for every standard replacement $u'$ of $u$, there exists a pattern $v$ so that replacing $u$ by $u'$ in $v$ creates a new occurrence of $u$.

Clearly, if we can find $u$ a subpattern of $w$ which is large enough and does not have Property A, we will have proved Theorem 6.1. We will prove some general periodicity facts about patterns with Property A, for which we will need some definitions.

*Definition 6.3.* A pattern $u$ is **purely periodic with period size** $p$ if there exist positive integers $p_i \leq p$, $1 \leq i \leq d$ such that $u$ is periodic with respect to $p_i \vec{e_i}$ for every $i$.

*Definition 6.4.* For any $j$, a **suboctant** of $\Gamma_j$ is defined to be any copy of $\Gamma_{\frac{j-3k}{2}}$ which shares a corner with $\Gamma_j \setminus \Gamma_j^{(k)}$ and is contained in $\Gamma_j \setminus \Gamma_j^{(k)}$. A **superoctant** of $\Gamma_j$ is defined to be any copy of $\Gamma_{\frac{j+k}{2}}$ which shares a a corner with $\Gamma_j$ and is contained in $\Gamma_j$.

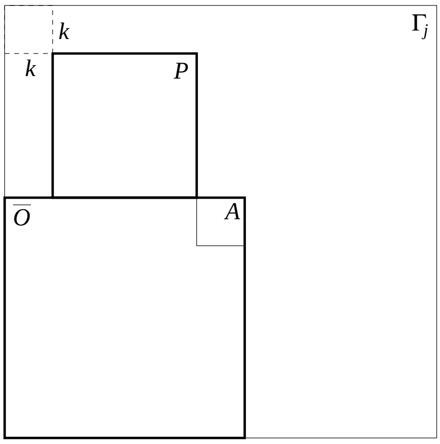


FIGURE 4. A suboctant $P$ and superoctant $\overline{O}$ of $\Gamma_j$

We note that any of these objects has an obvious associated corner of $\Gamma_j$. (For superoctants, this is the corner shared with $\Gamma_j$, and for suboctants, it is the closest corner of $\Gamma_j$.)

Note that the existence of a standard replacement $u'$ of $u$ with the property that a replacement of $u$ by $u'$ can create a new occurrence of $u$ yields information about periodicity of $u$. Say that the replacement occurs at $\Gamma_j$, and that the new occurrence occupies $S$ a copy of $\Gamma_j$. Define $\vec{v} := S - \Gamma_j$. Then since $\Gamma_j$ can be filled with $u'$ while $S$ is filled with $u$, and since $u$ and $u'$ differ on only a very small set, $u$ must be very nearly periodic with respect to $\vec{v}$. However, since the occurrence of $u$ at $S$ did not exist before the replacement, $u$ must not actually be periodic with respect to $\vec{v}$. The following lemma is just a stronger, more technical version of this phenomenon.

LEMMA 6.5. *If $n > N_{0.75}$, then for any $j \in [\frac{n}{2}, n]$ and any subpattern $u \in L_{\Gamma_j}(X)$ of $w$ with Property A, there exists a suboctant $P$ of $\Gamma_j$, contained in a superoctant $\overline{O}$ of $\Gamma_j$, such that $u|_P$ is purely periodic with period size less than $j^{\frac{1}{3}}$, but $u|_{\overline{O}}$ is not purely periodic with period size less than $2j^{\frac{1}{3}}$.*

*Proof.* Fix a subpattern $u \in L_{\Gamma_j}$ of $w$, and assume that $u$ has Property A. Define $k = k(j) = 3(2R + \ell)\lceil j^{1-\frac{1}{3d}} \rceil$ and denote by $A$ the central copy of $\Gamma_k$ in $\Gamma_j$ as above. Then for every standard replacement $u'$ of $u$, there exists a copy of $\Gamma_j$ which overlaps $\Gamma_j$, and which could be filled with $u$ at the same time that $\Gamma_j$ is filled with $u'$. We use $M$ to denote the set (possibly with repetitions) of these copies of $\Gamma_j$, and say that every $S \in M$ is associated to both the obvious standard replacement $u'$ of $u$ and to the copy of $\Gamma_{2R+\ell}$ on which $u'$ and $u$ disagree.

Since all standard replacements are created by changing $u$ only on letters in a particular copy of $\Gamma_{2R+\ell}$ within $\Gamma_j$, it must be the case that every element of $M$ contains some portion of its associated copy of $\Gamma_{2R+\ell}$. (Otherwise, the supposed "new" occurrence of $u$ would have already existed before the replacement.) Also, for every standard replacement $u'$, $a$ is a subpattern of $u'|_A$. Since $a$ is not a subpattern of $u$, none of the elements in $M$ entirely contains its associated copy of $\Gamma_{2R+\ell}$. We first make the claim that $M$ contains no repeated elements, i.e. that two distinct copies of $\Gamma_{2R+\ell}$ must have distinct elements of $M$ associated to them.

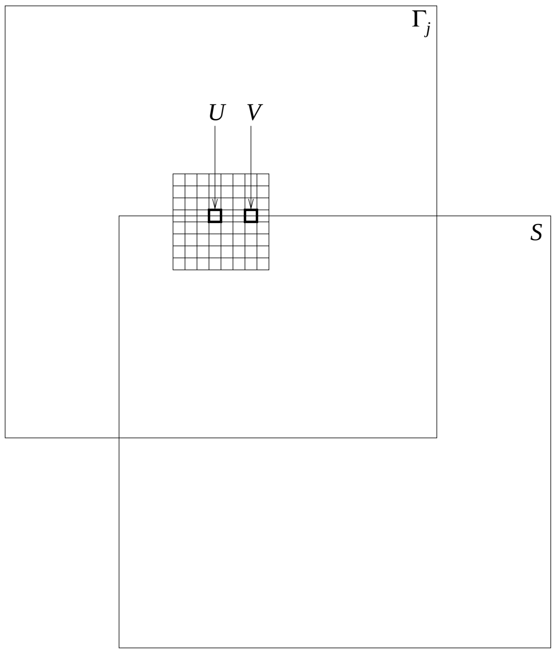


FIGURE 5. An element $S$ of $M$ associated to two standard replacements

In Figure 5, suppose for a contradiction that $S$ is associated to the standard replacements corresponding to each of $U$ and $V$, the two highlighted copies of $\Gamma_{2R+\ell}$. This means that there exists a pattern $v \in L_{\Gamma_j \cup S}(X)$ with $v|_S = u$ and $v|_{\Gamma_j} = u'$, where $u'$ is a standard replacement of $u$ associated to $U$. (This is the pattern that appears after $v|_{\Gamma_j}$ has been changed from $u$ to $u'$, and a new occurrence of $u$ has appeared at $S$.) $u$ and $u'$ must not agree on $U \cap S$; if they did, then the occurrence of $u$ in $S$ would have already existed before the replacement. Therefore, $v|_{\Gamma_j \setminus U} = u|_{\Gamma_j \setminus U}$ and $v|_{U \cap S} \neq u|_{U \cap S}$. Similarly, there exists a pattern

$v' \in L_{\Gamma_j \cup S}(X)$ with $v'|_S = u$, $v'|_{\Gamma_j \setminus V} = u|_{\Gamma_j \setminus V}$, and $v'|_{V \cap S} \neq u|_{V \cap S}$. Since $U$ and $V$ are disjoint, this implies that $v|_{U \cap S} \neq u|_{U \cap S}$ and $v'|_{U \cap S} = u|_{U \cap S}$, and so $v|_{U \cap S} \neq v'|_{U \cap S}$. But $v|_S = v'|_S = u$, and since $U \cap S \subseteq S$ we have a contradiction.

Now since there were more than $2^d j^{d-\frac{1}{3}}$ standard replacements of $u$, we know that $M$ consists of more than $2^d j^{d-\frac{1}{3}}$ distinct copies of $\Gamma_j$. Since every $S \in M$ has nonempty intersection with $\Gamma_j$ (and therefore contains some corner of $\Gamma_j$), there is some corner $\vec{q}$ of $\Gamma_j$ which is contained in more than $j^{d-\frac{1}{3}}$ of the copies of $\Gamma_j$ in $M$. Denote by $\vec{r}$ the opposite corner of $\Gamma_j$ to $\vec{q}$, and denote by $\overline{O}$ and $\overline{O'}$ the superoctants of $\Gamma_j$ containing $\vec{q}$ and $\vec{r}$ respectively. We wish to show that it is not true that both $u|_{\overline{O}}$ and $u|_{\overline{O'}}$ are purely periodic with period sizes less than $2j^{\frac{1}{3}}$.

We assume for a contradiction that $u|_{\overline{O}}$ and $u|_{\overline{O'}}$ are purely periodic with period sizes less than $2j^{\frac{1}{3}}$, and without loss of generality, we assume that $\overline{O}$ is the least superoctant lexicographically of $\Gamma_j$, i.e. $\vec{q} = \vec{\mathbf{1}}$. (This is without loss of generality because one can make this assumption simply by reflecting $\Gamma_j$ several times, which does not affect our proof.) Choose any $S \in M$ containing $\vec{q}$. In Figure 6, $u|_{\overline{O}}$ is purely periodic with periods $q_i\vec{e_i}$, $u|_{\overline{O'}}$ is purely periodic with periods $r_i\vec{e_i}$, and $U$ is the copy of $\Gamma_{2R+\ell}$ on which $u$ is changed to make the standard replacement of $u$ associated to $S$.

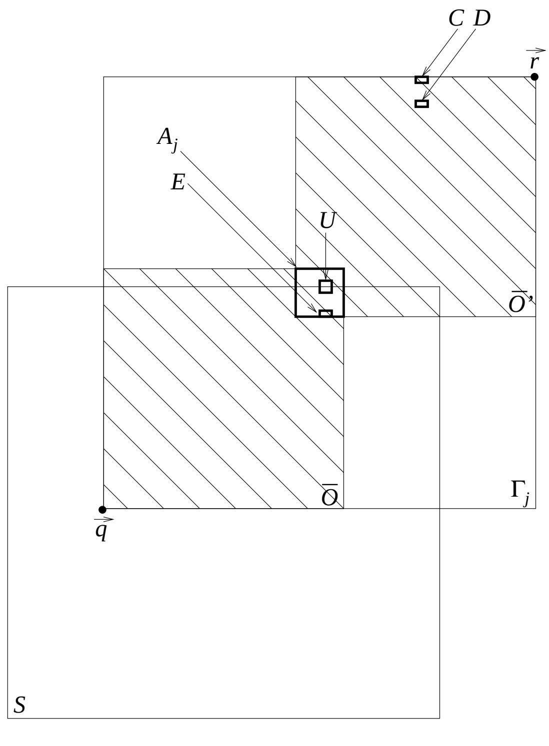


FIGURE 6. An element $S$ of $M$ which contains $\vec{q}$

Note that $u|_{\overline{O} \cup \overline{O'}}$ is periodic with respect to $q_i r_i \vec{e_i}$ for $1 \leq i \leq d$, and that $q_i, r_i < 2j^{\frac{1}{3}}$. We have denoted by $C$ the region $(U \cap S) + (\Gamma_j - S)$, i.e. $C$ in $\Gamma_j$ corresponds to $U \cap S$ in $S$. Since $\vec{q} = \vec{\mathbf{1}} \in S$, the vector $\Gamma_j - S$ has all nonnegative entries. Therefore, since $U \cap S \in \overline{O'}$, $C \subset \overline{O'}$ as well, and for all $1 \leq i \leq d$, $C$ consists of points whose $i$th coordinate is between $j - k + 1$ and $j$. We then choose $\vec{u}$ a multiple of $q_i r_i \vec{e_i}$ such that $C + \vec{u}$ is also contained in $\overline{O'}$ and $U + \vec{u}$ is a subset of $\overline{O}$ which is disjoint from $U$. To do this, we need some multiple of $q_i r_i$ which is greater than $k$, but still less than $\frac{j-k}{2}$. We know that such a multiple exists since $q_i r_i < 4j^{\frac{2}{3}}$, $k > 4j^{\frac{2}{3}}$, and $\frac{j-k}{2} > 2k$. (The last inequality is true since $n > N_{0.75}$.) Define $D = C + \vec{u}$ and $E = (U \cap S) + \vec{u}$. Since $S \in M$, there exists a pattern $y \in L_{\Gamma_j \cup S}(X)$ where $y|_S = u$, and $y|_{\Gamma_j}$ is a standard replacement for $u$ such that $y|_{\Gamma_j \setminus U} = u|_{\Gamma_j \setminus U}$ and $y|_{U \cap S} \neq u|_{U \cap S}$. Since $U \cap S$ in $S$ corresponds to $C$ in $\Gamma_j$, and since $y|_S = u$, it must be the case that $y|_{U \cap S} = u|_C$. By periodicity of $u|_{\overline{O'}}$ with respect to $\vec{u}$, $u|_C = u|_D$. Since $D$ in $\Gamma_j$ corresponds to $E$ in $S$, and since $y|_S = u$, $u|_D = y|_E$. Since $y|_{\Gamma_j \setminus U} = u|_{\Gamma_j \setminus U}$ and $E$ and $U$ are disjoint, $y|_E = u|_E$. Finally, by periodicity of $u|_{\overline{O}}$ with respect to $\vec{u}$, $u|_E = u|_{U \cap S}$. But we have then shown that $y|_{U \cap S} = u|_{U \cap S}$, a contradiction. Therefore, it is not the case that $u|_{\overline{O}}$ and $u|_{\overline{O'}}$ are purely periodic with period sizes less than $2j^{\frac{1}{3}}$.

Denote by $P$ and $P'$ the suboctants contained in $\overline{O}$ and $\overline{O'}$ respectively. Recall that $\vec{q}$ is the corner of $\Gamma_j$ shared by $\Gamma_j$ and $\overline{O}$ and $\vec{r}$ is the corner of $\Gamma_j$ opposite $\vec{q}$. It is clear upon observation that for any element $S$ of $M$ which contains $\vec{q}$, the corner of $S$ corresponding to $\vec{q}$ in $\Gamma_j$ must be contained in $\Gamma_j$. Fix any $\vec{e_i}$ in $\mathbb{Z}^d$. $\Gamma_j$ can be partitioned into the $j^{d-1}$ sets $\{\vec{w} + m\vec{e_i}\}_{0 \leq m < j}$ where $\vec{w}$ ranges over all $\vec{w} \in \Gamma_j$ with $\vec{w}_i = 1$, and there are more than $j^{d-\frac{1}{3}}$ distinct points in $\Gamma_j$ which are corners of elements of $M$ which correspond to $\vec{q}$ in $\Gamma_j$. By the pigeonhole principle, this implies that one of the sets $\{\vec{w} + m\vec{e_i}\}_{0 \leq m < j}$ contains more than $j^{\frac{2}{3}}$ of these points. Again by the pigeonhole principle, this implies that there are two elements of $M$ which contain $\vec{q}$, call them $S$ and $S'$, such that $S' - S$ is a multiple of $\vec{e_i}$ whose length is less than $j^{\frac{1}{3}}$. We make the notation $\vec{v_i} := S' - S$. We now claim that $u|_P$ and $u|_{P'}$ are periodic with respect to $\vec{v_i}$. We assume without loss of generality that $\vec{q} = \vec{\mathbf{1}}$ (meaning that $\vec{r} = j\vec{\mathbf{1}}$) and that $\vec{v_i}$ is a negative multiple of $\vec{e_i}$.

In Figure 7, $\vec{t}$ is any point of $\Gamma_j$ such that $\vec{t} + \vec{v_i}$ is also in $\Gamma_j$, $\vec{p}$ is the corner of $S$ corresponding to $\vec{r}$ in $\Gamma_j$, and $\vec{v'} = \vec{t} - (\vec{p} + \vec{v_i})$. We define $\vec{t'}$ to be $\vec{r} + \vec{v'}$, giving us two new points $\vec{t'}$ and $\vec{t'} + \vec{v_i}$. Since $S, S' \in M$, each may be filled with $u$ when $\Gamma_j$ is filled with the correct standard replacement of $u$. Denote by $U$ the copy of $\Gamma_{2R+\ell}$ on which $u$ is altered to make the standard replacement which allows $S$ to be filled with $u$, and by $V$ the copy of $\Gamma_{2R+\ell}$ on which $u$ is altered to make the standard replacement which allows $S'$ to be filled with $u$.

We will now show that if $\vec{t}, \vec{t} + \vec{v_i}, \vec{t'}, \vec{t'} + \vec{v_i} \in \Gamma_j \setminus (U \cup V)$, then $u(\vec{t}) = u(\vec{t} + \vec{v_i})$ and $u(\vec{t'}) = u(\vec{t'} + \vec{v_i})$. Assume that $\vec{t}, \vec{t} + \vec{v_i}, \vec{t'}, \vec{t'} + \vec{v_i} \in \Gamma_j \setminus (U \cup V)$. Then by noting that $S$ can be filled with $u$ when $\Gamma_j$ is filled with a standard replacement of $u$ which agrees with $w_j$ outside of $U \cup V$, we can infer that $u(\vec{t}) = u(\vec{t'} + \vec{v_i})$ (this is because $\vec{t} \in S$ corresponds to $\vec{t'} + \vec{v_i} \in \Gamma_j$), and by noting that $S'$ can be filled with $u$ when $\Gamma_j$ is filled with a standard replacement of $w_j$ which agrees with $w_j$

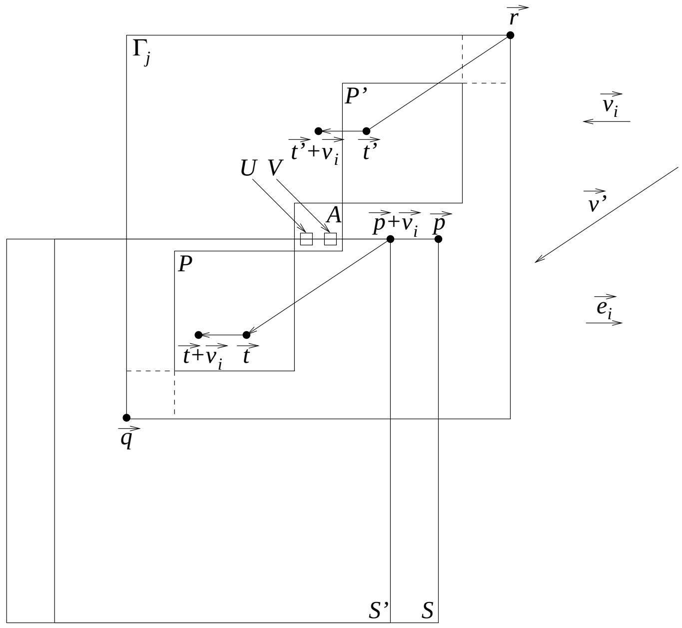


FIGURE 7. Elements $S, S'$ of $M$ whose difference $\vec{v_i}$ is a multiple of $\vec{e_i}$

outside of $U \cup V$, we can infer that $u(\vec{t}) = u(\vec{t'})$ and $u(\vec{t} + \vec{v_i}) = u(\vec{t'} + \vec{v_i})$. (This is because $\vec{t} \in S'$ corresponds to $\vec{t'} \in \Gamma_j$, and $\vec{t} + \vec{v_i} \in S'$ corresponds to $\vec{t'} + \vec{v_i} \in \Gamma_j$.) This implies that $u(\vec{t}) = u(\vec{t} + \vec{v_i})$ and $u(\vec{t'}) = u(\vec{t'} + \vec{v_i})$.

We claim that if $\vec{t}, \vec{t} + \vec{v_i} \in P$, then $\vec{t'}, \vec{t'} + \vec{v_i} \in \Gamma_j \setminus (U \cup V)$. Suppose that $\vec{t}, \vec{t} + \vec{v_i} \in P$. Since $S'$ contains some portion of $V$, but does not contain all of $V$, it also holds that $S'$ contains some portion of $A$, but not all of $A$. Therefore, some coordinate of $\vec{p} + \vec{v_i}$ is between $\frac{j-k}{2} + 1$ and $\frac{j+k}{2}$. The fact that $S'$ has nonempty intersection with $A$ also implies that all coordinates of $\vec{p} + \vec{v_i}$ are between $\frac{j-k}{2} + 1$ and $j$, since all coordinates of all elements of $A$ are at least $\frac{j-k}{2} + 1$. Since $\vec{t} \in P$, every coordinate of $\vec{t}$ is between $k + 1$ and $\frac{j-k}{2}$. This implies that every coordinate of $\vec{v'}$ is nonpositive and at least $-j + 1$, and also that one coordinate of $\vec{v'}$ is at least $-\frac{j-k}{2} + 1$. Then, $\vec{t'} = \vec{r} + \vec{v'} = j\mathbf{1} + \vec{v'}$ is in $\Gamma_j$, and has one coordinate at least $\frac{j+k}{2} + 1$, and therefore does not lie in $A$, and so is not in $U$ or $V$. Since $\vec{t} + \vec{v_i} \in P$, the same argument shows that $\vec{t'} + \vec{v_i}$ is in $\Gamma_j$, but does not lie in $U$ or $V$. Then, since $\vec{t}, \vec{t} + \vec{v_i}, \vec{t'}, \vec{t'} + \vec{v_i} \in \Gamma_j \setminus (U \cup V)$, by the previous paragraph, we know that $u(\vec{t}) = u(\vec{t} + \vec{v_i})$. Since this is true for any $\vec{t}, \vec{t} + \vec{v_i} \in P$, $u|_P$ is periodic with respect to $\vec{v_i}$.

A similar argument shows that if $\vec{t'}, \vec{t'} + \vec{v_i} \in P'$, then $\vec{t}, \vec{t} + \vec{v_i} \in P$, and is left to the reader. Again, since $\vec{t}, \vec{t} + \vec{v_i}, \vec{t'}, \vec{t'} + \vec{v_i} \in \Gamma_j \setminus (U \cup V)$, we know that $u(\vec{t'}) = u(\vec{t'} + \vec{v_i})$. Since this is true for any $\vec{t'}, \vec{t'} + \vec{v_i} \in P'$, $u|_{P'}$ is also periodic with respect to $\vec{v_i}$.

Since the above is true for all $i$, and since the length of $\vec{v_i}$ is less than $j^{\frac{1}{3}}$, $u|_P$ and $u|_{P'}$ are purely periodic with period sizes less than $j^{\frac{1}{3}}$. Clearly then one of the pairs $P, \overline{O}$ or $P', \overline{O'}$ satisfies the conditions of the lemma.

□

*Proof of Theorem 6.1.* Fix $d > 1$ and $X$ a strongly irreducible $\mathbb{Z}^d$ shift of finite type with uniform filling length $R$ containing more than one point. Choose any $n > N_{0.75}$ and $w \in L_{\Gamma_n}(X)$. If $w$ does not have Property A, then take $w' = w$, and we are done. So, we assume that $w$ has Property A, and recall that $\ell = \ell(n) = \left\lceil \left(\frac{d \ln n}{h^{top}(X)}\right)^{\frac{1}{d}} \right\rceil + 1$. Define $N_1 > \max(2^{4d}, N_{0.75})$ so that $n > N_1$ implies $4 \cdot 2^{d+2} 3^{2^d - 1} k(n) = 4 \cdot 2^{d+2} 3^{2^d} (2R + \ell) \lceil n^{1 - \frac{1}{3d}} \rceil < n^{1 - \frac{1}{4d}}$. Assume that $n > N_1$.

We will be inductively creating a sequence $w_j$ of subpatterns of $w$ with shape $\Gamma_{n_j}$, and will show that one of the first $2^d$ must not have Property A. This construction has two distinct steps. The first step is to take $w_1$ a subpattern of $w$ which is purely periodic on a large central subcube.

By Lemma 6.5, there is a suboctant $P$ of $\Gamma_n$, with associated corner $\vec{q}$ of $\Gamma_n$, such that $w|_P$ is purely periodic with period size less than $n^{\frac{1}{3}}$. Define $m_1 = 4 \cdot 3^{2^d - 1} k(n)$. Define $B$ to be the copy of $\Gamma_{n - m_1 - k(n)}$ which has $\vec{q}$ as a corner, and define $B' = B \setminus B^{(k(n))}$. Take $w_1$ to be the subpattern of $w$ occupying $B'$. Then if we define $C_1$ to be the central copy of $\Gamma_{m_1}$ in $w_1$, then $w_1|_{C_1}$ is a subpattern of $w|_P$, and is therefore purely periodic with period size less than $n^{\frac{1}{3}}$. (The verification of this is relatively straightforward, and we leave it to the reader.) $w_1$ has shape $\Gamma_{n_1}$, where $n_1 = n - m_1 - 3k(n)$.

We will now follow a slightly different procedure to generate the subpatterns $w_j$ of $w$ for $j > 1$. First, define $m_j = 4 \cdot 3^{2^d - j} k(n)$ for $1 \leq j \leq 2^d + 1$. Suppose that $w_1, w_2, \ldots, w_j$ have all been defined, that $j \leq 2^d$, and that $w_j$ has Property A. For $1 \leq i \leq j$, denote by $n_i$ the size of $w_i$, define $k_i = k(n_i) = 3(2R + \ell) \lceil n_i^{1 - \frac{1}{3d}} \rceil$, and denote the central copies of $\Gamma_{k_i}$ and $\Gamma_{m_i}$ within $\Gamma_{n_i}$ by $A_i$ and $C_i$ respectively. Since $j \leq 2^d$, $m_i \geq k_i$ and so $A_i \subseteq C_i$ for $1 \leq i \leq j$. Also, we claim that $n_j \geq \frac{n}{2}$ for all $1 \leq j \leq 2^d$ (which is necessary to apply Lemma 6.5); this will be verified by the end of the proof.

We need one last definition:

*Definition 6.6.* For any $1 \leq j \leq 2^d$, a ***j*-hyperoctant** of $\Gamma_{n_j}$ is a subcube which shares corners with both $\Gamma_{n_j}$ and $C_j$, is contained in $\Gamma_{n_j}$, and contains $C_j$.

Note that since $A_i \subseteq C_i$ for $1 \leq i \leq j \leq 2^d$, every $i$-hyperoctant of $\Gamma_{n_i}$ contains a unique superoctant of $\Gamma_{n_i}$. For every $1 \leq i \leq j$, denote by $T_i$ the list of points $\vec{\epsilon} \in \{0,1\}^d$ so that the restriction of $w_i$ to the $i$-hyperoctant containing the corner $\vec{\mathbf{1}} + (n_i - 1)\vec{\epsilon}$ of $\Gamma_{n_i}$ is purely periodic with period size less than $n^{\frac{1}{3}}$. In a slight abuse of notation, we will also refer to $T_i$ as a list of $i$-hyperoctants of $\Gamma_{n_i}$, using the obvious correspondence between points in $\{0,1\}^d$ and $i$-hyperoctants of $\Gamma_{n_i}$. Note that if we denote by $Q_1$ the 1-hyperoctant of $\Gamma_{n_1}$ containing the vertex of $\Gamma_{n_1}$ closest to $\vec{q}$ in $\Gamma_n$, then $w_1|_{Q_1}$ is a subpattern of $w|_P$ and is purely periodic with period size less than $n^{\frac{1}{3}}$. Therefore, $T_1$ is nonempty.

We will choose $n_{j+1}$ and $w_{j+1} \in L_{\Gamma_{n_{j+1}}}(X)$ a subpattern of $w_j$ so that once one defines $T_{j+1}$ in the obvious way, $T_{j+1} \supsetneq T_j$. By Lemma 6.5, there is a suboctant $P_j$ of $\Gamma_{n_j}$ so that $w_j|_{P_j}$ is purely periodic with period size less than $n_j^{\frac{1}{3}}$, and if we denote by $\overline{O_j}$ the superoctant of $\Gamma_{n_j}$ containing $P_j$, then $w_j|_{\overline{O_j}}$ is not purely periodic with period size less than $2n_j^{\frac{1}{3}}$. (Since $n_j \geq \frac{n}{2}$, this implies that $w_j|_{\overline{O_j}}$ is also not purely periodic with period size less than $n^{\frac{1}{3}}$.) Define $Q_j$ to be the $j$-hyperoctant of $\Gamma_{n_j}$ containing $\overline{O}_j$; clearly $Q_j \notin T_j$. Since the size $m_j$ of $C_j$ is at least $4k(n)$, and since $k_j = k(n_j) \leq k(n)$, $P_j \cap C_j$ is a cube whose size is at least $\frac{m_j}{3} = m_{j+1}$. Make the notation $C'_{j+1} = P_j \cap C_j$. Define $B_{j+1}$ to be the largest subcube of $\Gamma_{n_j}$ so that $C'_{j+1}$ is central in $B_{j+1}$, $P_j$ is contained in $B_{j+1}$, and $P_j$ and $B_{j+1}$ share a corner. Denote the size of $B_{j+1}$ by $n_{j+1}$, and the size of $C'_{j+1}$ by $m'_{j+1}$. Then $m'_{j+1} \geq m_{j+1}$, so we can take $C''_{j+1}$ to be the copy of $\Gamma_{m_{j+1}}$ central in $C'_{j+1}$ (and therefore in $B_{j+1}$ as well.)

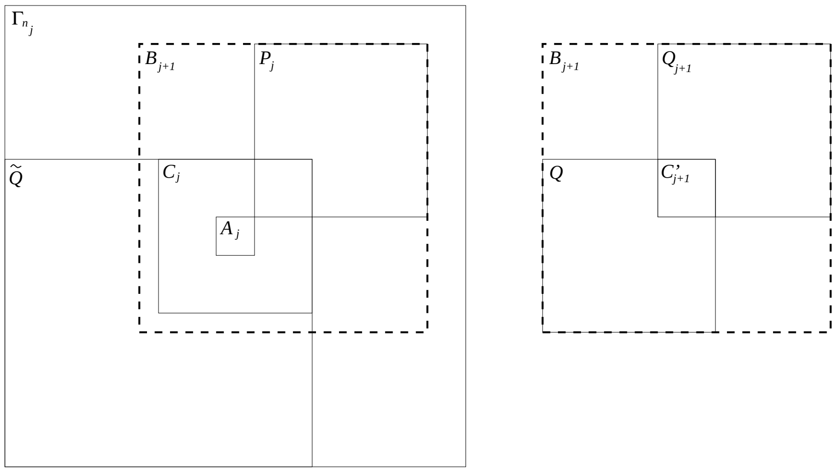


FIGURE 8. $\Gamma_{n_j}$ and $B_{j+1}$

The reader may verify that $n_{j+1} \geq n_j - (m_j + k_j)$. Define $w_{j+1} = w_j|_{B_{j+1}}$. To make the inductive construction work, we consider $w_{j+1}$ to have shape $\Gamma_{n_{j+1}}$ rather than the copy $B_{j+1}$, and denote by $C_{j+1}$ the subcube of $\Gamma_{n_{j+1}}$ corresponding to $C''_{j+1}$ in $B_j$. We can see from Figure 8 that for every $j+1$-hyperoctant $Q$ of $\Gamma_{n_{j+1}}$, $w_{j+1}|_Q$ is contained in $w_j|_{\widetilde{Q}}$ for the corresponding $j$-hyperoctant $\widetilde{Q}$ of $\Gamma_{n_j}$. Therefore, if we define $T_{j+1} \subseteq \{0,1\}^{2^d}$ to be the set of $j+1$-hyperoctants of $w_{j+1}$ which are purely periodic with period size less than $n^{\frac{1}{3}}$, then $T_{j+1} \supseteq T_j$. Also, if we denote by $Q_{j+1}$ the $j+1$-hyperoctant of $\Gamma_{n_{j+1}}$ corresponding to $Q_j$ in $\Gamma_{n_j}$, then $w_{j+1}|_{Q_{j+1}}$ is a subpattern of $w_j|_{P_j}$, and so since $w_j|_{P_j}$ was purely periodic with period size less than $n_j^{\frac{1}{3}}$, and since $n_j \leq n$, $Q_{j+1} \in T_{j+1}$. Since $Q_j \notin T_j$, this means that $|T_{j+1}| > |T_j|$.

Since $|T_j| \leq 2^d$ for all $j$, and since $T_1$ was nonempty by construction, it then cannot be the case that $w_1, w_2, \ldots, w_{2^d}$ all have Property A. Denote by $w'$ the first $w_j$ which does not have Property A. Then $w'$ is obtained by truncating $w$ at most $2^d$ times, where the first truncation reduces the size by $m_1 + 3k(n)$, and the $j$th truncation reduces the size by at most $k_j + m_j \leq m_1 + 3k(n)$ for $j > 1$. Since $k(n) < m_1$, the size $m$ of $w'$ is at least $n - 2^d(4m_1) = n - 4 \cdot 2^{d+2}3^{2^d-1}k(n)$. Then by our original assumption about the size of $n$, $m > n - n^{1-\frac{1}{4d}}$. (Since we assumed $n > 2^{4d}$, this also shows that $m > \frac{n}{2}$, and therefore that every $n_j$ is greater than $\frac{n}{2}$, as claimed earlier.) Since $w'$ does not have Property A, there exists $w''$ a standard replacement of $w'$ such that replacing $w'$ by $w''$ in some $x \in X$ cannot possibly create a new occurrence of $w'$. Thus, Theorem 6.1 is proved.

□

It is natural to wonder why we go to the trouble of dealing with subpatterns in the statement of Theorem 6.1; i.e., is it possible, instead of dealing with the intermediate step of taking a subpattern of $w$, to just choose $w' = w$? The answer is that there is an example of a strongly irreducible SFT $X$ and arbitrarily large patterns $w \in L(X)$ for which this is impossible.

Proposition 6.7. *For any $d$, there exists a strongly irreducible $\mathbb{Z}^d$ SFT $X$ with the property that for any $n > 1$, there exists $w \in L_{\Gamma_n}(X)$ such that for any $w'' \in L_{\Gamma_n}(X)$ with $w'' \neq w$, a replacement of $w$ by $w''$ could create a new occurrence of $w$.*

*Proof.* For any $d$ and any $n > 1$, take $X$ to be the full shift $\{0,1\}^{\mathbb{Z}^d}$, and define $w \in L_{\Gamma_n}(X)$ by $w(\vec{\mathbf{1}}) = 1$, and $w(\vec{v}) = 0$ if $\vec{v} \in \Gamma_n \setminus \{\vec{\mathbf{1}}\}$. We claim that for any $w'' \in L_{\Gamma_n}(X)$, $w'' \neq w$, there exists $x \in X$ such that $x|_{\Gamma_n} = w$, and replacing $x|_{\Gamma_n}$ by $w''$ creates a new occurrence of $w$. Suppose that $w''$ contains a 1, i.e. there is some $\vec{v} \in \Gamma_n$ such that $w''(\vec{v}) = 1$. Since $w'' \neq w$, we can assume that $\vec{v} \neq \vec{\mathbf{1}}$. Consider $x \in X$ defined by taking $x(\vec{\mathbf{1}}) = 1$ and $x(\vec{v}) = 0$ for all other $\vec{v} \in \mathbb{Z}^d$. Then, $x|_{\Gamma_n} = w$. Create a new $x' \in X$ by replacing $x|_{\Gamma_n}$ by $w''$. Then, it is not hard to check that $x'|_{\Gamma_n + (\vec{v} - \vec{\mathbf{1}})} = w$, and that $x|_{\Gamma_n + (\vec{v} - \vec{\mathbf{1}})}$ is the pattern consisting of all 0s, and so not equal to $w$. This means that the replacement involved in changing $x$ to $x'$ created a new occurrence of $w$. The only other possibility is that $w''$ is the pattern consisting of all 0s, i.e. $w''(\vec{v}) = 0$ for all $\vec{v} \in \Gamma_n$. If this is the case, then define $x \in X$ by taking $x(0, 1, \ldots, 1) = x(1, 1, \ldots, 1) = 1$ and $x(\vec{v}) = 0$ for all other $\vec{v} \in \mathbb{Z}^d$. Again, $x|_{\Gamma_n} = w$. Create $x' \in X$ by replacing $x|_{\Gamma_n}$ by $w''$. Then, it is not hard to check that $x'|_{\Gamma_n - \vec{e_1}} = w$, and that $x|_{\Gamma_n - \vec{e_1}}$ had two 1s, and is thus not equal to $w$. This means that in this case also, the replacement involved in changing $x$ to $x'$ created a new occurrence of $w$. □

Proposition 6.7 shows that the extra step of taking the subpattern $w'$ of $w$ in Theorem 6.1 is in fact necessary. However, for a "typical" pattern $w$, there exists a $w''$ which has the desired replacement properties, without need of passing to a subpattern.

PROPOSITION 6.8. *For any $d > 1$ and any strongly irreducible $\mathbb{Z}^d$ SFT $X$ containing more than one point, if for any $n$ we denote by $E_n(X)$ the set of patterns $w \in L_{\Gamma_n}(X)$ for which it is possible to take $w' = w$ in Theorem 6.1, then for any ergodic measure $\widetilde{\mu}$ of maximal entropy on $X$, $\lim_{n\to\infty} \widetilde{\mu}(E_n(X)) = 1$.*

*Proof.* Fix $d > 1$ and $X$ a strongly irreducible $\mathbb{Z}^d$ SFT with uniform filling length $R$ containing more than one point. We claim that for sufficiently large $n$, if $w$ does not contain any pair of equal disjoint subpatterns with shape $\Gamma_{\lfloor\sqrt{n}\rfloor}$, then $w \in E_n(X)$. Consider any $n$ and any $w \in L_{\Gamma_n}(X)$ with the property just described. Define $\ell = \ell(n) = \left\lceil \left(\frac{d \ln n}{h^{top}(X)}\right)^{\frac{1}{d}} \right\rceil + 1$, and create a standard replacement $w''$ of $w$ by changing $w$ only on a central copy of $\Gamma_{2R+\ell}$, which we denote by $K$. Then $w|_{\Gamma_n \setminus K} = w''|_{\Gamma_n \setminus K}$.

Suppose that $w \notin E_n(X)$. Then a replacement of $w$ by $w''$ could create a new occurrence of $w$. Then there exist $x \in X$ and copies $T$ and $T'$ of $\Gamma_n$ such that $x|_T = w$, and if we define by $x'$ the element of $X$ created by replacing $x|_T$ by $w''$, then $x'|_{T'} = w$, but $x|_{T'} \neq w$. This means that $x'|_T = w''$ and $x'|_{T'} = w$. Since the central copy of $\Gamma_l$ in $w''$ is occupied by $a$, this implies that $\|T' - T\|_\infty > \frac{n-\ell}{2}$, or else $x'|_{T'} = w$ would contain this occurrence of $a$. However, in order for $x'|_{T'}$ to be a newly created occurrence of $w$, $T'$ must have nonempty intersection with $K + (T - \Gamma_n)$, and so $\|T' - T\|_\infty < \frac{n+(2R+\ell)}{2}$. By this upper bound on $\|T' - T\|_\infty$, $T \cap T'$ contains a cube of size at least $\frac{n-(2R+\ell)}{2}$, which must, for large $n$, contain a copy of $\Gamma_{\lfloor\sqrt{n}\rfloor}$ disjoint from the very small sets $K + (T - \Gamma_n)$ and $K + (T' - \Gamma_n)$. Denote this copy of $\Gamma_{\lfloor\sqrt{n}\rfloor}$ by $U$. Since $x'|_T = w''$, $x'|_U = w''|_{U-(T-\Gamma_n)}$. Since $U$ is disjoint from $K + (T - \Gamma_n)$, $w''|_{U-(T-\Gamma_n)} = w|_{U-(T-\Gamma_n)}$. Since $x'|_{T'} = w$, $x'|_U = w|_{U-(T'-\Gamma_n)}$. Therefore, $w|_{U-(T-\Gamma_n)} = w|_{U-(T'-\Gamma_n)}$. But, since $\|T' - T\|_\infty > \frac{n-\ell}{2}$, which is greater than $\sqrt{n}$ for sufficiently large $n$, $U - (T - \Gamma_n)$ and $U - (T' - \Gamma_n)$ are disjoint copies of $\Gamma_{\lfloor\sqrt{n}\rfloor}$, and so $w$ contains equal disjoint subpatterns with shape $\Gamma_{\lfloor\sqrt{n}\rfloor}$, which is a contradiction.

Therefore, as long as $w|_S \neq w|_T$ for any pair $S$ and $T$ of disjoint copies of $\Gamma_{\lfloor\sqrt{n}\rfloor}$ in $\Gamma_n$, $w \in E_n(X)$. This implies that

$$L_{\Gamma_n}(X) \setminus E_n(X) \subseteq \bigcup_{\vec{v} \in L_{\Gamma_{\lfloor\sqrt{n}\rfloor}}(X)} \left( \bigcup_{\vec{u'} \in [-n,n]^d, \|\vec{u'}\|_\infty > \sqrt{n}} \left( \bigcup_{\vec{u} \in [-n,n]^d} ([v] \cap ([v] + \vec{u'})) + \vec{u} \right)\right). \quad (10)$$

By Lemma 4.6, for any $\vec{v} \in L_{\Gamma_{\lfloor\sqrt{n}\rfloor}}(X)$, any $\vec{u'}$ with $\|\vec{u'}\|_\infty > \sqrt{n}$, and any ergodic measure of maximal entropy $\widetilde{\mu}$ on $X$,

$$\widetilde{\mu}([v] \cap ([v] + \vec{u'})) \leq \frac{1}{\left| L_{\left(\Gamma_{\lfloor\sqrt{n}\rfloor} \cup (\Gamma_{\lfloor\sqrt{n}\rfloor} + \vec{u'})\right) \setminus \left(\Gamma_{\lfloor\sqrt{n}\rfloor} \cup (\Gamma_{\lfloor\sqrt{n}\rfloor} + \vec{u'})\right)^{(R)}}(X) \right|}.$$

Since $\|\vec{u'}\|_\infty > \sqrt{n}$,

$$\big(\Gamma_{\lfloor\sqrt{n}\rfloor} \cup (\Gamma_{\lfloor\sqrt{n}\rfloor} + \vec{u'})\big) \setminus \big(\Gamma_{\lfloor\sqrt{n}\rfloor} \cup (\Gamma_{\lfloor\sqrt{n}\rfloor} + \vec{u'})\big)^{(R)} \\ = \big((\Gamma_{\lfloor\sqrt{n}\rfloor}) \setminus (\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big) \cup \Big(\big((\Gamma_{\lfloor\sqrt{n}\rfloor}) \setminus (\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big) + \vec{u'}\Big).$$

Since $\|\vec{u'}\|_\infty > \sqrt{n}$, $d\Big(\big((\Gamma_{\lfloor\sqrt{n}\rfloor}) \setminus (\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big), \big((\Gamma_{\lfloor\sqrt{n}\rfloor}) \setminus (\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big) + \vec{u'}\Big) > \sqrt{n} > R$ for sufficiently large $n$, and so by strong irreducibility,

$$\Big|L_{\big((\Gamma_{\lfloor\sqrt{n}\rfloor})\setminus(\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big)\cup\big(\big((\Gamma_{\lfloor\sqrt{n}\rfloor})\setminus(\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big)+\vec{u'}\big)}(X)\Big| \\ = \big|L_{\big((\Gamma_{\lfloor\sqrt{n}\rfloor})\setminus(\Gamma_{\lfloor\sqrt{n}\rfloor})^{(R)}\big)}(X)\big|^2,$$

which is at least $e^{2h^{top}(X)(\sqrt{n}-2R)^d}$ by Lemma 4.11. Combining this with shift-invariance of $\widetilde{\mu}$ and (10), we see that

$$\widetilde{\mu}(L_{\Gamma_n}(X) \setminus E_n(X)) \leq |L_{\Gamma_{\lfloor\sqrt{n}\rfloor}}(X)|(2n+1)^{2d}e^{-2h^{top}(X)(\sqrt{n}-2R)^d}.$$

Again by Lemma 4.11, $|L_{\Gamma_{\lfloor\sqrt{n}\rfloor}}(X)| \leq e^{h^{top(X)}(\sqrt{n}+R)^d}$. Therefore,

$$\widetilde{\mu}(L_{\Gamma_n}(X) \setminus E_n(X)) \leq (2n+1)^{2d}e^{-h^{top}(X)(2(\sqrt{n}-2R)^d-(\sqrt{n}+R)^d)},$$

which clearly approaches zero as $n \to \infty$.

□

## 7. *The proof of the lower bound in Theorem 1.2*

We use a different tactic to prove a lower bound for $h^{top}(X) - h^{top}(X_w)$. We in a sense proceed in the opposite way from the proof of the upper bound: we will, for $k > n$, define a map $\psi_k$ which sends any pattern in $L_{\Gamma_k}(X_w)$ to a subset of $L_{\Gamma_k}(X)$ such that for any $u \neq u' \in L_{\Gamma_k}(X_w)$, $\psi_k(u)$ and $\psi_k(u')$ are disjoint.

The $\psi_k$-image of any $u \in L_{\Gamma_k}(X_w)$ consists of patterns obtained by introducing some occurrences of $w$ into $u$. A natural way to define this map is by choosing a pattern $f'$ which agrees on the boundary of thickness $t$ with $w$ and replacing occurrences of $f'$ by $w$. We then choose an ergodic measure of maximal entropy $\widetilde{\mu}_w$ on $X_w$, and for small $\epsilon$, bound the size of images $\psi_k(u)$ from below for all $u \in A_{k,\epsilon,\widetilde{\mu}_w,f'}$. The disjointness of these images will yield a lower bound on $\frac{|L_{\Gamma_k}(X)|}{|A_{k,\epsilon,\widetilde{\mu}_w,f'}|}$, which will in turn yield the desired lower bound on $h^{top}(X) - h^{top}(X_w)$ as $\epsilon \to 0$ by Lemma 4.8. There are three issues with this approach which must be addressed.

Firstly, new occurrences of $w$ could be created in this process, which could cause some problems with overcounting (two different patterns in $L_{\Gamma_k}(X_w)$ could, after performing some replacements of $f'$ by $w$, become the same pattern.) This cannot be avoided by passing to a subpattern of $w$; for example, if $w$ is a block of 0s, then any subpattern of $w$ is also a block of 0s, and then any introduction of this subpattern could also introduce many new occurrences of itself. For this reason, the pattern which we will introduce occurrences of is not $w$ itself, but a pattern $f$ which consists of $w$ surrounded by a "shell" which admits very few periods. (The construction of this shell $w_{o,d}$ is a bit technical, and so is deferred to Section 8.)

Secondly, in order to bound $|\psi_k(u)|$ from below for "typical" $u$, we need a lower bound on the number of occurrences of $f'$ in $u$. This necessitates a lower bound on $\widetilde{\mu_w}([f'])$. The results of Section 4 are not useful for these estimates since the type of $X_w$ could be very large, maybe even equal to the size of $w$. We have no elegant workaround for this problem, and instead will just choose $f'$ to be any pattern which has larger-than-average measure with respect to $\widetilde{\mu}_w$.

Thirdly, if we wish to replace $f'$ by $f$, then $f'$ and $f$ must agree on the boundary of thickness $t$. However, we have no control over the boundary of thickness $t$ of $f'$, so this would force the lower bound in Theorem 1.2 to depend on $t$. We would like the bound to be independent of $t$, so we make a slight modification: $f'$ will be chosen to have size $2R$ larger than the size of $f$, and $\psi_k$ will actually introduce occurrences of $f$ in central subcubes of occurrences of $f'$. This way, no $t$ need appear.

Fix $d > 1$, $X$ a strongly irreducible $\mathbb{Z}^d$ shift of finite type $t$ with uniform filling length $R$ containing more than one point, and define $W = 8R + 14$. Fix any $n > (3d-3)W$ and $w \in L_{\Gamma_n}(X)$. Take $o$ to be the smallest integer such that $(o-4)W > n + 2R$. Then, $(o-4)W > (3d-3)W$, and so $o \geq 3d+2$. We also define $n' = oW + 2R$. By the definition of $o$, $n' \leq n + 4R + 5W \leq n + 44R + 70$. Take $\widetilde{\mu}_w$ to be any ergodic measure of maximal entropy on $X_w$. Since

$$\sum_{w' \in L_{\Gamma_{n'}}(X_w)} \widetilde{\mu}_w([w']) = \widetilde{\mu}_w\left(\bigcup_{w' \in L_{\Gamma_{n'}}(X_w)} [w']\right) = \widetilde{\mu}_w(X_w) = 1,$$

there must exist some pattern $f' \in L_{\Gamma_{n'}}(X_w)$ such that $\widetilde{\mu}_w([f']) \geq \frac{1}{|L_{\Gamma_{n'}}(X_w)|}$. We note that since $X_w \subset X$, $|L_{\Gamma_{n'}}(X_w)| \leq |L_{\Gamma_{n'}}(X)|$. By Lemma 4.11, $|L_{\Gamma_{n'}}(X)| \leq e^{h^{top}(X)(n'+R)^d}$. Therefore, $\widetilde{\mu}_w([f']) \geq e^{-h^{top}(X)(n'+R)^d}$.

In Figure 9, we show a pattern $f \in L_{\Gamma_{oW}}(X)$ constructed as follows: $\Gamma_{oW}^{(2W)}$ is filled with $w_{o,d}$ as constructed in Theorem 8.1 in Section 8, and the central copy of $\Gamma_n$ is filled with $w$. The important property of $w_{o,d}$ is that if two copies of $w_{o,d}$ overlap, their central (empty) portions of shape $\Gamma_{(o-4)W}$ are disjoint. The remaining shaded portion is filled using strong irreducibility to create a pattern $f \in L(X)$.

We now describe our map $\psi_k$. Consider any $k > n'$ and $u \in L_{\Gamma_k}(X_w)$. Denote by $K$ the number of occurrences of $f'$ in $u$. We choose a set of disjoint occurrences of $f'$ using a simple algorithm: choose any occurrence of $f'$ to begin, call it $f'^{(1)}$. Less than $2^d n'^d$ occurrences of $f'$ overlap $f'^{(1)}$, and so we choose any occurrence of $f'$ which does not overlap $f'^{(1)}$, and call it $f'^{(2)}$. Choose any occurrence of $f'$ which does not overlap $f'^{(1)}$ or $f'^{(2)}$ and call it $f'^{(3)}$, and continue in this fashion. In this way, we can choose a set $Q$ of disjoint occurrences of $f'$ in $u$, where $|Q| \geq \frac{K}{2^d n'^d}$. For any $Q' \subseteq Q$, define a pattern $u'_{Q'}$ as follows. For each occurrence of $f'$ in $Q'$,

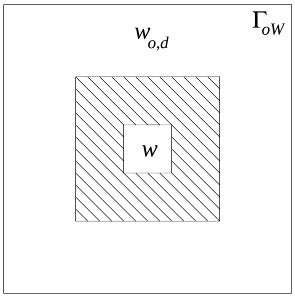


FIGURE 9. $f$

if its shape is $U$ a copy of $\Gamma_{n'}$, use strong irreducibility to replace the central copy of $\Gamma_{n'-2R}$ in $U$ by $f$. Define $\psi_k(u) = \{u'_{Q'}\}_{Q' \subseteq Q}$. Then clearly $|\psi_k(u)| \geq 2^{\frac{K}{2^d n'^d}}$.

Now, fix any $\epsilon > 0$. We will consider the restriction of $\psi_k$ to $A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$. (Definition 4) For any $u \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$, $K > k^d(e^{-h^{top}(X)(n'+R)^d} - \epsilon)$, and so the cardinality of $\psi_k(u)$ is then greater than $2^{k^d 2^{-d} n'^{-d}(\exp(-h^{top}(X)(n'+R)^d)-\epsilon)}$.

LEMMA 7.1. *For any $u \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$ and $u' \in \psi_k(u)$, $u'|_U = f$ for a copy $U$ of $\Gamma_{n'-2R}$ if and only if $U$ is centrally located in a copy of $\Gamma_{n'}$ on which a replacement was made in changing $u$ to $u'$.*

*Proof.* Suppose for a contradiction that for some $u \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$ and $u' \in \psi_k(u)$, $u'$ contains an occurrence of $f$ which occupies a copy of $\Gamma_{n'-2R}$, which we call $B'$, which was not central in a copy of $\Gamma_{n'}$ which was occupied by one of the replaced occurrences of $f'$ in $u$. Denote by $B''$ the central copy of $\Gamma_n$ of $B'$ and by $B$ the copy of $\Gamma_{n'}$ in which $B'$ is central. Then $u'|_{B''} = w$. Since $u \in L(X_w)$, $u|_{B''} \neq w$. This implies that one of the replacements made had nonempty intersection with $B''$, otherwise $u|_{B''} = u'|_{B''} = w$, a contradiction to the fact that $u \in L(X_w)$. Choose a copy of $\Gamma_{n'}$ where such a replacement was made, and denote it by $C$ and its central copy of $\Gamma_{n'-2R}$ by $C'$. By our hypothesis, $B$ was not one of the replaced copies of $\Gamma_{n'}$, and so $B \neq C$. Since all of the replacements made were disjoint, $u'|_{C'} = f$. We know that $u'|_{B'} = f$ as well. Since $C \cap B'' \neq \varnothing$, $\|B - C\|_\infty \leq n + \frac{n'-n}{2} = \frac{n'+n}{2}$. However, $\frac{n'+n}{2} = \frac{n'-oW}{2} + \frac{oW+n}{2}$. Since $n' = oW + 2R$, $\frac{n'-oW}{2} + \frac{oW+n}{2} = R + \frac{oW+n}{2}$. It was part of the definition of $o$ that $(o-4)W > n + 2R$, so $R < \frac{(o-4)W-n}{2}$. Therefore, $R + \frac{oW+n}{2} < \frac{(o-4)W-n}{2} + \frac{oW+n}{2} = (o-2)W$. We have then shown that $\|B - C\|_\infty < (o-2)W$. Since $B'$ is central in $B$ and $C'$ is central in $C$, $B' - C' = B - C$ and so $\|B' - C'\|_\infty < (o-2)W$ as well. We make one more notation: denote $E' = B'^{(2W)}$ and $F' = C'^{(2W)}$. Since $B \neq C$, $E' \neq F'$. Then, since $u'|_{B'} = u'|_{C'} = f'$, $u'|_{E'} = u'|_{F'} = w_{o,d}$. Also, $E' - F' = B' - C'$, so $\|E' - F'\|_\infty < (o-2)W$, which implies that $E'$ has nonempty intersection with the central copy of $\Gamma_{(o-4)W}$ in $F'$, a contradiction to the defining property of $w_{o,d}$. Our original assumption was then wrong, and $u'|_U = f$ only if $U$ is centrally located in a copy of $\Gamma_{n'}$ on which a replacement was made in changing $u$ to $u'$.

The converse is easy: since all replacements were disjoint, of course $u'|_U = f$ for every $U$ centrally located in a copy of $\Gamma_{n'}$ where a replacement was made.

□

LEMMA 7.2. *For any $u, v \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$, if $u \neq v$, then $\psi_k(u) \cap \psi_k(u') = \varnothing$.*

*Proof.* Fix any $k > n'$ and $\epsilon > 0$. Fix $u \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$ and $u' \in \psi_k(u)$. By Lemma 7.1, $u'$ uniquely determines the set $Q'$ of occurrences of $f'$ in $u$ which were replaced to make $u'$. However, trivially $u = u'$ outside of the occurrences of $f'$ in $Q$, and so $u$ is uniquely determined by $u'$. This means that for $u, v \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$, if $u \neq v$, then $\psi_k(u) \cap \psi_k(v) = \varnothing$.

□

*Proof of Theorem 1.2.* For any $u \in A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)$, we know that $|\psi_k(u)| \geq 2^{k^d 2^{-d} n'^{-d}(\exp(-h^{top}(X)(n'+R)^d)-\epsilon)}$. Then by Lemma 7.2,

$$|L_{\Gamma_k}(X)| > 2^{k^d 2^{-d} n'^{-d}(\exp(-h^{top}(X)(n'+R)^d)-\epsilon)} |A_{k,\epsilon,\widetilde{\mu}_w,f'}(X_w)|.$$

Take natural logarithms of both sides, divide by $k^d$, and let $k \to \infty$. By Lemma 4.8 and the definition of topological entropy, this yields

$$h^{top}(X) \geq (\ln 2)(2^{-d} n'^{-d}(e^{-h^{top}(X)(n'+R)^d} - \epsilon)) + h^{top}(X_w).$$

Since $\epsilon$ was arbitrary, we allow it to approach zero, and so

$$h^{top}(X) - h^{top}(X_w) \geq (\ln 2) 2^{-d} n'^{-d} e^{-h^{top}(X)(n'+R)^d}.$$

Since $d > 1$, there exists $N_6 > N_5$ so that for $n > N_6$, this is less than or equal to

$$h^{top}(X) - h^{top}(X_w) \geq e^{-h^{top}(X)(n'+R+1)^d},$$

and by replacing $n'$ by its maximum possible value $n + 44R + 70$,

$$h^{top}(X) - h^{top}(X_w) \geq \frac{1}{e^{h^{top}(X)(n+44R+71)^d}}.$$

Combining this with the already proven upper bound on $h^{top}(X) - h^{top}(X_w)$, we see that for $n > N_6$,

$$\frac{1}{e^{h^{top}(X)(n+44R+71)^d}} \leq h^{top}(X) - h^{top}(X_w) \leq \frac{D_X}{e^{h^{top}(X)(n-2R)^d}}.$$

□

## 8. *The construction of $w_{j,d}$*

The main goal of this section is to prove the following theorem, which was necessary in Section 7.

THEOREM 8.1. *For any $d > 1$, let $X$ be a strongly irreducible $\mathbb{Z}^d$ subshift with uniform filling length $R$ and more than one point, and let $W = 8R + 14$. Then for all sufficiently large $j$, there is a pattern $w_{j,d} \in L_{\Gamma^{(2W)}_{jW}}(X)$ such that there cannot exist two overlapping occurrences of $w_{j,d}$ where one has nonempty intersection with the empty central copy of $\Gamma_{(j-4)W}$ in the other.*

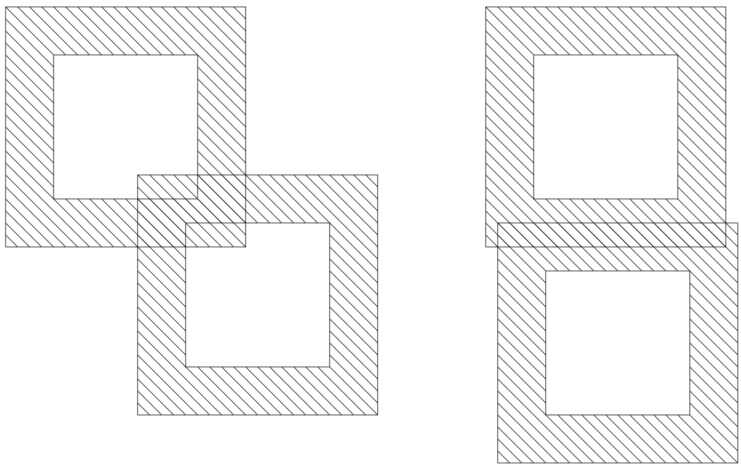

FIGURE 10. Disallowed (left) and allowed (right) pairs of overlapping $w_{j,d}$

The outline of the proof of Theorem 8.1 is as follows. First, for any $d > 1$, we construct a pattern $b_{j,d}$ with shape $\Gamma^{(2)}_j$ on the alphabet $\{0, 1\}$ with a similar property to the one $w_{j,d}$ has in the statement of Theorem 8.1. We will then use $b_{j,d}$ to construct $w_{j,d}$ by "blowing up" $b_{j,d}$ by a factor of $W$, i.e. using each letter in $b_{j,d}$ to determine a pattern occupying a copy of $\Gamma_W$ in $w_{j,d}$. For this, we need two types of patterns in $L_{\Gamma_W}(X)$, to correspond to 0s and 1s in $b_{j,d}$. We do this by finding a small pattern $y \in L(X)$ so that $X_y$ is nonempty, and then 0s and 1s in $b_{j,d}$ correspond to patterns in $L_{\Gamma_W}(X)$ with no occurrences of $y$, and many occurrences of $y$, respectively.

The first step in the construction of $b_{j,d}$ is the construction of an aperiodic pattern $a_{j,d-1}$.

LEMMA 8.2. *For any fixed $d$ and $j \geq 3d + 5$, there exists an aperiodic pattern $a_{j,d}$ in $\{0, 1\}^{\Gamma_j}$.*

*Proof.* Fix any $d$ and $j \geq 3d + 5$. For any $\vec{p} \in \Gamma_j$, define $a_{j,d}(\vec{p}) = 1$ if for some $1 \leq i \leq d - 1$, $p_i \notin \{1, j\}$. This leaves only $a_{j,d}(\vec{p})$ where each of $p_1, p_2, \ldots, p_{d-1}$ are 1 or $j$ to be defined. We think of this undefined portion as a set of $2^{d-1}$ onedimensional patterns of length $j$: for every $(p_1, p_2, \ldots, p_{d-1}) \in \{1, j\}^{d-1}$, we think of the yet-to-be-defined $a_{j,d}(p_1, p_2, \ldots, p_{d-1}, 1), a_{j,d}(p_1, p_2, \ldots, p_{d-1}, 2), \ldots,$ $a_{j,d}(p_1, p_2, \ldots, p_{d-1}, j)$ as the letters of a $j$-letter pattern. Now, to fill in these portions, we define $2^{d-1}$ $j$-letter patterns, which we will call $h_0, h_1, \ldots, h_{2^{d-1}-1}$. We do this in the following way: the first four letters of any $h_i$ are defined to be 0001, and the final four letters are defined to be 0111. The $3d-3$ letters following the initial 0001 in any $h_i$ are defined as follows: concatenate $d-1$ three-letter patterns, determined by $i$'s $d-1$-digit binary expansion: each 0 in the binary expansion corresponds to the pattern 001, and each 1 in the binary expansion corresponds to 011. The remaining $j - (3d + 5)$ letters of $h_i$ which precede the final 0111 are alternating 0s and 1s, beginning with a 0.

An example should be helpful: suppose that $d = 3$ and $j = 18$. Then we create four 18-letter patterns $h_0$, $h_1$, $h_2$, and $h_3$. The initial four letters of $h_0$ are 0001. The next $3d - 3 = 6$ letters are dependent on the two-digit binary expansion of the subscript 0: since it is 00, the next six letters are 001001. The next $j - (3d + 5) = 4$ letters are alternating 0s and 1s beginning with a 0, i.e. 0101, and the final four letters are 0111. This gives $h_0 = 000100100101010111$. Using similar reasoning, we see that $h_1 = 000100101101010111$, $h_2 = 000101100101010111$, and $h_3 = 000101101101010111$.

We claim that the patterns $h_0, h_1, \ldots h_{2^{d-1}-1}$ have the property that no two may overlap each other, i.e. if the final $k$ letters of $h_i$ are equal to the initial $k$ letters of $h_{i'}$ for some $k > 0$, then $k = j$ and $i = i'$. Suppose not; then for some $k > 0$ and $0 \leq i, i' < 2^{d-1}$, the initial $k$ letters of $h_i$ are the same as the final $k$ letters of $h_{i'}$. If $k \in \{1, 2\}$, then the first $k$ letters of $h_i$ are 0 or 00, which cannot be the case since $h_{i'}$ does not end with either of those patterns. So, $k \geq 3$. Then the initial $k$ letters of $h_i$ begin with 000, and since the only place that 000 occurs in $h_{i'}$ is at the beginning, this implies that $k = j$, and since $\{h_i\}_{i=0}^{2^{d-1}-1}$ contains $2^{d-1}$ distinct patterns, that $i = i'$.

Define any bijection $\upsilon$ from $\{1, j\}^{d-1}$ to $[0, 2^{d-1}-1]$, and for any $\vec{p} = (p_1, p_2, \ldots, p_{d-1}) \in \{1, j\}^{d-1}$, define $a_{j,d}(p_1, p_2, \ldots, p_{d-1}, i') = h_{\upsilon(\vec{p})}(i')$ for $1 \leq i' \leq j$. We now make the claim that $a_{j,d}$ is aperiodic. Suppose not; then there exists $\vec{v}$ with $\|\vec{v}\|_\infty < n$ such that $a_{j,d}(\vec{r}) = a_{j,d}(\vec{r} + \vec{v})$ for all $\vec{r} \in \Gamma_j$ such that $\vec{r} + \vec{v} \in \Gamma_j$. Since $-\vec{v}$ is also a period of $a_{j,d}$, we may assume without loss of generality that $v_d \geq 0$. We choose a corner $\vec{q}$ of $\Gamma_j$ based on $\vec{v}$: for each $1 \leq i \leq d$, if $v_i \geq 0$, $q_i = 1$. If $v_i < 0$, $q_i = j$. In this way, we ensure that $\vec{q} + \vec{v} \in \Gamma_j$, and therefore that $a_{j,d}(\vec{q}) = a_{j,d}(\vec{q} + \vec{v})$. Since $q_i \in \{1, j\}$ for $1 \leq i \leq d-1$ and $q_d = 1$, by construction $a_{j,d}(\vec{q}) = 0$. Therefore, $a_{j,d}(\vec{q} + \vec{v}) = 0$. However, again due to construction, the only 0s in $a_{j,d}$ lie at points whose first $d-1$ coordinates are either 1 or $j$. This implies that the first $d-1$ coordinates of $\vec{q} + \vec{v}$ are either 1 or $j$. Let's denote by $h_i$ the $j$-letter pattern where $h_i(k) = a_{j,d}(\vec{q} + (k-1)\vec{e_d})$ for $1 \leq k \leq j$, and by $h_{i'}$ the $j$-letter pattern where $h_{i'}(k) = a_{j,d}(\vec{q} + \vec{v} + (k - 1 - v_d)\vec{e_d})$. Due to the supposed periodicity of $a_{j,d}$, the first $j - v_d$ letters of $h_i$ are the same as the final $j - v_d$ letters of $h_{i'}$. Since no two distinct patterns in $\{h_n\}_{n=1}^{2^{d-1}}$ may overlap, $v_d = 0$ and $h_i = h_{i'}$. This implies that the first $d-1$ coordinates of $\vec{q}$ are the same as the first

$d-1$ coordinates of $\vec{q}+\vec{v}$, and therefore that the first $d-1$ coordinates of $\vec{v}$ are zero, implying that $\vec{v}=0$, a contradiction. Thus, $a_{j,d}$ is aperiodic.

□

LEMMA 8.3. *For any $d>1$ and any $j \geq 3d+2$, there exists a pattern $b_{j,d} \in \{0,1\}^{\Gamma_j^{(2)}}$ with the property that there cannot exist two overlapping occurrences of $b_{j,d}$ such that one has nonempty intersection with the central copy of $\Gamma_{j-2}$ in the other.*

*Proof.* We use $a_{j,d-1}$ as a tool to create $b_{j,d}$. We define $b_{j,d}(\vec{p})=0$ if $p_d=1$ or 2, and $b_{j,d}(\vec{p})=1$ if $3 \leq p_d \leq j-1$. If $p_d=j$, then $b_{j,d}(\vec{p})=a_{j,d-1}(p_1,\ldots,p_{d-1})$. Suppose that $b_{j,d}$ does not have the property claimed in Lemma 8.3. Then $b_{j,d}$ has a period $\vec{v}$ with $\|v\|_\infty \leq j-2$. We can again assume without loss of generality that $v_d \geq 0$. Choose a point $\vec{q}$ of $\Gamma_j^{(2)}$ as follows: for each $1 \leq i \leq d-2$, take $q_i=1$ if $v_i \geq 0$, and $q_i=j-1$ if $v_i<0$. Choose $q_{d-1}=j-1-v_{d-1}$ if $v_{d-1} \geq 0$, and $q_{d-1}=1-v_{d-1}$ if $v_{d-1}<0$. Finally, take $q_d=1$. For any $\vec{r} \in \mathbb{Z}^d$ all of whose coordinates are zero or one, the first $d-1$ coordinates of $\vec{q}+\vec{r}$ are between 1 and $j$, and the $d$th coordinate is 1 or 2, implying that $\vec{q}+\vec{r} \in \Gamma_j^{(2)}$. This means that a copy of $\Gamma_2$ with its least corner lexicographically at $\vec{q}$ is a subset of $\Gamma_j^{(2)}$, call it $S$. Since $S$ is composed entirely of points whose $d$th coordinate is 1 or 2, $b_{j,d}|_S$ is filled with 0s. Again, for any point $\vec{r}$ all of whose coordinates are zero or one, the first $d-2$ coordinates of $\vec{q}+\vec{v}+\vec{r}$ are between 1 and $j$, the $(d-1)$th coordinate of $\vec{q}+\vec{v}+\vec{r}$ is 1, 2, $j-1$, or $j$, and the $d$th coordinate of $\vec{q}+\vec{v}+\vec{r}$ is between 1 and $j$. Therefore, any such $\vec{q}+\vec{v}+\vec{r}$ is in $\Gamma_j^{(2)}$, implying that a copy $S+\vec{v}$ of $\Gamma_2$ with its least corner lexicographically at $\vec{q}+\vec{v}$ is a subset of $\Gamma_j^{(2)}$ as well. By the supposed periodicity of $b_{j,d}$ with respect to $\vec{v}$, $b_{j,d}|_{S+\vec{v}}$ must be filled with 0s as well. However, by construction, for any copy of $\Gamma_2$ which is filled with 0s in $b_{j,d}$, the $d$th coordinate of its least corner lexicographically is 1. Since $q_d=1$, it must be the case that $v_d=0$. This implies that $a_{j,d-1}$ is periodic with period $(v_1,v_2,\ldots,v_{d-1})$. Since $a_{j,d-1}$ is aperiodic by Lemma 8.2, $v_i=0$ for $1 \leq i \leq d-1$ as well, and so $\vec{v}=\vec{0}$, a contradiction. Therefore, $b_{j,d}$ has the claimed property.

□

These constructions have both been about patterns in the full shift on two symbols. We must now turn to our SFT $X$. First, we will be constructing a pattern $y \in L_{\Gamma_{3R+7}}(X)$ for which $X_y$ is nonempty. Obviously, finding a pattern $y$ such that $X_y$ is nonempty could be done by using the already proven upper bound for $h^{top}(X)-h^{top}(X_y)$, but this may require $y$ to have shape with large size. We would like the shape of $y$ to have a small size to prove as tight a lower bound as possible in Theorem 1.2. We first need two preliminary results.

LEMMA 8.4. *For any $d$ and $n$, define $H_{n,d}=\{\vec{v} \in \Gamma_{2n+1}-n\vec{\mathbf{1}} \ :\ \vec{v}$ is less than $\vec{0}$ lexicographically$\}$, $G_{n,k,d}=\{\vec{v} \in \mathbb{Z}^d \ :\ v_i>0$ for $1 \leq i \leq d, \sum_{i=1}^d n^{i-1}v_i \leq k\}$, and $K_{n,k,d}=(\Gamma_{k+n}-n\vec{\mathbf{1}}) \diagdown \Gamma_k$. For any $d$ and $n$, if $X$ is a $\mathbb{Z}^d$ subshift such that $x|_{H_{n,d}}$ forces $x(\vec{0})$ for all $x \in X$, then $x|_{K_{n,k,d}}$ forces $x|_{G_{n,k,d}}$ for any $k$ and all $x \in X$.*

*Proof.* As this is the only place in the paper where the dimension $d$ of a cube may not be clear, for this proof we adopt the notation $\Gamma_{n,d}$ for the cube $\Gamma_n$ with dimension $d$. We also define $I_{n,d} = \{\vec{v} \in (\Gamma_{2n+1,d} - n\vec{\mathbf{1}}) \ : \ v_d < 0\}$ for any $n, d$. We quickly note a few useful facts about these sets which can be checked by the reader: $H_{n,d+1} = I_{n,d} \cup (H_{n,d} \times \{0\})$, and $G_{n,k,d+1} = \bigcup_{i=1}^{\lfloor \frac{k}{n^d} \rfloor} (G_{n,k-in^d,d} \times \{i\})$.

The proof will be by induction on $d$. For $d = 1$, Lemma 8.4 is fairly easy to check (and is in fact a classical theorem due to Hedlund and Morse ([**4**])); it amounts to showing that if any $n$ consecutive letters of any $x$ force the next, then any $n$ consecutive letters of $x$ force the next $k$ for any $k$. But this is clear; if $x|_{[1,n]}$ forces $x(n+1)$, then $x|_{[2,n+1]}$ forces $x(n+2)$, and we can proceed like this indefinitely. Thus, Lemma 8.4 is proven for $d = 1$. Now suppose that it is true for a fixed $d$, and we will prove it for $d + 1$.

Suppose that $X$ is a $\mathbb{Z}^d$ subshift such that $x|_{H_{n,d+1}}$ forces $x(\vec{0})$ for all $x \in X$. Fix any $x \in X$. We wish to show that $x|_{K_{n,k,d+1}}$ forces $x|_{G_{n,k,d+1}}$. Since $G_{n,k,d+1} = \bigcup_{i=1}^{\lfloor \frac{k}{n^d} \rfloor} (G_{n,k-in^d,d} \times \{i\})$, it suffices to show that $x|_{K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{i\})}$ forces $x|_{G_{n,k-(j+1)n^d,d} \times \{j+1\}}$ for every $0 \leq j < \lfloor \frac{k}{n^d} \rfloor$. Fix any $0 \leq j < \lfloor \frac{k}{n^d} \rfloor$, and suppose that $x|_{K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{i\})}$ is given. We will show that $x|_{G_{n,k-(j+1)n^d,d} \times \{j+1\}}$ is forced.

Consider any $\vec{v} = (\vec{v'}, j+1) \in G_{n,k-(j+1)n^d,d} \times \{j+1\}$. We first claim that $\vec{v} + I_{n,d+1} \subseteq K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{j\})$. Since $I_{n,d+1} = (\Gamma_{2n+1,d} - n\vec{\mathbf{1}}) \times [-n, -1]$, it suffices to show that $\vec{v'} + (\Gamma_{2n+1,d} - n\vec{\mathbf{1}}) \subseteq (K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{j\})) \cap (\mathbb{Z}^d \times \{i\})$ for any $i \in [j-n+1, j]$.

$$\text{Since } (K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{j\})) \cap (\mathbb{Z}^d \times \{i\}) = \begin{cases} (K_{n,k,d} \cup \Gamma_{k,d}) \times \{i\} & \text{if } -n+1 \leq i \leq 0 \\ (K_{n,k,d} \cup G_{n,k-in^d,d}) \times \{i\} & \text{if } i > 0, \end{cases}$$

and since $G_{n,k-(i+1)n^d,d} \subseteq G_{n,k-in^d,d} \subseteq \Gamma_{k,d}$ for all $i > 0$, it suffices to show that $\vec{v'} + (\Gamma_{2n+1,d} - n\vec{\mathbf{1}}) \subseteq K_{n,k,d} \cup G_{n,k-jn^d,d}$.

Consider any $\vec{u} \in \Gamma_{2n+1,d} - n\vec{\mathbf{1}}$. By definition of $G_{n,k-(j+1)n^d,d}$, $\sum_{i=1}^{d} v'_i n^{i-1} \leq k - (j+1)n^d$, and $v'_i > 0$ for $1 \leq i \leq d$. By choice of $\vec{u}$, $-n \leq u_i \leq n$ for $1 \leq i \leq d$. Therefore, $-n < (\vec{v'} + \vec{u})_i$ for $1 \leq i \leq d$, and $\sum_{i=1}^{d} (\vec{v'} + \vec{u})_i n^{i-1} \leq (k-(j+1)n^d) + (\sum_{i=1}^{d-1} n^i) < k - jn^d$. There are then two cases; if any coordinate of $\vec{v'} + \vec{u}$ is nonpositive, then $\vec{v'} + \vec{u} \in K_{n,k,d}$, and if all coordinates are positive, then by definition, $\vec{v'} + \vec{u} \in G_{n,k-jn^d,d}$. So, indeed $\vec{v'} + (\Gamma_{2n+1,d} - n\vec{\mathbf{1}}) \subseteq K_{n,k,d} \cup G_{n,k-jn^d,d}$. As argued above, this means that $\vec{v} + I_{n,d+1} \subseteq K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{i\})$ for any $\vec{v} \in G_{n,k-(j+1)n^d,d} \times \{j+1\}$.

Since $H_{n,d+1} = I_{n,d+1} \cup (H_{n,d} \times \{0\})$, and since $x|_{K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{i\})}$ has been given, we can then conclude that for any $\vec{v} \in (G_{n,k-(j+1)n^d,d} \times \{j+1\})$, $x|_{\vec{v}+(H_{n,d} \times \{0\})}$ forces $x(\vec{v})$. But then by the inductive hypothesis, $x|_{G_{n,k-(j+1)n^d,d}} \times \{j+1\}$ is forced by $x|_{K_{n,k-(j+1)n^d,d+1}} \times \{j+1\}$, and since $K_{n,k-(j+1)n^d,d} \times \{j+1\} \subset$

$K_{n,k,d+1}$, $x|_{G_{n,k-(j+1)n^d,d}} \times \{j+1\}$ is forced by $x|_{K_{n,k,d+1} \cup \bigcup_{i=1}^{j} (G_{n,k-in^d,d} \times \{i\})}$ as well. Since $j$ was arbitrary here, as described above this shows that $x|_{K_{n,k,d+1}}$ forces $x|_{G_{n,k,d+1}}$. Since $x \in X$ was arbitrary, we have verified the inductive hypothesis for $d+1$, and so we are done.

□

PROPOSITION 8.5. *For any $d$, any $\mathbb{Z}^d$ subshift $X$ with $h^{top}(X) > 0$, and any finite $S \subset \mathbb{Z}^d$, $|L_S(X)| > |S|$.*

*Proof.* We write $S = \{\vec{s_1}, \vec{s_2}, \ldots, \vec{s_{|S|}}\}$ where $\vec{s_i}$ comes before $\vec{s_{i+1}}$ in the usual lexicographical order for $1 \leq i < |S|$, and make the notation $S_i = \{\vec{s_1}, \ldots, \vec{s_i}\}$ for all $1 \leq i \leq |S|$. Suppose for a contradiction that $|L_S(X)| \leq |S|$. Since $h^{top}(X) > 0$, $|L_{S_1}(X)| > 1$. It must then be the case that for some $1 \leq j < |S|$, $|L_{S_{j+1}}(X)| \leq |L_{S_j}(X)|$. Since $S_j \subset S_{j+1}$, this means that for every pattern $w \in L_{S_j}(X)$, there is a unique way to extend it to a pattern in $L_{S_{j+1}}(X)$. In other words, for any $x \in X$, $x|_{S_j}$ forces $x(\vec{s_{j+1}})$. By shift-invariance of $X$, for any $x \in X$, $x|_{S_j - \vec{s_{j+1}}}$ forces $x(\vec{0})$. Since $T := S_j - \vec{s_{j+1}}$ is finite, take $N > \text{diam}(T)$. Then, $T$ consists of elements of $\mathbb{Z}^d$ within a $d$-distance of less than $N$ from $\vec{0}$ and lexicographically less than $\vec{0}$. It is then clear that $T \subseteq H_{N,d}$, and so we note that $x|_{H_{N,d}}$ forces $x(\vec{0})$.

Then by Lemma 8.4, $x|_{K_{N,k,d}}$ forces $x|_{G_{N,k,d}}$ for all $k$ and any $x \in X$. This means that $|L_{G_{N,k,d}}(X)| \leq |L_{K_{N,k,d}}(X)|$ for any $k$. Note that $\Gamma_{\lfloor \frac{k}{dN^d} \rfloor} \subseteq G_{N,k,d}$ for all $k$. Therefore, $|L_{\Gamma_n}(X)| \leq |L_{G_{N,ndN^d,d}}(X)| \leq |L_{K_{N,ndN^d,d}}(X)|$ for all $n$. Since $|K_{N,ndN^d,d}| = (ndN^d + N)^d - (ndN^d)^d \leq 2Nd(ndN^d)^{d-1} = Cn^{d-1}$ for large $n$ and a constant $C$ independent of $n$, we see that $|L_{\Gamma_n}(X)| \leq |A|^{Cn^{d-1}}$ for all large $n$, and so by the definition of topological entropy, $h^{top}(X) = \lim_{n\to\infty} \frac{\ln |L_{\Gamma_n}(X)|}{n^d} \leq \limsup_{n\to\infty} \frac{Cn^{d-1} \ln |A|}{n^d} = 0$. Therefore, $h^{top}(X) = 0$, a contradiction to the hypotheses of the theorem. Our initial assumption was therefore wrong, and so $|L_S(X)| > |S|$ for all finite shapes $S$.

□

LEMMA 8.6. *For any $d > 1$ and any strongly irreducible $\mathbb{Z}^d$ SFT $X$ with uniform filling length $R$ containing more than one point, there exists $y \in L_{\Gamma_{3R+7}}(X)$ so that $X_y \neq \varnothing$.*

*Proof.* Fix $d > 1$ and a strongly irreducible $\mathbb{Z}^d$ SFT $X$ containing more than one point. By Lemma 4.11, $h^{top}(X) > 0$. Define $S_1 = \{1, 2\} \times \{1, 2, \ldots, R+4\}^{d-1} \subset \Gamma_{R+4}$ and $S_2 = \{R+3, R+4\} \times \{1, 2, \ldots, R+4\}^{d-1} \subset \Gamma_{R+4}$. Then $\rho(S_1, S_2) > R$, and so by strong irreducibility, $|L_{\Gamma_{R+4}}(X)| \geq |L_{S_1}(X)||L_{S_2}(X)|$, which is greater than $4(R+4)^{2d-2}$ by Proposition 8.5. Since $R + 4 \geq 4$,

$$4(R+4)^{2d-2} \geq 4^{d-1}(R+4)^d \geq 2^d(R+4)^d = (2R+8)^d > (2R+4)^d.$$

Therefore, $|L_{\Gamma_{R+4}}(X)| > (2R+4)^d$. Consider any $y \in L_{\Gamma_{3R+7}}(X)$. Since there are $(2R+4)^d$ copies of $\Gamma_{R+4}$ contained in $\Gamma_{3R+7}$, there are at most $(2R+4)^d$

different patterns in $L_{\Gamma_{R+4}}(X)$ which are subpatterns of $y$. Therefore, there exists $z \in L_{\Gamma_{R+4}}(X)$ such that $z$ is not a subpattern of $y$. Then, again by using strong irreducibility, we can construct $x \in X$ such that for any $\vec{v} \in \mathbb{Z}^d$, $x|_{\Gamma_{R+4}+(2R+4)\vec{v}} = z$. (See Figure 11.) Then, for any $S$ a copy of $\Gamma_{3R+7}$, $x|_S$ contains a $z$, and therefore $x|_S \neq y$. Thus, $y$ is not a subpattern of $x$, and so $X_y$ contains at least one point and is nonempty.

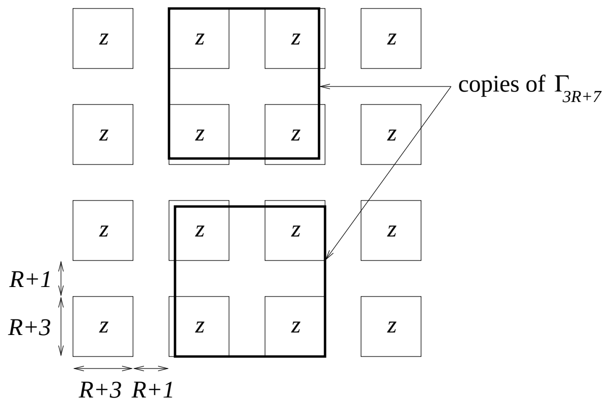


FIGURE 11. A point $x \in X_y$

□

*Proof of Theorem 8.1.* Fix $d > 1$, a strongly irreducible $\mathbb{Z}^d$ subshift $X$ with uniform filling length $R$ containing more than one point, and $j \geq 3d+2$. Let $W = 8R+14$. We will use $y$ and $b_{j,d}$ to create $w_{j,d}$. To do this, we first partition $\Gamma_{jW}^{(2W)}$ into disjoint copies of $\Gamma_W$. The disjoint copies of $\Gamma_W$ then have an obvious bijective correspondence to the points of $\Gamma_j^{(2)}$, illustrated in Figure 12.

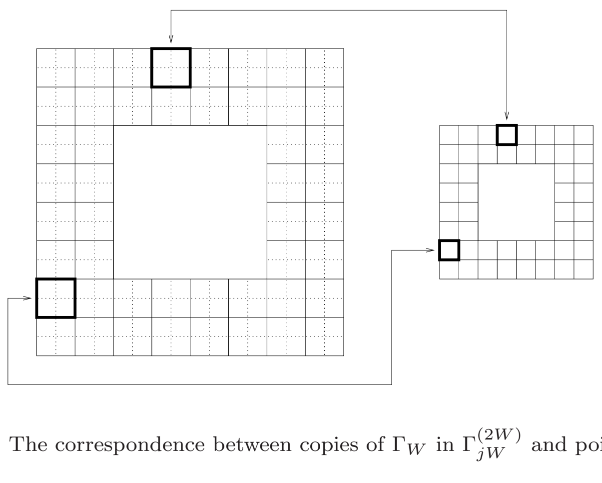

FIGURE 12. The correspondence between copies of $\Gamma_W$ in $\Gamma_{jW}^{(2W)}$ and points in $\Gamma_j^{(2)}$

We then use each entry of $b_{j,d}$ to assign entries of $w_{j,d}$ in the corresponding copy of $\Gamma_W$ in $\Gamma_{jW}^{(2W)}$. For $\vec{p} \in \Gamma_j^{(2)}$, if $b_{j,d}(\vec{p}) = 0$, then the lexicographically least copy of $\Gamma_{W-R}$ in the $\Gamma_W$ corresponding to $\vec{p}$ is filled with any pattern in $L_{\Gamma_{W-R}}(X_y)$, which clearly contains no occurrences of $y$. If $b_{j,d}(\vec{p}) = 1$, then the lexicographically least copy of $\Gamma_{W-R}$ in the $\Gamma_W$ corresponding to $\vec{p}$ has $2^d$ occurrences of $y$ placed inside it, each one sharing a corner with it.

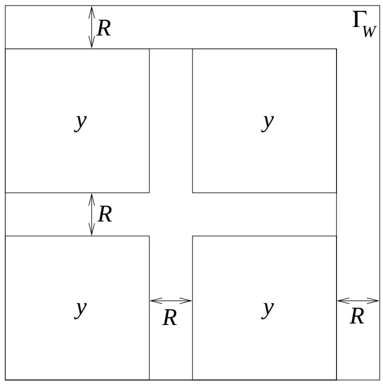


FIGURE 13. How a copy of $\Gamma_W$ is filled if $b_{j,d}(\vec{p}) = 1$

The remainder of $\Gamma_{jW}^{(2W)}$ is then filled to make the pattern $w_{j,d}$ by using strong irreducibility of $X$, since all of the filled portions are a distance of at least $R+1$ from each other. We claim that this pattern $w_{j,d} \in L_{\Gamma_{jW}^{(2W)}}(X)$ has the property described in Theorem 8.1. Suppose not; then there exist two overlapping occurrences of $w_{j,d}$ such that one has nonempty overlap with the empty central copy of $\Gamma_{(j-4)W}$ in the other, which implies that $w_{j,d}$ is periodic with respect to some $\vec{v} \neq 0$ with $\|\vec{v}\|_\infty < (j-2)W$. We then define $\vec{v'}$ by defining $v'_i$ to be the closest multiple of $W$ to $v_i$ for $1 \leq i \leq d$. If two are equally close, choose either. Clearly $\|\vec{v} - \vec{v}'\|_\infty \leq \frac{W}{2}$. We make the notation $\vec{v''} := \vec{v} - \vec{v'}$.

Each coordinate of $\vec{v'}$ is divisible by $W$, and so $\frac{\vec{v'}}{W}$ has integer coordinates. Since $\|\vec{v}'\|_\infty \leq (j-2)W$, $\|\frac{\vec{v'}}{W}\|_\infty \leq j-2$. This implies that either $\vec{v'} = 0$ or $\frac{\vec{v'}}{W}$ is the difference between two overlapping occurrences of $\Gamma_j^{(2)}$, one of which has nonempty intersection with the central copy of $\Gamma_{j-2}$ in the other. Assume for now that the latter is the case. Then, by the definition of $b_{n,d}$, this implies that there exist $\vec{q}, \vec{q} + \frac{\vec{v'}}{W} \in \Gamma_j^{(2)}$ such that $b_{j,d}(\vec{q}) \neq b_{j,d}(\vec{q} + \frac{\vec{v'}}{W})$. Without loss of generality, we assume that $b_{j,d}(\vec{q}) = 1$ and $b_{j,d}(\vec{q} + \frac{\vec{v'}}{W}) = 0$.

Let's call $S$ the lexicographically least copy of $\Gamma_{W-R}$ in the copy of $\Gamma_W$ in $\Gamma_{jW}^{(2W)}$ corresponding to $\vec{q}$ in $\Gamma_j^{(2)}$ and call $T$ the lexicographically least copy of $\Gamma_{W-R}$ in the copy of $\Gamma_W$ in $\Gamma_{jW}^{(2W)}$ corresponding to $\vec{q} + \frac{\vec{v'}}{W}$ in $\Gamma_j^{(2)}$. Then $T - S = \vec{v'}$, and since $w_{j,d}$ is periodic with respect to $\vec{v}$, $w_{j,d}|_{S \cap (T - \vec{v})} = w_{j,d}|_{(S+\vec{v}) \cap T}$. Note that $T - \vec{v} = (T - \vec{v'}) - \vec{v''} = S - \vec{v''}$. Since $\|\vec{v''}\|_\infty \leq \frac{W}{2}$, $S \cap (S - \vec{v''}) \neq \varnothing$, and so

$S \cap (T - \vec{v}) \neq \varnothing$ as well. In fact, $S \cap (T - \vec{v})$ is a copy of $P_{s_1,\ldots,s_d}$ which shares a corner with $S$, and for which $s_i \geq (W - R) - \frac{W}{2} = 3R + 7$ for all $1 \leq i \leq d$. Therefore, $S \cap (T - \vec{v})$ contains one of the $2^d$ copies of $\Gamma_{3R+7}$ in $S$ which share a corner with $S$. Since $b_{j,d}(\vec{q}) = 1$, and since $S$ is the lexicographically least copy of $\Gamma_{W-R}$ in the copy of $\Gamma_W$ in $\Gamma_{jW}^{(2W)}$ corresponding to $\vec{q}$ in $\Gamma_j^{(2)}$, every one of these subcubes of $S$ contains an occurrence of $y$ in $w_{j,d}$. Therefore, $w_{j,d}|_{S\cap(T-\vec{v})}$ has $y$ as a subpattern. However, since $(S + \vec{v}) \cap T \subseteq T$, $w_{j,d}|_{(S+\vec{v})\cap T}$ is a subpattern of $w_{j,d}|_T$. Since $b_{j,d}(\vec{q} + \frac{\vec{v'}}{W}) = 0$, and since $T$ is the lexicographically least copy of $\Gamma_{W-R}$ in the copy of $\Gamma_W$ in $\Gamma_{jW}^{(2W)}$ corresponding to $\vec{q} + \frac{\vec{v'}}{W}$ in $\Gamma_j^{(2)}$, $w_{j,d}|_T$ contains no occurrences of $y$. Therefore, $w_{j,d}|_{(S+\vec{v})\cap T}$ contains no occurrences of $y$ either. Since $w_{j,d}|_{S\cap(T-\vec{v})} = w_{j,d}|_{(S+\vec{v})\cap T}$, we have a contradiction.

The only remaining case is when $\vec{v'} = 0$, i.e. $\|\vec{v}\|_\infty \leq \frac{W}{2}$. We can then simply take an integer multiple $n\vec{v}$ of $\vec{v}$ such that some coordinate of $n\vec{v}$ is greater than $\frac{W}{2}$, but at most $W$. Then, $n\vec{v}$ is also a period of $w_{j,d}$, and we can repeat the argument above for the same contradiction. We arrive at a contradiction in either case, and so $w_{j,d}$ has the claimed property.

□

## 9. *An application to an undecidability question*

One of the complexities of $\mathbb{Z}^d$ SFTs for $d > 1$ is that there does not exist an algorithm which takes as input an alphabet $A$ and a finite set $\mathcal{F}$ of finite forbidden patterns, and decides whether or not the associated SFT $(A^{\mathbb{Z}^d})_\mathcal{F}$ is nonempty. (See [**1**] and [**14**] for details.) One application of Theorem 6.1 is a condition under which $(A^{\mathbb{Z}^d})_\mathcal{F}$ has positive topological entropy, and so in particular is nonempty.

THEOREM 9.1. *For any alphabet $A$ and any $d > 1$, there exist $F, G \in \mathbb{N}$ such that for any $m > 0$ and any finite set of patterns $\mathcal{F}_m = \{w_k \in L_{\Gamma_{n_k}}(X) \;:\; 1 \leq k \leq m\}$ satisfying $n_1 > G$ and $n_k \geq F(n_{k-1})^{24d^2}$ for $1 < k \leq m$, $h^{top}((A^{\mathbb{Z}^d})_{\mathcal{F}_m}) > 0$.*

To prove this, we will need the following lemma:

LEMMA 9.2. *For any $d > 1$, there exists $C \in \mathbb{N}$ so that for any strongly irreducible $\mathbb{Z}^d$ shift $X = (A^{\mathbb{Z}^d})_\mathcal{F}$ of finite type $t$ with uniform filling length $R$ containing more than one point, any $n > \max(C(R+1)^{24d^2}, 5t)$, and any $w \in L_{\Gamma_n}(X)$, there is some $w' \in L_{\Gamma_m}(X)$ a subpattern of $w$ such that $X_{w'}$ contains more than one point and is strongly irreducible with uniform filling length at most $2n + R$.*

*Proof.* Suppose that such a shift $X$ is given. By Theorem 6.1, there exists $N_1$ such that for any $w \in L_{\Gamma_n}(X)$ with $n > N_1$, there exist $w', w'' \in L_{\Gamma_m}(X)$ such that $w'$ is a subpattern of $w$ and replacing an occurrence of $w'$ by $w''$ in an element of $X$ cannot possibly create a new occurrence of $w'$. By an examination of the proof of Theorem 6.1, we see that a sufficient condition for the existence of $w'$ and $w''$ with these properties is that

$$n > \max(C'\Big(\frac{\ln n}{h^{top}(X)} + R\Big)^{12d}, 5t)$$

for some constant $C'$ depending only on $d$. By Lemma 4.11, $h^{top}(X) > \frac{\ln 2}{(R+1)^d}$. The reader may then check that there exists a constant $C$ depending only on $d$ so that $n > \max(C(R+1)^{24d^2}, 5t)$ implies the displayed formula, and thus the existence of $w'$ and $w''$. By the same argument as that used in the proof of the upper bound of Theorem 1.2, this implies that $X_{w'}$ is nonempty: to create a point of $X_{w'}$, simply begin with a point $x \in X$, choose an ordering of $\mathbb{Z}^2$ with a least element, and replace each occurrence of $w'$ in $x$ by $w''$ in order. In fact, it is easy to choose $w''$ containing at least two different letters: just make sure that the subword $a$ of $w''$ from the proof of Theorem 6.1 contains at least two letters. This means that $|X_{w'}| > 1$: any point $x' \in X_{w'}$ obtained by repeated replacements of $w'$ by $w''$ in some point $x \in X$ contains at least two letters, meaning that some two of its shifts are unequal.

Consider any two shapes $S, T \subseteq \mathbb{Z}^d$ such that $\rho(S,T) > 2n + R$, and any patterns $y \in L_S(X_{w'})$ and $z \in L_T(X_{w'})$. Since $y \in L_S(X_{w'})$, there exists $y' \in L_{S \cup (S^c)^{(n)}}(X_{w'})$ such that $y'|_S = y$. Similarly, there exists $z' \in L_{T \cup (T^c)^{(n)}}(X_{w'})$ such that $z'|_T = z$. For any $\vec{p} \in S \cup (S^c)^{(n)}$, by definition there exists $\vec{p'} \in S$ such that $\|\vec{p} - \vec{p'}\|_\infty \leq n$. Similarly, for any $\vec{q} \in T \cup (T^c)^{(n)}$, there exists $\vec{q'} \in T$ such that $\|\vec{q} - \vec{q'}\|_\infty \leq n$. But $\rho(S,T) > 2n + R$, so $\|\vec{p'} - \vec{q'}\|_\infty > 2n + R$. By the triangle inequality, $\|\vec{p} - \vec{q}\|_\infty > R$. Therefore, since $\vec{p}, \vec{q}$ were arbitrary, $\rho(S \cup (S^c)^{(n)}, T \cup (T^c)^{(n)}) > R$, and so by strong irreducibility of $X$ there exists $x \in X$ such that $x|_{S \cup (S^c)^{(n)}} = y'$ and $x|_{T \cup (T^c)^{(n)}} = z'$. We now fix any ordering of the elements of $\mathbb{Z}^d$ with a least element (say lexicographically with respect to polar coordinates) and replace each element of $w'$ by $w''$ in turn with respect to this order. In this way, we will eventually arrive at $x' \in X$ which has no occurrences of $w'$ and is thus an element of $X_{w'}$. Note that since $x|_{S \cup (S^c)^{(n)}} = y'$ and $x|_{T \cup (T^c)^{(n)}}$ are patterns in $L(X_{w'})$, they had no occurrences of $w'$. Therefore, any of the replaced occurrences of $w'$ had nonempty intersection with $\big((S \cup (S^c)^{(n)}) \cup (T \cup (T^c)^{(n)})\big)^c$. Consider such a replaced occurrence which occupies $U$ a copy of $\Gamma_m$. From the fact just noted, there exists $\vec{p} \in U$ such that $\vec{p} \notin S \cup (S^c)^{(n)} \cup T \cup (T^c)^{(n)}$. This implies that $\rho(\{\vec{p}\}, S) > n$ and $\rho(\{\vec{p}\}, T) > n$. Therefore, since the size of $U$ is $m \leq n$, $U$ is disjoint from both $S$ and $T$. Since $U$ is the location of an arbitrary replaced occurrence of $w'$, this implies that $x$ remained unchanged on $S$ and $T$ throughout the process of changing it to $x'$, and so $x'|_S = x|_S = y$ and $x'|_T = x|_T = z$. By definition, we have shown that $X_{w'}$ contains more than one point and is strongly irreducible with uniform filling length at most $2n + R$.

□

*Proof of Theorem 9.1.* By Lemma 9.2, there exists $G$ such that for any $n_1 > G$ and $w_1 \in A^{\Gamma_{n_1}}$, there is a subpattern $w_1'$ of $w_1$ such that $(A^{\mathbb{Z}^d})_{\{w_1'\}}$ contains more than one point and is strongly irreducible with uniform filling length $R_1 < 4n_1$. Take $F$ to be $5^{24d^2}C$, where $C$ is from Lemma 9.2.

We prove this theorem by induction. Our inductive hypothesis is that for any $\mathcal{F}_m$ as described in the hypotheses, $(A^{\mathbb{Z}^d})_{\mathcal{F}_m}$ contains a strongly irreducible SFT with more than one point with uniform filling length less than $4n_m$. The base case $m = 1$ is verified by the definition of $G$ above. Now suppose the hypothesis to be true for $m$, and consider any $\mathcal{F}_{m+1} = \{w_1, w_2, \ldots, w_{m+1}\}$ satisfying the hypotheses of Theorem 9.1.

By the inductive hypothesis, $(A^{\mathbb{Z}^d})_{\{w_1,w_2,\ldots,w_m\}}$ contains a strongly irreducible shift $X_m$ of finite type $t_m$ with more than one point and uniform filling length $R_m < 4n_m$. We first claim that $n_{m+1} > \max(C(R_m+1)^{24d^2}, 5t_m)$. By the hypotheses of Theorem 9.1, $n_{m+1} > F(n_m)^{24d^2} = 5^{24d^2}C(n_m)^{24d^2} > C(5n_m)^{24d^2} > C(R_m+1)^{24d^2}$. Since the largest size of a pattern in $\mathcal{F}$ is $n_m$, $t_m \leq n_m$, and so $n_{m+1} > F(n_m)^{24d^2} > 5t_m$.

We now have two cases: either $w_{m+1} \in L(X_m)$ or not. If $w_{m+1} \in L(X_m)$, then since $n_{m+1} > \max(C(R_m+1)^{24d^2}, 5t_m)$, by Lemma 9.2 there is $w'_{m+1}$ a subpattern of $w_{m+1}$ such that $(X_m)_{w'_{m+1}}$ contains more than one point and is strongly irreducible with uniform filling length less than $2n_{m+1} + R_m < 4n_{m+1}$. Then $(X_m)_{w'_{m+1}} \subseteq (X_m)_{w_{m+1}} \subseteq (A^{\mathbb{Z}^d})_{\mathcal{F}_{m+1}}$, and the inductive hypothesis is verified for $m+1$.

If $w_{m+1} \notin L(X_m)$, then $(X_m)_{w_{m+1}} = X_m$, and so since $X_m = (X_m)_{w_{m+1}} \subseteq (A^{\mathbb{Z}^d})_{\mathcal{F}_{m+1}}$, in this case $(A^{\mathbb{Z}^d})_{\mathcal{F}_{m+1}}$ contains a strongly irreducible SFT with more than one point and uniform filling length $R_m < 4n_m < 4n_{m+1}$, and the induction is complete. We have then shown that for any $m$, $(A^{\mathbb{Z}^d})_{\mathcal{F}_m}$ contains a strongly irreducible shift with more than one point, and therefore by Lemma 4.11 its topological entropy is positive.

□

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