Tutorial: Ramsey theory in Forcing - Day 2

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Day 2 Overview

Yesterday we saw how certain σ-closed forcings have dense subsets forming topological Ramsey spaces, and how ultrafilters forced by a Ramsey space have complete combinatorics and simpler methods for finding their Ramsey degrees.

The motivation for this is today's tutorial:

1. Rudin-Keisler and Tukey orders on ultrafilters.
2. New Ramsey-classification Theorems finding canonical equivalence relations on fronts and barriers.
3. Applications to find exact initial Rudin-Keisler and Tukey structures on ultrafilters.

Rudin-Keisler order on ultrafilters

$\mathcal{V} \le_{RK} \mathcal{U} \Leftrightarrow$ there is a function $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{U})$,

where $h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$.

Tukey order on ultrafilters

Let $\mathcal{U}, \mathcal{V}$ be ultrafilters.

Def. $\mathcal{X} \subseteq \mathcal{U}$ is **cofinal** in $(\mathcal{U}, \supseteq)$ iff for each $U \in \mathcal{U}$, there is an $X \in \mathcal{X}$ such that $X \subseteq U$; i.e. $\mathcal{X}$ is a filter base for $\mathcal{U}$.

Def. $\mathcal{V}$ is **Tukey reducible** to $\mathcal{U}$ ($\mathcal{V} \leq_T \mathcal{U}$) $\Leftrightarrow$ there is a **cofinal map** from $\mathcal{U}$ into $\mathcal{V}$: $\exists f : \mathcal{U} \to \mathcal{V}$ mapping each base for $\mathcal{U}$ to a base for $\mathcal{V}$.

Def. The **Tukey type** of $\mathcal{U}$ is the Tukey equivalence class of $\mathcal{U}$.

Guiding Questions in Tukey Theory of Ultrafilters

Fact. $\mathcal{V} \leq_{RK} \mathcal{U}$ implies $\mathcal{V} \leq_T \mathcal{U}$, but not vice versa.

1. What is the structure of the Tukey types of ultrafilters?
2. How closely related are the Tukey and Rudin-Keisler orderings?
3. What is the structure of the RK classes inside a Tukey type?

At the top

There is no maximal Rudin-Keisler type of an ultrafilter.

In constrast, Isbell showed that in ZFC there is always a Tukey maximal ultrafilter.

Open Problem. Is there a model of ZFC in which every ultrafilter has the maximal Tukey type?

Structures in RK and Tukey types

There are two main types of results: embeddings of structures and exact structures.

A recent result of Raghavan and Shelah shows that under MA(σ-centered), the structure of the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ embeds into the Tukey types of p-points.

By **initial Tukey structure** we mean a $\leq_T$-closed collection of Tukey types of ultrafilters. These are exact structures rather than embeddings of structures. Topological Ramsey spaces are useful for finding initial Tukey structures, as we shall see today.

Ramsey Ultrafilters are Tukey-minimal

Def. An ultrafilter $\mathcal{U}$ is **Ramsey** if for each $c : [\omega]^2 \to 2$, there is a $U \in \mathcal{U}$ such that c is monochromatic on $[U]^2$.

Def. The **Fubini product** of $\mathcal{U}$ and $\mathcal{V}$ is

$$\{X \subseteq \omega \times \omega : \{i \in \omega : \{j \in \omega : (i,j) \in X\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

Thm. (Todorcevic in [Raghavan/Todorcevic 12]) If $\mathcal{U}$ is Ramsey, $\mathcal{V}$ is non-principal, and $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ is isomorphic to a countable Fubini iterate of $\mathcal{U}$. Thus, Ramsey ultrafilters are Tukey minimal among the nonprincipal ultrafilters. Furthermore, there are exactly $\aleph_1$ RK-classes in $[\mathcal{U}]_T$.

Initial Structures known so far

The following are the initial Tukey and Rudin-Keisler structures and classification of the Rudin-Keisler types within the Tukey types obtained so far.

Thm. (D./Todorcevic) (CH, MA or in a forcing extension). For each $1 \leq \alpha < \omega_1$, there is an ultrafilter $\mathcal{U}_\alpha$ such that

1. The initial Tukey structure below $\mathcal{U}_\alpha$ is exactly the linear order $(\alpha + 1)^*$.
2. The initial Rudin-Keisler structure below $\mathcal{U}_\alpha$ is exactly the linear order $(\alpha + 1)^*$.
3. For each $\mathcal{V} \leq_T \mathcal{U}_\alpha$, the Tukey type of $\mathcal{V}$ consists precisely of the isomorphism classes of iterated Fubini products of ultrafilters from among a fixed countable collection of rapid p-points, which are represented by the canonical equivalence relations. Thus, there are exactly $\aleph_1$ many RK-classes in $[\mathcal{V}]_T$, and we know its structure.

Initial structures for hypercube ultrafilters

The topological Ramsey space $\mathcal{H}^2$ is dense in the n-square forcing in [Blass 73]. The hypercube space $\mathcal{H}^k$ uses blocks consisting of k-dimensional hypercubes.

Thm. (D./Mijares/Trujillo) For each $k \geq 2$, there is a topological Ramsey space $\mathcal{H}^k$ which forces an ultrafilter $\mathcal{U}_k$ such that

1. The initial Tukey structure below $\mathcal{U}_k$ is the Boolean algebra $\mathcal{P}(k)$.
2. The initial Rudin-Keisler structure below $\mathcal{U}_k$ is also $\mathcal{P}(k)$.
3. If $\mathcal{V} \leq_T \mathcal{U}_k$, then the Tukey type $[\mathcal{V}]_T$ consists of all isomorphism types of Fubini iterates of certain p-points represented by canonical projection maps.

Initial Tukey and Rudin-Keisler structures differing

Let $\mathcal{K}_0, \ldots, \mathcal{K}_n$ be any collection of Fraïssé classes of finite relational structures with the Ramsey property (and the OPFAP).

Thm. (D./Mijares/Trujillo) (CH, MA or by forcing) There is a p-point $\mathcal{U}$ such that

1. The initial Tukey structure below $\mathcal{U}$ is exactly $\mathcal{P}(n+1)$.
2. The set of isomorphism types of the product $\mathcal{K}_0 \times \cdots \times \mathcal{K}_n$, partially ordered by embedding, is realized as the initial RK structure below $\mathcal{U}$.
3. If $\mathcal{V} \leq_T \mathcal{U}$, then the Tukey type $[\mathcal{V}]_T$ consists of all RK types of Fubini iterates of p-points essentially coded by members of $\mathcal{K}_0 \times \cdots \times \mathcal{K}_n$.

Initial Tukey and RK structures of non-p-points

Extending the construction of Fin $\otimes$ Fin recursively to $\text{Fin}^{\otimes k}$, we obtain the forcings $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$.

Thm. (D.) there is an ultrafilter (non-p-point) $\mathcal{W}_k$ forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$ such that

1. The initial Tukey structure below $\mathcal{W}_k$ is exactly a chain of length k.
2. The initial RK structure below $\mathcal{W}_k$ is exactly a chain of length k.

Initial Tukey and RK structures of size $\mathfrak{c}$ for non-p-points

Extending the recursive construction of $\mathrm{Fin}^{\otimes k}$ to Fin^B, for B a uniform barrier on ω, let $\mathcal{W}_B$ denote the ultrafilter forced by $\mathcal{P}(B)/\mathrm{Fin}^B$.

Thm. (D.) $\mathcal{W}_B$ is not a p-point and

1. The initial RK structure below $\mathcal{G}_B$ is a linear order of size $\mathfrak{c}$ which is isomorphic to a certain non-standard model of ω.
2. The initial Tukey structure below $\mathcal{G}_B$ contains a copy of the initial Rudin-Keisler structure below $\mathcal{G}_B$, but also contains more.

Now we give an idea of how these results were obtained.

Todorcevic: A Ramsey ultrafilter is Tukey minimal

The proof proceeds by letting $\mathcal{V} \leq_T \mathcal{U}$, a Ramsey ultrafilter, and

1. Turning a cofinal map from $\mathcal{U}$ to $\mathcal{V}$ into a RK-map on a front using 'Continuous Cofinal Maps Theorem' in [D./Todorcevic 11],
2. Analysing the induced RK map in terms of canonical equivalence relations using Pudlák-Rödl Canonization Theorem.

This proof outline can be extended to topological Ramsey spaces.

Exact Tukey and RK Structures: General Proof Outline

Let $\mathcal{R}$ be a topological Ramsey space, $\mathcal{U}$ be the filter forced by $(\mathcal{R}, \leq^*)$, and suppose $\mathcal{V} \leq_T \mathcal{U}$. Let $f : \mathcal{U} \to \mathcal{V}$ be a monotone cofinal map. Wlog assume $\mathcal{V}$ is an ultrafilter on base set ω.

(1) Prove f is continuous in the metric topology on $\mathcal{R}$ when restricted below some member of $\mathcal{U}$.

(2) Then f is approximated by a finitary map $\hat{f} : \mathcal{AR} \to [\omega]^{<\omega}$.

(3) Let $\mathcal{F}$ be the front $\{a \in \mathcal{AR}$ minimal such that $\hat{f}(a) \neq \emptyset\}$.

(4) Define $g : \mathcal{F} \to \omega$ by $g(a) = \min \hat{f}(a)$, for $a \in \mathcal{F}$.

(5) Given $X \in \mathcal{U}$, define

$$\mathcal{F}|X = \{a \in \mathcal{F} : \exists k < \omega \, (a \leq_{\text{fin}} r_k(X))\}.$$

Exact Tukey and RK Structures: General Proof Outline

(6) Define an $\mathcal{U} \restriction \mathcal{F}$ to be the filter on base set $\mathcal{F}$ generated by the set of all $\mathcal{F}|X$, $X \in \mathcal{U}$. Prove $\mathcal{U} \restriction \mathcal{F}$ is an ultrafilter.

(7) Prove that $g(\mathcal{U} \restriction \mathcal{F}) = \mathcal{V}$. Thus, the ultrafilter $\mathcal{U} \restriction \mathcal{F}$ is RK equivalent to $\mathcal{V}$.

(8) Prove a Ramsey-classification Theorem for equivalence relations on fronts.

(9) Apply this to the function g on $\mathcal{F}$, since $g : \mathcal{F} \to \omega$ induces an equivalence relation on $\mathcal{F}$.

(10) Decode $g(\mathcal{F} \restriction \mathcal{U})$. Often, but not always, this is isomorphic to a Fubini iterate of p-points which are determined by the products of substructures on the blocks.

A general proof of (1) - (7) for a large class of spaces is given in [DMT].

The remainder of this day's tutorial concentrates on canonical equivalence relations on fronts and some specific examples.

We begin at the beginning.

Erdős-Rado Theorem

Extending Ramsey's Theorem to colorings with infinitely many colors amounts to extending it to equivalence relations.

An equivalence relation E on $[\omega]^k$ is **canonical** iff there is some $I \subseteq k$ such that $\mathrm{E} = \mathrm{E}_I$, where for $a = \{a_0, \dots, a_{k-1}\}$, $b = \{b_0, \dots, b_{k-1}\} \in [\omega]^k$,

$$a \ \mathrm{E}_I \ b \text{ iff } \forall i \in I,\ a_i = b_i.$$

Erdős-Rado Canonization Theorem. For each $k \geq 1$ and each equivalence relation E on $[\omega]^k$, there is an infinite $M \subseteq \omega$ such that $\mathrm{E} \restriction [M]^k$ is canonical.

Remark. E_I can be thought of as a projection map π_I, where $\pi_I(a) = \{a_i : i \in I\}$. Then $a \ \mathrm{E}_I \ b$ iff $\pi_I(a) = \pi_I(b)$.

Exercise. The Erdős-Rado Theorem implies Ramsey's Theorem.

Fronts and Barriers on $[\omega]^\omega$

Def. $\mathcal{F} \subseteq [\omega]^{<\omega}$ is a **front** on $[\omega]^\omega$ iff

(i) $\forall X \in [\omega]^\omega$, $\exists a \in \mathcal{F}$ such that $a \sqsubset X$; and

(ii) $\mathcal{F}$ is Nash-Williams: For $a, b \in \mathcal{F}$, $a \not\sqsubset b$.

$\mathcal{B} \subseteq [\omega]^{<\omega}$ is a **barrier** if (i) holds and also

(ii') $\mathcal{B}$ is Schreier: For $a \neq b \in \mathcal{B}$, $a \not\subseteq b$.

Galvin's Lemma. Any front is a barrier when restricted to some small enough infinite subset of ω.

Uniform fronts of rank $\alpha < \omega_1$

$[\omega]^k$ is the uniform front of rank k.

The **Schreier barrier**

$$\mathcal{S} = \{a \in [\omega]^{<\omega} : |a| = \min(a) + 1\}$$

is a uniform front of rank ω.

Uniform fronts of higher rank are made recursively from those of lower rank.

Extension of Erdős-Rado Theorem to all fronts

Def. For a front $\mathcal{F}$ and $M \in [\omega]^\omega$, $\mathcal{F}|M = \{a \in \mathcal{F} : a \subseteq M\}$.

Def. For a front $\mathcal{F}$, a map $\varphi : \mathcal{F} \to [\mathbb{N}]^{<\omega}$ is **irreducible** if φ is

(a) *inner*, i.e. $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$, and

(b) *Nash-Williams*, i.e. for each $a, b \in \mathcal{F}$, $\varphi(a) \not\sqsubset \varphi(b)$.

Pudlak-Rödl Canonization Thm. For every front (barrier) $\mathcal{F}$ on ω and every equivalence relation E on $\mathcal{F}$, there is an infinite $M \subseteq \omega$ such that $\mathrm{E} \restriction (\mathcal{F}|M)$ is represented by an irreducible map defined on $\mathcal{F}|M$.

Exercise. For a uniform barrier of finite rank, the Pudlák-Rödl Theorem gives back the Erdős-Rado Theorem.

Fronts on $(\mathcal{R}, \leq, r)$

Like the Ellentuck space, abstract topological Ramsey spaces have a notion of front.

Def. $\mathcal{F} \subseteq \mathcal{AR}$ is a **front** on $\mathcal{R}$ iff

(i) $\forall X \in \mathcal{R}$, $\exists a \in \mathcal{F}$ such that $a \sqsubset X$; and

(ii) $\mathcal{F}$ is Nash-Williams: For $a, b \in \mathcal{F}$, $a \not\sqsubset b$.

The finite rank fronts are of the form $\mathcal{AR}_k$ for some $k < \omega$. Recursively, one can extend the definition to infinite rank fronts.

A similar theorem to Galvin's Lemma allows us to interchange fronts and barriers in general topological Ramsey spaces, so we use fronts since they are simpler.

Now we focus on canonical equivalence relations for new topological Ramsey spaces.

Canonical Equivalence Relations for Products of Fraïssé Classes

The Erdős-Rado Theorem can be extended to products from large class of Fraïssé classes of ordered relational structures with the Ramsey property, as long as they have the Order-Preserving Free Amalgamation Property.

Thm. (D. in [DMT]) For finite products of Fraïssé classes of ordered relational structures with the Ramsey property and the OPFAP, canonical equivalence relations are given by canonical projection maps regarding only indices, exactly as for the Erdős-Rado Theorem.

By work of Nešetřil and Rödl (77 and 83), the Fraïssé classes of finite graphs, finite graphs omitting k-cliques, and other classes satisfy the conditions of the previous theorem.

We now go through how canonical projection maps decode the RK types of all ultrafilters Tukey reducible to the generic ultrafilter forced by the Ramsey space where blocks are from $\mathcal{K}_3$, the finite triangle-free graphs.

Board work.

We now turn an interesting class of non-p-points.

The forcing $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$

Recall that $\mathcal{P}(\omega)/\mathrm{Fin}$ forces a Ramsey ultrafilter.

$\mathrm{Fin} \otimes \mathrm{Fin} = \{X \subseteq \omega \times \omega : \forall^\infty i \in \omega\ \{j \in \omega : (i,j) \in X\} \text{ is finite}\}$.

Let $\mathcal{G}_2$ denote the generic ultrafilter forced by $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$.

Thm. (Blass/D./Raghavan) $\mathcal{G}_2$ is not a p-point, is a weak p-point, satisfies $\mathcal{U} \to (\mathcal{U})^2_{k,4}$, has exactly one RK-predecessor $\pi_0(\mathcal{G}_2)$, is not Tukey maximum, and is Tukey strictly above its projected Ramsey ultrafilter $\pi_0(\mathcal{G}_2)$.

This left open what exactly is the initial Tukey structure below $\mathcal{G}_2$.

2-Dimensional Ellentuck Space dense in $(\mathrm{Fin} \times \mathrm{Fin})^+$

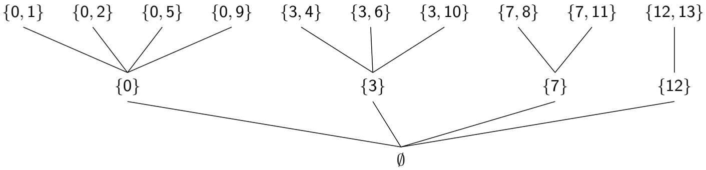

Figure: $\mathbb{W}_2 \subseteq [\omega]^2$; o.t.$(\mathbb{W}_2) = \omega^2$

$X \in \mathcal{E}_2$ iff $X \subseteq \mathbb{W}_2$ tree-isomorphic to $\mathbb{W}_2$ (so o.t.$(X) = \omega^2$) and respects the order of the labels on the nodes.

$Y \leq X$ iff $Y \subseteq X$.

$\mathcal{E}_2$ is a dense subset of $(\mathrm{Fin} \times \mathrm{Fin})^+$.

slide 85 and 86

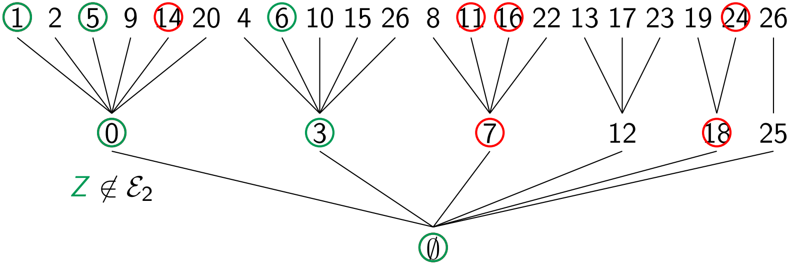

Figure: Two members X and Y of $\mathcal{E}_2$ with $Y \leq X$, and a $Z \notin \mathcal{E}_\in$.

The 2-dimensional Ellentuck spaces

Thm. (D.) $\mathcal{E}_2$ is a topological Ramsey space, and $(\mathcal{E}_2, \leq^*)$ is forcing equivalent to $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$.

The Tukey and RK structures below $\mathcal{G}_2$ are obtained using the canonical projection maps, similarly as in $\mathcal{A}_3$.

The initial Tukey structure below $\mathcal{G}_2$ consists only of $\mathcal{G}_2$ and the projected Ramsey ultrafilter below it.

Likewise the initial RK structure below $\mathcal{G}_2$ is also a chain of length 2.

$\mathcal{E}_3$, dense in $\mathcal{P}(\omega^3)/\mathrm{Fin}^{\otimes 3}$

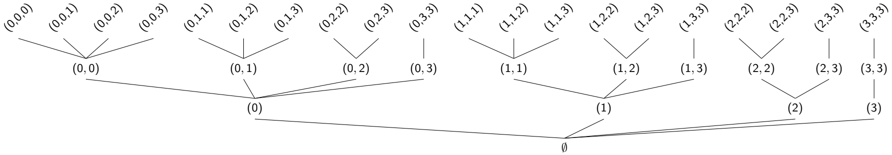

Figure: $\omega^{\not\leq 3}$

$$\emptyset \prec (0) \prec (0,0) \prec (0,0,0) \prec (0,0,1) \prec (0,1) \prec (0,1,1) \prec (1) \prec (1,1) \prec$$

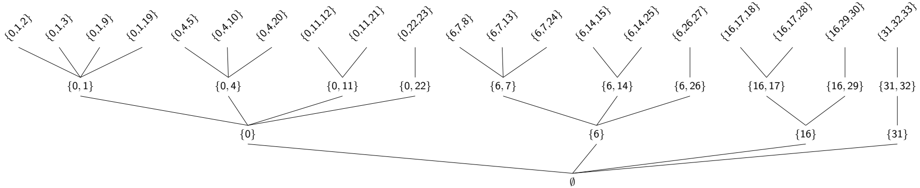

Figure: The maximum member of $\mathcal{E}_3$, $\mathbb{W}_3 \subseteq [\omega]^3$

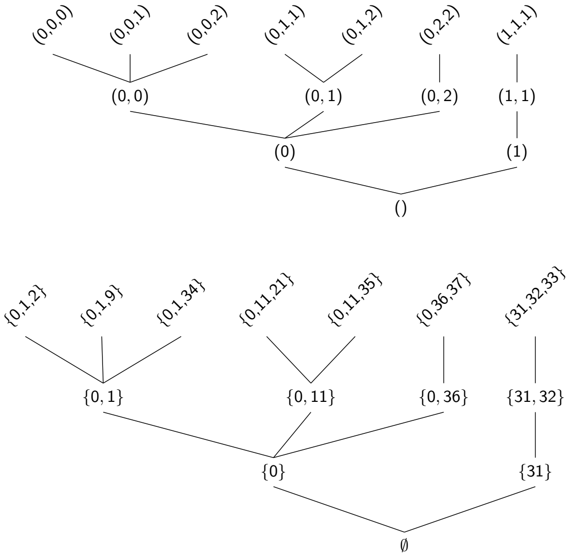

Figure: $r_7(Y)$, a typical finite approximation to a member of $\mathcal{E}_3$

The first infinite dimensional Ellentuck space

Let $\mathcal{S}$ denote the *Schreier barrier* $\{a \in [\omega]^{<\omega} : |a| = \min(a) + 1\}$.

$X \subseteq \mathcal{S}$ is in $\mathrm{Fin}^{\mathcal{S}}$ iff for all but finitely many i, $X_i \in \mathrm{Fin}^{\otimes i}$.

$$X_i = \{a \in X : \min(a) = i\}$$

$\mathrm{Fin}^{\mathcal{S}}$ is a σ-ideal on $\mathcal{S}$.

$\mathcal{P}(\mathcal{S})/\mathrm{Fin}^{\mathcal{S}}$ forces an ultrafilter $\mathcal{G}_{\mathcal{S}}$ on base set $\mathcal{S}$.

We use the form of $\mathcal{S}$ to make our template structure of finite non-decreasing sequences of natural numbers.

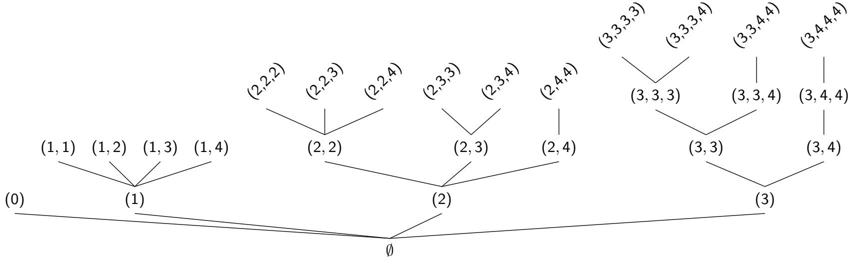

Figure: $\omega^{\not\downarrow\mathcal{S}}$

$() \prec (0) \prec (1) \prec (1,1) \prec (1,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (1,3) \prec (2,2,3) \prec (2,3) \prec (2,3,3) \prec (3) \prec (3,3) \prec (3,3,3) \prec (3,3,3,3) \prec (1,4) \prec \dots$

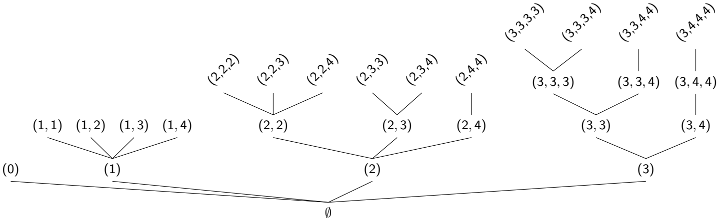

Figure: $\omega^{\not\downarrow\mathcal{S}}$

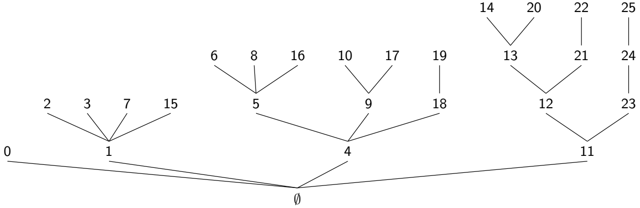

Figure: $\mathbb{W}_{\mathcal{S}}$

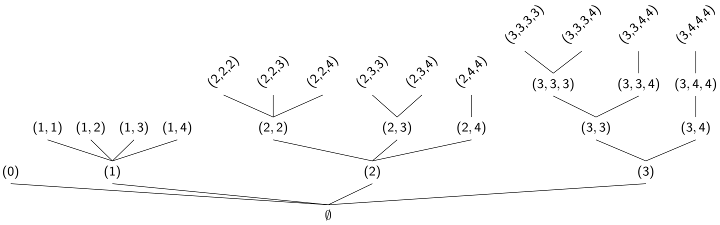

Figure: $\omega^{\not\downarrow\mathcal{S}}$

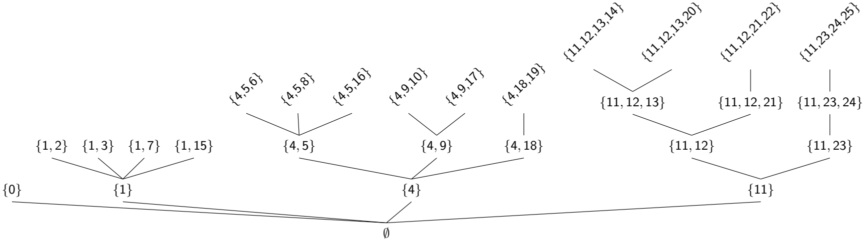

Figure: $\mathbb{W}_{\mathcal{S}}$

This construction can continue over all uniform barriers $\mathcal{B}$ of countable rank to obtain a topological Ramsey space $\mathcal{E}_{\mathcal{B}}$ dense in the forcing $\mathcal{P}(\mathcal{B})/\mathrm{Fin}^{\mathcal{B}}$.

These spaces include $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$ for all $\alpha < \omega_1$.

Ramsey degrees for ultrafilters forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$

The Ramsey space structure of the high-dimensional Ellentuck spaces make it possible, with work, to find the Ramsey degrees for their forced ultrafilters.

Thm. (Navarro Flores) Let $r(\mathcal{E}_k, 2)$ denote the number r such that

$$\mathcal{G}_k \to (\mathcal{G}_k)^2_{k,r}.$$

1. $r(\mathcal{E}_3, 2) = 14$.
2. $r(\mathcal{E}_4, 2) = 49$.
3. $r(\mathcal{E}_5, 2) = 175$.
4. $r(\mathcal{E}_6, 2) = 642$.
5. $r(\mathcal{E}_7, 2) = 2378$.

Moreover, there is a recursive formula for finding $r(\mathcal{E}_k, 2)$ for any $k < \omega$.

On Thursday, we will look at the Halpern-Läuchli Theorem, topological Ramsey spaces of strong trees, and applications to universal relational structures.

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