Strong coding trees and applications to Ramsey theory on infinite graphs

Natasha Dobrinen
University of Denver

50 Years of Set Theory in Toronto

Thanks!

To the organizers for the invitation.

To the Fields Institute for support to participate in this conference.

To National Science Foundation for Grant DMS-1600781 for supporting this research.

To Stevo for taking me as an unofficial postdoc in 2008, catalyzing my research into Ramsey theory.

Finite Ramsey's Theorem

Finite Ramsey Theorem. (Ramsey, 1929) Given $k, m, r \geq 1$, there is an $n \geq m$ such that for each coloring of the k-element subsets of n into r colors, there is an $X \subseteq n$ of size m such that the coloring takes one color on the k-element subsets of X.

$$(\forall k, m, r \geq 1) \ \ (\exists n \geq m) \ \ n \to (m)^k_r$$

Infinite Ramsey's Theorem

Infinite Ramsey's Theorem. (Ramsey, 1929) Given $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite subset $X \subseteq \omega$ such that c is constant on $[X]^k$.

$$(\forall k, r \geq 1) \;\; \omega \to (\omega)^k_r$$

Graph Interpretation: For $k \geq 1$, given a complete k-hypergraph on infinitely many vertices and a coloring of the k-hyperedges into finitely many colors, there is an infinite complete sub-hypergraph in which all k-hyperedges have the same color.

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is Ramsey if each non-empty open set $\mathcal{O} \subseteq [\omega]^\omega$ contains another non-empty open subset $\mathcal{O}' \subseteq \mathcal{O}$ such that either $\mathcal{O}' \subseteq \mathcal{X}$ or else $\mathcal{O}' \cap \mathcal{X} = \emptyset$.

Nash-Williams Theorem. (1965) Clopen sets are Ramsey.

Galvin-Prikry Theorem. (1973) Borel sets are Ramsey.

Silver Theorem. (1970) Analytic sets are Ramsey.

Ellentuck Theorem. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \to_* (\omega)^\omega$$

Fraïssé Classes and Their Limits

A collection $\mathcal{K}$ of finite structures forms a Fraïssé class if it satisfies the Hereditary Property, the Joint Embedding Property, and the Amalgamation Property.

The Fraïssé limit of a Fraïssé class $\mathcal{K}$, denoted Flim($\mathcal{K}$) or $\mathbb{K}$, is (up to isomorphism) the ultrahomogeneous structure with Age($\mathbb{K}$) $= \mathcal{K}$.

Examples. Finite linear orders $\mathcal{LO}$; $\mathrm{Flim}(\mathcal{LO}) = \mathbb{Q}$.

Finite graphs $\mathcal{G}$; $\mathrm{Flim}(\mathcal{G}) =$ Rado graph.

Finite Structural Ramsey Theory

For structures A, B, write $A \leq B$ iff A embeds into B.

A Fraïssé class $\mathcal{K}$ has the Ramsey property if

$$(\forall A \leq B \in \mathcal{K})\ (\forall r \geq 1)\ \ \mathrm{Flim}(\mathcal{K}) \to (B)^A_r$$

Some classes of finite structures with the Ramsey property:
Linear orders, complete graphs, Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Recent Motivation: Kechris-Pestov-Todorcevic correspondence between Ramsey property and extreme amenability.

Infinite Structural Ramsey Theory

Let $\mathcal{K}$ be a Fraïssé class and $\mathbb{K} = \text{Flim}(\mathcal{K})$.

(KPT 2005) For $A \in \mathcal{K}$, $T(A, \mathcal{K})$ is the least number T, if it exists, such that for each $k \geq 1$ and any coloring of the copies of A in $\mathbb{K}$, there is a substructure $\mathbb{K}' \leq \mathbb{K}$, isomorphic to $\mathbb{K}$, in which the copies of A have no more than T colors.

$$(\forall k \geq 1) \ \ \mathbb{K} \to (\mathbb{K})^A_{k, T(A,\mathcal{K})}$$

$\mathbb{K}$ has finite big Ramsey degrees if $T(A, \mathcal{K})$ is finite, for each $A \in \mathcal{K}$.

Motivation. Problem 11.2 in (KPT 2005) and (Zucker 2019).

Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S(2)}$, $\mathbf{S(3)}$. (Laflamme, NVT, Sauer 2010)
- The random k-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)
- Profinite graphs. (Huber-Geschke-Kojman, and Zheng 2018)

Infinite Dimensional Structural Ramsey Theory

(KPT 2005) Given $\mathbb{K} = \mathrm{Flim}(\mathcal{K})$ and some natural topology on $\mathbb{I}_{\mathbb{K}} := \binom{\mathbb{K}}{\mathbb{K}}$,

$$\mathbb{K} \to_* (\mathbb{K})^{\mathbb{K}}$$

means that all "definable" subsets of $\mathbb{I}_{\mathbb{K}}$ are Ramsey.

Motivation. Problem 11.2 in (KPT 2005).

Examples. The Baire space $[\omega]^\omega = \mathbb{I}_\omega$. Any topological Ramsey space. Very little known about infinite dimensional Ramsey theory for Fraïssé structures.

Several results on big Ramsey degrees use

(1) Trees to code structures.

(2) Milliken's Ramsey theorem for strong trees, and variants.

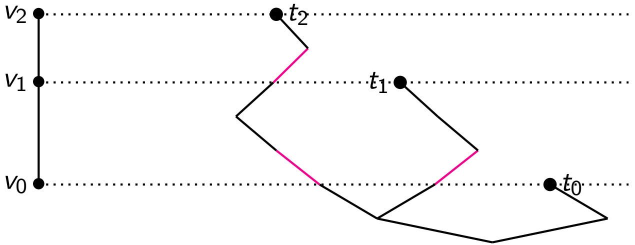
Using Trees to Code Binary Relational Structures

Rationals. $(\mathbb{Q}, <)$ can be coded by $2^{<\omega}$.

Graphs. Let A be a graph with vertices $\langle v_n : n < N \rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Strong Trees

For $t \in 2^{<\omega}$, the length of t is $|t| = \mathrm{dom}(t)$.

$T \subseteq 2^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

For $t \in T$, the height of t is $\mathrm{ht}_T(t) = \mathrm{o.t.}\{u \in T : u \subset t\}$.

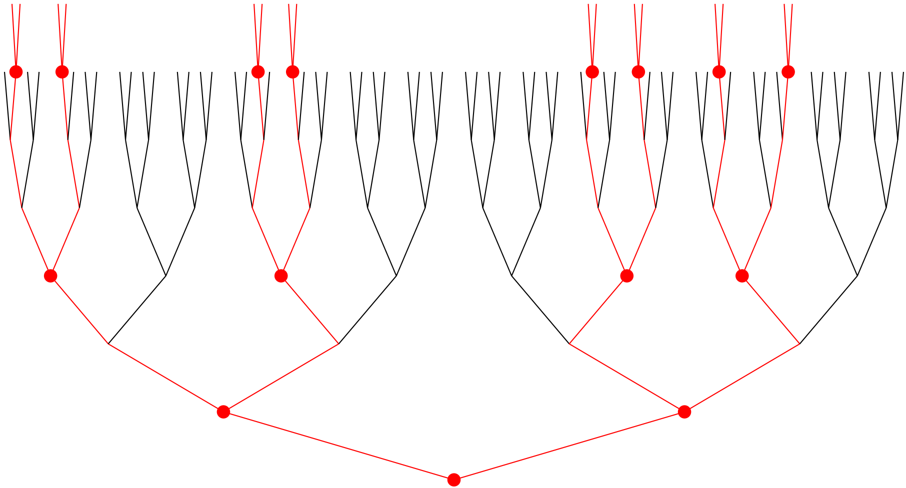
$T(n) = \{t \in T : \mathrm{ht}_T(t) = n\}$.

For $t \in T$, $\mathrm{Succ}_T(t) = \{u \restriction (|t|+1) : u \in T \text{ and } u \supset t\}$.

$S \subseteq T$ is a strong subtree of T iff for some $\{m_n : n < N\}$ $(N \leq \omega)$,

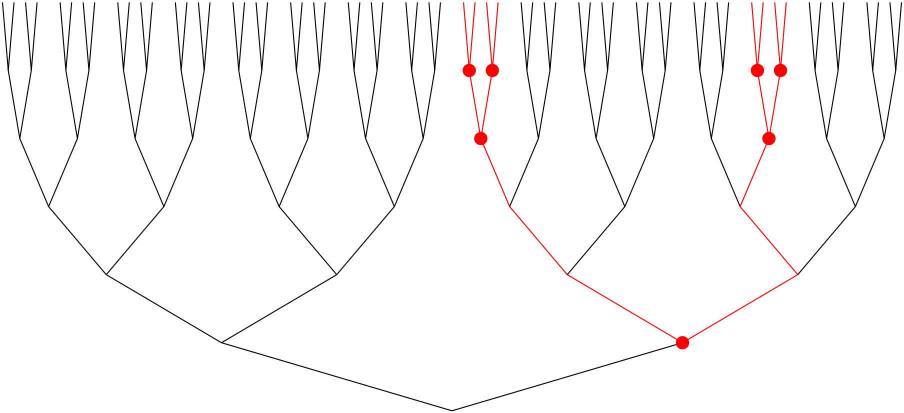
1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < N$, $s \in S(n)$ and $u \in \mathrm{Succ}_T(s)$, there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $0, 1, 3, 6, \ldots$

Example: A Strong Subtree $U \subseteq 2^{<\omega}$



The nodes in U are of lengths $1, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees

A *k-strong tree* is a finite strong tree with k levels.

Thm. (Milliken 1979) Let $T \subseteq 2^{<\omega}$ be a strong tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

Big Ramsey degrees of the Rado graph

Fact. (Henson 1971) Vertices have big Ramsey degree 1.

Thm. (Erdős-Hajnal-Pósa 1975) Edges have big Ramsey degree ≥ 2.

Thm. (Pouzet-Sauer 1996) Edges have big Ramsey degree exactly 2.

Thm. (Sauer 2006) All finite graphs have finite big Ramsey degree.

Idea: Since the Rado graph is bi-embeddable with the graph coded by all nodes in $2^{<\omega}$, one can use Milliken's Theorem on strong trees and later take out a copy of the Rado graph to deduce upper bounds for its big Ramsey degrees.

Thm. Actual big Ramsey degrees found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

My induction to studying Henson graphs

In early 2012, while reading Stevo's book on Ramsey Spaces, I got interested in whether the ultrahomogeneous universal triangle-free graph has finite big Ramsey degrees.

Later that year, at the Fields Thematic Semester, I asked Stevo about it and he told me one would need a new Milliken theorem, but no one knew what that should be.

$\mathcal{H}_k$: The k-Clique-Free Random Graph

For $k \geq 3$, a k-clique, denoted K_k, is a complete graph on k vertices.

The the k-clique-free Henson graph, $\mathcal{H}_k$, is the Fraïssé limit of the Fraïssé class of finite K_k-free graphs.

Thus, $\mathcal{H}_k$ is the ultrahomogenous K_k-free graph which is universal for all k-clique-free graphs on countably many vertices.

Henson graphs were constructed by Henson in 1971.

Henson Graphs: History of Results

Thm. (Henson 1971) For each $k \geq 3$, $\mathcal{H}_k$ is weakly indivisible.

Thm. (Nešetřil-Rödl 1977/83) The Fraïssé class of finite ordered K_k-free graphs has the Ramsey property.

Thm. (Komjáth-Rödl 1986) $\mathcal{H}_3$ is indivisible.

Thm. (El-Zahar-Sauer 1989) For all $k \geq 4$, $\mathcal{H}_k$ is indivisible.

Thm. (Sauer 1998) Edges have big Ramsey degree 2 in $\mathcal{H}_3$.

Thm. (Dobrinen 2017 and 2019) For each $k \geq 3$, $\mathcal{H}_k$ has finite big Ramsey degrees.

New Methods

Problem for Henson graphs: no Milliken theorem, and no nicely definable structure which is bi-embeddable with $\mathcal{H}_k$.

Question. How do you make a tree that codes a K_k-free graph which branches enough to carry some Ramsey theory?

Key Ideas in the proof that Henson graphs have finite big Ramsey degrees include

(1) Trees with coding nodes.

(2) Use forcing mechanism to obtain (in ZFC) new Milliken-style theorems for trees with coding nodes.

Other ideas and methods were developed, but we will concentrate mostly on these two today.

Structure of Proof

Proof Strategy: Follow the outline of Sauer's proof, but start with the end in mind and develop what is needed from scratch.

I Develop strong $\mathcal{H}_k$-coding trees which code $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for strictly similar finite antichains.

This is an analogue of Milliken's Theorem for strong trees. The proof uses forcing for a ZFC result, building on ideas of Harrington for the Halpern-Läuchli Theorem.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_k$.

Similar to the end of Sauer's proof.

Trees with Coding Nodes

A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c \rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n-th coding node in T, usually denoted c_n^T.

K_k-Free Branching Criterion

Note: A collection of coding nodes $\{c_{n_i} : i < k\}$ in T codes a k-clique iff $i < j < k \longrightarrow c_{n_j}(|c_{n_i}|) = 1$.

A tree T with coding nodes $\langle c_n : n < N \rangle$ satisfies the K_k-Free Branching Criterion (k-FBC) if for each non-maximal node $t \in T$, $t^\frown 0 \in T$ and

(∗) $t^\frown 1$ is in T iff adding $t^\frown 1$ as a coding node to T would not code a k-clique with coding nodes in T of shorter length.

Henson's Criterion for building $\mathcal{H}_k$

Henson gave a criterion for building $\mathcal{H}_k$, interpreted to our setting here:

A tree with coding nodes satisfies $(A_k)^{\texttt{tree}}$ iff

(i) T satisfies the K_k-Free Criterion.

(ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i - 1$, and each finite subset of ω appears as F_i for infinitely many indices i. Given $i < \omega$, if for each subset $J \subseteq F_i$ of size $k - 1$, $\{c_j : j \in J\}$ does not code a $(k-1)$-clique, then there is some $n \geq i$ such that for all $j < i$, $c_n(l_j) = 1$ iff $j \in F_i$.

Thm. (D.) Suppose T is a tree with no maximal nodes satisfying the K_k-Free Branching Criterion, and the set of coding nodes dense in T. Then T satisfies $(A_k)^{\texttt{tree}}$, and hence codes $\mathcal{H}_k$.

Strong K_3-Free Tree

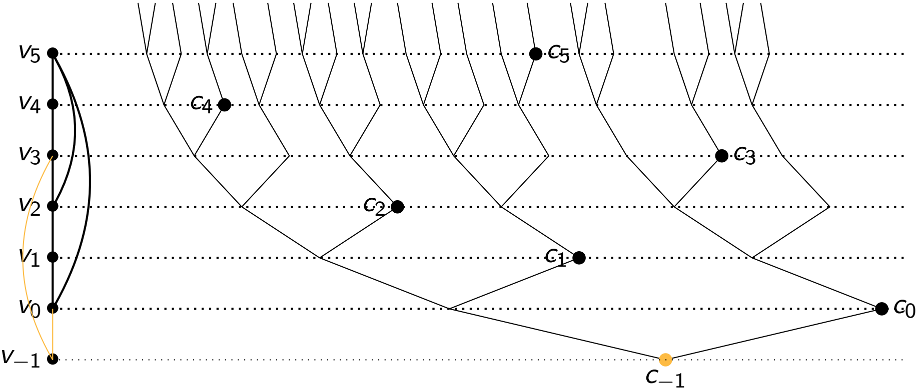


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree

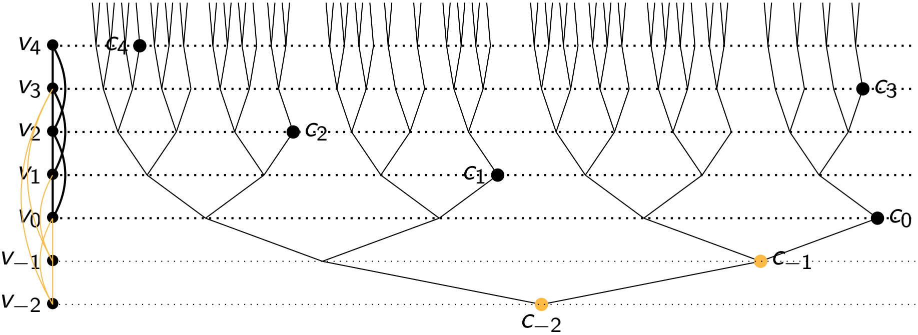


Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

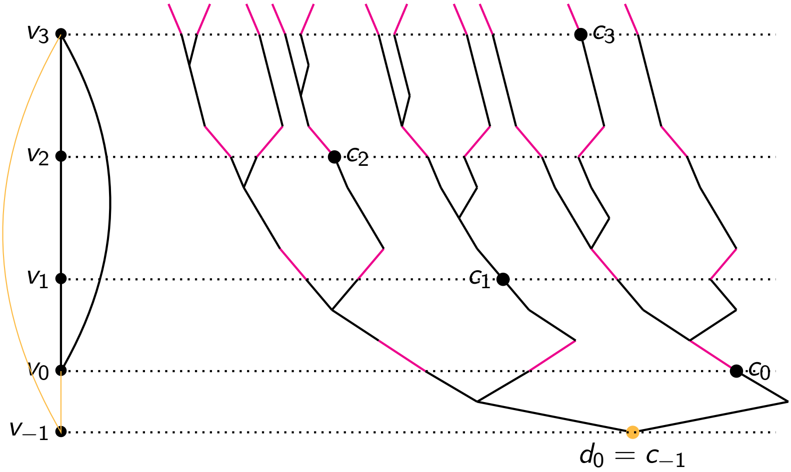
Almost sufficient

One can develop almost all the Ramsey theory one needs on strong K_k-free trees

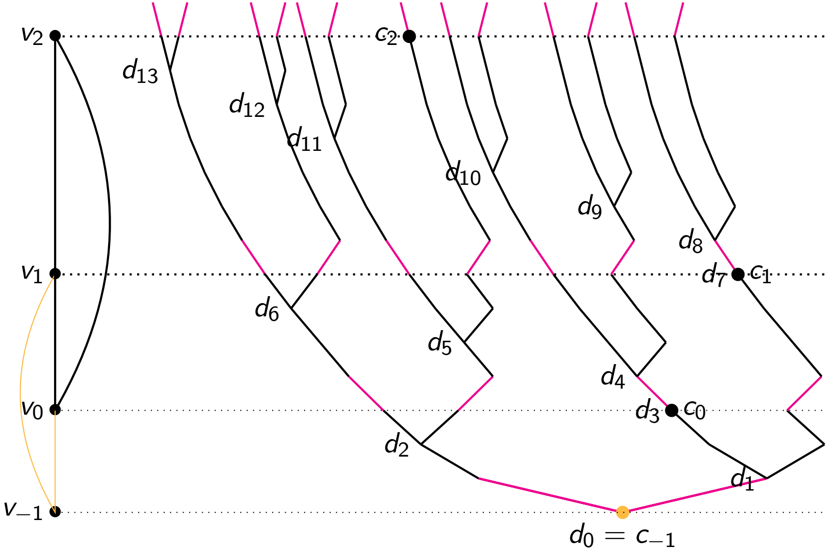
except for vertex colorings: there is a bad coloring of coding nodes.

Solution: Skew the levels of interest.

Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$

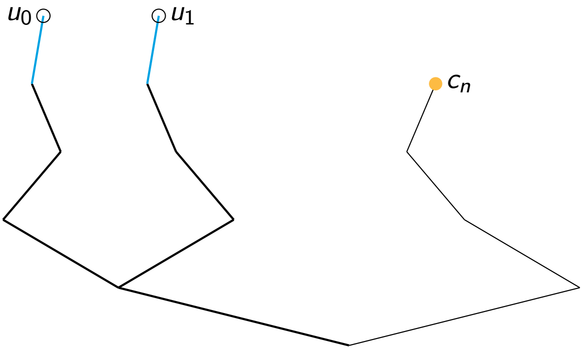


Pre-a-Clique: A key concept

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level set $X \subseteq \mathbb{T}_k$ of size at least two, with nodes of length ℓ_X, has a pre-a-clique if there are $a - 2$ coding nodes in $\mathbb{T}_k$ coding an $(a - 2)$-clique, and each node in X has passing number 1 by each of these coding nodes.

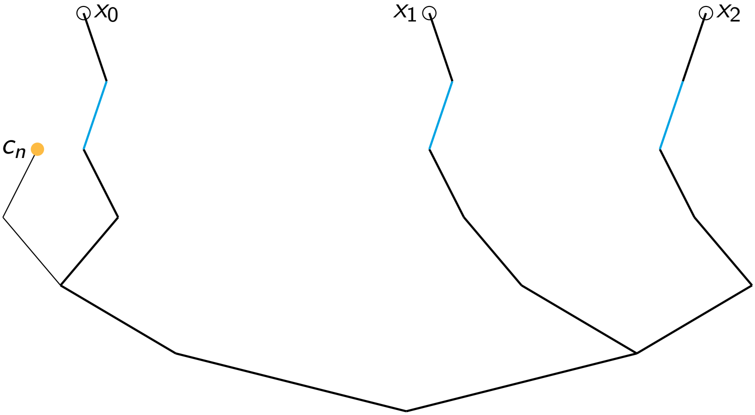
The Point. Pre-a-cliques for $a \in [3, k]$ code entanglements that affect how nodes in X can extend inside $\mathbb{T}$.

A level set U with a pre-3-clique



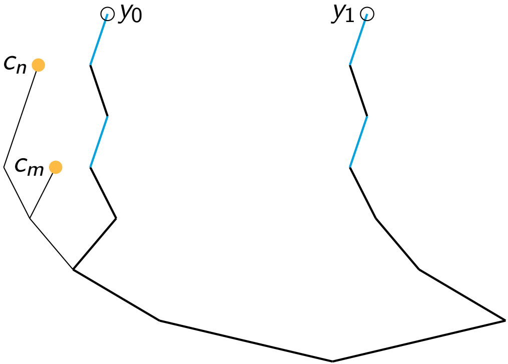
The yellow node is a coding node in $\mathbb{T}_k$ not in U.

A level set X with a pre-3-clique



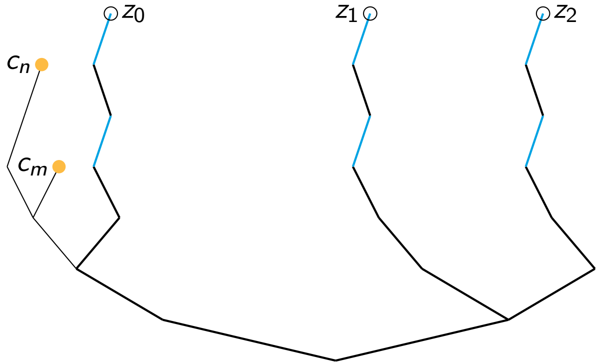
The yellow node is a coding node in $\mathbb{T}_k$ not in X.

A level set Y with a pre-4-clique



The yellow node is a coding node in $\mathbb{T}_k$ not in Y.

A level set Z with a pre-4-clique



The yellow node is a coding node in $\mathbb{T}_k$ not in Z.

The Space of Strong $\mathcal{H}_k$-Coding Trees $\mathcal{T}_k$

Two subtrees S and T of $\mathbb{T}_k$ are stably isomorphic iff there is a strong similarity map $f : S \to T$ which preserves maximal new pre-cliques in each interval. Such a map f is a stable isomorphism.

Idea: Stable isomorphisms preserve

1. the structure of the trees with respect to tree and lexicographic orders
2. placement of coding nodes
3. passing numbers at levels of coding nodes
4. whether or not an interval has new pre-cliques.

$\mathcal{T}_k$ is the collection of all subtrees of $\mathbb{T}_k$ which are stably isomorphic to $\mathbb{T}_k$.

The members of $\mathcal{T}_k$ are called strong $\mathcal{H}_k$-coding trees.

Part II: Ramsey Theorem for Strictly Similar Finite Antichains

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

(b) Then weave together to obtain an analogue of Milliken's Theorem.

(c) New notion of envelope.

Ramsey Theorem for Strictly Similar Antichains

Thm. (D.) Let Z be a finite antichain of coding nodes in a strong $\mathcal{H}_k$-coding tree $T \in \mathcal{T}_k$, and suppose h colors of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.

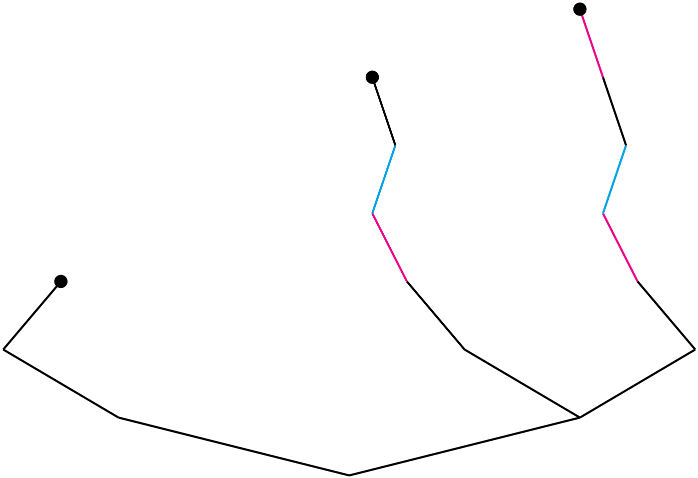
Some Examples of Strict Similarity Types for $k = 3$

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

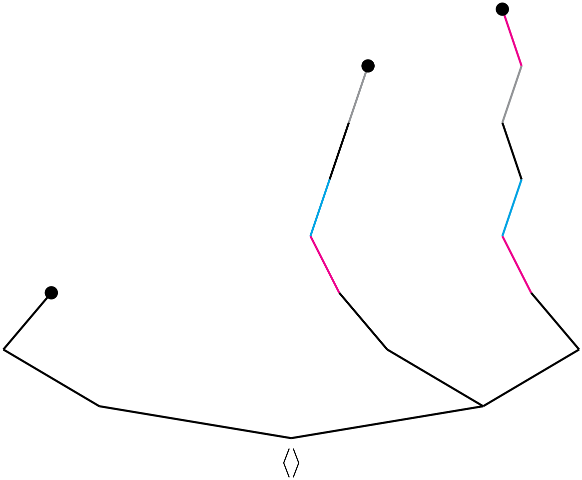
G a graph with three vertices and no edges

A tree A coding G



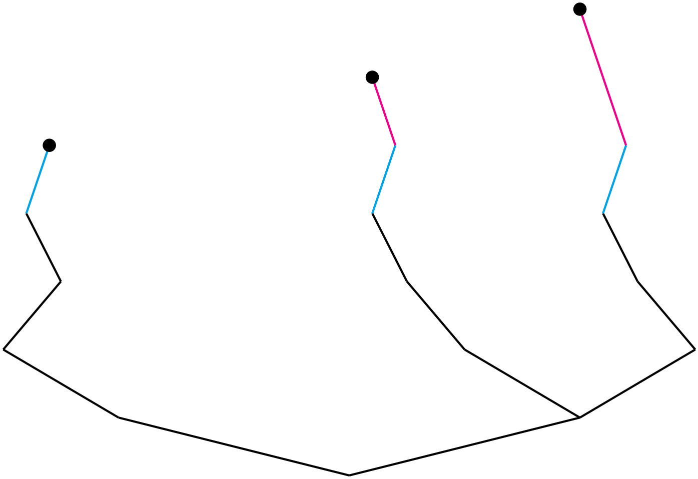
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



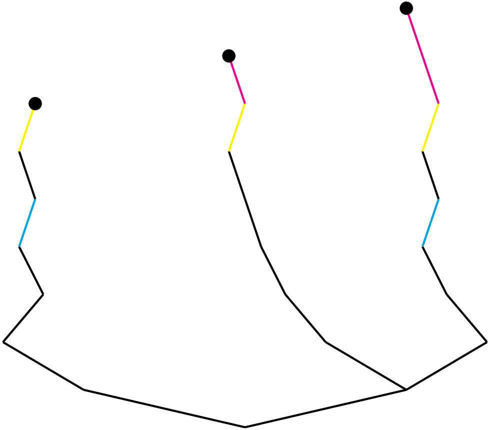
The tree C codes G

C is not strictly similar to A.

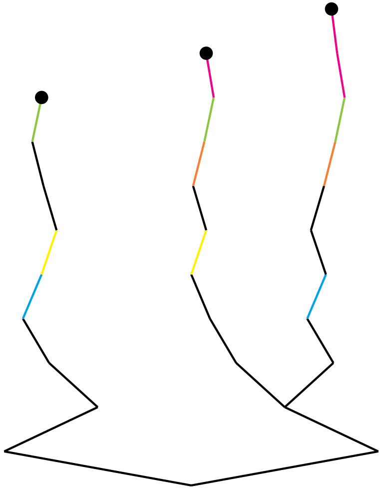


The tree D codes G

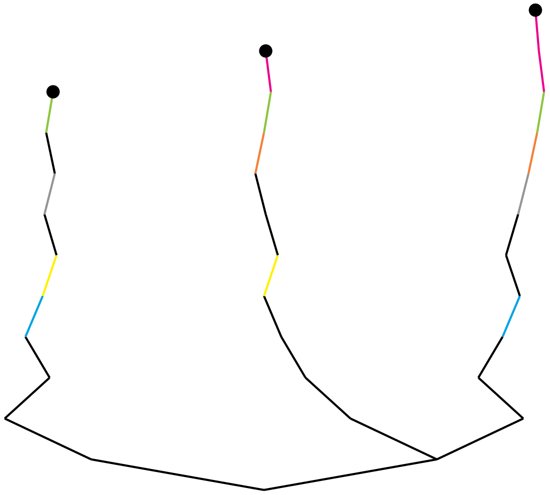
D is not strictly similar to either A or C.



The tree E codes G and is not strictly similar to A - D



The tree F codes G and is strictly similar to E



Envelopes and Witnessing Coding Nodes

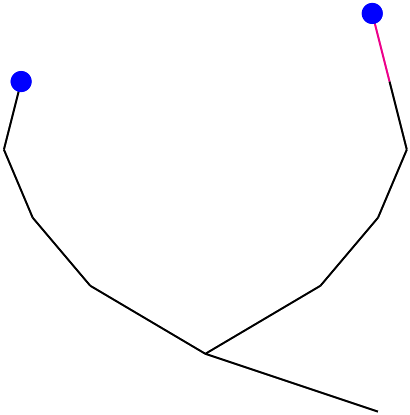
Envelopes add some neutral coding nodes to a finite tree so that all new pre-cliques are witnessed by a coding node.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

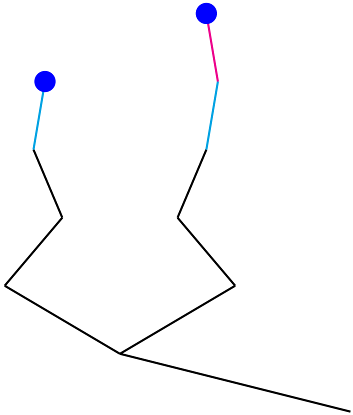
We now give some examples of envelopes.

H codes a non-edge



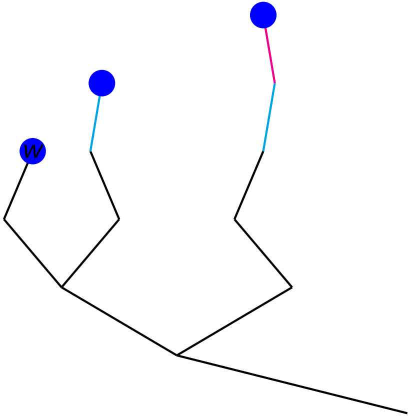
H is its own envelope.

I codes a non-edge



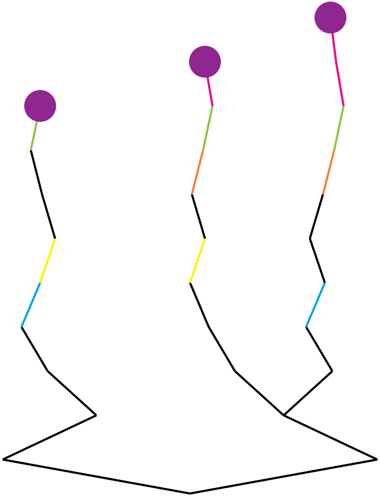
I is not its own envelope.

An Envelope $\mathbf{E}(I)$

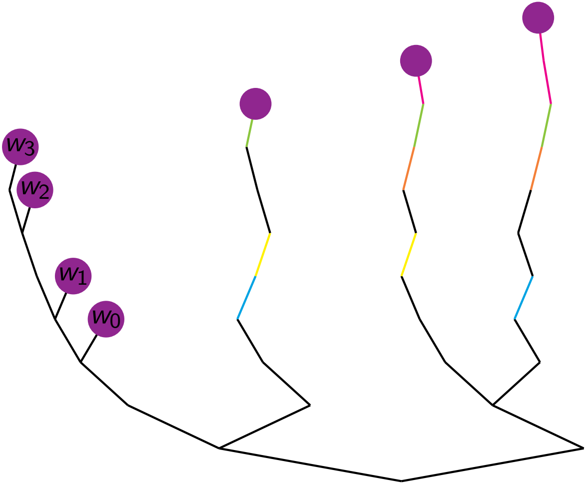


An envelope of I.

The antichain E from before

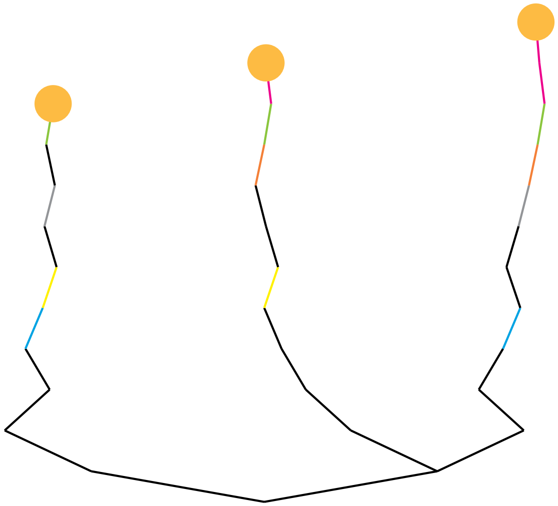


An envelope $\mathbf{E}(E)$

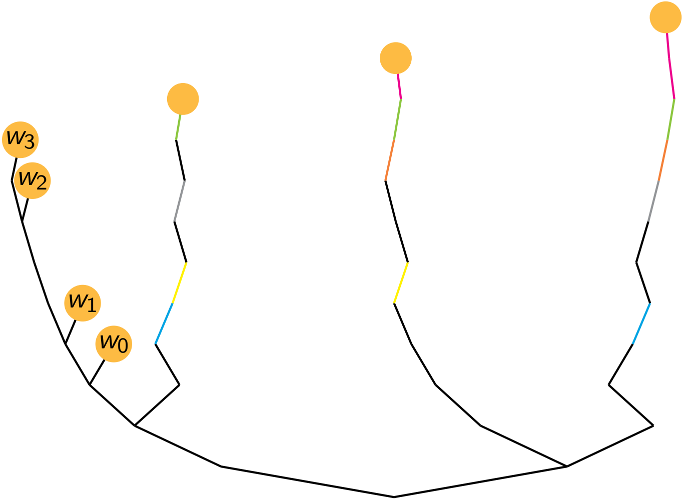


The coding nodes $w_0, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E

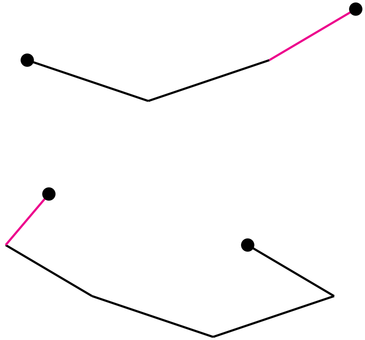


$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



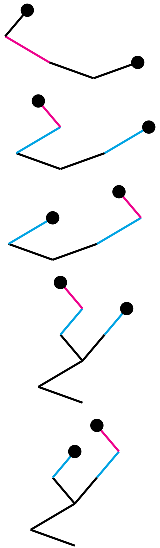
The coding nodes $w_0, \ldots, w_3$ make an envelope of F.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



These are their own envelopes.
$T(Edge, \mathcal{G}_3) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types in $\mathcal{H}_3$ (D.)



Infinite Dimensional Ramsey Theory of the Rado Graph

Thm. (Galvin-Prikry 1973) Every Borel subset of the Baire space is Ramsey.

$$\omega \xrightarrow{\text{Borel}} (\omega)^{\omega}.$$

Question. (KPT 2005) Which Fraïssé structures have infinite dimensional Ramsey theory for definable subsets?

For the Rado graph, $\mathcal{R}$, the natural topology would be the one induced by ordering the vertices of $\mathcal{R}$ in order-type ω, and viewing $\binom{\mathcal{R}}{\mathcal{R}}$ as a subspace of the product space 2^{ω} with the Tychonoff topology.

By the work of Laflamme, Sauer and Vuksanovic, we would need to restrict to copies of the Rado graph which are strongly similar.

The Space of Strong Rado Coding Trees $(\mathcal{T}_{\mathcal{R}}, \leq, r)$

Let $\langle u_n : n < \omega \rangle$ be a well-ordering of $2^{<\omega}$.

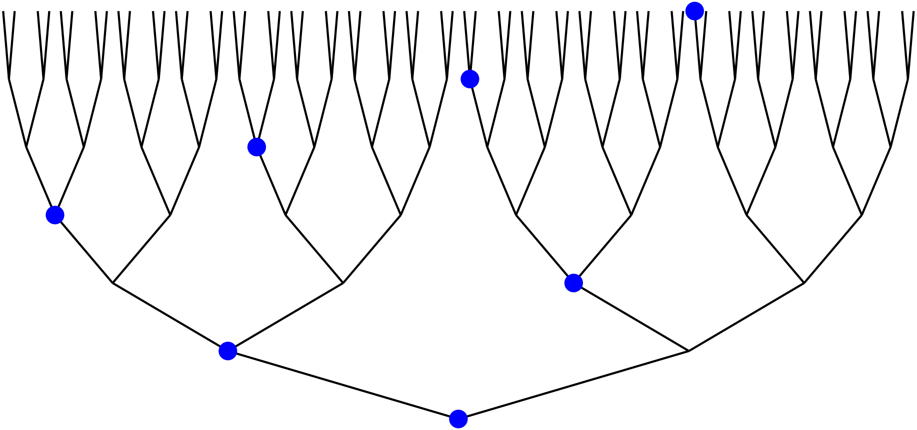
Define $\mathbb{T}_{\mathcal{R}} = (2^{<\omega}, \omega; \subseteq, <, c)$, where for each $n < \omega$, $c(n)$ is the lexicographically least node in 2^n extending u_n.

$\mathcal{T}_{\mathcal{R}}$ consists of all trees with coding nodes $(T, \omega; \subseteq, <, c^T)$, where

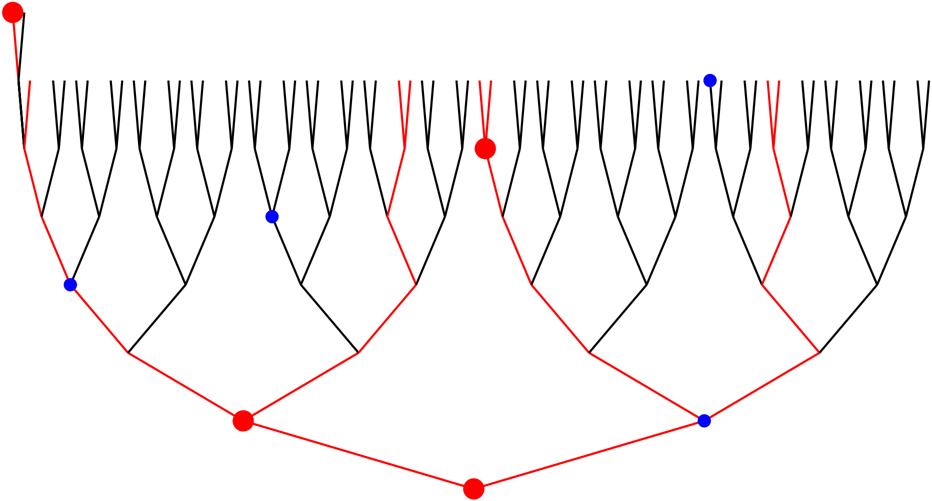
1. T is a strong subtree of $2^{<\omega}$; and
2. The strong tree isomorphism $\varphi : \mathbb{T}_{\mathcal{R}} \to T$ has the property that for each $n < \omega$, $\varphi(c(n)) = c^T(n)$.

The members of $\mathcal{T}_{\mathcal{R}}$ are called strong Rado coding trees.

Strong Rado Coding Tree $\mathbb{T}_{\mathcal{R}}$



A Strong Rado Coding Tree $T \in \mathcal{T}_{\mathcal{R}}$



Thm. (D.) Every Borel subset of $\mathcal{T}_{\mathcal{R}}$ has the Ramsey property.

So there is a topological space of Rado graphs which has infinite dimensional Ramsey theory.

Why only Borel and not Property of Baire?

Similarly to strong $\mathcal{H}_k$-coding trees, the collection of strong Rado trees form a space satisfying all of Todorcevic's Axioms for topological Ramsey spaces, except for A.3(2).

A "forced" Halpern-Läuchli-style theorem provides a means for fusion arguments in the style of Galvin-Prikry.

Question. Is there a topological Ramsey space of Rado graphs?

Future Directions

1. Extend methods to other infinite structures with forbidden configurations. In-progress: Ultrahomogeneous partial order, metric spaces, bowtie-free graph, etc.

2. Trees with coding nodes and these forcing arguments have allowed the development of infinite dimensional Ramsey theory on copies of the Rado graph. Extend these methods to other structures with finite big Ramsey degrees. It looks like they extend to the Henson graphs.

References

Dobrinen, *The Ramsey theory of the universal homogeneous triangle-free graph* (2017) (Submitted).

Dobrinen, *Ramsey theory of the Henson graphs* (2019) (Preprint).

Dobrinen, *Borel of Rado graphs and Ramsey's theorem* (2019) (Submitted).

Ellentuck, *A new proof that analytic sets are Ramsey*, JSL (1974).

Erdős-Rado, *A partition calculus in set theory*, Bull. AMS (1956).

Galvin-Prikry, *Borel sets and Ramsey's Theorem*, JSL (1973).

Halpern-Läuchli, *A partition theorem*, TAMS (1966).

Henson, *A family of countable homogeneous graphs*, Pacific Jour. Math. (1971).

References

Laflamme-Nguyen Van Thé-Sauer, *Partition properties of the dense local order and a colored version of Milliken's Theorem*, Combinatorica (2010).

Laflamme-Sauer-Vuksanovic, *Canonical partitions of universal structures*, Combinatorica (2006).

Larson, J. *Counting canonical partitions in the Random graph*, Combinatorica (2008).

Larson, J. *Infinite combinatorics*, Handbook of the History of Logic (2012).

Laver, *Products of infinitely many perfect trees*, Jour. London Math. Soc. (1984).

Kechris-Pestov-Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis (2005).

Milliken, *A Ramsey theorem for trees*, Jour. Combinatorial Th., Ser. A (1979).

Nešetřil-Rödl, *Partitions of finite relational and set systems*, Jour. Combinatorial Th., Ser. A (1977).

References

Nešetřil-Rödl, *Ramsey classes of set systems*, Jour. Combinatorial Th., Ser. A (1983).

Nguyen Van Thé, *Big Ramsey degrees and divisibility in classes of ultrametric spaces*, Canadian Math. Bull. (2008).

Pouzet-Sauer, *Edge partitions of the Rado graph*, Combinatorica (1996).

Sauer, *Edge partitions of the countable triangle free homogeneous graph*, Discrete Math. (1998).

Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica (2006).

Todorcevic, *Introduction to Ramsey spaces* (2010).

Zucker, *Big Ramsey degrees and topological dynamics*, Groups Geom. Dyn. (2019).

Thanks for your attention!

Happy 50th Birthday, Set Theory in Toronto!

Part II: Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

A closer look

Ideas:

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

(b) Then weave together to obtain an analogue of Milliken's Theorem.

Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and $A \subseteq B \subseteq T$ finite valid subtrees of T with WP, and $\max(A) \subseteq \max(B)$.

Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.

Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.

Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP.

Assume moreover that $0^{(l_{\tilde{X}})} \in \tilde{X}$.

Case (a). $\tilde{X}$ contains a splitting node.

Case (b). $\tilde{X}$ contains a coding node.

$$\mathrm{Ext}_T(A,\tilde{X}) = \{X \subseteq T : X \sqsupseteq \tilde{X} \text{ is a level set, } A \cup X \cong A \cup \tilde{X}, \\ \text{and } A \cup X \text{ is valid in } T\}.$$

Ramsey Theorem for Level Set Colorings

Thm. Assume the previous set-up.
Given any coloring $h : \mathrm{Ext}_T(A, \tilde{X}) \rightarrow 2$, there is a strong coding tree $S \in [B, T]$ such that h is monochromatic on $\mathrm{Ext}_S(A, \tilde{X})$.
If $\tilde{X}$ has a coding node, then the strong coding tree S is, moreover, taken to be in $[r_{m_0-1}(B'), T]$, where m_0 is the integer for which there is a $B' \in r_{m_0}[B, T]$ with $\tilde{X} \subseteq \max(B')$.

Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.
2. If d^A_{m+1} is a coding node, A has at most one new pre-clique in this interval.
3. If Y is a new pre-clique in this interval, then each proper subset of Y has a new pre-clique in some interval $(|d^A_j|, |d^A_{j+1}|]$, where $j < m$.

Lem. (D.) If $A \subseteq \mathbb{T}_k$ has the Strict Witnessing Property and $B \cong A$, then B also has the Strict Witnessing Property.

Any B stably isomorphic to A is a copy of A.

Ramsey Theorem for Finite Trees with SWP

Thm. (D.) Let $T \in \mathcal{T}_k$ and A be a finite subtree of T with the Strict Witnessing Property. Let c be a coloring of all copies of A in T. Then there is a strong $\mathcal{H}_k$-coding tree $S \leq T$ in which all copies of A in S have the same color.

This is an analogue of Milliken's Theorem for strong coding trees.

Forcing Arguments for Colorings of Level Sets

Case (i): level set X contains a splitting node.
List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \leq d$.

Fix κ large enough so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \upharpoonright l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new pre-cliques in T.

$q \le p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \ge l_p$, and

(i) $q(d) \supset p(d)$, and $q(i,\delta) \supset p(i,\delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) $\mathrm{ran}(q \upharpoonright \vec{\delta}_p)$ has no new pre-cliques above $\mathrm{ran}(p)$.

Case (i): Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}$, a name for the levels in $\dot{b}_d$.

For $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \ldots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle$.

- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

Case (i): The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p(d), &\langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ &\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ &\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

Case (i): f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*, \ k_{\vec{\alpha}} = k^*, \text{ and } (\forall i < d)\, \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf uses homogeneity of f.

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

Pf uses 'sliding' idea.

Case (i): A compatible set

Lem 3. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d\rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$.
Then for each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} =: t_i^*.$$

The $t_0^*, \dots, t_d^*$ provide good starting nodes for constructing the tree homogeneous for the coloring on $\mathrm{Ext}_T(A, \tilde{X})$.

Case (i): building a tree homog. for level set coloring

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

(2) These forcings are not simply Cohen forcings; the partial orderings are stronger in order to guarantee that the new levels we obtain by forcing are extendible inside T to another strong coding tree.

(3) The assumption that $A \cup \tilde{X}$ satisfies the Witnessing Property is necessary.

Case (ii): Coloring level sets with a coding node

This case is harder.

After obtaining a Ramsey theorem for level sets extending a given finite tree, there is a third case, using another forcing, to homogenize over monochromatic cones.

After this, much induction produces the Milliken analogue: The Ramsey Theorem for trees with the Strict Witnessing Property.

Envelopes are then used to obtain the final Ramsey Theorem for Strict Similarity Types.