# Barren Extensions

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joint work with

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# Overview

In their paper, "A barren extension," Henle, Mathias and Woodin proved that assuming $\omega \to (\omega)^\omega$,

1. Forcing with $([\omega]^\omega, \subseteq^*)$ adds no new sets of ordinals.
2. Under an additional assumption, $([\omega]^\omega, \subseteq^*)$ preserves all strong partition cardinals.

In joint work with Hathaway, we extend these results to a large collection of $\sigma$-closed forcings which add ultrafilters with weak partition properties.

These ultrafilters can have rich Rudin-Keisler and Tukey structures below them, with a Ramsey ultrafilter at the bottom.

Part I: Barren Extensions

# A Barren Extension

$\omega \to (\omega)^\omega$ means that for each $c : [\omega]^\omega \to 2$, there is an $N \in [\omega]^\omega$ such that $c$ is constant on $[N]^\omega$. This holds, for instance, in the $L(\mathbb{R})$ of $V^{\mathrm{Coll}(\omega,<\kappa)}$, where $\kappa$ is strongly inaccessible (Mathias), and in $L(\mathbb{R})$ in the presence of a supercompact in $V$ (Shelah-Woodin).

**Thm.** (Henle-Mathias-Woodin) Let $M$ be a transitive model of ZF + $\omega \to (\omega)^\omega$ and let $N$ be a forcing extension via $([\omega]^\omega, \subseteq^*)$. Then $M$ and $N$ have the same sets of ordinals; moreover every sequence in $N$ of elements of $M$ lies in $M$.

Note: $([\omega]^\omega, \subseteq^*)$ forces a Ramsey ultrafilter.

Question: Which other $\sigma$-closed forcings adding ultrafilters have similar properties?

# Ultrafilters with Weak Partition Relations

$$\mathcal{U} \to (\mathcal{U})^2_{l,r}$$

means that for each $X \in \mathcal{U}$ and $c : [X]^2 \to l$, there is a $U \subseteq X$ in $\mathcal{U}$ such that $c$ takes at most $r$ colors on $[U]^2$.

The least $r$ such that for all $l$, $\mathcal{U} \to (\mathcal{U})^2_{l,r}$ is the Ramsey degree of $\mathcal{U}$, denoted $r(\mathcal{U})$.

Examples

$\mathcal{P}(\omega)/\mathrm{Fin}$, equiv. $([\omega]^\omega, \subseteq^*)$, forces a Ramsey ultrafilter $\mathcal{U}$: $r(\mathcal{U}) = 1$.

A forcing of Laflamme produces a weakly Ramsey ultrafilter $\mathcal{U}_1$: $r(\mathcal{U}_1) = 2$.

(Laflamme) There is a hierarchy forcings $\mathbb{P}_\alpha$ $(\alpha < \omega_1)$ which produce ultrafilters $\mathcal{U}_\alpha$. For $k < \omega$, $r(\mathcal{U}_k) = 2^k$.

## Examples

(Navarro Flores): For each $k \geq 1$, $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ forces an ultrafilter $\mathcal{G}_k$ with $r(\mathcal{G}_k) = \sum_{i<k} 3^i$. (Blass for $k = 2$)

(Blass): $n$-square forcing produces an ultrafilter with $r(\mathcal{U}) = 5$.

(Baumgartner-Taylor): For $k \geq 2$, $\mathbb{Q}_k$ produces a $k$-arrow/not $(k+1)$-arrow ultrafilter $\mathcal{A}_k$: $\mathcal{A}_k \rightarrow (\mathcal{A}_k, k)^2$ but $\mathcal{A}_k \not\rightarrow (\mathcal{A}_k, k+1)^2$.

(D.-Mijares-Trujillo): Fraïssé classes can be used to generalize the previous two constructions to produce ultrafilters with various Ramsey degrees. Their Rudin-Keisler structures can be as complex as Fraïssé classes.

# Barren Extensions

**Thm.** (D.-Hathaway) Assuming a supercompact cardinal, let $\mathcal{U}$ be any of the above ultrafilters forced over $L(\mathbb{R})$. Then $L(\mathbb{R})[\mathcal{U}]$ has the same sets of ordinals as $L(\mathbb{R})$. Moreover it adds no new functions from any ordinal to $L(\mathbb{R})$.

Remark. This theorem holds for many other ultrafilters as well, including stable ordered union. The main tool is topological Ramsey spaces (dense inside these forcings) endowed with $\sigma$-closed partial orders which behave similarly to $([\omega]^\omega, \subseteq^*)$.

# The Essence of this HMW Theorem

$\mathbb{P} = \langle P, \leq, \leq^* \rangle$ is strongly coarsened if

1. $\forall x, y \in P, \quad x \leq y \longrightarrow x \leq^* y$, and
2. $\forall x \in P \ \ \forall y \leq^* x \ \ \exists z \leq x$ such that $z =^* y$.

For $x \in P$, let $[x] = \{y \in P : y \leq x\}$ and $[x]^* = \{y \in P : y \leq^* x\}$.

Examples: $([\omega]^\omega, \subseteq, \subseteq^*)$

More generally, $([\omega]^\omega, \subseteq, \subseteq^{\mathcal{I}})$ where $\mathcal{I}$ is a $\sigma$-closed ideal on $\mathcal{P}(\omega)$.

For many topological Ramsey spaces $(\mathcal{R}, \leq, r)$, there is a naturally related $\sigma$-closed partial order $\leq^*$ which strongly coarsens $\leq$.

# Left-Right Axiom - key properties of $([\omega]^\omega, \subseteq, \subseteq^*)$

A strongly coarsened poset $\mathbb{P} = \langle P, \leq, \leq^* \rangle$ satisfies the Left-Right Axiom (LRA) iff there are functions $\mathrm{L} : P \to P$ and $\mathrm{R} : P \to P$ such that the following are satisfied:

1. $\forall x \in P, \;\; \mathrm{L}(x), \mathrm{R}(x) \leq^* x$.
2. $\forall x \in P \;\; \exists y, z \leq x$ such that $\mathrm{L}(y) =^* \mathrm{R}(z)$ and $\mathrm{R}(y) =^* \mathrm{L}(z)$.
3. For each $p, x, y \in P$ with $x, y \leq p$, there is $z \leq p$ such that
   a) $\mathrm{L}(z) \leq^* x$
   b) $\mathrm{L}(\mathrm{R}(z)) \leq^* x$
   c) $\mathrm{R}(\mathrm{R}(z)) \leq^* y$.

Remark. All of the partial orders mentioned on slides 5 and 6 contain dense subsets forming Ramsey spaces which satisfy the LRA.

# Barren Extensions - general theorem

**Thm.** (D.-Hathaway) Let $M$ be a transitive model of ZF. Suppose $\mathbb{P} = \langle P, \leq, \leq^* \rangle \in M$ is a strongly coarsened poset satisfying

1. the Left-Right Axiom, and
2. for each $x \in P$ and every coloring $c : [x]^* \to 2$, there is some $y \leq^* x$ such that $c \restriction [y]$ is constant.

Let $N$ be a forcing extension of $M$ via $\langle P, \leq^* \rangle$. Then $M$ and $N$ have the same sets of ordinals; moreover, every sequence in $N$ of elements of $M$ lies in $M$.

Remark. Condition (2) is like $\omega \to (\omega)^\omega$. It holds in $L(\mathbb{R})$ for several classes of topological Ramsey spaces which form dense subsets of $\sigma$-closed forcings adding ultrafilters, in the presence of a supercompact cardinal.

Part II: Preservation of Strong Partition Cardinals

# Strong Partition Cardinals Preserved by $([\omega]^\omega, \subseteq^*)$

$\kappa \to (\kappa)^\lambda_\mu$ means that for each $c : [\kappa]^\lambda \to \mu$, there is a $K \in [\kappa]^\kappa$ such that $c$ is constant on $[K]^\lambda$.

**Thm.** (Henle-Mathias-Woodin) (ZF + EP + LU) Suppose

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^\lambda_\mu$, and
3. there is a surjection from $[\omega]^\omega$ onto $[\kappa]^\kappa$.

Then $\kappa \to (\kappa)^\lambda_\mu$ holds in the extension via $([\omega]^\omega, \subseteq^*)$.

# EP and LU

A subset $A \subseteq [\omega]^\omega$ is invariant if $(p \in A$ and $p' =^* p) \longrightarrow p' \in A$.

For $a \in [\omega]^{<\omega}$ and $p \in [\omega]^\omega$, let

$$[a, p] = \{q \in [\omega]^\omega : a \sqsubset q \wedge q \subseteq p\}$$

$X \subseteq [\omega]^\omega$ is Completey Ramsey (CR) if $\forall \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that

$$\text{(a) } [a, q] \subseteq X \quad \text{or} \quad \text{(b) } [a, q] \cap X = \emptyset.$$

$X \subseteq [\omega]^\omega$ is $\mathrm{CR}^+$ if $\forall \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that (a) holds;
$X$ is $\mathrm{CR}^-$ if $\forall \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that (b) holds.

LU: For any relation $R \subseteq [\omega]^\omega \times \mathcal{P}(\omega)$ such that $\forall p\ \exists y\ R(p, y)$, the set $\{x : R \text{ is uniformized on } [x]^\omega\}$ is $\mathrm{CR}^+$.

EP: The intersection of any well-ordered collection of $\mathrm{CR}^+$ sets is $\mathrm{CR}^+$.

# Preserving Strong Partition Cardinals over $L(\mathbb{R})$

**Thm.** (Henle-Mathias-Woodin) (AD + $V = L(\mathbb{R})$)
If $0 < \lambda = \omega \cdot \lambda \leq \kappa$, $2 \leq \mu < \kappa$, and $\kappa \to (\kappa)^{\lambda}_{\mu}$, then

$$L(\mathbb{R})[\mathcal{U}] \models \kappa \to (\kappa)^{\lambda}_{\mu},$$

where $\mathcal{U}$ is the Ramsey ultrafilter forced by $([\omega]^{\omega}, \subseteq^*)$ over $L(\mathbb{R})$.

Remark. AD + $V = L(\mathbb{R})$ imply LU, EP, and (3) in the previous rendition of this theorem.

# Extension to Topological Ramsey Spaces

Topological Ramsey spaces are triples $(\mathcal{R}, \leq, (r_n)_{n<\omega})$, where $\leq$ is a partial order and $r$ is a finite approximation map; basic open sets are of the form

$$[a, p] = \{q \in \mathcal{R} : \exists n < \omega\, (a = r_n(p)) \text{ and } q \leq p\}.$$

A subset $X \subseteq \mathcal{R}$ is (Completely) Ramsey if for each $\emptyset \neq [a, p]$ there is some $q \in [a, p]$ such that

$$\text{(a) } [a, q] \subseteq X \quad \text{ or else } \quad \text{(b) } [a, q] \cap X = \emptyset.$$

The defining property of a topological Ramsey space is that all subsets with the property of Baire are Ramsey.

The Ellentuck space $\mathcal{E} = ([\omega]^\omega, \subseteq, (r_n)_{n<\omega})$ has approximation maps $r_n(x) = \{x_i : i < n\}$, where $\{x_i : i < \omega\}$ is the strictly increasing enumeration of $x \in [\omega]^\omega$.

# Abstractions of $\mathrm{CR}^+$, $\mathrm{CR}^-$, $\omega \to (\omega)^\omega$, EP, and LU

The structure of topological Ramsey spaces, as roughly $\omega$-sequences of finite structures, often produces many of the same properties as the forcing $([\omega]^\omega, \subseteq^*)$.

$R(\mathcal{R})$: For each $c : \mathcal{R} \to 2$, exists $p \in \mathcal{R}$ such that $c$ is constant on $[p]$.

$X \subseteq \mathcal{R}$ is invariant $\mathrm{R}^+$ if

1. invariant: $(p \in X$ and $p' =^* p) \longrightarrow p' \in X$, and
2. $\mathrm{R}^+$: $\forall p \in \mathcal{R}\ \exists q \leq p$ such that $[q] \subseteq X$.

$\mathrm{R}^+$-Invariance Axiom: Suppose $\langle \mathcal{R}, \leq, \leq^*, r \rangle$ is a coarsened topological Ramsey space, where $\leq^*$ is $\sigma$-closed. $\mathcal{R}$ satisfies the $\mathrm{R}^+$-Invariance Axiom if for each invariant $R^+$ set $X \subseteq \mathcal{R}$ and each $p \in \mathcal{R}$ and $n < \omega$, there is a $q \in [r_n(p), p]$ such that $q \in S$.

# Abstractions of $\mathrm{CR}^+$, $\mathrm{CR}^-$, $\omega \to (\omega)^\omega$, EP, and LU

For $\mathbb{P} = \langle \mathcal{R}, \leq, \leq^*, r \rangle$:

EP($\mathbb{P}$): Given any well-ordered sequence $\langle C_\alpha \subseteq P : \alpha < \kappa \rangle$ of invariant $\mathrm{R}^+$ sets, the intersection of the sequence is again invariant $\mathrm{R}^+$.

LU$^*$($\mathbb{P}$): Uniformization relative to some invariant cube $[p]^*$ for relations $R \subseteq \mathcal{R} \times {}^\omega 2$.

LCU($\mathbb{P}$): Continuous uniformization for relations $R \subseteq \mathcal{R} \times {}^\omega 2$ relative to some cube $[p]$.

Similar to Todorcevic's Ramsey Uniformization Theorem for relations on $[\omega]^\omega \times X$ where $X$ is a Polish space.

# Preserving Strong Partition Cardinals - general theorem

**Thm.** (D.-Hathaway) Suppose $\mathbb{P} = \langle \mathcal{R}, \leq, \leq^* \rangle$ is a coarsened Ramsey space, where $\leq^*$ is $\sigma$-closed, satisfying $R(\mathcal{R})$ and R$^+$-IA. Assume EP($\mathbb{P}$), LU$^*$($\mathbb{P}$), and

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^\lambda_\mu$,
3. there is a surjection from ${}^\omega 2$ onto $[\kappa]^\kappa$.

Then $\langle \mathcal{R}, \leq^* \rangle$ forces $\kappa \to (\kappa)^\lambda_\mu$ holds in the forcing extension.

## Preserving Strong Partition Cardinals - simple version

**Thm.** (D.-Hathaway) Suppose that there is a supercompact cardinal in $V$. In $L(\mathbb{R})$, suppose

1. $(\mathcal{R}, \leq, \leq^*)$ is a coarsened topological Ramsey space satisfying the $\mathrm{R}^+$-Invariance Axiom,
2. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
3. $\kappa \to (\kappa)^\lambda_\mu$.

If $\mathcal{U}$ is the generic (ultra-)filter forced by $\langle \mathcal{R}, \leq^* \rangle$ over $L(\mathbb{R})$, then $L(\mathbb{R})[\mathcal{U}] \models \kappa \to (\kappa)^\lambda_\mu$.

Remark. The (Ramsey spaces dense in the) forcings on slides 5 and 6 all satisfy the $\mathrm{R}^+$-Invariance Axiom, as well as others.

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