Ramsey theory of the Henson graphs

Natasha Dobrinen

University of Denver

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The Ramsey theory of the universal homogeneous triangle-free graph, 65 pp, submitted,

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The Ramsey theory of Henson graphs, 68 pp, preprint.

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This work commenced during the Newton Institute HIF Programme (2015).

Finite Ramsey Theorem

Finite Ramsey Theorem. (Ramsey, 1929) $k, m, r \geq 1$ with $m \geq k$, there is an $n \geq m$ such that for each coloring $c : [n]^k \to r$, there is an $X \in [n]^m$ such that c is monochromatic on $[X]^k$.

Finite Structural Ramsey Theory

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Some Fraïssé classes of finite structures with the Ramsey property:
Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Small Ramsey Degrees

A Fraïssé class $\mathcal{K}$ has small Ramsey degrees if for each $A \in \mathcal{K}$ there is an integer $t(A, \mathcal{K})$ such that for each $B \in \mathcal{K}$ with $A \leq B$, there is a $C \in \mathcal{K}$ with $B \leq C$ so that for each $r \geq 1$ and each coloring $f : \binom{C}{A} \to r$, there is a $B' \in \binom{C}{B}$ such that f takes at most $t(A, \mathcal{K})$ colors on $\binom{B'}{A}$.

$$\forall A \in \mathcal{K} \ \ \exists t(A, \mathcal{K}) \geq 1 \ \ \forall B \in \mathcal{K} \ \ \exists C \in \mathcal{K} \ \ \forall r \geq 1, \ \ C \to (B)^A_{r, t(A, \mathcal{K})}.$$

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The classes of finite graphs, hypergraphs, graphs omitting k-cliques, and others.

Infinite Ramsey Theorem

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Note: For $n = 2$, this can also be stated in terms of coloring edges in an infinite complete graph into finitely many colors finding an infinite complete graph with all edges having the same color.

Ramsey's Theorem and Logic

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Find a procedure for determining whether any given formula is valid.

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Ramsey applied his theorem to solve this problem for formulas with only universal quantifiers in front (Π_1).

Big Ramsey Degrees of Infinite Structures

Where combinatorics, set theory, model theory, and topology meet.

Big Ramsey Degrees of Infinite Structures

Let $\mathcal{S}$ be an infinite structure. For a finite substructure $\mathrm{A} \le \mathcal{S}$, let $T(A, \mathcal{S})$ denote the least number, if it exists, such that for each coloring c of $\binom{\mathcal{S}}{A}$ into finitely many colors, there is an $\mathcal{S}' \in \binom{\mathcal{S}}{\mathcal{S}}$ such that c takes no more than $T(A, \mathcal{S})$ colors on $\binom{\mathcal{S}'}{A}$.

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Infinite structures known to have finite big Ramsey degrees: The infinite complete graph (Ramsey 1929); the rationals (Devlin 1979); the Rado graph and random tournament (Sauer 2006); the countable ultrametric Urysohn space (Nguyen Van Thé 2008); the $\mathbb{Q}_n$ and the tournaments **S(2)**, **S(3)** (Laflamme, NVT, Sauer 2010), and a few others.

Ramsey Theory and Topological Dynamics

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(Zucker 2019) Characterized universal completion flows of Aut(Flim $\mathcal{K}$) whenever Flim $\mathcal{K}$ admits a big Ramsey structure (big Ramsey degrees).

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Infinite structures with forbidden configurations have presented particular difficulties.

The Problem: Lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

Guiding Example: Big Ramsey degrees of the Rado graph

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The previously known big Ramsey structures have at their core Milliken's Ramsey Theorem for strong trees.

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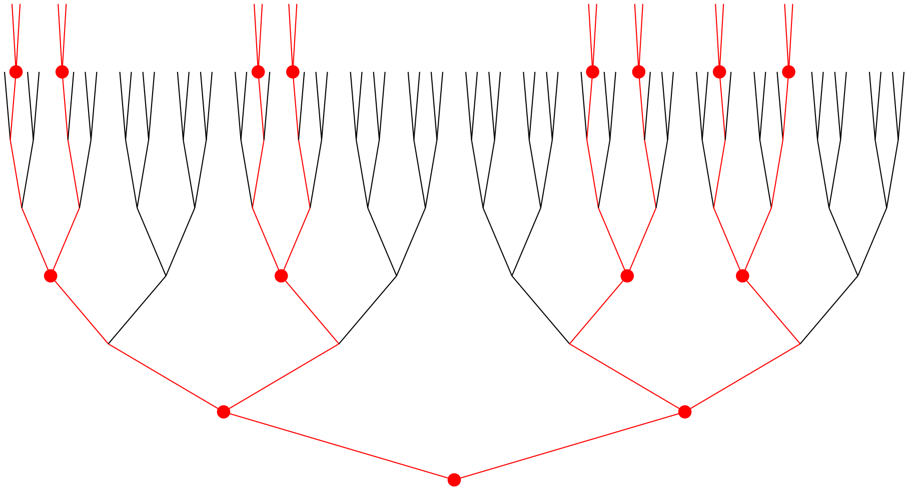
$S \subseteq T$ is a strong subtree of T iff there is an infinite set $\{m_n : n < \omega\}$ such that

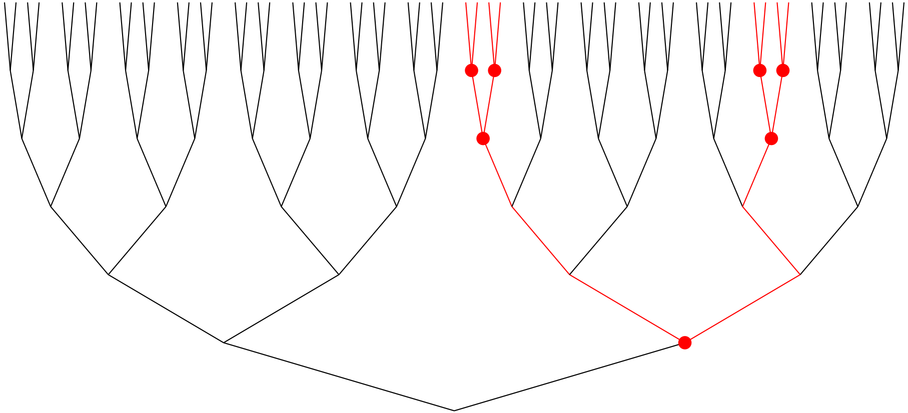
1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < \omega$, $s \in S(n)$ and $u \in \mathrm{Succ}_T(s)$, there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $S \subseteq 2^{<\omega}$



The nodes in S are of lengths $0, 1, 3, 6, \ldots$

Example: A Strong Subtree $U \subseteq 2^{<\omega}$



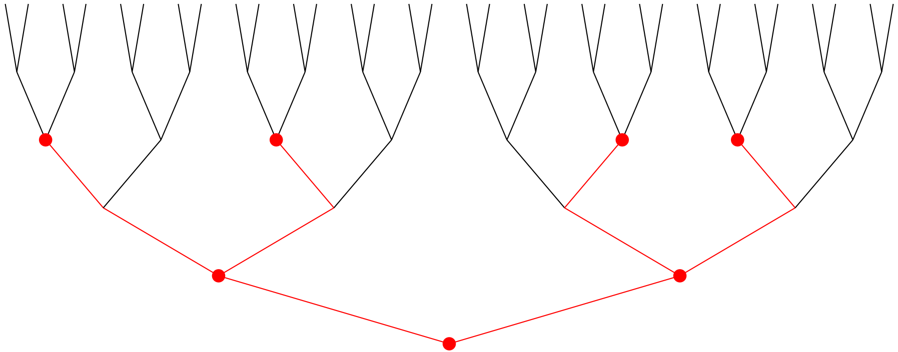
The nodes in U are of lengths $1, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $T \subseteq \omega^{<\omega}$ be a finitely branching tree with no terminal nodes. Let $k \geq 0$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

Ex: Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

Given a coloring c of all 3-strong trees in $2^{<\omega}$ into red and blue:



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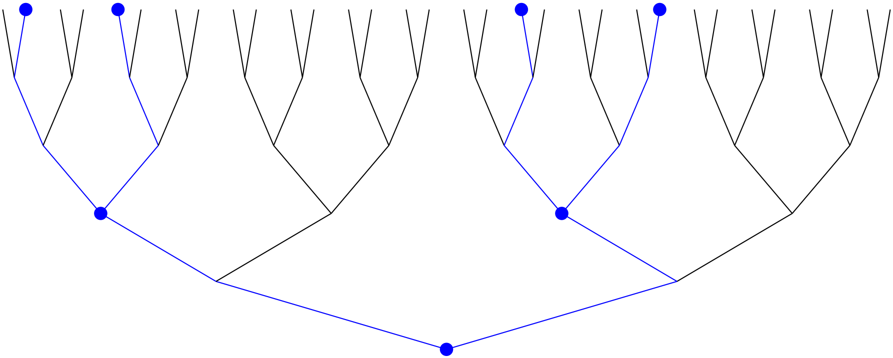

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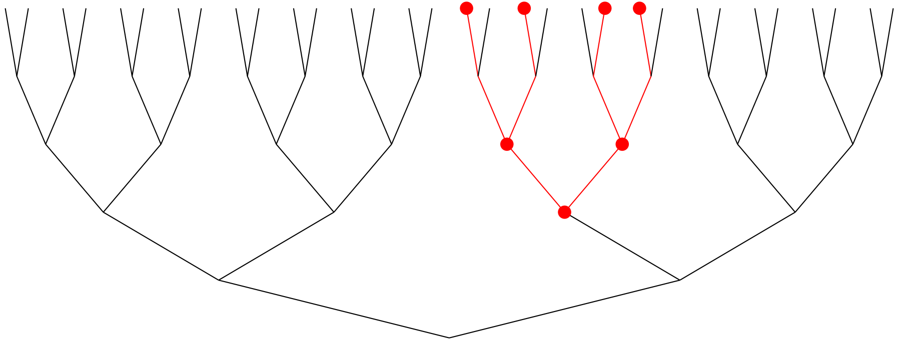

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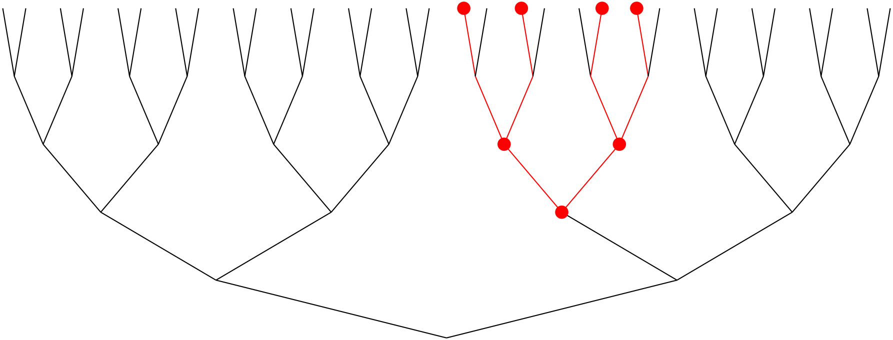

Milliken's Theorem guarantees a strong subtree in which all 3-strong subtrees have the same color.

Remark. Milliken's space $\mathcal{M}$ of infinite strong trees forms a topological Ramsey space.

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How is Milliken's Theorem applied to get upper bounds for the Ramsey degrees of the Rado graph?

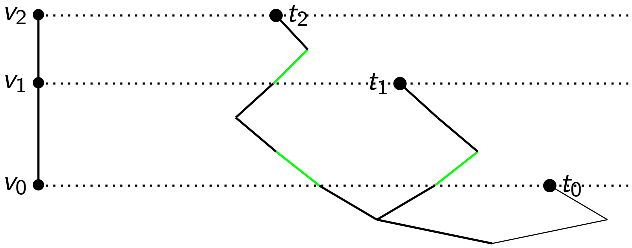
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1.$$

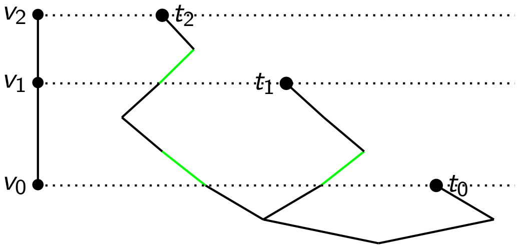
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Diagonal Trees Code Graphs

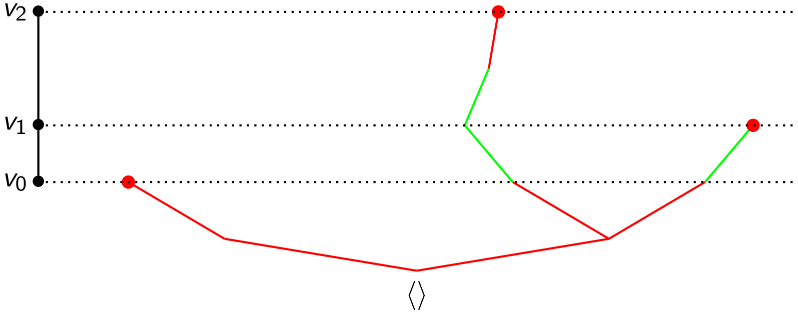
A tree T is diagonal if there is at most one meet or terminal node per level.

T is strongly diagonal if passing numbers at splitting levels are all 0 (except for the right extension of the splitting node).

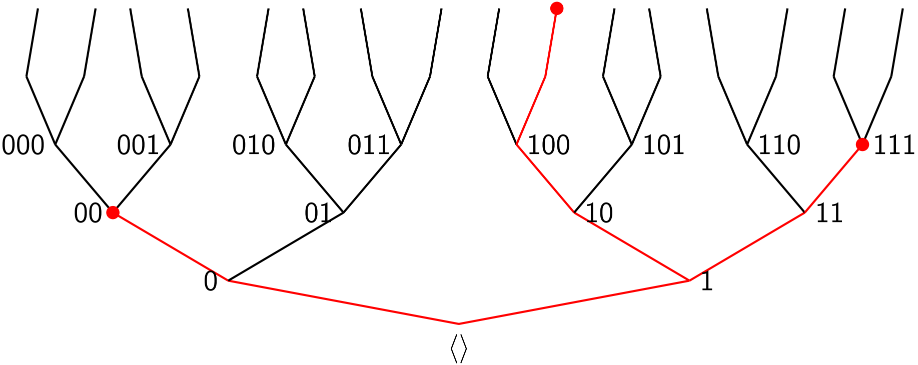


Every graph can be coded by the terminal nodes of a diagonal tree. Moreover, there is a strongly diagonal tree which codes $\mathcal{R}$.

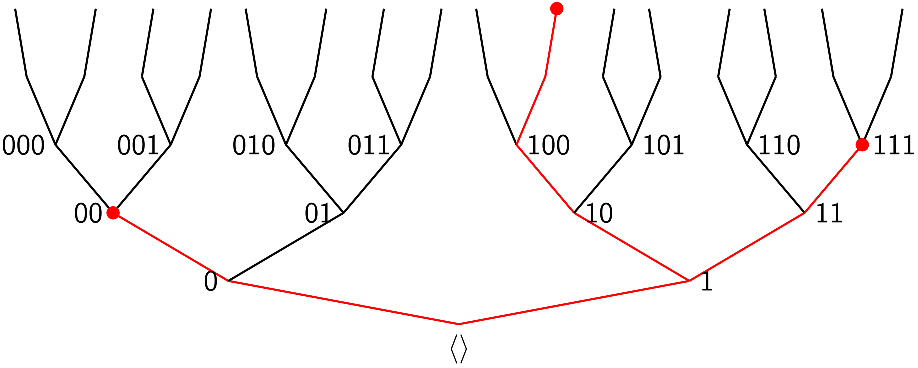
A Different Strongly Diagonal Tree Coding a Path



Strongly diagonal trees can be enveloped into strong trees



Another strong tree envelope



Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

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3. Each strongly diagonal tree can be enveloped into a finite strong tree.
4. Apply Milliken's Theorem finitely many times to obtain one color for each type.
5. Choose a strongly diagonal antichain coding the Rado graph.

Henson Graphs

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Main Thm. (D.) The Henson graphs have finite big Ramsey degrees.

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There progress halted. Why?

Main Obstacles to Big Ramsey Degrees of $\mathcal{H}_k$

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"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, Habilitation 2013)

Our proof strategy

Follow the outline of Sauer's proof of upper bounds for big Ramsey degrees of the Rado graph, constructing new analogues at each stage.

Main Theorem: Ramsey Theory for Henson Graphs

Theorem. (D.) Let $k \geq 3$. For each finite k-clique-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{G}_k)$ such that for any coloring of all copies of A in $\mathcal{H}_k$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_k$, with $\mathcal{H} \cong \mathcal{H}_k$, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{G}_k)$ colors.

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Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

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Similar to the end of Sauer's proof.

Part I: Strong $\mathcal{H}_k$-Coding Trees

Idea: Want correct analogue of strong trees for setting of $\mathcal{H}_k$.

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Problem: How to make sure K_k is never encoded
but branching is as thick as possible?

First Approach: Strong K_k-Free Trees

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• Make a Branching Criterion so that a node s splits iff all its extensions will never code K_k with coding nodes at or below the level of s.

K_k-Free Branching Criterion

For $a \geq 2$, given an index set I of size a, a collection of coding nodes $\{c_i : i \in I\}$ in T codes an a-clique iff for each pair $i < j$ in I, $c_j(l_i) = 1$.

K_k-Free Branching Criterion

For $a \geq 2$, given an index set I of size a, a collection of coding nodes $\{c_i : i \in I\}$ in T codes an a-clique iff for each pair $i < j$ in I, $c_j(l_i) = 1$.

A tree T with coding nodes $\langle c_n : n < N \rangle$ satisfies the K_k-Free Branching Criterion (k-FBC) if for each non-maximal node $t \in T$,

(a) $t^\frown 0$ is always in T, and

(b) $t^\frown 1$ is in T iff adding $t^\frown 1$ as a coding node to T would not code a k-clique with coding nodes in T of shorter length.

Henson's Criterion for building $\mathcal{H}_k$

Henson proved that a countable graph $\mathcal{H}$ is universal for countable K_k-free graphs if and only if $\mathcal{H}$ satisfies the property (A_k):

(i) $\mathcal{H}$ does not admit any k-cliques,

(ii) If V_0, V_1 are disjoint finite sets of vertices of $\mathcal{H}$ and $\mathcal{H}|V_0$ does not admit any $(k-1)$-cliques, then there is another vertex which is connected in $\mathcal{H}$ to every member of V_0 and to no member of V_1.

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For trees with coding nodes, this becomes $(A_k)^{\texttt{tree}}$:

(i) T satisfies the K_k-Free Criterion.

(ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i - 1$, and each finite subset of ω appears as F_i for infinitely many indices i. Given $i < \omega$, if for each subset $J \subseteq F_i$ of size $k-1$, $\{c_j : j \in J\}$ does not code a $(k-1)$-clique, then there is some $n \geq i$ such that for all $j < i$, $c_n(l_j) = 1$ iff $j \in F_i$.

Thm. Let T be a tree with no maximal nodes and coding nodes dense in T, and satisfying the K_k-Free Branching Criterion. Then T satisfies $(A_k)^{\mathtt{tree}}$, and hence codes $\mathcal{H}_k$.

Strong K_3-Free Tree

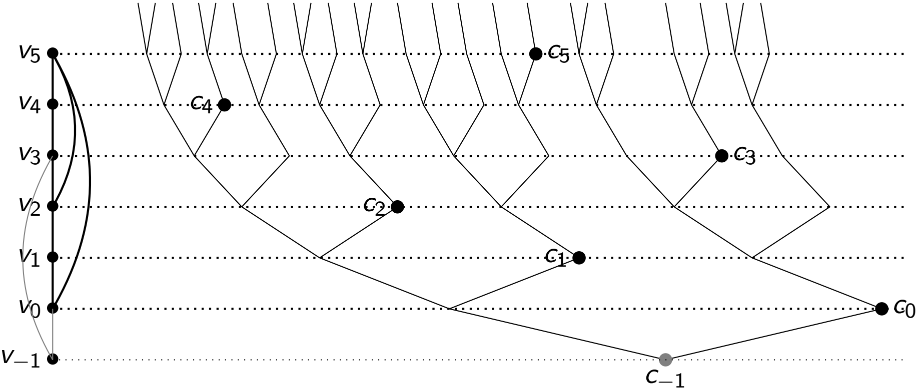

Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree

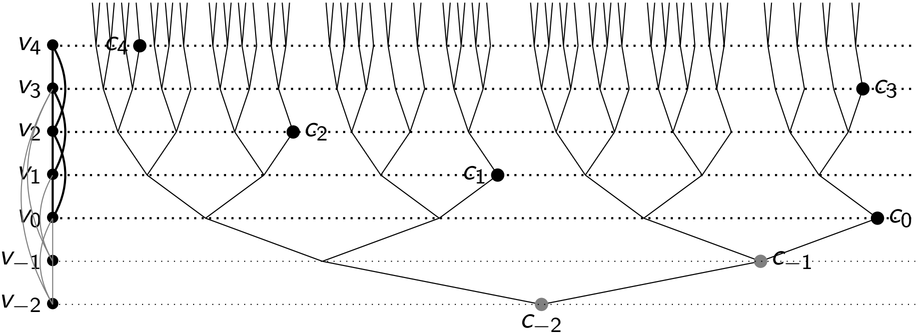

Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

Almost sufficient

One can develop almost all the Ramsey theory one needs on strong triangle-free trees

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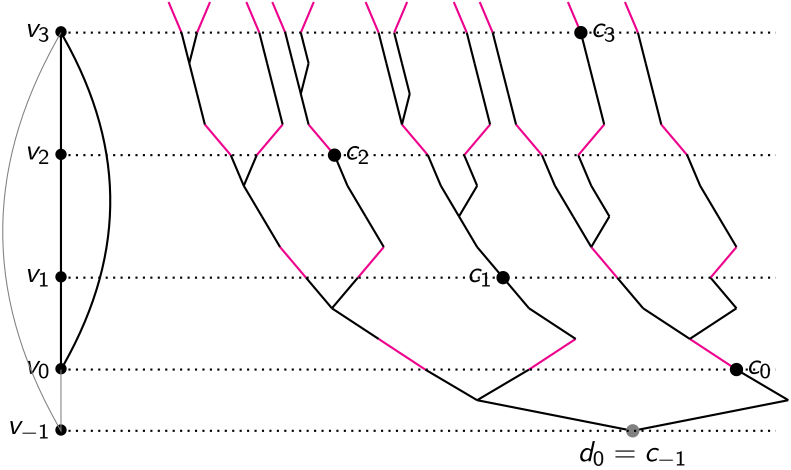
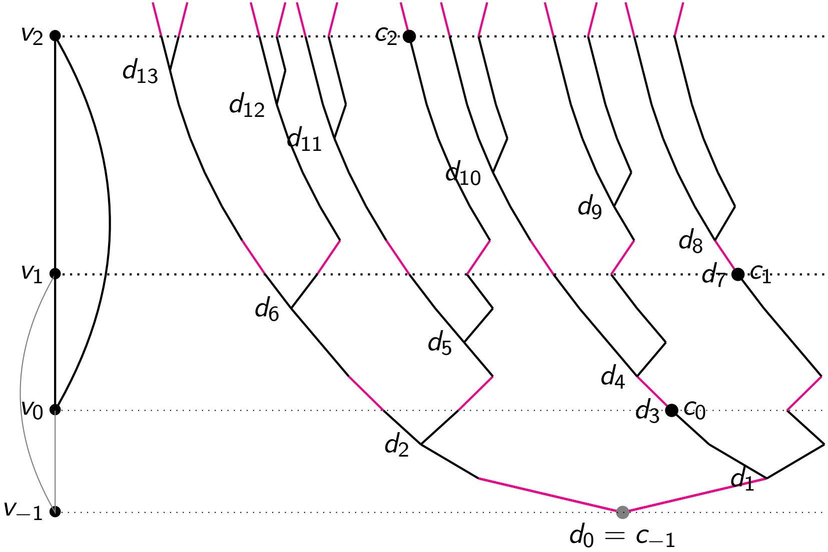
except for vertex colorings: there is a bad coloring of coding nodes.

Refined Approach: Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



Skew the levels of interest.

Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$



Strong Similarity Map

Let $k \geq 3$ be given and let $S, T \subseteq \mathbb{T}_k$ be meet-closed subsets. A bijection $f : S \to T$ is a strong similarity map if for all nodes $s, t, u, v \in S$, the following hold:

1. f preserves lexicographic order.
2. f preserves meets, and hence splitting nodes.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves coding nodes.
6. f preserves passing numbers at coding nodes.

Mutual Pre-a-Clique: A key concept

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level subset X of $\mathbb{T}_k$ of size at least two has a (mutual) pre-a-clique if $\exists \mathcal{I} \subseteq [\omega]^{a-2}$ such that, letting $i_* = \max(\mathcal{I})$ and $l_* = |c^k_{i_*}|$:

1. $l_* \leq l_X$, and there are exactly the same number of nodes in the level set $X \restriction l_*$ as in X;
2. The set $\{c^k_i : i \in \mathcal{I}\}$ codes a $(a-2)$-clique;
3. Each node in X^+ has passing number 1 at c^k_i, for each $i \in \mathcal{I}$.

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The Point. Pre-a-cliques for $a \in [3, k]$ code entanglements that affect how nodes can extend.

Stable Map

Let S and T be strongly similar subtrees of $\mathbb{T}_k$ with $M \leq \omega$ critical nodes. The strong similarity map $f : T \to S$ is stable if for each $m \in [1, M)$, the following holds:

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For each $a \in [3, k]$, a level subset $X \subseteq T \restriction |d^T_m|$ has a maximal new pre-a-clique in T in the interval $(|d^T_{m-1}|, |d^T_m|]$ if and only if $f[X]$ has a maximal new pre-a-clique in S in the interval $(|d^S_{m-1}|, |d^S_m|]$.

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For each $a \in [3, k]$, a level subset $X \subseteq T \restriction |d_m^T|$ has a maximal new pre-a-clique in T in the interval $(|d_{m-1}^T|, |d_m^T|]$ if and only if $f[X]$ has a maximal new pre-a-clique in S in the interval $(|d_{m-1}^S|, |d_m^S|]$.

We say that S and T are stably isomorphic and write $S \cong T$.

The Space of Strong $\mathcal{H}_k$-Coding Trees: $(\mathcal{T}_k, \leq, r)$

$\mathcal{T}_k$ is the collection of all subtrees of $\mathbb{T}_k$ which are stably isomorphic to $\mathbb{T}_k$.

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$S \leq T$ iff S is a subtree of T

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The space $\mathcal{T}_k$ is very near a topological Ramsey space.

A structural characterization of members of $\mathcal{T}_k$

A subtree T of $\mathbb{T}_k$ has the Witnessing Property (WP) if for each $a \in [3, k]$, each new pre-a-clique in T takes place in some interval in T of the form $(|d^T_{m_n-1}|, |c^T_n|]$ and is witnessed by a set of coding nodes *in* T.

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Lem. A tree $T \subseteq \mathbb{T}_k$ is a member of $\mathcal{T}_k$ iff T is strongly similar to $\mathbb{T}_k$ and has the Witnessing Property.

Extension Lemmas

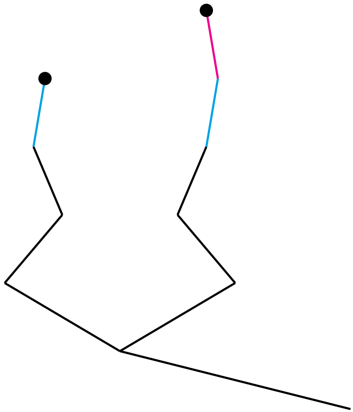
Not every finite subtree A of some $T \in \mathcal{T}_k$ can be extended within T as desired.

A series of extension lemmas shows that whenever A has the Witnessing Property and free level sets, then A is extendible as desired within T.

We call such finite trees valid in T.

A subtree of $\mathbb{T}_3$ in which WP fails

It has a pre-3-clique not witnessed by a coding node.



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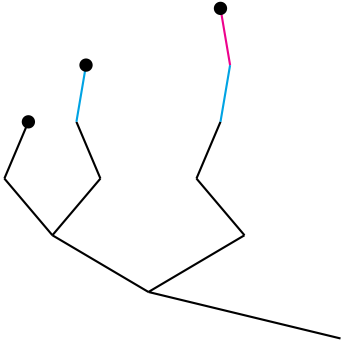

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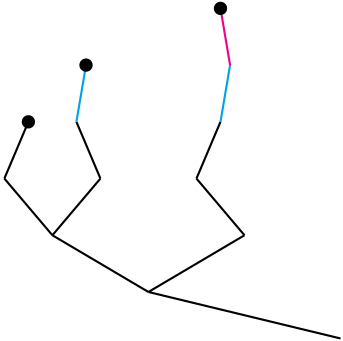

This gives the basic idea of WP.

Part II: A Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

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Ideas:

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

(b) Then weave together to obtain an analogue of Milliken's Theorem.

(a) Set-up for level set colorings

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Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP.

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$$\begin{aligned}\mathrm{Ext}_T(A, \tilde{X}) = \{X \subseteq T : X \sqsupseteq \tilde{X} \text{ is a level set, } A \cup X \cong A \cup \tilde{X},\\ \text{and } A \cup X \text{ is valid in } T\}.\end{aligned}$$

(a) Ramsey Theorem for Level Set Colorings

Thm. Assume the previous set-up.
Given any coloring $h : \mathrm{Ext}_T(A, \tilde{X}) \to 2$, there is a strong coding tree $S \in [B, T]$ such that h is monochromatic on $\mathrm{Ext}_S(A, \tilde{X})$.
If $\tilde{X}$ has a coding node, then the strong coding tree S is, moreover, taken to be in $[r_{m_0-1}(B'), T]$, where m_0 is the integer for which there is a $B' \in r_{m_0}[B, T]$ with $\tilde{X} \subseteq \max(B')$.

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We will go over the idea of the 'forcing' proof later, if time.

(b) Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

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3. If Y is a new pre-clique in this interval, then each proper subset of Y has a new pre-clique in some interval $(|d^A_j|, |d^A_{j+1}|]$, where $j < m$.

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Lem. If $A \subseteq \mathbb{T}_k$ has the Strict Witnessing Property and $B \cong A$, then B also has the Strict Witnessing Property.

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Lem. If $A \subseteq \mathbb{T}_k$ has the Strict Witnessing Property and $B \cong A$, then B also has the Strict Witnessing Property.

Any B stably isomorphic to A is a copy of A.

(b) Ramsey Theorem for Finite Trees with SWP

Thm. Let $T \in \mathcal{T}_k$ and A be a finite subtree of T with the Strict Witnessing Property. Let c be a coloring of all copies of A in T. Then there is a strong $\mathcal{H}_k$-coding tree $S \leq T$ in which all copies of A in S have the same color.

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This is an analogue of Milliken's Theorem for strong coding trees.

Part III: Ramsey Theorem for Strictly Similar Finite Antichains

Ramsey Theorem for Strictly Similar Antichains

Thm. Let Z be a finite antichain of coding nodes in an incremental tree $T \in \mathcal{T}_k$, and suppose h colors of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an incremental strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.

Ramsey Theorem for Strictly Similar Antichains

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New Concepts: incremental new pre-cliques, strict similarity, envelopes to transform an antichain to a tree with SWP.

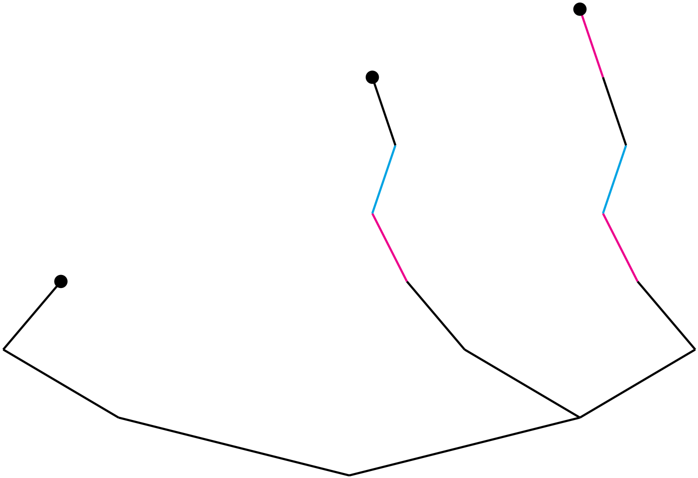
Some Examples of Strict Similarity Types for $k = 3$

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

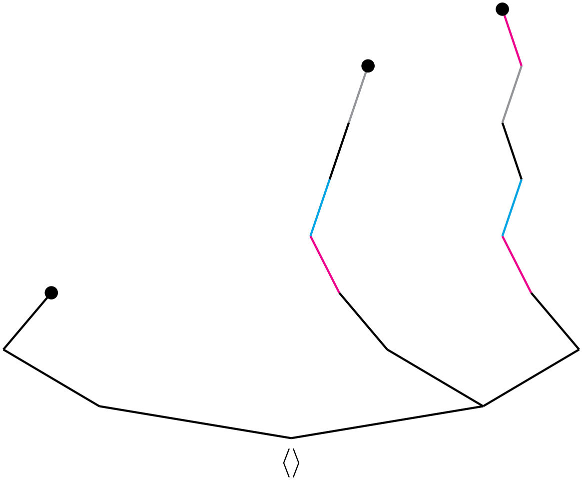
G a graph with three vertices and no edges

A tree A coding G - not WP but still a valid strict similarity type



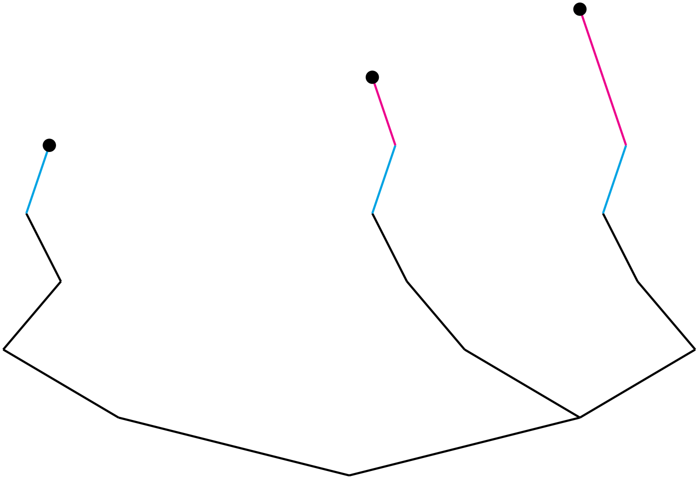
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



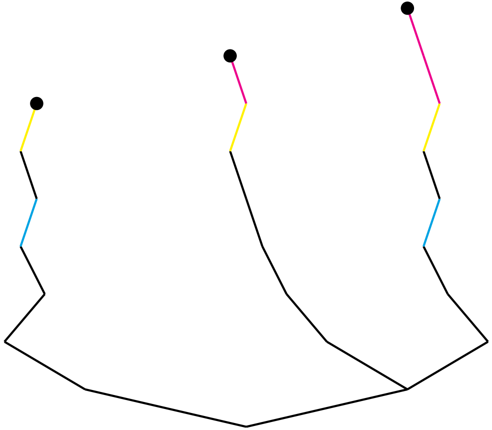
The tree C codes G

C is not strictly similar to A.

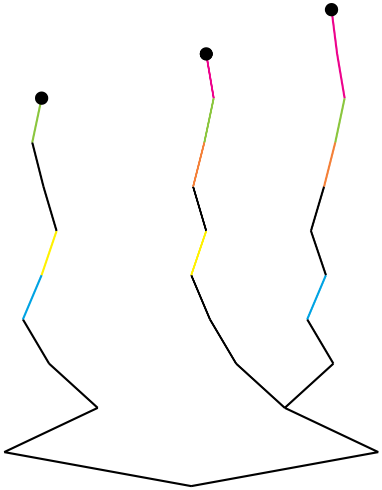


The tree D codes G

D is not strictly similar to either A or C.

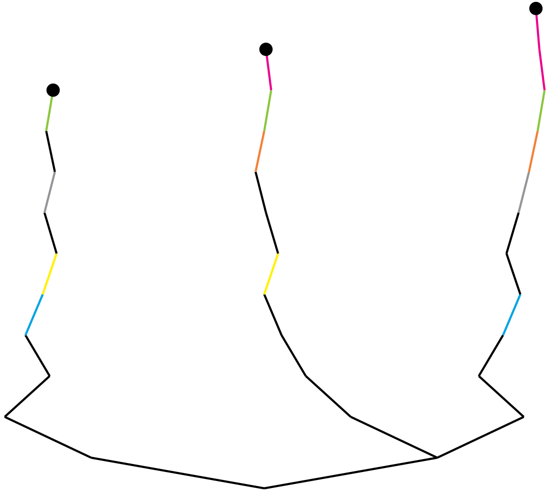


The tree E codes G and is not strictly similar to A - D



E is incremental. More on that later.

The tree F codes G and is strictly similar to E



F is also incremental.

Envelopes and Witnessing Coding Nodes

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Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

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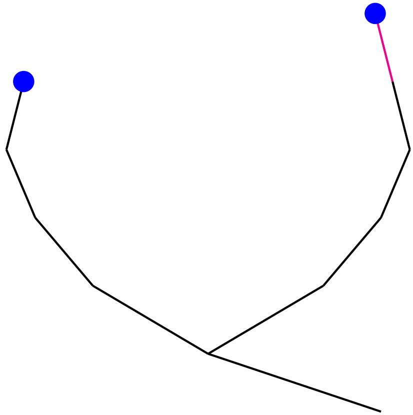
Envelopes add some neutral coding nodes to a finite tree to make it satisfy the Strict Witnessing Property.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

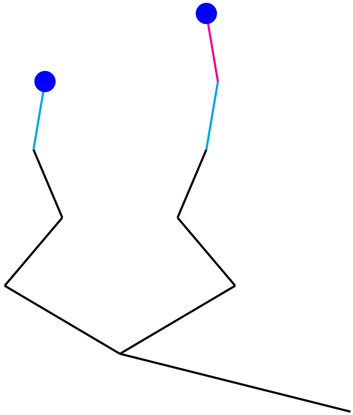
We now give some examples of envelopes.

H codes a non-edge



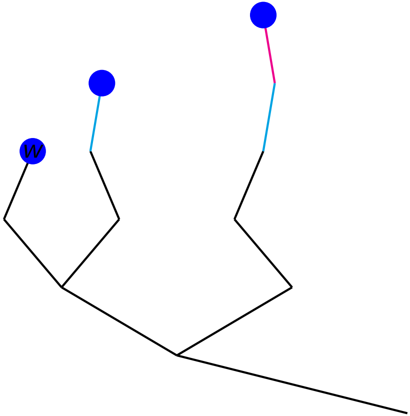
This satisfies the SWP, so H is its own envelope.

I codes a non-edge



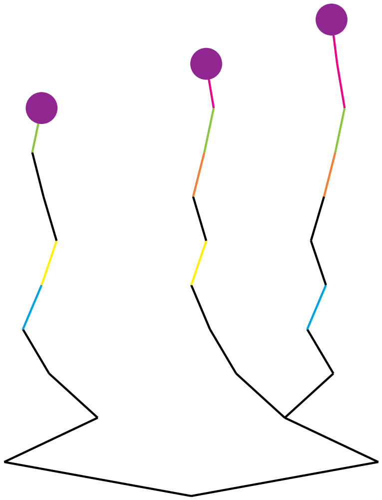
I does not satisfy the WP.

An Envelope $\mathbf{E}(I)$

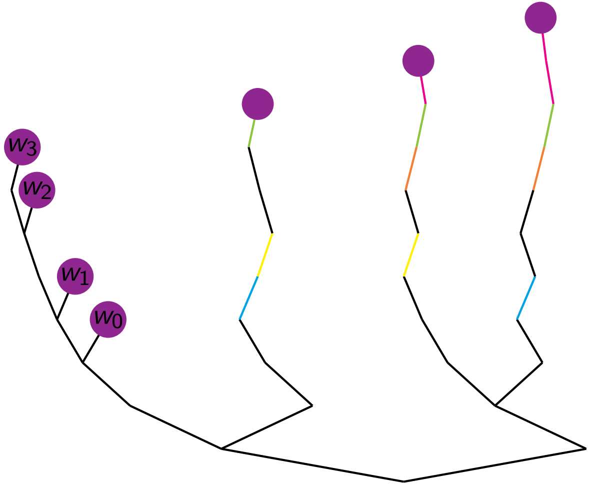


The witnessing coding node w is added to make an envelope.

The incremental tree E from before

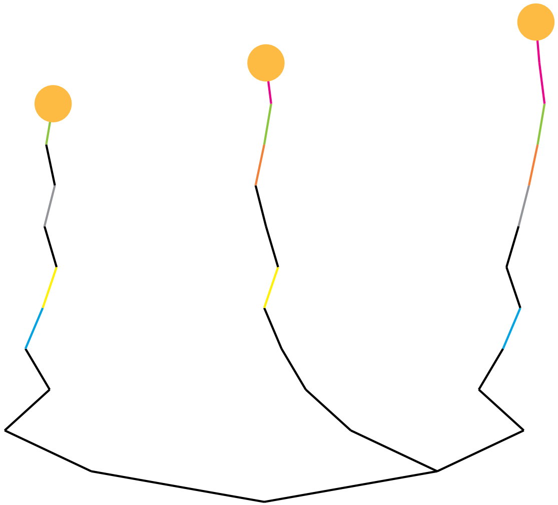


An envelope $\mathbf{E}(E)$

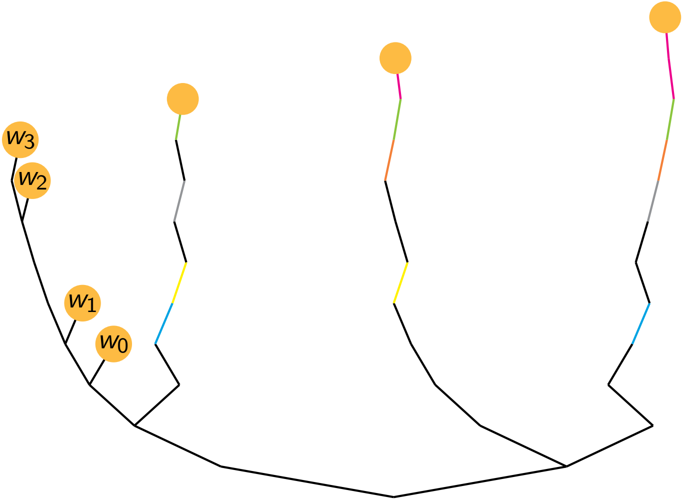


The witnessing coding nodes $w_1, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E



$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



The witnessing coding nodes $w_0, \ldots, w_3$ make an envelope of F.

Part IV: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.

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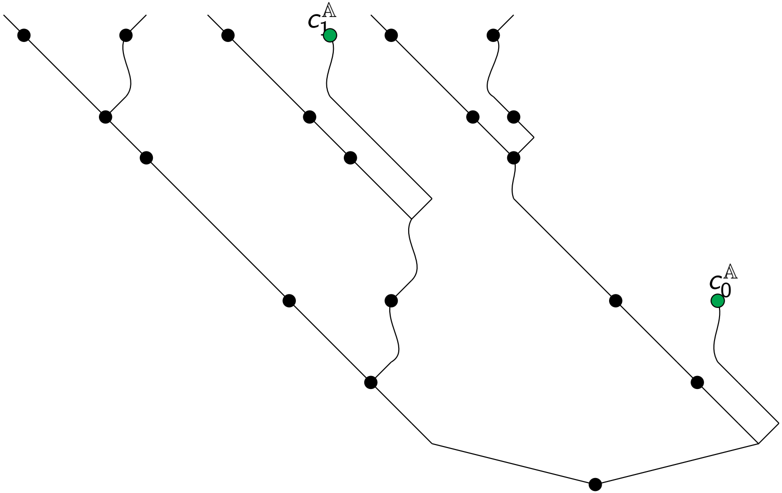
4. Apply the Ramsey Theorem from Part III, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take an antichain of coding nodes, $\mathbb{A}$ in S, which codes $\mathcal{H}_k$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_k$ coded by $\mathbb{A}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of (incremental) strict similarity types of antichains coding G.

An antichain $\mathbb{A}$ of coding nodes of S coding $\mathcal{H}_3$

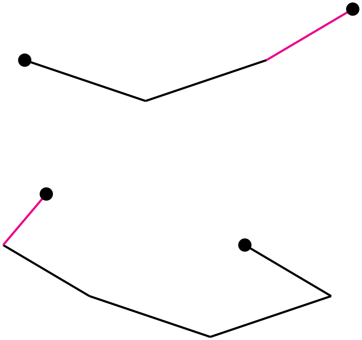


The tree minus the antichain of $c_n^{\mathbb{A}}$'s is isomorphic to $\mathbb{T}_3$.

Proving the lower bounds in general for big Ramsey degrees of $\mathcal{H}_k$ is a work in progress.

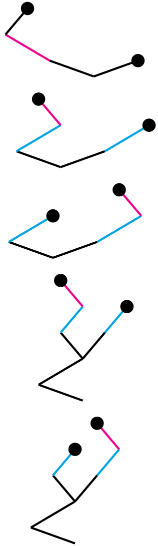
Big Ramsey degrees for edges and non-edges have been computed.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



These are their own envelopes.
$T(Edge, \mathcal{G}_3) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types in $\mathcal{H}_3$ (D.)



The beginning of a more general theory

The techniques developed for Henson graphs are very broad and likely to extend to a large class of Fraïssé structures with forbidden configurations.

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The techniques developed for Henson graphs are very broad and likely to extend to a large class of Fraïssé structures with forbidden configurations.

I am currently working to extend this research to big Ramsey degrees of the homogeneous partial order, homogeneous bowtie-free graph, and others.

Remarks

If time, we present ideas of the forcing argument for Halpern-Läuchli analogues.

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II(a) - Case (i): level set X contains a splitting node

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \leq d$.

Fix κ large enough so that $\kappa \rightarrow (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new pre-cliques in T.

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(iii) $\mathrm{ran}(p)$ has no pre-determined new pre-cliques in T.

$q \leq p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \geq l_p$, and

(i) $q(d) \supset p(d)$, and $q(i,\delta) \supset p(i,\delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) $\mathrm{ran}(q \restriction \vec{\delta}_p)$ has no new pre-cliques above $\mathrm{ran}(p)$.

II(a) - Case (i): Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

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Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}$, a name for the levels in $\dot{b}_d$.

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For $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \ldots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle$.

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- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

II(a) - Case (i): The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

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For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2)}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

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For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p(d), \langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle,& \\ \langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle,& \\ \langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle,& \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order.

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where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

II(a) - Case (i): f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

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Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and}\ (\forall i < d)\,\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

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Pf uses homogeneity of f.

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Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

II(a) - Case (i): f gives fixed ranges and color

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Pf uses 'sliding' idea.

II(a) - Case (i): f gives fixed ranges and color

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II(a) - Case (i): A compatible set

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The $t_0^*, \dots, t_d^*$ provide good starting nodes for constructing the tree homogeneous for the coloring on $\mathrm{Ext}_T(A, \tilde{X})$.

II(a) - Case (i): building a tree homog. for level set coloring

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(3) The assumption that $A \cup \tilde{X}$ satisfies the Witnessing Property is necessary.