Ramsey theory of homogeneous structures

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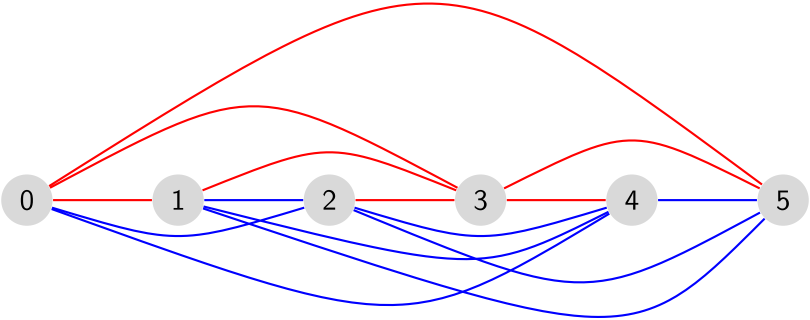
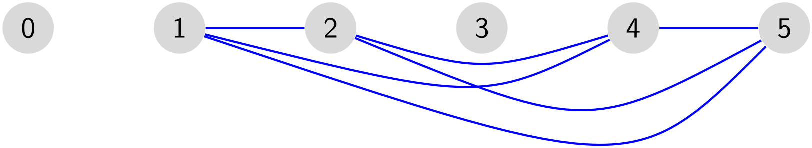
Ramsey's Theorem for Pairs of Natural Numbers

Given a coloring of pairs of natural numbers into red and blue:



Ramsey's Theorem for Pairs of Natural Numbers

There is an infinite subset M such that all pairs of numbers in M have the same color.



This can also be stated in terms of finding a complete infinite graph with all edges having the same color.

Ramsey's Theorem and Logic

Theorem. (Ramsey, 1929) Given $k, r \geq 1$ and a coloring

$$c : [\mathbb{N}]^k \to r,$$

there is an infinite $M \subseteq \mathbb{N}$ such that c takes only one color on $[M]^k$.

This theorem appears in Ramsey's paper, *On a problem of formal logic*, and is motivated by Hilbert's Entscheidungsproblem:

Find a procedure for determining whether any given formula is valid.

Ramsey applied his theorem to solve this problem for formulas with only universal quantifiers in front (Π_1).

One direction of extending Ramsey's Theorem is to trees.

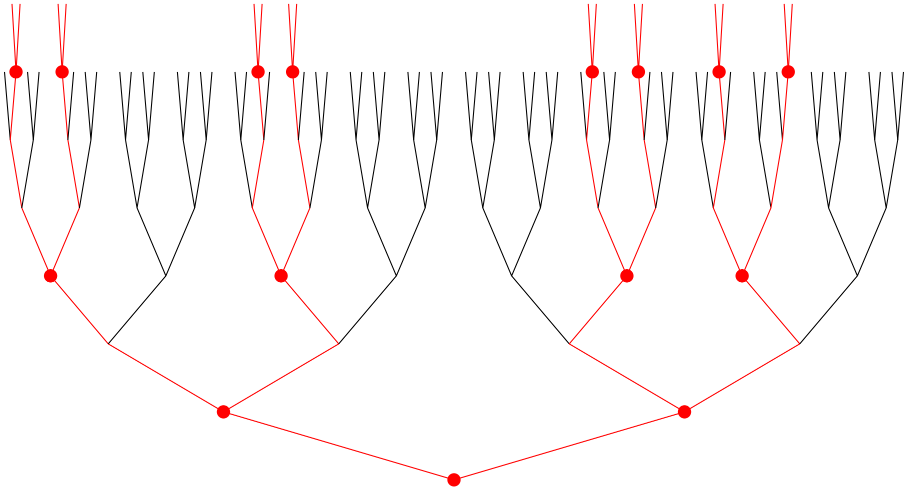
Several routes have been taken, and we will concentrate on the one on strong trees.

Ramsey Theory for Strong Trees

A tree $S \subseteq T = 2^{<\omega}$ is an infinite strong subtree of T iff it is (strongly) isomorphic to $2^{<\omega}$.

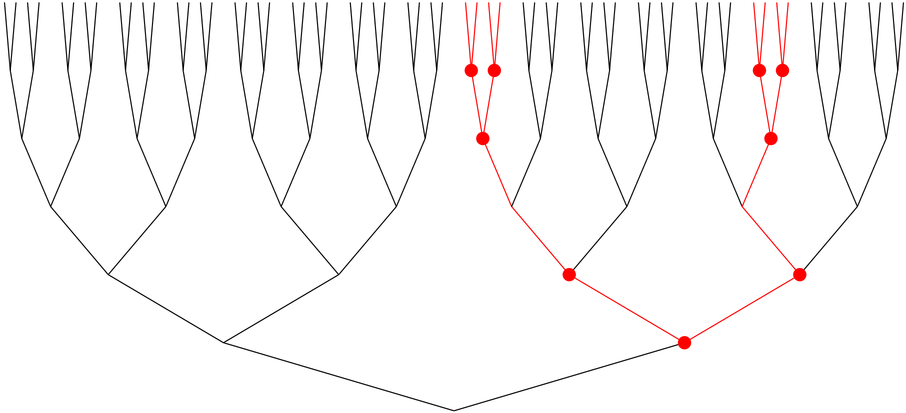
(This is a special case, but is sufficient for this talk.)

Example, An Infinite Strong Subtree $S \subseteq 2^{<\omega}$



The nodes in S are of lengths $0, 1, 3, 6, \ldots$

Example, An Infinite Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $1, 2, 4, 5, \ldots$.

Halpern-Läuchli Theorem - strong tree version for 2 trees

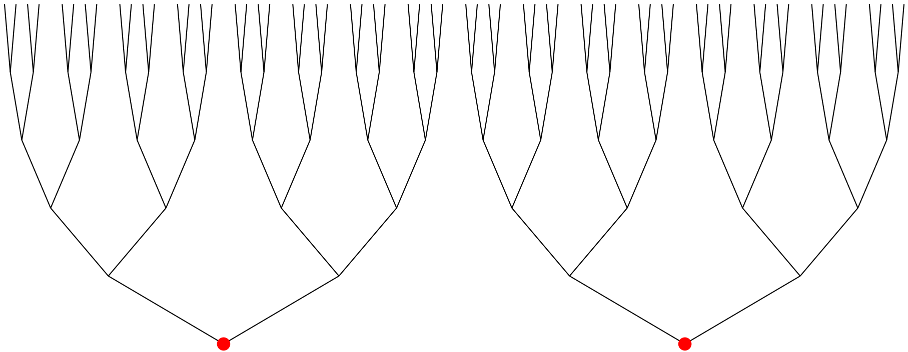
Theorem. (Halpern-Läuchli 1966) Let $r \geq 2$, and $T_0 = T_1 = 2^{<\omega}$. Given a coloring of the product of level sets of the T_i into r colors,

$$c : \bigcup_{n<\omega} T_0(n) \times T_1(n) \to r,$$

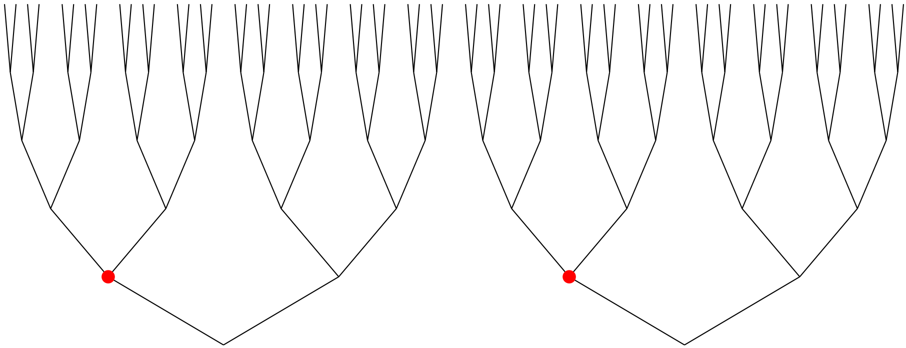
there are infinite strong trees $S_i \leq T_i$ and an infinite sets of levels $M \subseteq \omega$ where the splitting in S_i occurs, such that f is constant on $\bigcup_{m \in M} S_0(m) \times S_1(m)$.

This was found as a key lemma while proving that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (See [Halpern-Lévy 1971].)

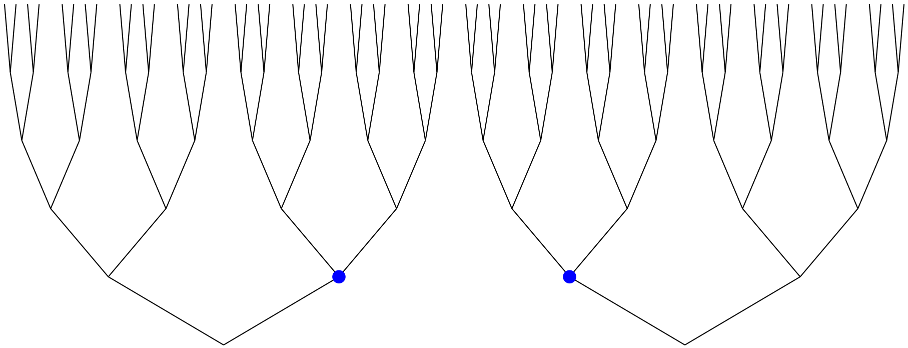
Coloring Products of Level Sets: $T_0(0) \times T_1(0)$



Coloring Products of Level Sets: $T_0(1) \times T_1(1)$



Coloring Products of Level Sets: $T_0(1) \times T_1(1)$



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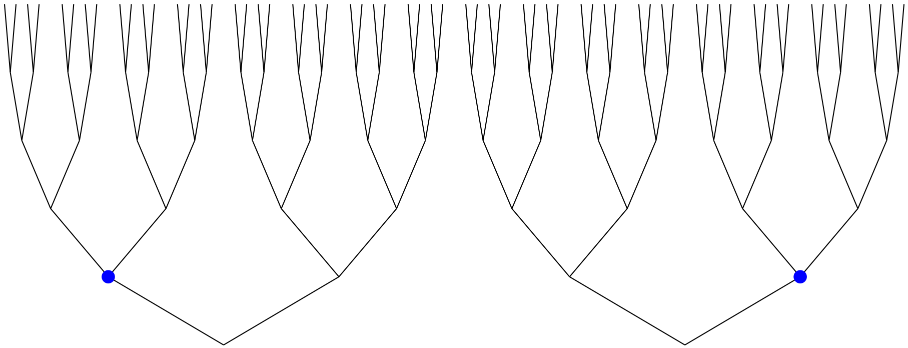
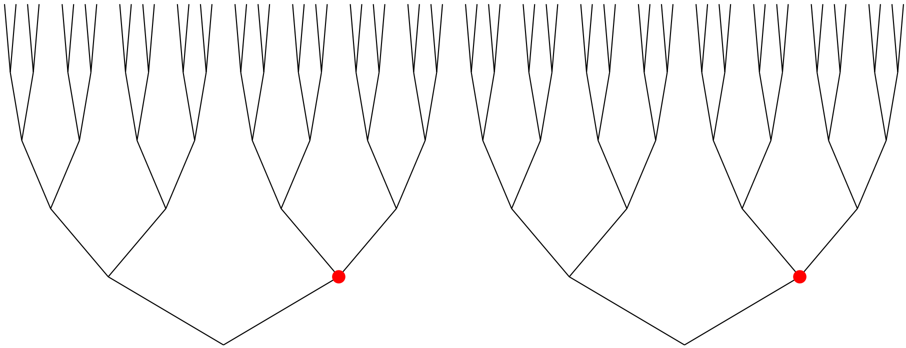
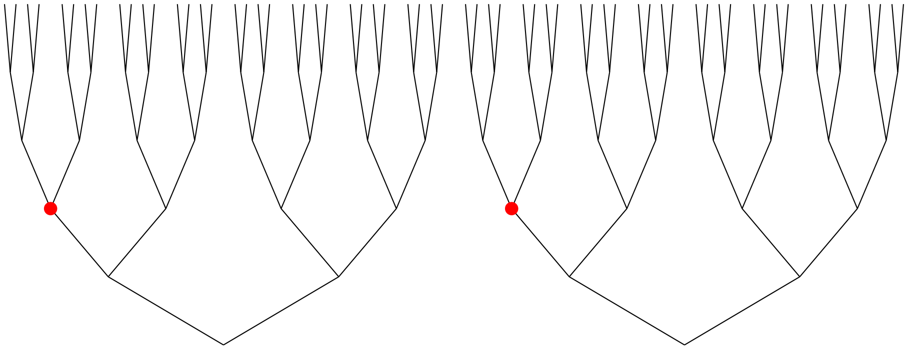


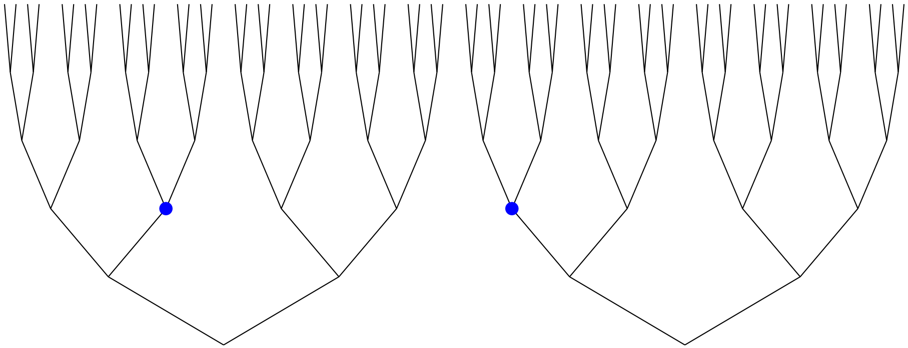
Coloring Products of Level Sets: $T_0(1) \times T_1(1)$



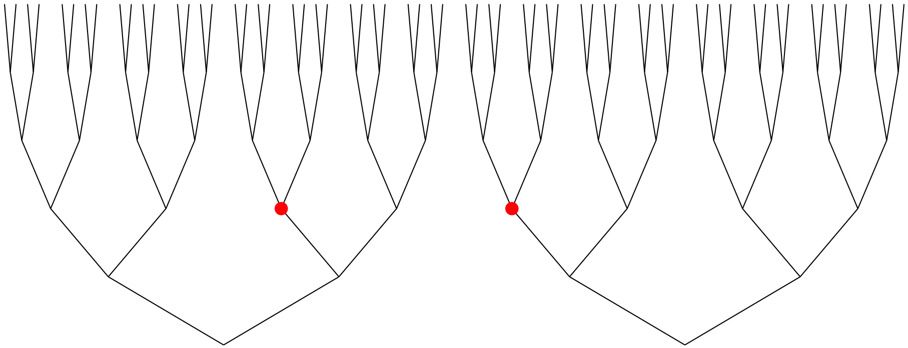
Coloring Products of Level Sets: $T_0(2) \times T_1(2)$



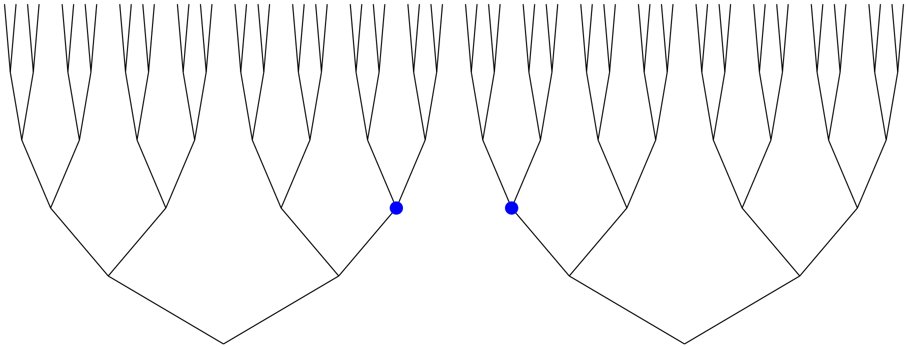
Coloring Products of Level Sets: $T_0(2) \times T_1(2)$



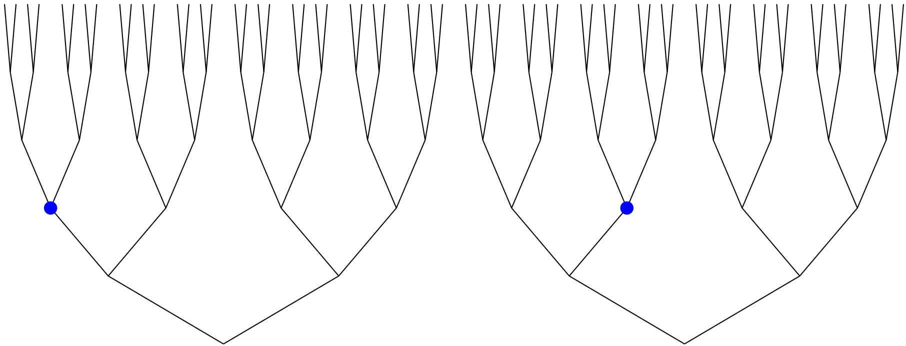
Coloring Products of Level Sets: $T_0(2) \times T_1(2)$



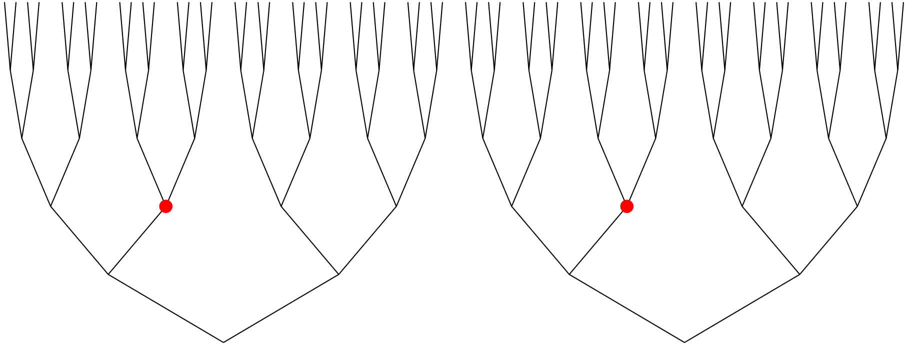
Coloring Products of Level Sets: $T_0(2) \times T_1(2)$



Coloring Products of Level Sets: $T_0(2) \times T_1(2)$

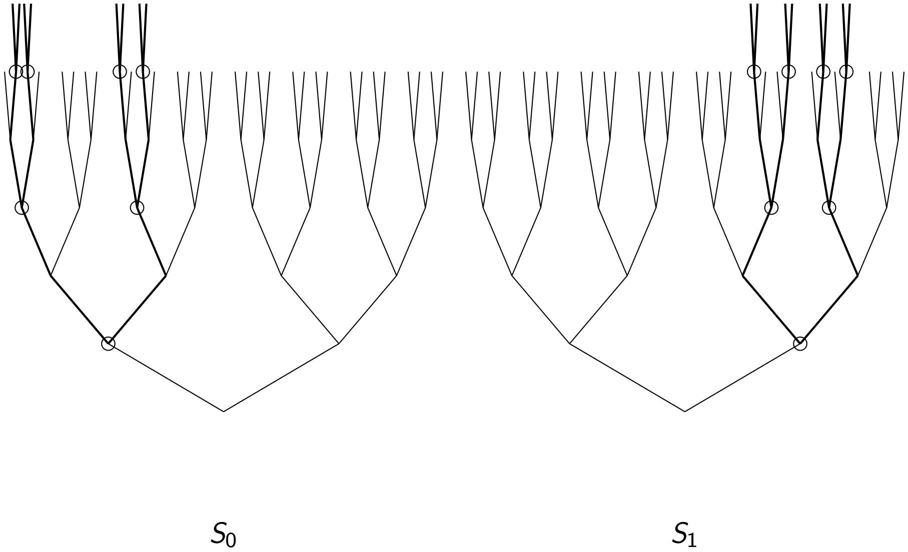


Coloring Products of Level Sets: $T_0(2) \times T_1(2)$



Etc.

HL gives Strong Subtrees with 1 color for level products



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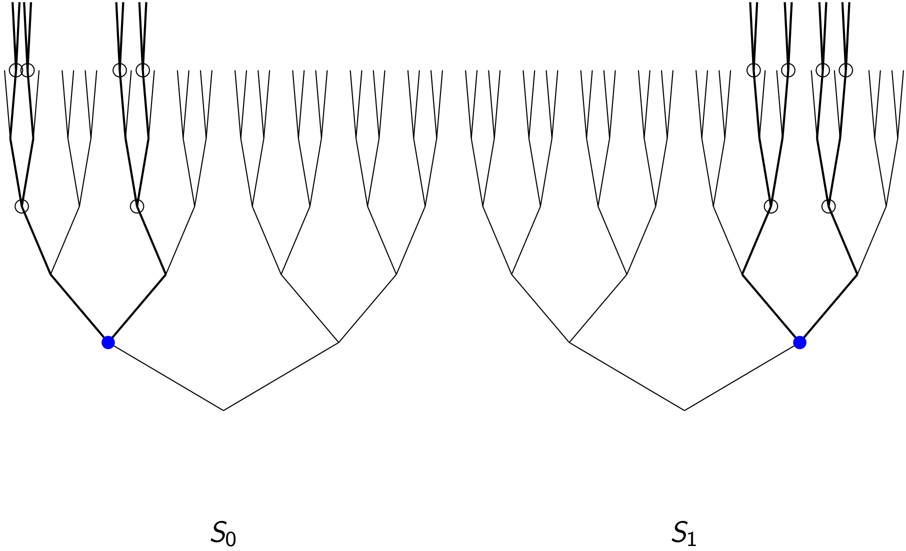

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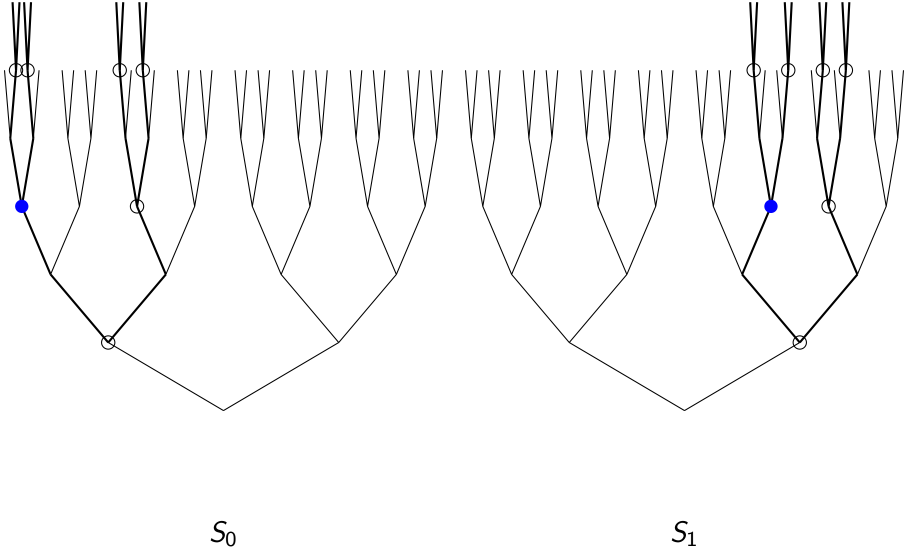

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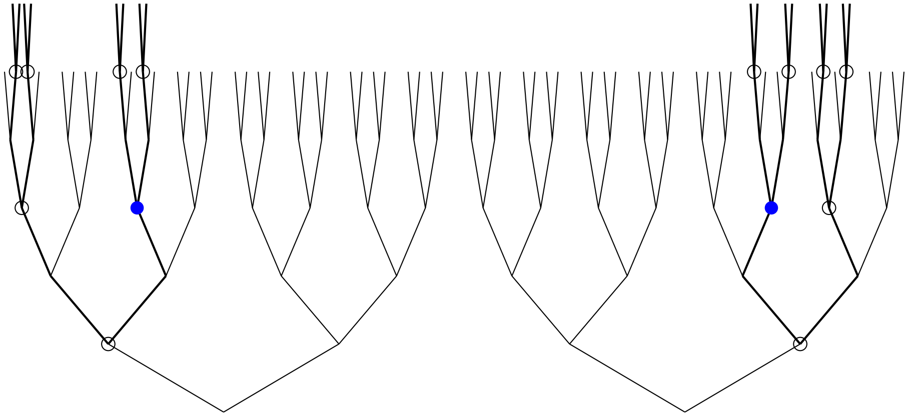
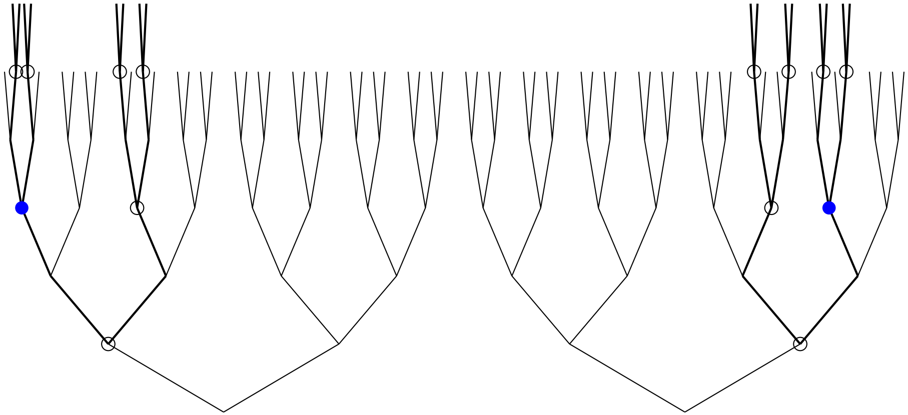


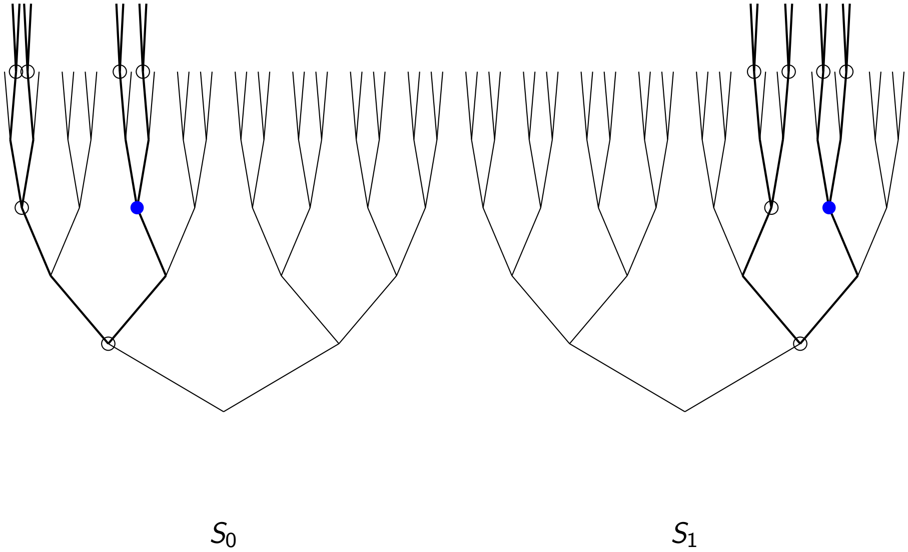
S_0 $\qquad$ S_1

HL gives Strong Subtrees with 1 color for level products



S_0 S_1

HL gives Strong Subtrees with 1 color for level products



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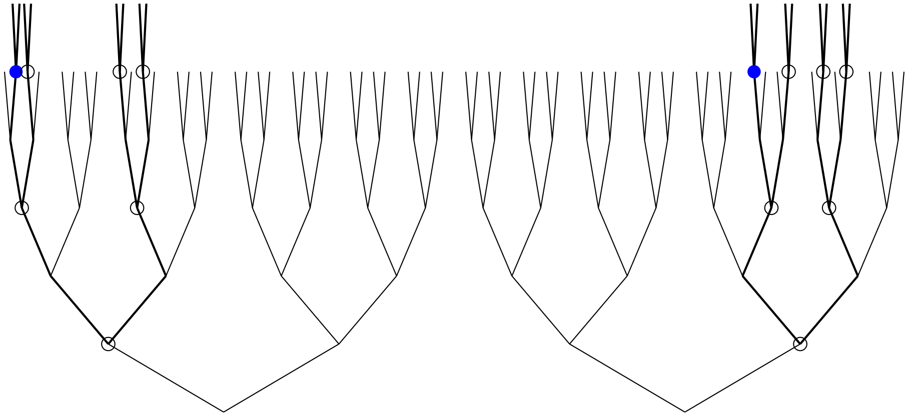
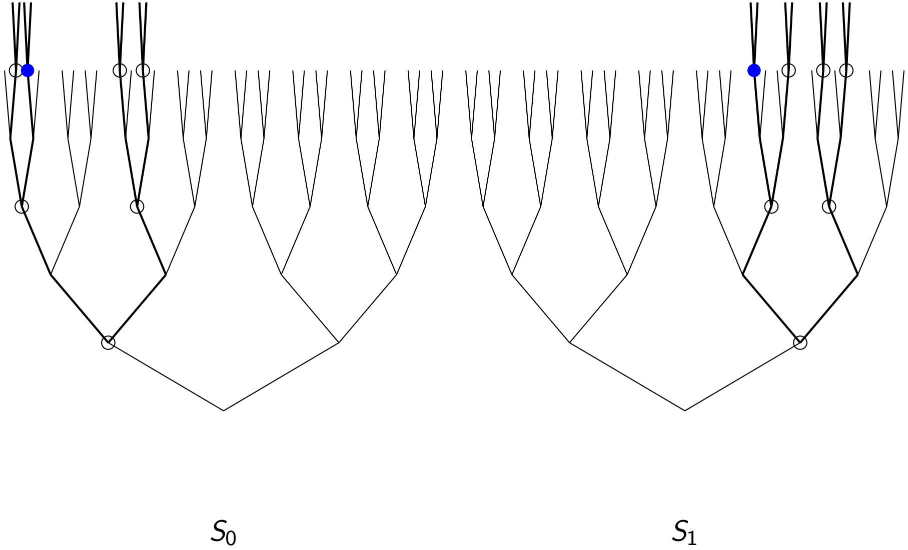


S_0 S_1

HL gives Strong Subtrees with 1 color for level products



Connection with Forcing

Harrington devised a proof of the Halpern-Läuchli Theorem that uses the method of forcing, though without ever moving into a generic extension of the ground model. This will be important later.

The next theorem is proved by induction from the Halpern-Läuchli Theorem for any finite number of trees.

A Ramsey Theorem for Strong Trees

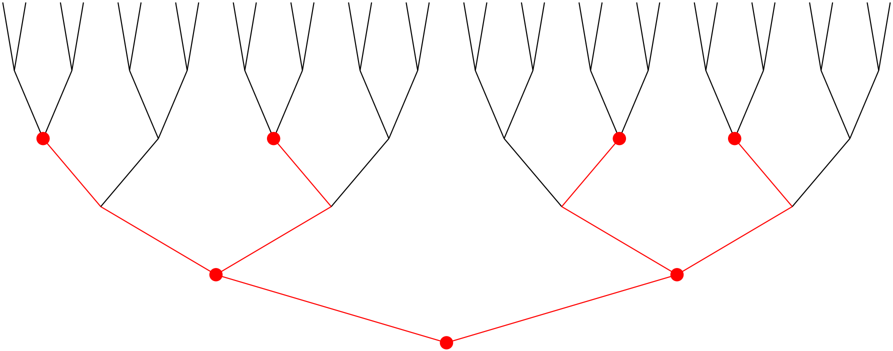
A *k-strong subtree* is the truncation of an infinite strong tree to k many levels.

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all k-strong subtrees of $2^{<\omega}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

Milliken's theorem for 2-strong trees directly implies the Halpern-Läuchli Theorem.

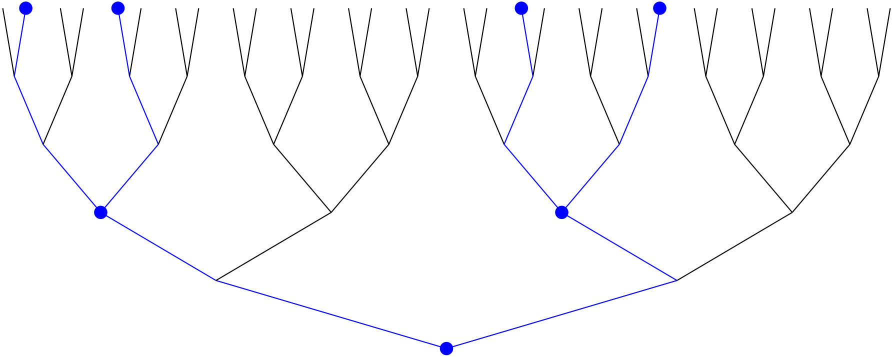
Milliken's Theorem for 3-Strong Trees

takes a coloring all subtrees of $2^{<\omega}$ like this:



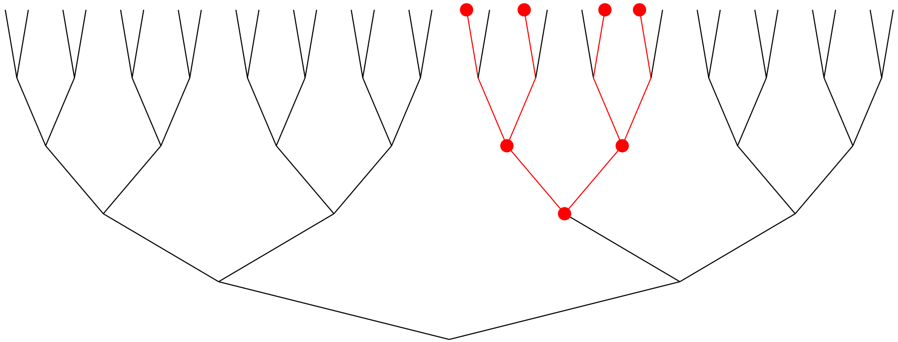
Milliken's Theorem for 3-Strong Trees

and this:



Milliken's Theorem for 3-Strong Trees

and this



and finds an infinite strong subtree in which all 3-strong subtrees have the same color.

Applications of this will be seen shortly.

Extensions of Ramsey's Theorem to Structures

What happens if we try to extend Ramsey's Theorem to infinite structures $\mathcal{S}$, where the subsets allowed must have induced structure isomorphic to $\mathcal{S}$?

Example: The Rationals.

Any finite coloring of the singletons in $\mathbb{Q}$ is monochromatic on a subset isomorphic to $\mathbb{Q}$. However,

Theorem. (Sierpinski) There is a coloring of pairs of rationals into two colors such that any subset $\mathbb{Q}' \subseteq \mathbb{Q}$, which is again a dense linear order without endpoints, takes both colors on its pairsets.

Decades later, Milliken's Theorem was seen to be the structural heart of this phenomenon.

Milliken on Ramsey Theory of the Rationals

The rationals can be coded as the nodes in $2^{<\omega}$. Applying Milliken's Theorem one finds:

Fact. Given any $n \geq 2$, there is a number $T(n, \mathbb{Q}) \geq 2$ such that any coloring of $[\mathbb{Q}]^n$ into finitely many colors can be reduced to no more than $T(n, \mathbb{Q})$ colors on a substructure $\mathbb{Q}'$ isomorphic to $\mathbb{Q}$.

With more work, Devlin (building on Laver's work) found the exact numbers: these are tangent numbers! These numbers $T(n, \mathbb{Q})$ are called big Ramsey degrees. They are deduced from the number of types of trees that can code an n-tuple of rationals in $2^{<\omega}$.

Ramsey Theory of the Rado Graph

The Rado graph $\mathcal{R}$ is the universal homogeneous graph on countably many vertices. It is $\mathcal{R}$ the Fraïssé limit of the class of finite graphs.

Fact. (Folklore) Given a coloring of the vertices of the Rado graph into finitely many colors, there is a subgraph which is again Rado in which all vertices have the same color.

The big Ramsey degree for vertex colorings in the Rado graph is 1.

Example: Ordered graph A embeds into ordered graph B.

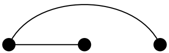


Figure: Ordered Graph A

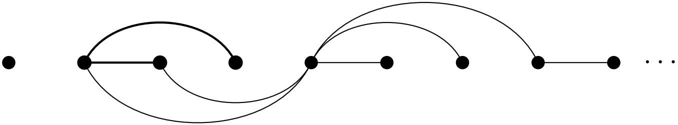
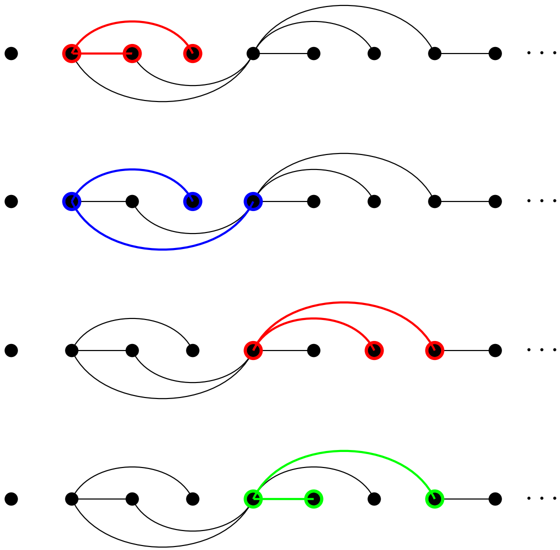


Figure: Ordered Graph B

Some copies of A in B



Ramsey Theory of the Rado Graph

• Edges have big Ramsey degree 2. (Pouzet/Sauer 1996).

• All finite graphs have finite big Ramsey degree. (Sauer 2006) In this paper is also the set-up for

• Actual degrees were found structurally in (LSV 2006) and computed in (J. Larson 2008).

How was Milliken's Theorem used?

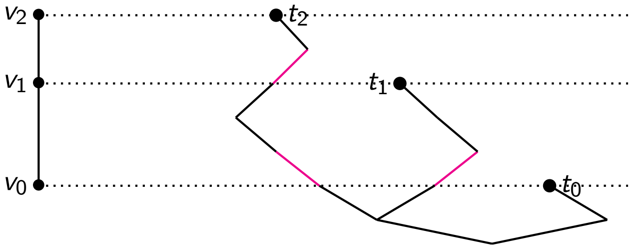
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

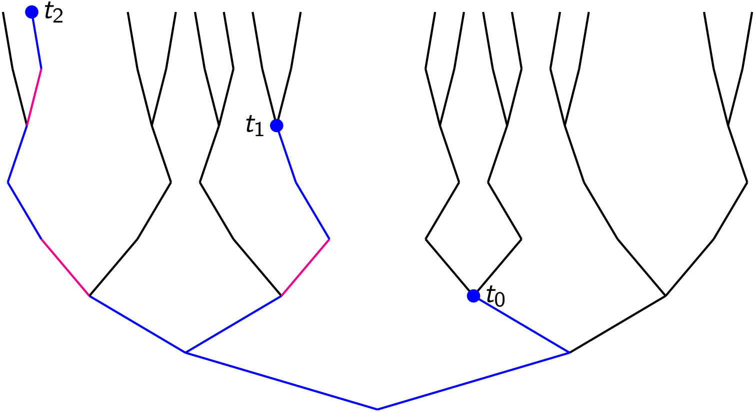
A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1.$$

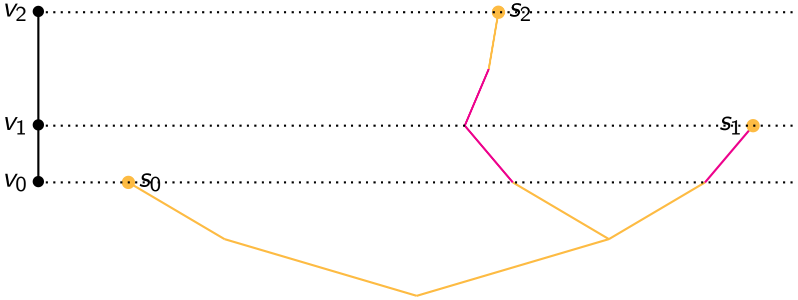
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



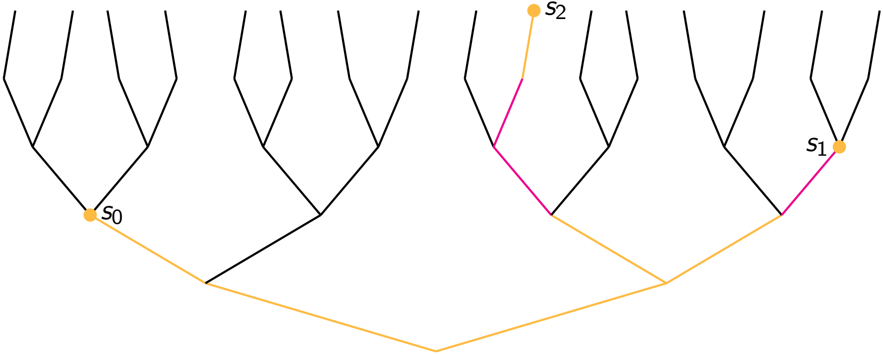
A Strong Tree Envelope



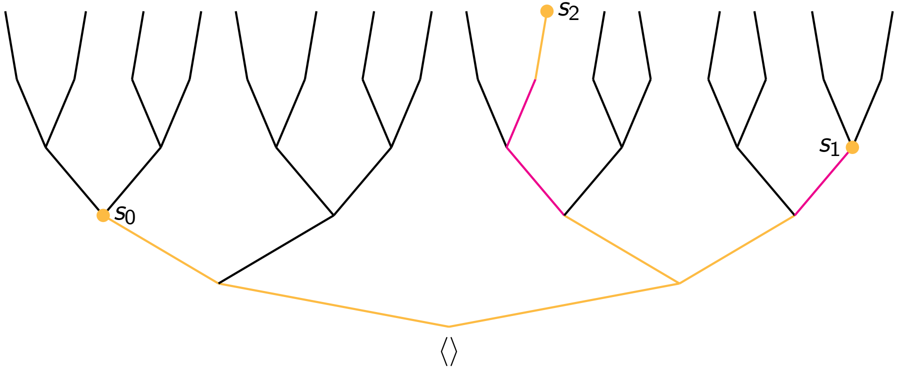
A Different Antichain Coding a Path of Length 2



A Strong Tree Envelope



A different strong tree envelope



Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of (diagonal) antichains.
3. Each strongly diagonal antichain can be enveloped into finitely many strong trees.
4. Apply Milliken's Theorem finitely many times to obtain one color for each (strong similarity) type.
5. Choose a strongly diagonal antichain coding the Rado graph.
6. Show that each type persists in each subgraph which is random to obtain exact numbers.

Structures known to have big Ramsey degrees

• the natural numbers (Ramsey 1929) (all big Ramsey degrees are 1)

• the rationals (Galvin, Laver, Devlin 1979)

• the Rado graph and similar binary relational structures (Sauer 2006)

• the countable ultrametric Urysohn space (Nguyen Van Thé 2008)

• the dense local order, circular tournament, $\mathbb{Q}_n$ (Laflamme, NVT, Sauer 2010).

The crux of all but two of these proofs is Milliken's Theorem (or variant).

(The Urysohn space result uses Ramsey's Theorem.)

Missing Piece: Forbidden Configurations

No Fraïssé structure with forbidden configurations had a complete analysis of its Big Ramsey Degrees.

The Problem: Lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

This problem is addressed starting with my submitted paper, *The Ramsey theory of the universal homogeneous triangle-free graph*, 48 pp, and work-in-progress extending it to all Henson graphs.

The methods developed therein are flexible and should apply, after modifications, to a large collection of homogeneous structures with forbidden configurations.

Why study Ramsey Theory of Homogeneous Structures?

• Natural extension of structural Ramsey theory on finite structures, and is in line with Ramsey's original theorem.

• Connections with topological dynamics - universal completion flows.

• Possible connections with model theory.

The Triangle-free Henson Graph $\mathcal{H}_3$: History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_3$ is the Fraïssé limit of the class of finite triangle-free graphs.

• Henson constructed $\mathcal{H}_3$ and proved it is weakly indivisible in 1971.

• The Fraïssé class of finite ordered triangle-free graphs has the Ramsey property. (Nešetřil-Rödl 1973)

• $\mathcal{H}_3$ is indivisible: Vertex colorings of $\mathcal{H}_3$ have big Ramsey degree 1. (Komjáth/Rödl 1986)

• $\mathcal{H}_3$ has big Ramsey degree 2 for edges. (Sauer 1998)

There progress halted due to lack of broadscale techniques.

Main Theorem: $\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T_{\mathcal{K}_3}(\mathrm{A})$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T_{\mathcal{K}_3}(\mathrm{A})$ colors.

Thanks to the following:
2011 Laver outlined Harrington's 'forcing proof' of Halpern-Läuchli for me.
2012 and 2013 Todorcevic and Sauer both mention the lack of an appropriate Milliken Theorem as the main obstacle to the solution.

Structure of Proof: Three Main Parts

I Develop new notion of strong coding tree to represent $\mathcal{H}_3$.

II Prove a Ramsey Theorem for strictly similar finite antichains.

The proof uses ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem, and obtains a Milliken-style theorem.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Part I: Strong Coding Trees

Idea: Want correct analogue of strong trees for setting of $\mathcal{H}_3$.

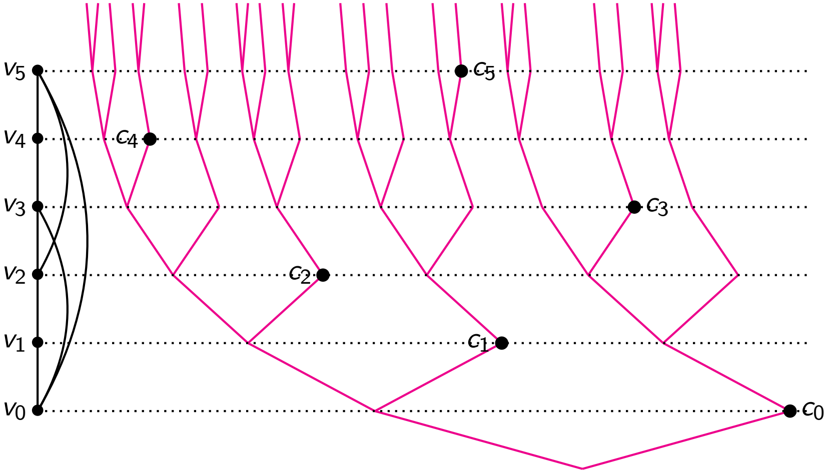
Problem: How to make sure triangles are never encoded but branching is as thick as possible?

First Approach: Strong Triangle-Free Trees

• Use a unary predicate for distinguishing certain nodes to code vertices of a given graph (called coding nodes).

• Make a Branching Criterion so that a node s splits iff all its extensions will never code a triangle with coding nodes at or below the level of s.

Strong triangle-free tree $\mathbb{S}$

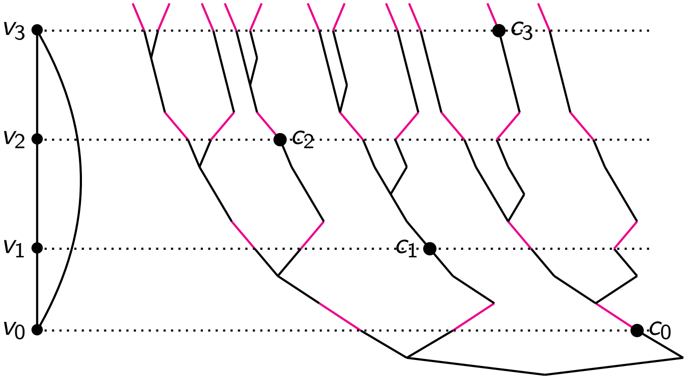


Almost sufficient

One can develop almost all the Ramsey theory one needs on strong triangle-free trees

except for vertex colorings: there is a bad coloring of coding nodes.

Refined Approach: Strong coding tree $\mathbb{T}$



Skew the levels of interest.

The Space of Strong Coding Trees: $\mathcal{T}_3$

$\mathcal{T}_3$ is the collection of all subtrees of $\mathbb{T}$ which are strongly similar to $\mathbb{T}$.

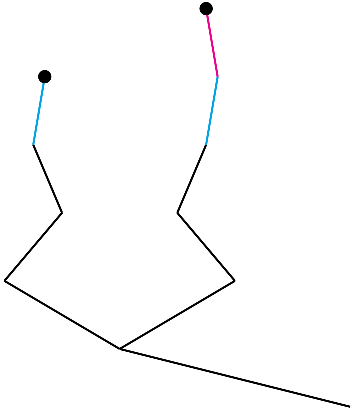
Extension Criterion: A finite subtree A of a strong coding tree $T \in \mathcal{T}_3$ can be extended to a strong coding subtree of T whenever A is strongly similar to an initial segment of $\mathbb{T}$ and all entanglements of A are witnessed - no types are lost.

The criteria guaranteeing this are

1. Pre-Triangle Criterion: All new sets of parallel 1's in A are witnessed by a coding node in A 'nearby'.
2. A if free in T: A has no pre-determined new parallel 1's in T.

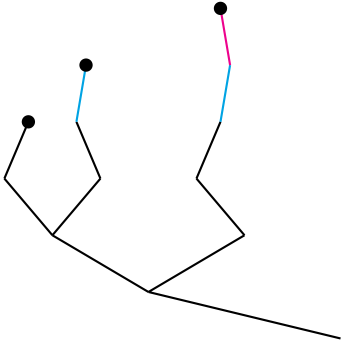
A subtree of $\mathbb{T}$ in which Pre-Triangle Criterion fails

It has parallel 1's not witnessed by a coding node (PTC fails).



A subtree of $\mathbb{T}$ in which PTC holds

Its parallel 1's are witnessed by a coding node.



This gives the basic idea of PTC, though more subtleties are involved.

Part II: A Ramsey Theorem for Strictly Similar Finite Antichains.

Idea: Strict similarity takes into account tree isomorphism and placements of coding nodes and new sets of parallel 1's.

It persists upon taking subtrees in $\mathcal{T}_3$.

Ramsey Theorem for Strong Coding Trees

Theorem. (D.) Let A be a finite subtree of a strong coding tree T, and let c be a coloring of all strictly similar copies of A in T.

Then there is a strong coding tree $S \leq T$ in which all strictly similar copies of A in S have the same color.

This is an analogue of Milliken's Theorem for strong coding trees.

Strict similarity is a strong version of isomorphism, and forms an equivalence relation.

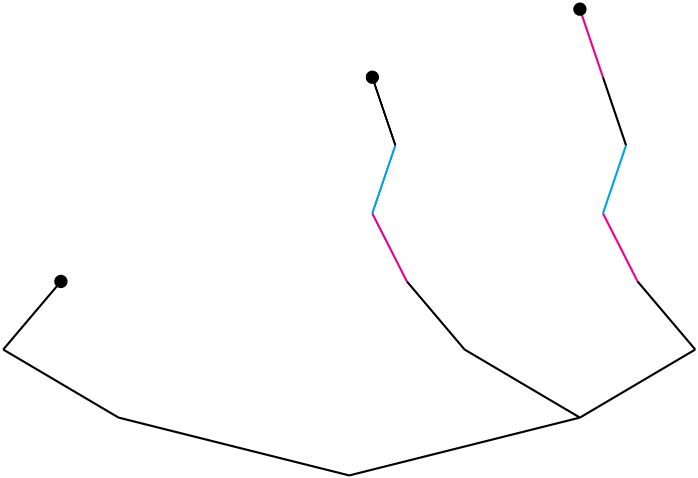
Some Examples of Strict Similarity Types

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

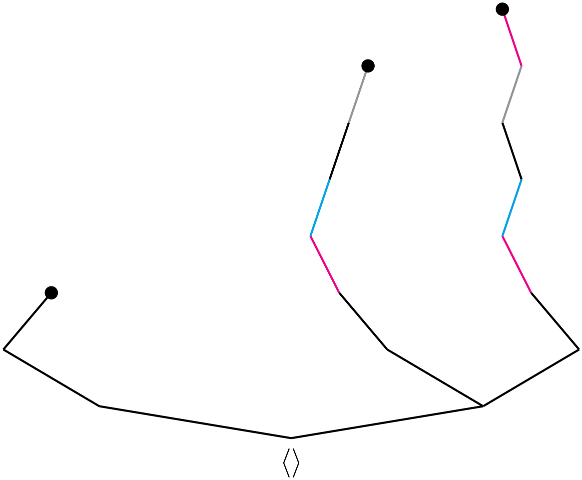
G a graph with three vertices and no edges

A tree A coding G - not P1C but still a valid strict similarity type



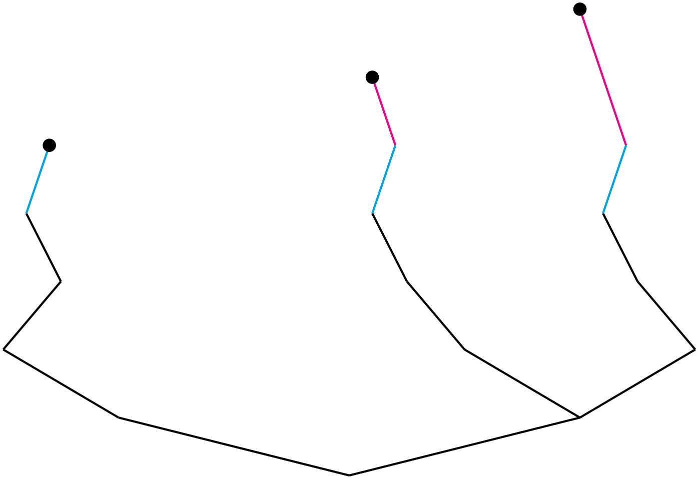
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



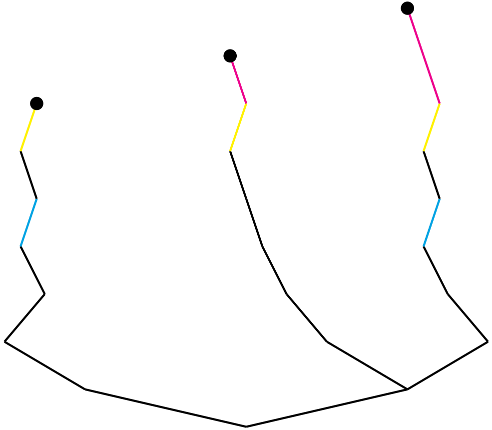
The tree C codes G

C is not strictly similar to A.

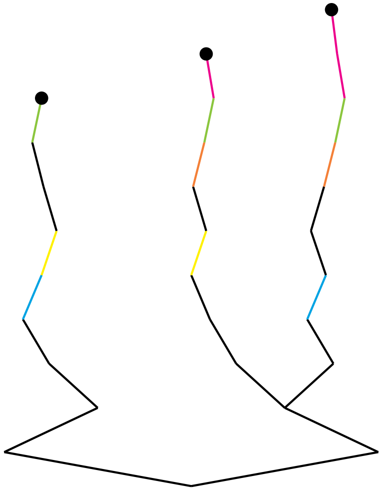


The tree D codes G

D is not strictly similar to either A or C.

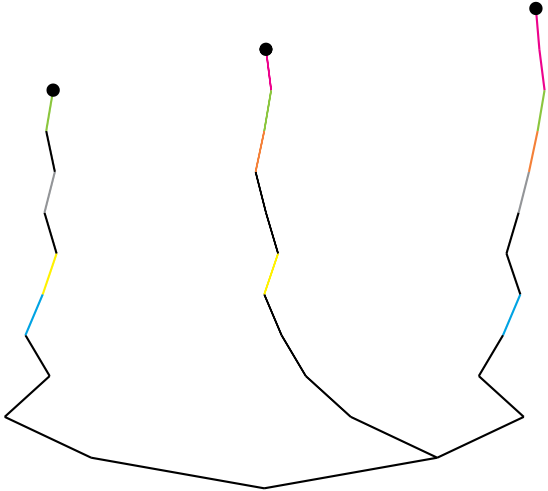


The tree E codes G and is not strictly similar to A - D



E is incremental. More on that later.

The tree F codes G and is strictly similar to E



F is also incremental.

Part III: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for $T_{\mathcal{K}_3}(G)$

1. Let G be a finite triangle-free graph, and let f color the copies of G in $\mathcal{H}_3$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.
4. Apply the Ramsey Theorem from Part II, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take an antichain of coding nodes, $\mathbb{A}$ in S, which codes $\mathcal{H}_3$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_3$ coded by $\mathbb{A}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of (incremental) strict similarity types of antichains coding G.

Reducing the Upper Bounds

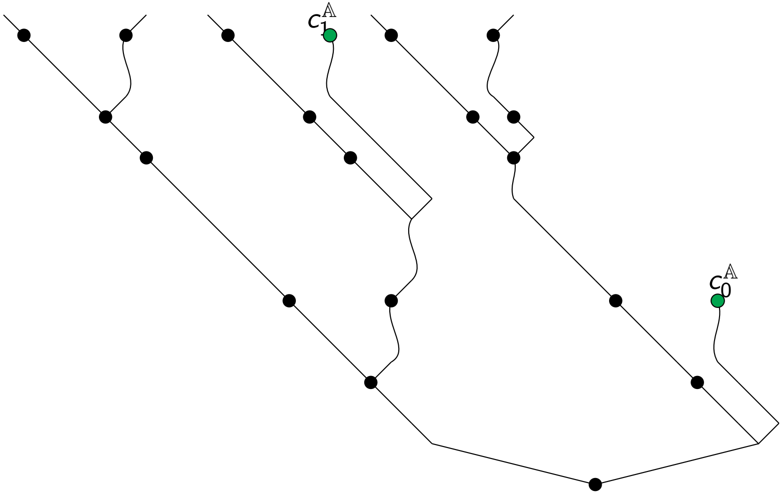
A strong tree U with coding nodes is incremental if whenever a new set of parallel 1's appears in U, all of its subsets appear as parallel 1's at a lower level.

The trees A, B, E, and F are incremental.

The trees C and D are not incremental.

We can take S in the previous slide to be an incremental strong coding tree.

An antichain $\mathbb{A}$ of coding nodes of S coding $\mathcal{H}_3$



The tree minus the antichain of $c_n^{\mathbb{A}}$'s is isomorphic to $\mathbb{T}$.

Part II Expanded: Ideas behind the proof of the Ramsey Theorem for Strictly Similar Finite Trees

(a) Prove new Halpern-Läuchli style Theorems for strong coding trees.
– Three new forcings are needed, but the proofs take place in ZFC.

(b) Prove a new Ramsey Theorem for finite trees satisfying the Strict Pre-Triangle Criterion.
– An analogue of Milliken's Theorem.

(c) New notion of envelope.
– Turns an antichain into a tree satisfying Strict Pre-Triangle Criterion.

(a) Halpern-Läuchli-style Theorem

Thm. (D.) Given a strong coding tree T and

1. B a finite, valid strong coding subtree of T;
2. A a finite subtree of B with $\max(A) \subseteq \max(B)$; and
3. X a level set extending A into T with $A \cup X$ satisfying the PTC and valid in T.

Color all end-extensions Y of A in T for which $A \cup Y$ is strictly similar to $A \cup X$ into finitely many colors.

Then there is a strong coding tree $S \leq T$ end-extending B such that all level sets Y in S with $A \cup Y$ strictly similar to $A \cup X$ have the same color.

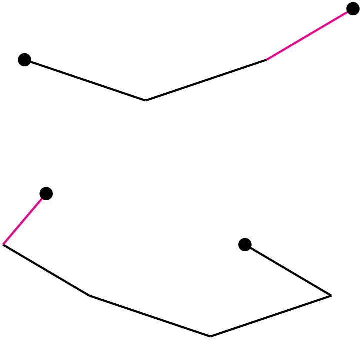
Remark. The proof uses three different forcings and Harrington-style ideas. The forcings are best thought of as conducting unbounded searches for finite objects in ZFC.

Remarks

Proving the lower bounds in general for big Ramsey degrees of $\mathcal{H}_3$ is a work in progress.

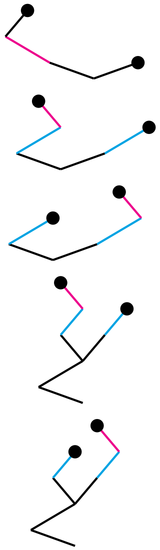
Big Ramsey degrees for edges and non-edges have been computed.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



$T_{\mathcal{H}_3}(Edge) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types (D.)



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