MINIMAL ZERO ENTROPY SUBSHIFTS CAN BE UNRESTRICTED ALONG ANY SPARSE SET

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Abstract. We present a streamlined proof of a result essentially present in [5], namely that for every set $S = \{s_1, s_2, \ldots\} \subset \mathbb{N}$ of zero Banach density and finite set A, there exists a minimal zero-entropy subshift (X, σ) so that for every sequence $u \in A^{\mathbb{Z}}$, there is $x_u \in X$ with $x_u(s_n) = u(n)$ for all $n \in \mathbb{N}$. Informally, points from minimal subshifts can achieve completely arbitrary behavior upon restriction to a set of zero Banach density.

As a corollary, this provides counterexamples to the Polynomial Sarnak Conjecture of [1] which are significantly more general than some recently provided in [3] and [4] and shows that no similar result can hold under only the assumptions of minimality and zero entropy.

1. Introduction

The well-known **Sarnak conjecture** states that the Möbius function μ is uncorrelated with all deterministic sequences. A sequence is called **deterministic** if it is the image under a continuous function of the trajectory of a point in a **topological dynamical system** with zero **entropy** (see Section 2 for definitions of this and other concepts not defined in this introduction). More formally,

Conjecture 1 (Sarnak Conjecture). *If (X, T) is a topological dynamical system with zero entropy, $x_0 \in X$, and $f \in C(X)$, then*

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) f(T^n x_0) \to 0.$$

Although this problem is still open, there are many recent works on the topic, which have made significant progress and resolved it for some classes of dynamical systems. In [1], a potential stronger 'polynomial' (meaning that only polynomial iterates of x_0 are taken rather than all) version of the Sarnak Conjecture was conjectured. In order to rule out some degenerate examples, the assumption of **minimality** was added on (X, T), meaning that for every $x \in X$, the set $\{T^n x\}$ is dense.

Conjecture 2 (Polynomial Sarnak Conjecture ([1], Conjecture 2.3)). *If (X, T) is a minimal topological dynamical system with zero entropy, $x_0 \in X$, $f \in C(X)$, and $p : \mathbb{N} \to \mathbb{N}_0$ is a polynomial, then*

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) f(T^{p(n)} x_0) \to 0.$$

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This conjecture is now known to be false; recently Kanigowski, Lemańczyk, and Radziwiłł([3]) and Lian and Shi ([4]) have separately provided counterexamples. However, these counterexamples are specific to the case $p(n) = n^2$ (though they could perhaps be generalized) and make strong usage of the nice arithmetic properties of this function. The first is a skew product and the second is a symbolically defined dynamical system called a Toeplitz subshift.

The purpose of this note is to show that even much weaker versions of Conjecture 2 are false, because minimal zero entropy systems can achieve **any** possible behavior (i.e., not just correlation with μ) along **any** prescribed set $S \subset \mathbb{N}$ of zero **Banach density** (i.e., not just the image of a polynomial). One such result had already been proved by the author in [5], which already immediately refutes the Polynomial Sarnak Conjecture.

Theorem 3 ([5], Corollary 5.1)**.** *Assume that $d \in \mathbb{N}$, (w_n) is an increasing sequence of positive integers where $w_{n+1} < (w_{n+1} - w_n)^{d+1}$ for large enough n, and (z_n) is any sequence in $\mathbb{T} := \mathbb{Z}/\mathbb{N}$. Then there exists a totally minimal, totally uniquely ergodic, topologically mixing zero entropy map S on $\mathbb{T}^{2d+4}$ so that, if π is projection onto the final coordinate, $\pi(S^{w_n}\mathbf{0}) = z_n$ for sufficiently large n.*

(We don't further work with the properties of unique ergodicity and topological mixing, and so don't provide definitions here. However, we do note that Theorem 3 shows that even adding these hypotheses to Conjecture 2 would not make it true.) We note that the entropy of the transformation S was never mentioned in [5]. However, S is defined as a suspension flow of a product of a toral rotation and a skew product T under a roof function $1 < g < 3$. The skew product T is of the form $(x_1, x_2, x_3, \dots, x_m) \mapsto (x_1 + \alpha, x_2 + f(x_1), x_3 + x_2, \dots, x_m + x_{m-1})$ for a continuous self-map f of $\mathbb{T}$. Since its first coordinate is an irrational rotation, known to have zero entropy, the map T also has zero entropy by Abramov's skew product entropy formula. Then S has zero entropy as well, by Abramov's suspension flow entropy formula.

Remark 4. Here are a few more relevant facts about the construction from [5]:

(1) The map S is **distal**, meaning that for all $x \neq y$, $\{d(T^n x, T^n y)\}_n$ is bounded away from 0.
(2) The roof function g is C^∞ and the function f, though not C^∞ as constructed in [5], can easily be made so; it is just a uniformly convergent infinite series of 'bump functions,' which can easily be chosen C^∞.

The second fact may be of interest since the authors of [3] prove a positive result for convergence along prime iterates of similar skew products $(x, y) \mapsto (x + \alpha, y + f(x))$ under the assumption that the function f is real analytic, provide some counterexamples with continuous f, and ask whether this assumption could be weakened to C^∞. Though the constructions are not exactly the same, and though the primes absolutely do not satisfy the assumption of Theorem 3, (2) might suggest that C^∞ is not always sufficient for good averaging of skew products along sparse sequences.

We note that Theorem 3 clearly applies to any sequence $w_n = p(n)$ for a non-constant polynomial $p : \mathbb{N} \to \mathbb{N}_0$ (possibly omitting finitely many terms), and so, by simply defining z_n to be $\frac{1}{2}$ when $\mu(n) = 1$ and 0 otherwise, one achieves

$$\frac{1}{N}\sum_{n=1}^{N}\mu(n)\pi(S^{p(n)}\mathbf{0}) = \frac{0.5|\mu^{-1}(\{1\}) \cap \{1,\dots,N\}|}{N},$$

which does not approach 0 as $N \to \infty$, disproving the Polynomial Sarnak Conjecture for every nonconstant p. The same is true of any function p with polynomial growth, even for degree less than 2, e.g. $p(n) = \lfloor n^{1.01} \rfloor$. However, Theorem 3 does not apply to more slowly growing p such as $\lfloor n \ln n \rfloor$. The author proved a different result (Corollary 3.1) in [5] using **subshifts**; a subshift is a closed shift-invariant subset of $A^{\mathbb{Z}}$ (for some finite alphabet A) endowed with the left-shift transformation. Corollary 3.1 of [5] states that given any sequence of zero Banach density (regardless of growth rate), there exists a minimal subshift whose points can achieve arbitrary behavior along that sequence. However, entropy was not mentioned there, and although the proof there can indeed yield a zero entropy subshift, it's not easy to verify; the construction is quite complicated in order to achieve (X,T) which is totally minimal, totally uniquely ergodic, and topologically mixing.

In this note, we present a streamlined self-contained proof of the following result, which shows that minimal zero entropy subshifts can realize arbitrary behavior along any sequence of zero Banach density.

Theorem 5. *For any $S = \{s_1, s_2, \dots\} \subset \mathbb{N}$ with $d^*(S) = 0$ and any finite alphabet A, there exists a minimal zero entropy subshift $X \subset A^{\mathbb{Z}}$ so that for every $u \in A^{\mathbb{N}}$, there is $x_u \in X$ where $x_u(s_n) = u(n)$ for all $s \in S$.*

We note that this proves that even with substantially weaker hypotheses, nothing in the spirit of the Polynomial Sarnak Conjecture can hold under only the assumptions of minimality and zero entropy. Even if p is only assumed to have range of zero Banach density and $\rho : \mathbb{N} \to \mathbb{Z}$ is only assumed to have $\limsup \frac{1}{N}\sum_{n=1}^{N}|\rho(n)| > 0$ (equivalently, ρ takes nonzero values on a set of positive upper density), one can define a subshift X on $\{-1,0,1\}$ and $x_u \in X$ as in Theorem 5 for $u(n) = \mathrm{sgn}(\rho(n))$. Then, for $f \in C(X)$ defined by $x \mapsto x(0)$, the limit supremum of the averages

$$\frac{1}{N}\sum_{n=1}^{N}\rho(n)f(\sigma^{p(n)}x_u) = \frac{1}{N}\sum_{n=1}^{N}\rho(n)x_u(p(n)) = \frac{1}{N}\sum_{n=1}^{N}\rho(n)u(n) = \frac{1}{N}\sum_{n=1}^{N}\rho(n)\mathrm{sgn}(\rho(n)) = \frac{1}{N}\sum_{n=1}^{N}|\rho(n)|$$

is positive by assumption.

We remark that when $\rho = \mu$ is the Möbius function, this means that

$$\frac{1}{N}\sum_{n=1}^{N}\mu(n)f(\sigma^{p(n)}x_u)$$

can be made to approach $\frac{6}{\pi^2}$ (for x_u in a minimal zero-entropy subshift), a slight improvement of [4] which showed that it could attain values arbitrarily close to $\frac{6}{\pi^2}$.

2. Definitions

A **topological dynamical system** (X,T) is defined by a compact metric space X and homeomorphism $T : X \to X$. A **subshift** is a topological dynamical system defined by some finite set A (called the **alphabet**) and the restriction of the **left shift** map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $(\sigma x)(n) = x(n+1)$ to some closed and σ-invariant $X \subset A^{\mathbb{Z}}$ (with the induced product topology). A subshift (X, σ) is **minimal** if for every $x \in X$, $\{\sigma^n x\}_{n \in \mathbb{Z}}$ is dense in X.

A **word** over A is any finite string of symbols from A; a word $w = w(1) \dots w(n)$ is said to be a **subword** of a word or infinite sequence x if there exists i so that $w(1) \dots w(n) = x(i+1) \dots x(i+n)$. The **language** $L(X)$ of a subshift (X, σ) is the set of all subwords of sequences in X, and for any $n \in \mathbb{N}$ we denote $L_n(X) = L(X) \cap A^n$. For two words $u = u(1) \dots u(m)$ and $v = v(1) \dots v(n)$, denote by uv their **concatenation** $u(1) \dots u(m)v(1) \dots v(n)$.

We do not give a full definition of **topological entropy** here, but note that it is a number $h(X, T) \in [0, \infty]$ associated to any TDS (X, T) which is conjugacy-invariant. We will only need the following definition for subshifts: for any (X, σ),

$$h(X, \sigma) = \lim \frac{\ln |L_n(X)|}{n}.$$

The **Banach density** of a set $S \subset \mathbb{N}$ is

$$d^*(S) := \lim_{n \to \infty} \sup_{k \in \mathbb{N}} \frac{|S \cap \{k, \dots, k+n-1\}|}{n}.$$

3. Proof of Theorem 5

Proof. As in [5], we adapt the block-concatenation construction of Hahn and Katznelson ([2]).

We construct X iteratively via auxiliary sequences m_k of odd positive integers, $A_k \subset A^{m_k}$, and $w_k \in A_k$. Define $m_0 = 1$, $A_0 = A$, and $w_0 = 0$ (which we assume without loss of generality to be in A). Now, suppose that m_k, A_k, and w_k are defined. Define $m_{k+1} > \max(3m_k|A_k|, 12(\ln 2)(4/3)^{k+1})$ to be an odd multiple of $3m_k$ large enough that $|S \cap I|/|I| < (3m_k)^{-1}$ for all intervals I of length m_{k+1} (using the fact that $d^*(S) = 0$). Define A_{k+1} to be the set of all concatenations of $\frac{m_{k+1}}{m_k}$ words in A_k in which every word in A_k is used at least once and in which at least one-third of the concatenated words are equal to w_k. Define Y_k to be the set of shifts of biinfinite (unrestricted) concatenations of words in A_k, define $Y = \bigcap_k Y_k$, and define X to be the subshift of Y consisting of sequences in which every subword is a subword of some w_k.

We claim that (X, σ) is minimal. Indeed, consider any $x \in X$ and $w \in L(X)$. By definition, w is a subword of w_k for some k. By definition, w_k is a subword of every word in A_{k+1}. Finally, x is a shift of a concatenation of words in A_{k+1}, each of which contains w_k, and therefore w. So, x contains w, and since $w \in L(X)$ was arbitrary, the orbit of x is dense. Since $x \in X$ was arbitrary, (X, σ) **is minimal**.

We also claim that (X, σ) has zero entropy. We see this by bounding $|A_k|$ from above. For every k, each word in A_{k+1} is defined by an ordered (m_{k+1}/m_k)-tuple of words in A_k, where at least one-third are w_k. The number of such tuples can be bounded from above by

$$\binom{m_{k+1}/m_k}{m_{k+1}/3m_k} |A_k|^{2m_{k+1}/3m_k} \leq 2^{m_{k+1}/m_k} |A_k|^{2m_{k+1}/3m_k}.$$

Therefore,

$$\frac{\ln |A_{k+1}|}{m_{k+1}} \leq \frac{\ln 2}{m_k} + \frac{2}{3} \frac{\ln |A_k|}{m_k}.$$

Now, it's easily checked that $\frac{\ln |A_k|}{m_k} \leq \ln |A| (3/4)^k$ for all k by induction. The base case $k = 0$ is immediate. For the inductive step, if we assume that $\frac{\ln |A_k|}{m_k} \leq \ln |A|(3/4)^k$, then recalling that $m_k > 12(\ln 2)(4/3)^k$,

$$\frac{\ln |A_{k+1}|}{m_{k+1}} < \frac{1}{12}(3/4)^k + \frac{2}{3}\ln |A|(3/4)^k \leq \frac{\ln |A|}{12}(3/4)^k + \frac{2}{3}\ln |A|(3/4)^k = \ln |A|(3/4)^{k+1}.$$

Therefore, for all k, $|A_k| \leq e^{\ln |A|(3/4)^k m_k}$. Finally, we note that every word in $L_{m_k}(X)$ is a subword of a concatenation of a pair of words in A_k, so determined by such a pair and by the location of the first letter. Therefore, $|L_{m_k}(X)| \leq m_k |A_k|^2 < m_k e^{2 \ln |A|(3/4)^k m_k}$. This clearly implies that

$$h(X) = \lim_{k \to \infty} \frac{\ln |L_{m_k}(X)|}{m_k} \leq \limsup_{k \to \infty} \frac{\ln m_k}{m_k} + 2 \ln |A|(3/4)^k = 0,$$

i.e. **X has zero entropy**.

It remains, for $u \in A^{\mathbb{N}}$, to construct $x_u \in X$ with $x_u(s_n) = u(n)$ for all $s_n \in S$. The construction of x_u proceeds in steps, where it is continually assigned letters from A on portions of $\mathbb{Z}$, with undefined portions labeled by $*$. Formally, define $x^{(0)} \in A \sqcup \{*\}^{\mathbb{Z}}$ by $x^{(0)}(s_n) = u(n)$ for $s \in S$ and $*$ for all other locations.

Now partition $\mathbb{Z}$ into the intervals $((i - 0.5)m_1, (i + 0.5)m_1)$ (herein, all intervals are assumed to be intersected with $\mathbb{Z}$). For every i for which $S \cap ((i - 0.5)m_1, (i + 0.5)m_1) \neq \varnothing$, consider the m_1-letter word $x^{(0)}(((i - 0.5)m_1, (i + 0.5)m_1))$. By definition of m_1, $|S \cap ((i - 0.5)m_1, \ldots, (i + 0.5)m_1)| < m_1/3m_0 = m_1/3$, and so at most one-third of the letters in this word are non-$*$. Fill the remaining locations by assigning the first $m_1/3$ as $w_0 = 0$. At least $m_1/3$ letters remain, which is larger than $|A_0| = |A|$ by definition of m_1. Fill those in an arbitrary way which uses all letters from A at least once. The resulting m_1-letter word is in A_1 by definition, call it $w_i^{(1)}$. Now, define $x^{(1)}$ by setting $x^{(1)}(((i - 0.5)m_1, (i + 0.5)m_1)) = w_i^{(1)}$ for all i as above (i.e. those for which $S \cap ((i - 0.5)m_1, (i + 0.5)m_1) \neq \varnothing$) and $*$ elsewhere. Note that $x^{(1)}$ is an infinite concatenation of words in A_1 and blocks of $*$ of length m_1 and that $x^{(1)}$ contains $*$ on any interval $((i - 0.5)m_1, (i + 0.5)m_1)$ which is disjoint from S.

Now, suppose that $x^{(k)}$ has been defined as an infinite concatenation of words in A_k and blocks of $*$ of length m_k which contains $*$ on any interval $((i - 0.5)m_k, (i + 0.5)m_k)$ which is disjoint from S. We wish to extend $x^{(k)}$ to $x^{(k+1)}$ by changing some $*$ symbols to letters in A. Consider any i for which $S \cap ((i - 0.5)m_{k+1}, \ldots, (i + 0.5)m_{k+1}) \neq \varnothing$. The portion of $x^{(k)}$ occupying that interval is a concatenation of words in A_k and blocks of $*$ of length m_k (we use here the fact that m_{k+1} is odd), and the number which are words in A_k is bounded from above by the number of $j \in ((i - 0.5)m_{k+1}/m_k, (i + 0.5)m_{k+1}/m_k)$ for which $((j - 0.5)m_k, (j + 0.5)m_k)$ is not disjoint from S, which in turn is bounded from above by $|S \cap ((i - 0.5)m_{k+1}, (i + 0.5)m_{k+1})|$, which by definition of m_{k+1} is less than $m_{k+1}/3m_k$. Therefore, at least two-thirds of the concatenated m_k-blocks comprising $x^{(k)}(((i - 0.5)m_{k+1}, (i + 0.5)m_{k+1}))$ are blocks of $*$. Fill the first $m_{k+1}/3m_k$ of these with w_k. Then at least $m_{k+1}/3m_k$ blocks remain, which is more than $|A_k|$ by definition of m_{k+1}. Fill these in an arbitrary way which uses each word in $|A_k|$ at least once. By definition, this creates a word in A_{k+1}, which we denote by $w_i^{(k+1)}$. Define $x^{(k+1)}(((i - 0.5)m_{k+1}, (i + 0.5)m_{k+1})) = w_i^{(k+1)}$ for any i as above (i.e. those for which $S \cap ((i - 0.5)m_{k+1}, (i + 0.5)m_{k+1}) \neq \varnothing$) and as $*$ elsewhere. Note that $x^{(k+1)}$

is an infinite concatenation of words in A_{k+1} and blocks of $*$ of length m_{k+1} which contains $*$ on any interval $((i-0.5)m_{k+1},(i+0.5)m_{k+1})$ which is disjoint from S.

We now have defined $x^{(k)} \in (A \sqcup \{*\})^{\mathbb{Z}}$ for all $k \in \mathbb{N}$. Since each is obtained from the previous by changing some $*$s to letters from A, they approach a limit x_u which agrees with $x^{(0)}$ on all locations where $x^{(0)}$ had letters from A, i.e. $x_u(s_n) = u(n)$ for all $n \in \mathbb{N}$. Since $S \neq \varnothing$, $S \cap (-0.5m_k, 0.5m_k) \neq \varnothing$ for all large enough k, and so $x^{(k)}((-0.5m_k, 0.5m_k))$ has no $*$, meaning that $x_u \in A^{\mathbb{Z}}$.

It remains only to show that $x_u \in X$. By definition, x_u is a concatenation of words in A_k for every k, so $x_u \in Y = \bigcap_k Y_k$ as in the definition of X. Finally, every subword w of x_u is contained in $x_u((-0.5m_k, 0.5m_k))$ for large enough k, and this word is in A_k by definition. Since all words in A_k are subwords of w_{k+1}, w is also. Therefore by definition, $\mathbf{x_u} \in \mathbf{X}$ and $\mathbf{x_u(s_n)} = \mathbf{u(n)}$ for all n, completing the proof. □

Remark 6. We observe that the assumption of zero Banach density cannot be weakened in Theorem 5. Assume for a contradiction that $S \subset \mathbb{N}$ has $d^*(S) = \alpha > 0$, and that every $u \in A^{\mathbb{N}}$ could be assigned x_u as in Theorem 5. By definition of Banach density, there exist intervals I_n with lengths approaching infinity so that $|S \cap I_n|/|I_n| > \alpha/2$ for all n. For every n, since all possible assignments of letters from A to locations in $S \cap I_n$ give rise to sequences in X, $|L_{|I_n|}(X)| \geq 2^{|S \cap I_n|} > |A|^{\alpha |I_n|/2}$. Then,

$$h(X) = \lim_n \frac{\ln |L_{|I_n|}(X)|}{|I_n|} \geq \limsup \frac{\ln |A|^{\alpha |I_n|/2}}{|I_n|} = \alpha(\ln |A|)/2 > 0.$$

Therefore, no such X, minimal or otherwise, can have zero entropy.

References

[1] Tanja Eisner. A polynomial version of Sarnak's conjecture. *C. R. Math. Acad. Sci. Paris*, 353(7):569–572, 2015.

[2] Frank Hahn and Yitzhak Katznelson. On the entropy of uniquely ergodic transformations. *Trans. Amer. Math. Soc.*, 126:335–360, 1967.

[3] Adam Kanigowski, Mariusz Lemańczyk, and Maksym Radziwiłł. Prime number theorem for analytic skew products. Annals of Mathematics, to appear, 2023.

[4] Zhengxing Lian and Ruxi Shi. A counter-example for polynomial version of Sarnak's conjecture. *Adv. Math.*, 384:Paper No. 107765, 14, 2021.

[5] Ronnie Pavlov. Some counterexamples in topological dynamics. *Ergodic Theory Dynam. Systems*, 28(4):1291–1322, 2008.

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