Ramsey theory and infinite graphs

Natasha Dobrinen

University of Denver

University of Oxford

Analytic Topology Seminar

January 16, 2019

Research supported by NSF Grants DMS 1301665 and 1600781

This talk will highlight some of the main concepts in my papers,

The Ramsey theory of the universal homogeneous triangle-free graph, 65 pp, submitted,

The Ramsey theory of Henson graphs, 68 pp, preprint,

This talk will highlight some of the main concepts in my papers,

The Ramsey theory of the universal homogeneous triangle-free graph, 65 pp, submitted,

The Ramsey theory of Henson graphs, 68 pp, preprint, and

in work in progress on *Definable colorings of Rado graphs*.

This talk will highlight some of the main concepts in my papers,

The Ramsey theory of the universal homogeneous triangle-free graph, 65 pp, submitted,

The Ramsey theory of Henson graphs, 68 pp, preprint, and

in work in progress on *Definable colorings of Rado graphs*.

This work commenced during the Newton Institute HIF Programme (2015).

Finite Ramsey Theorem

Finite Ramsey Theorem. (Ramsey, 1929) $k, m, r \geq 1$ with $m \geq k$, there is an $n \geq m$ such that for each coloring $c : [n]^k \to r$, there is an $X \in [n]^m$ such that c is monochromatic on $[X]^k$.

Finite Structural Ramsey Theory

$\binom{B}{A}$ denotes the set of copies of A in B.

Finite Structural Ramsey Theory

$\binom{B}{A}$ denotes the set of copies of A in B.

A Fraïssé class $\mathcal{K}$ has the Ramsey property if for each pair $A \leq B$ in $\mathcal{K}$ and $r \geq 1$, there is some C in $\mathcal{K}$ such that for each coloring $f : \binom{C}{A} \rightarrow r$, there is a $B' \in \binom{C}{B}$ such that f takes one color on $\binom{B'}{A}$.

Finite Structural Ramsey Theory

$\binom{B}{A}$ denotes the set of copies of A in B.

A Fraïssé class $\mathcal{K}$ has the Ramsey property if for each pair $A \leq B$ in $\mathcal{K}$ and $r \geq 1$, there is some C in $\mathcal{K}$ such that for each coloring $f : \binom{C}{A} \to r$, there is a $B' \in \binom{C}{B}$ such that f takes one color on $\binom{B'}{A}$.

$$\forall A \leq B \in \mathcal{K}, \quad \forall r \geq 1, \quad \exists C \in \mathcal{K} \text{ such that } C \to (B)^A_r$$

Finite Structural Ramsey Theory

$\binom{B}{A}$ denotes the set of copies of A in B.

A Fraïssé class $\mathcal{K}$ has the Ramsey property if for each pair $A \leq B$ in $\mathcal{K}$ and $r \geq 1$, there is some C in $\mathcal{K}$ such that for each coloring $f : \binom{C}{A} \to r$, there is a $B' \in \binom{C}{B}$ such that f takes one color on $\binom{B'}{A}$.

$$\forall A \leq B \in \mathcal{K}, \ \ \forall r \geq 1, \ \ \exists C \in \mathcal{K} \ \text{ such that } \ C \to (B)^A_r$$

Some Fraïssé classes of finite structures with the Ramsey property: Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Small Ramsey Degrees

A Fraïssé class not satisfying the Ramsey Property may still possess some Ramseyness.

Small Ramsey Degrees

A Fraïssé class not satisfying the Ramsey Property may still possess some Ramseyness.

A Fraïssé class $\mathcal{K}$ has small Ramsey degrees if for each $A \in \mathcal{K}$ there is an integer $t(A, \mathcal{K})$ such that for each $B \in \mathcal{K}$ with $A \leq B$, and each $r \geq 1$, for each coloring $f : \binom{\mathbb{K}}{A} \to r$, there is a $B' \in \binom{\mathbb{K}}{B}$ such that f takes at most $t(A, \mathcal{K})$ colors on $\binom{B'}{A}$.

$$\forall A \in \mathcal{K} \ \ \exists t(A, \mathcal{K}) \geq 1 \ \ \forall B \in \mathcal{K} \ \ \forall r \geq 1, \ \mathbb{K} \to (B)^{A}_{r, t(A,\mathcal{K})}.$$

Small Ramsey Degrees

A Fraïssé class not satisfying the Ramsey Property may still possess some Ramseyness.

A Fraïssé class $\mathcal{K}$ has small Ramsey degrees if for each $A \in \mathcal{K}$ there is an integer $t(A, \mathcal{K})$ such that for each $B \in \mathcal{K}$ with $A \leq B$, and each $r \geq 1$, for each coloring $f : \binom{\mathbb{K}}{A} \to r$, there is a $B' \in \binom{\mathbb{K}}{B}$ such that f takes at most $t(A, \mathcal{K})$ colors on $\binom{B'}{A}$.

$$\forall A \in \mathcal{K} \;\; \exists t(A, \mathcal{K}) \geq 1 \;\; \forall B \in \mathcal{K} \;\; \forall r \geq 1, \; \mathbb{K} \to (B)^A_{r, t(A,\mathcal{K})}.$$

Here $\mathbb{K} = \mathrm{Flim}(\mathcal{K})$.

Small Ramsey Degrees

A Fraïssé class not satisfying the Ramsey Property may still possess some Ramseyness.

A Fraïssé class $\mathcal{K}$ has small Ramsey degrees if for each $A \in \mathcal{K}$ there is an integer $t(A, \mathcal{K})$ such that for each $B \in \mathcal{K}$ with $A \leq B$, and each $r \geq 1$, for each coloring $f : \binom{\mathbb{K}}{A} \to r$, there is a $B' \in \binom{\mathbb{K}}{B}$ such that f takes at most $t(A, \mathcal{K})$ colors on $\binom{B'}{A}$.

$$\forall A \in \mathcal{K} \;\; \exists t(A, \mathcal{K}) \geq 1 \;\; \forall B \in \mathcal{K} \;\; \forall r \geq 1, \; \mathbb{K} \to (B)^A_{r, t(A,\mathcal{K})}.$$

Here $\mathbb{K} = \mathrm{Flim}(\mathcal{K})$.

Some Fraïssé classes of finite structures with small Ramsey degrees:
The classes of finite graphs, hypergraphs, graphs omitting k-cliques, and others.

Infinite Ramsey Theorem

Infinite Ramsey's Theorem. (Ramsey, 1929) Given $n, r \geq 1$ and a coloring $c : [\mathbb{N}]^n \to r$, there is an infinite subset $N \subseteq \mathbb{N}$ such that c is monochromatic on $[N]^n$.

Infinite Ramsey Theorem

Infinite Ramsey's Theorem. (Ramsey, 1929) Given $n, r \geq 1$ and a coloring $c : [\mathbb{N}]^n \to r$, there is an infinite subset $N \subseteq \mathbb{N}$ such that c is monochromatic on $[N]^n$.

Note: For $n = 2$, this can also be stated in terms of coloring edges in an infinite complete graph by two colors finding an infinite complete graph with all edges having the same color.

Ramsey's Theorem and Logic

Ramsey's Theorem appears in his paper, *On a problem of formal logic*, and was motivated by Hilbert's Entscheidungsproblem:

Find a procedure for determining whether any given formula is valid.

Ramsey's Theorem and Logic

Ramsey's Theorem appears in his paper, *On a problem of formal logic*, and was motivated by Hilbert's Entscheidungsproblem:

Find a procedure for determining whether any given formula is valid.

Ramsey applied his theorem to solve this problem for formulas with only universal quantifiers in front (Π_1).

Infinite Structures with Analogues of Ramsey's Theorem?

Infinite Ramsey's Theorem. Given $n, r \geq 1$ and a coloring $c : [\mathbb{N}]^n \to r$, there is an infinite subset $N \subseteq \mathbb{N}$ such that c is monochromatic on $[N]^n$.

Infinite Structures with Analogues of Ramsey's Theorem?

Infinite Ramsey's Theorem. Given $n, r \geq 1$ and a coloring $c : [\mathbb{N}]^n \to r$, there is an infinite subset $N \subseteq \mathbb{N}$ such that c is monochromatic on $[N]^n$.

Note. $(N, <) \cong (\mathbb{N}, <)$.

Infinite Structures with Analogues of Ramsey's Theorem?

Infinite Ramsey's Theorem. Given $n, r \geq 1$ and a coloring $c : [\mathbb{N}]^n \to r$, there is an infinite subset $N \subseteq \mathbb{N}$ such that c is monochromatic on $[N]^n$.

Note. $(N, <) \cong (\mathbb{N}, <)$.

Question. Which infinite structures possess analogues of Infinite Ramsey's Theorem, where we require the substructure to be isomorphic to the original infinite structure?

Ramsey Theory of $(\mathbb{Q}, <)$

• Any finite coloring of the singletons in $\mathbb{Q}$ is monochromatic on a subset isomorphic to $\mathbb{Q}$.

Ramsey Theory of $(\mathbb{Q}, <)$

• Any finite coloring of the singletons in $\mathbb{Q}$ is monochromatic on a subset isomorphic to $\mathbb{Q}$. However,

• There is a coloring of pairs of rationals into two colors such that any subset $\mathbb{Q}' \subseteq \mathbb{Q}$, which is again a dense linear order without endpoints, takes both colors on its pairsets. (Sierpiński)

Ramsey Theory of $(\mathbb{Q}, <)$

• Any finite coloring of the singletons in $\mathbb{Q}$ is monochromatic on a subset isomorphic to $\mathbb{Q}$. However,

• There is a coloring of pairs of rationals into two colors such that any subset $\mathbb{Q}' \subseteq \mathbb{Q}$, which is again a dense linear order without endpoints, takes both colors on its pairsets. (Sierpiński)

• Given any $n \geq 2$, there is a number $T(n, \mathbb{Q}) \geq 2$ such that any coloring of $[\mathbb{Q}]^n$ into finitely many colors can be reduced to no more than $T(n, \mathbb{Q})$ colors on a substructure $\mathbb{Q}'$ isomorphic to $\mathbb{Q}$. (Laver)

Ramsey Theory of $(\mathbb{Q}, <)$

- Any finite coloring of the singletons in $\mathbb{Q}$ is monochromatic on a subset isomorphic to $\mathbb{Q}$. However,

- There is a coloring of pairs of rationals into two colors such that any subset $\mathbb{Q}' \subseteq \mathbb{Q}$, which is again a dense linear order without endpoints, takes both colors on its pairsets. (Sierpiński)

- Given any $n \geq 2$, there is a number $T(n, \mathbb{Q}) \geq 2$ such that any coloring of $[\mathbb{Q}]^n$ into finitely many colors can be reduced to no more than $T(n, \mathbb{Q})$ colors on a substructure $\mathbb{Q}'$ isomorphic to $\mathbb{Q}$. (Laver)

- The exact numbers $T(n, \mathbb{Q})$ are tangent numbers! (Devlin 1979)

Big Ramsey Degrees of Infinite Structures

Where combinatorics, set theory, model theory, and topology meet.

Big Ramsey Degrees of Infinite Structures

Given structures A, B with A embedding into B ($A \leq B$), let $\binom{B}{A}$ denote the set of all copies of A in B.

Big Ramsey Degrees of Infinite Structures

Given structures A, B with A embedding into B ($A \leq B$), let $\binom{B}{A}$ denote the set of all copies of A in B.

Let $\mathcal{S}$ be an infinite structure.

Big Ramsey Degrees of Infinite Structures

Given structures A, B with A embedding into B ($A \leq B$), let $\binom{B}{A}$ denote the set of all copies of A in B.

Let $\mathcal{S}$ be an infinite structure.

For a finite substructure $\mathrm{A} \leq \mathcal{S}$, let $T(A, \mathcal{S})$ denote the least number, if it exists, such that for each coloring c of $\binom{\mathcal{S}}{A}$ into finitely many colors, there is an $\mathcal{S}' \in \binom{\mathcal{S}}{\mathcal{S}}$ such that c takes no more than $T(A, \mathcal{S})$ colors on $\binom{\mathcal{S}'}{A}$.

Big Ramsey Degrees of Infinite Structures

Given structures A, B with A embedding into B ($A \leq B$), let $\binom{B}{A}$ denote the set of all copies of A in B.

Let $\mathcal{S}$ be an infinite structure.

For a finite substructure $\mathrm{A} \leq \mathcal{S}$, let $T(A, \mathcal{S})$ denote the least number, if it exists, such that for each coloring c of $\binom{\mathcal{S}}{A}$ into finitely many colors, there is an $\mathcal{S}' \in \binom{\mathcal{S}}{\mathcal{S}}$ such that c takes no more than $T(A, \mathcal{S})$ colors on $\binom{\mathcal{S}'}{A}$.

(Kechris, Pestov, Todorcevic, 2005) $\mathcal{S}$ has finite big Ramsey degrees if for each finite $A \leq \mathcal{S}$, $T(A, \mathcal{S})$ exists

Infinite structures known to have finite big Ramsey degrees: The infinite complete graph (Ramsey 1929); the rationals (Devlin 1979); the Rado graph and random tournament (Sauer 2006); the countable ultrametric Urysohn space (Nguyen Van Thé 2008); the $\mathbb{Q}_n$ and the tournaments **S(2)**, **S(3)** (Laflamme, NVT, Sauer 2010), and a few others. (graphs in blue)

Big Ramsey Structures and Topological Dynamics

Infinite structures known to have finite big Ramsey degrees: The infinite complete graph (Ramsey 1929); the rationals (Devlin 1979); the Rado graph and random tournament (Sauer 2006); the countable ultrametric Urysohn space (Nguyen Van Thé 2008); the $\mathbb{Q}_n$ and the tournaments **S(2)**, **S(3)** (Laflamme, NVT, Sauer 2010), and a few others. (graphs in blue)

(Zucker 2017) Characterized universal completion flows of $\mathrm{Aut}(\mathrm{Flim}\ \mathcal{K})$ whenever $\mathrm{Flim}\ \mathcal{K}$ admits a big Ramsey structure (big Ramsey degrees).

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Few Fraïssé structures have been shown to have big Ramsey degrees,

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Few Fraïssé structures have been shown to have big Ramsey degrees,

and

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Few Fraïssé structures have been shown to have big Ramsey degrees,

and

Fraïssé structure with forbidden configurations have presented particular difficulties.

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Few Fraïssé structures have been shown to have big Ramsey degrees,

and

Fraïssé structure with forbidden configurations have presented particular difficulties.

The Problem: Lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

Missing Link: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey property or small Ramsey degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Few Fraïssé structures have been shown to have big Ramsey degrees,

and

Fraïssé structure with forbidden configurations have presented particular difficulties.

The Problem: Lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

This is addressed in two papers, whose work is presented today.

Key Example: Big Ramsey degrees of the Rado graph

The Rado graph $\mathcal{R}$ is the universal ultrahomogeneous graph on countably many vertices.

Key Example: Big Ramsey degrees of the Rado graph

The Rado graph $\mathcal{R}$ is the universal ultrahomogeneous graph on countably many vertices.

- Vertices have big Ramsey degree 1. (Folklore)

Key Example: Big Ramsey degrees of the Rado graph

The Rado graph $\mathcal{R}$ is the universal ultrahomogeneous graph on countably many vertices.

- Vertices have big Ramsey degree 1. (Folklore)
- Edges have big Ramsey degree 2. (Pouzet-Sauer 1996).

Key Example: Big Ramsey degrees of the Rado graph

The Rado graph $\mathcal{R}$ is the universal ultrahomogeneous graph on countably many vertices.

- Vertices have big Ramsey degree 1. (Folklore)
- Edges have big Ramsey degree 2. (Pouzet-Sauer 1996).
- All finite graphs have finite big Ramsey degree. (Sauer 2006)

Key Example: Big Ramsey degrees of the Rado graph

The Rado graph $\mathcal{R}$ is the universal ultrahomogeneous graph on countably many vertices.

• Vertices have big Ramsey degree 1. (Folklore)

• Edges have big Ramsey degree 2. (Pouzet-Sauer 1996).

• All finite graphs have finite big Ramsey degree. (Sauer 2006)

• Actual degrees were found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

Key Example: Big Ramsey degrees of the Rado graph

The Rado graph $\mathcal{R}$ is the universal ultrahomogeneous graph on countably many vertices.

• Vertices have big Ramsey degree 1. (Folklore)

• Edges have big Ramsey degree 2. (Pouzet-Sauer 1996).

• All finite graphs have finite big Ramsey degree. (Sauer 2006)

• Actual degrees were found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

The previously known big Ramsey structures have at their core Milliken's Ramsey Theorem for strong trees.

Strong Trees

For $t \in \omega^{<\omega}$, $|t| = \mathrm{dom}(t) =:$ the length of t.

Strong Trees

For $t \in \omega^{<\omega}$, $|t| = \mathrm{dom}(t) =:$ the length of t.

$T \subseteq \omega^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

Strong Trees

For $t \in \omega^{<\omega}$, $|t| = \mathrm{dom}(t) =:$ the length of t.

$T \subseteq \omega^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

For $t \in T \subseteq \omega^{<\omega}$, $\mathrm{height}_T(t) = \mathrm{o.t.}\{u \in T : u \subset t\}$.

Strong Trees

For $t \in \omega^{<\omega}$, $|t| = \mathrm{dom}(t) =:$ the length of t.

$T \subseteq \omega^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

For $t \in T \subseteq \omega^{<\omega}$, $\mathrm{height}_T(t) = \mathrm{o.t.}\{u \in T : u \subset t\}$.

$T(n) = \{t \in T : \mathrm{height}_T(t) = n\}$.

Strong Trees

For $t \in \omega^{<\omega}$, $|t| = \mathrm{dom}(t) =:$ the length of t.

$T \subseteq \omega^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

For $t \in T \subseteq \omega^{<\omega}$, $\mathrm{height}_T(t) = \mathrm{o.t.}\{u \in T : u \subset t\}$.

$T(n) = \{t \in T : \mathrm{height}_T(t) = n\}$.

For $t \in T$, $\mathrm{Succ}_T(t) = \{u \in \widehat{T} : u \supset t \text{ and } |u| = |t| + 1\}$.

Strong Trees

For $t \in \omega^{<\omega}$, $|t| = \mathrm{dom}(t) =:$ the length of t.

$T \subseteq \omega^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

For $t \in T \subseteq \omega^{<\omega}$, $\mathrm{height}_T(t) = \mathrm{o.t.}\{u \in T : u \subset t\}$.

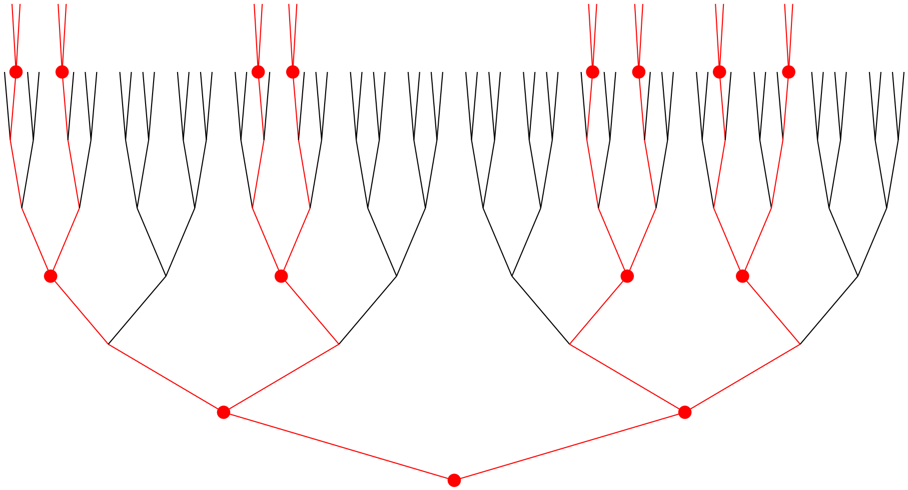
$T(n) = \{t \in T : \mathrm{height}_T(t) = n\}$.

For $t \in T$, $\mathrm{Succ}_T(t) = \{u \in \widehat{T} : u \supset t \text{ and } |u| = |t| + 1\}$.

$S \subseteq T$ is a strong subtree of T iff there is an infinite set $\{m_n : n < \omega\}$ such that

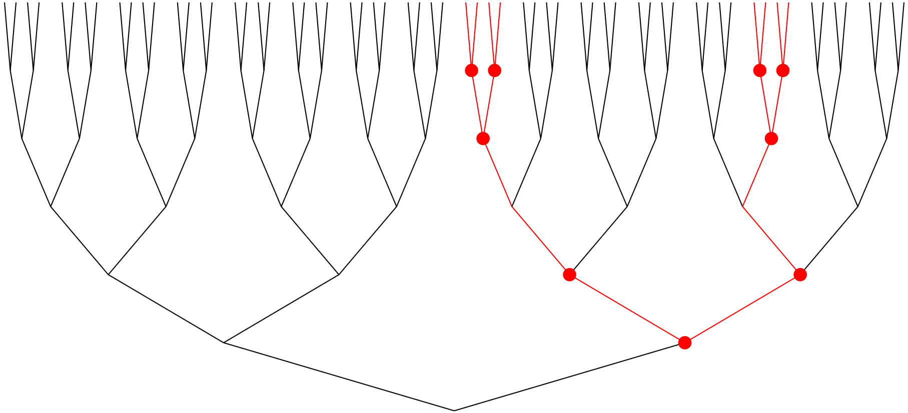
1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < \omega$, $s \in S(n)$ and $u \in \mathrm{Succ}_T(s)$, there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $S \subseteq 2^{<\omega}$



The nodes in S are of lengths $0, 1, 3, 6, \ldots$

Example: A Strong Subtree $U \subseteq 2^{<\omega}$



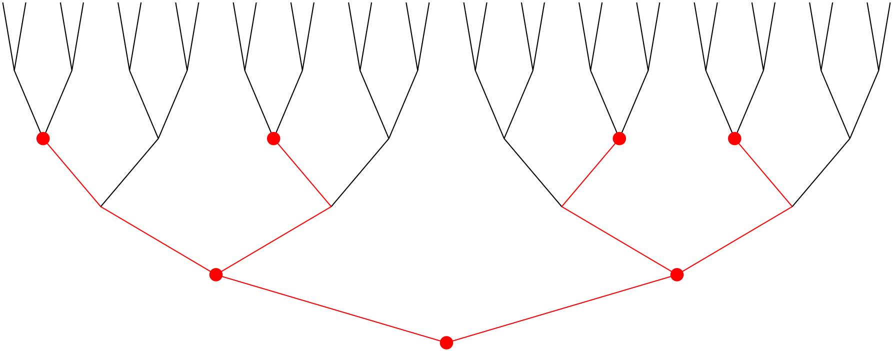
The nodes in U are of lengths $1, 2, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees

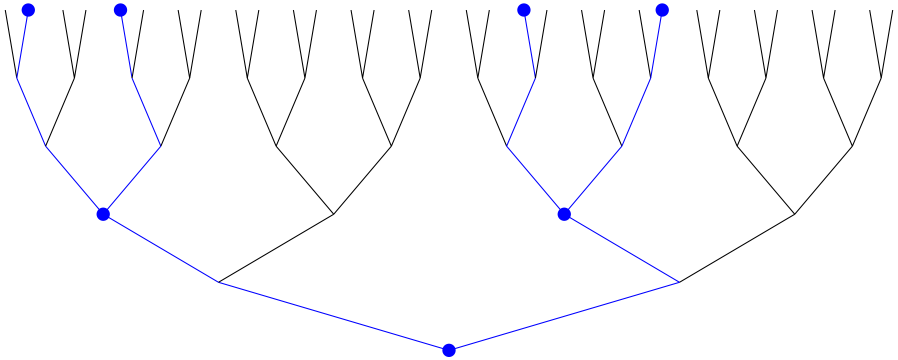
Thm. (Milliken 1979) Let $T \subseteq \omega^{<\omega}$ be a finitely branching tree with no terminal nodes. Let $k \geq 0$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

Ex: Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

Given a coloring c of all 3-strong trees in $2^{<\omega}$ into red and blue:

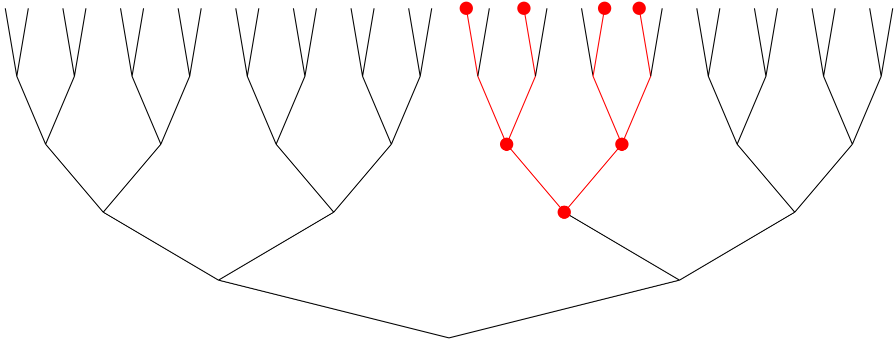


Ex: Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$



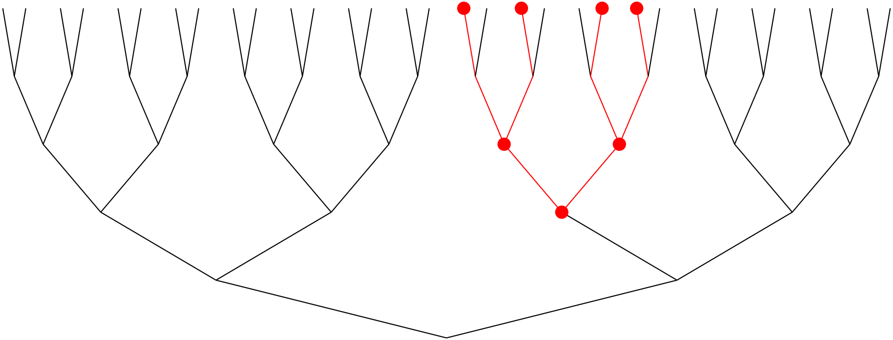
Ex: Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

and this



Ex: Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

and this



Milliken's Theorem guarantees a strong subtree in which all 3-strong subtrees have the same color.

Remark. Milliken's space $\mathcal{M}$ of infinite strong trees forms a topological Ramsey space.

Remark. Milliken's space $\mathcal{M}$ of infinite strong trees forms a topological Ramsey space.

This means that, similarly to the Baire space, there is a topology so that all definable subsets of $\mathcal{M}$ have the Ramsey property.

Remark. Milliken's space $\mathcal{M}$ of infinite strong trees forms a topological Ramsey space.

This means that, similarly to the Baire space, there is a topology so that all definable subsets of $\mathcal{M}$ have the Ramsey property.

How is Milliken's Theorem applied to get upper bounds for the Ramsey degrees of the Rado graph?

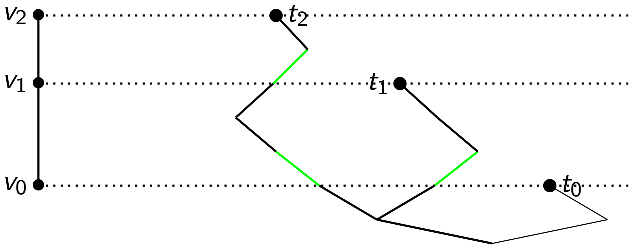
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

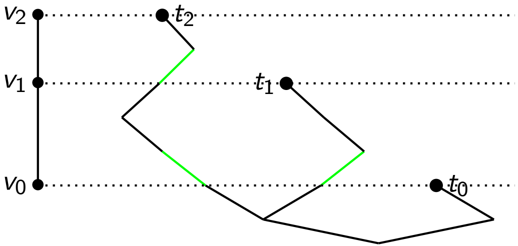
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Diagonal Trees Code Graphs

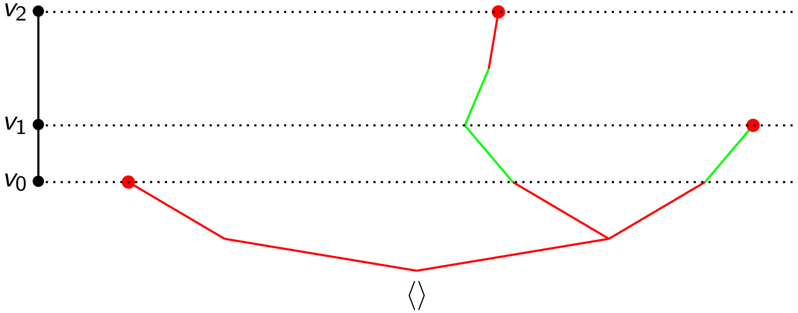
A tree T is diagonal if there is at most one meet or terminal node per level.

T is strongly diagonal if passing numbers at splitting levels are all 0 (except for the right extension of the splitting node).

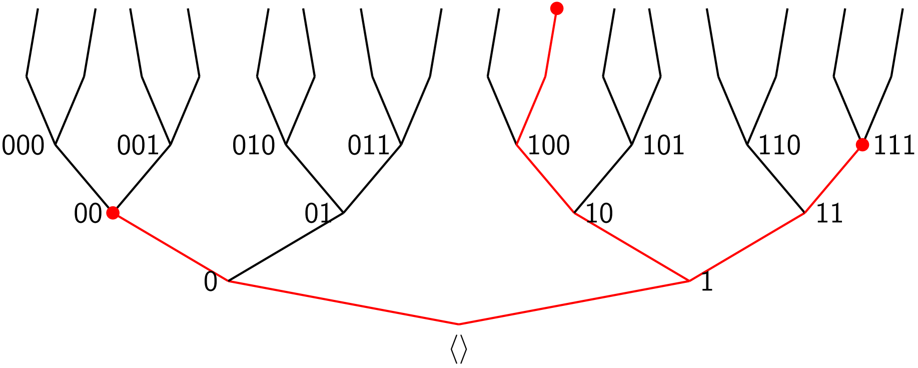


Every graph can be coded by the terminal nodes of a diagonal tree. Moreover, there is a strongly diagonal tree which codes $\mathcal{R}$.

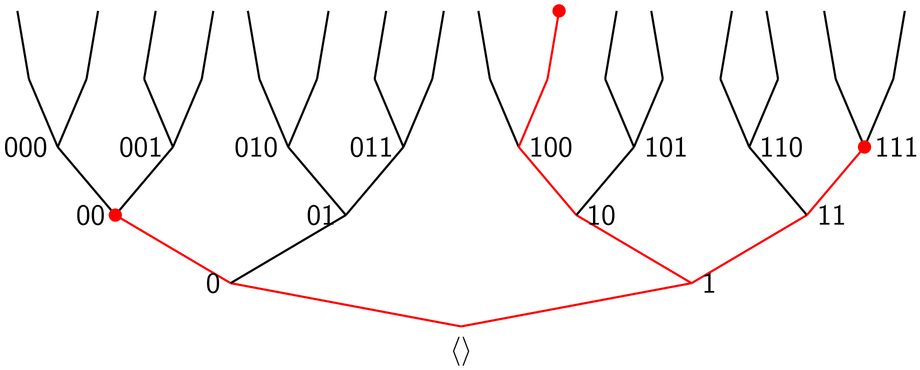
A Different Strongly Diagonal Tree Coding a Path



Strongly diagonal trees can be enveloped into strong trees



Another strong tree envelope



Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.

Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.

2. Each finite graph can be coded by finitely many strong similarity types of strongly diagonal trees.

Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of strongly diagonal trees.
3. Each strongly diagonal tree can be enveloped into a finite strong tree.

Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of strongly diagonal trees.
3. Each strongly diagonal tree can be enveloped into a finite strong tree.
4. Apply Milliken's Theorem finitely many times to obtain one color for each type.

Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of strongly diagonal trees.
3. Each strongly diagonal tree can be enveloped into a finite strong tree.
4. Apply Milliken's Theorem finitely many times to obtain one color for each type.
5. Choose a strongly diagonal antichain coding the Rado graph.

Henson Graphs

For $k \geq 3$, the k-clique-free Henson graph, $\mathcal{H}_k$, is the universal ultrahomogenous k-clique-free graph.

Henson Graphs

For $k \geq 3$, the k-clique-free Henson graph, $\mathcal{H}_k$, is the universal ultrahomogenous k-clique-free graph.

Main Thm. (D.) The Henson graphs have finite big Ramsey degrees.

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

• Henson constructed the graphs $\mathcal{H}_k$ for $k \geq 3$, and proved they are weakly indivisible in 1971.

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

• Henson constructed the graphs $\mathcal{H}_k$ for $k \geq 3$, and proved they are weakly indivisible in 1971.

• The Fraïssé class of finite ordered k-clique-free graphs $\mathcal{G}_k^<$ has the Ramsey property. (Nešetřil-Rödl, 1977/83)

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

• Henson constructed the graphs $\mathcal{H}_k$ for $k \geq 3$, and proved they are weakly indivisible in 1971.

• The Fraïssé class of finite ordered k-clique-free graphs $\mathcal{G}_k^<$ has the Ramsey property. (Nešetřil-Rödl, 1977/83)

• Vertices in $\mathcal{H}_3$ have big Ramsey degree 1. (Komjáth-Rödl, 1986)

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

• Henson constructed the graphs $\mathcal{H}_k$ for $k \geq 3$, and proved they are weakly indivisible in 1971.

• The Fraïssé class of finite ordered k-clique-free graphs $\mathcal{G}_k^<$ has the Ramsey property. (Nešetřil-Rödl, 1977/83)

• Vertices in $\mathcal{H}_3$ have big Ramsey degree 1. (Komjáth-Rödl, 1986)

• For all $k \geq 4$, $\mathcal{H}_k$ has big Ramsey degree 1. (El-Zahar-Sauer, 1989)

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

• Henson constructed the graphs $\mathcal{H}_k$ for $k \geq 3$, and proved they are weakly indivisible in 1971.

• The Fraïssé class of finite ordered k-clique-free graphs $\mathcal{G}_k^<$ has the Ramsey property. (Nešetřil-Rödl, 1977/83)

• Vertices in $\mathcal{H}_3$ have big Ramsey degree 1. (Komjáth-Rödl, 1986)

• For all $k \geq 4$, $\mathcal{H}_k$ has big Ramsey degree 1. (El-Zahar-Sauer, 1989)

• $\mathcal{H}_3$ has big Ramsey degree 2 for edges. (Sauer, 1998)

History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_k$ is the Fraïssé limit of the class of finite k-clique-free graphs.

• Henson constructed the graphs $\mathcal{H}_k$ for $k \geq 3$, and proved they are weakly indivisible in 1971.

• The Fraïssé class of finite ordered k-clique-free graphs $\mathcal{G}_k^<$ has the Ramsey property. (Nešetřil-Rödl, 1977/83)

• Vertices in $\mathcal{H}_3$ have big Ramsey degree 1. (Komjáth-Rödl, 1986)

• For all $k \geq 4$, $\mathcal{H}_k$ has big Ramsey degree 1. (El-Zahar-Sauer, 1989)

• $\mathcal{H}_3$ has big Ramsey degree 2 for edges. (Sauer, 1998)

There progress halted. Why?

Main Obstacles to Big Ramsey Degrees of $\mathcal{H}_k$

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Main Obstacles to Big Ramsey Degrees of $\mathcal{H}_k$

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Said the same thing, plus, "There is no simply representable triangle-free graph which is bi-embeddable with $\mathcal{H}_3$." (Sauer, 2013)

Main Obstacles to Big Ramsey Degrees of $\mathcal{H}_k$

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Said the same thing, plus, "There is no simply representable triangle-free graph which is bi-embeddable with $\mathcal{H}_3$." (Sauer, 2013)

"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, Habilitation 2013)

Our proof strategy

Follow the outline of Sauer's proof of upper bounds for big Ramsey degrees of the Rado graph, constructing new analogues at each stage.

Main Theorem: Ramsey Theory for Henson Graphs

Theorem. (D.) Let $k \geq 3$. For each finite k-clique-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{G}_k)$ such that for any coloring of all copies of A in $\mathcal{H}_k$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_k$, with $\mathcal{H} \cong \mathcal{H}_k$, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{G}_k)$ colors.

Structure of Proof: Four Main Parts

- Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

Structure of Proof: Four Main Parts

- Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

  These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for trees with the Strict Witnessing Property.

Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for trees with the Strict Witnessing Property.

This is an analogue of Milliken's Theorem for strong trees.

Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for trees with the Strict Witnessing Property.

This is an analogue of Milliken's Theorem for strong trees.

III Prove a Ramsey Theorem for strictly similar finite antichains.

Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for trees with the Strict Witnessing Property.

This is an analogue of Milliken's Theorem for strong trees.

III Prove a Ramsey Theorem for strictly similar finite antichains.

This is obtained by a new notion of envelope.

Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for trees with the Strict Witnessing Property.

This is an analogue of Milliken's Theorem for strong trees.

III Prove a Ramsey Theorem for strictly similar finite antichains.

This is obtained by a new notion of envelope.

IV Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Structure of Proof: Four Main Parts

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for trees with the Strict Witnessing Property.

This is an analogue of Milliken's Theorem for strong trees.

III Prove a Ramsey Theorem for strictly similar finite antichains.

This is obtained by a new notion of envelope.

IV Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Similar to the end of Sauer's proof.

Part I: Strong $\mathcal{H}_k$-Coding Trees

Idea: Want correct analogue of strong trees for setting of $\mathcal{H}_k$.

Part I: Strong $\mathcal{H}_k$-Coding Trees

Idea: Want correct analogue of strong trees for setting of $\mathcal{H}_k$.

Problem: How to make sure K_k is never encoded
but branching is as thick as possible?

First Approach: Strong K_k-Free Trees

• Work with trees with an extra unary predicate which distinguishes certain nodes to code vertices of a given graph (called coding nodes).

First Approach: Strong K_k-Free Trees

• Work with trees with an extra unary predicate which distinguishes certain nodes to code vertices of a given graph (called coding nodes).

• Make a Branching Criterion so that a node s splits iff all its extensions will never code K_k with coding nodes at or below the level of s.

K_k-Free Branching Criterion

For $a \geq 2$, given an index set I of size a, a collection of coding nodes $\{c_i : i \in I\}$ in T codes an a-clique iff for each pair $i < j$ in I, $c_j(l_i) = 1$.

K_k-Free Branching Criterion

For $a \geq 2$, given an index set I of size a, a collection of coding nodes $\{c_i : i \in I\}$ in T codes an a-clique iff for each pair $i < j$ in I, $c_j(l_i) = 1$.

A tree T with coding nodes $\langle c_n : n < N \rangle$ satisfies the K_k-Free Branching Criterion (k-FBC) if for each non-maximal node $t \in T$, $t^\frown 0$ is always in T, and $t^\frown 1$ is in T iff adding $t^\frown 1$ as a coding node to T would not code a k-clique with coding nodes in T of shorter length.

Henson's Criterion for building $\mathcal{H}_k$

Henson proved that a countable graph $\mathcal{H}$ is universal for countable K_k-free graphs if and only if $\mathcal{H}$ satisfies the property (A_k):

(i) $\mathcal{H}$ does not admit any k-cliques,

(ii) If V_0, V_1 are disjoint finite sets of vertices of $\mathcal{H}$ and $\mathcal{H}|V_0$ does not admit any $(k-1)$-cliques, then there is another vertex which is connected in $\mathcal{H}$ to every member of V_0 and to no member of V_1.

Henson's Criterion for building $\mathcal{H}_k$

Henson proved that a countable graph $\mathcal{H}$ is universal for countable K_k-free graphs if and only if $\mathcal{H}$ satisfies the property (A_k):

(i) $\mathcal{H}$ does not admit any k-cliques,

(ii) If V_0, V_1 are disjoint finite sets of vertices of $\mathcal{H}$ and $\mathcal{H}|V_0$ does not admit any $(k-1)$-cliques, then there is another vertex which is connected in $\mathcal{H}$ to every member of V_0 and to no member of V_1.

For trees with coding nodes, this becomes $(A_k)^{\texttt{tree}}$:

(i) T satisfies the K_k-Free Criterion.

(ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i - 1$, and each finite subset of ω appears as F_i for infinitely many indices i. Given $i < \omega$, if for each subset $J \subseteq F_i$ of size $k-1$, $\{c_j : j \in J\}$ does not code a $(k-1)$-clique, then there is some $n \geq i$ such that for all $j < i$, $c_n(l_j) = 1$ iff $j \in F_i$.

Thm. Let T be a tree with no maximal nodes and coding nodes dense in T, and satisfying the K_k-Free Branching Criterion. Then T satisfies $(A_k)^{\mathtt{tree}}$, and hence codes $\mathcal{H}_k$.

Strong K_3-Free Tree

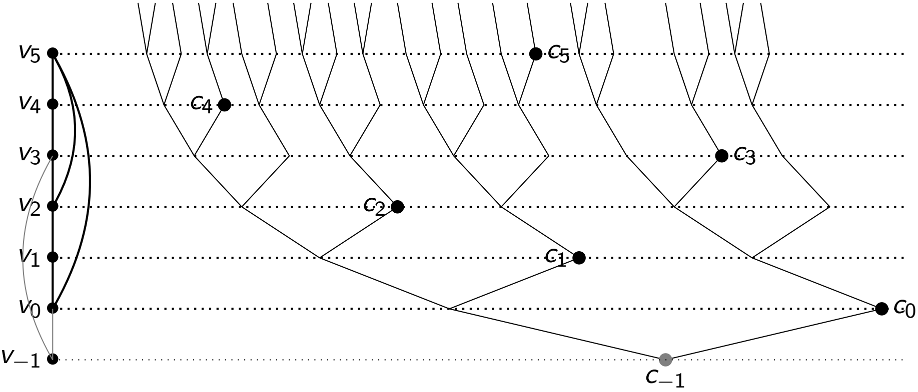


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree

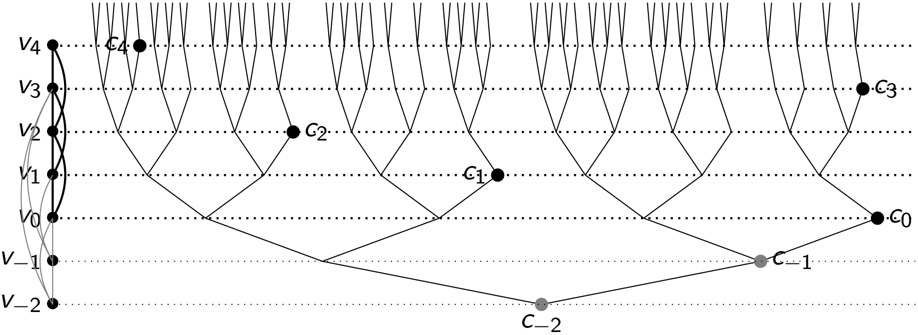


Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

Almost sufficient

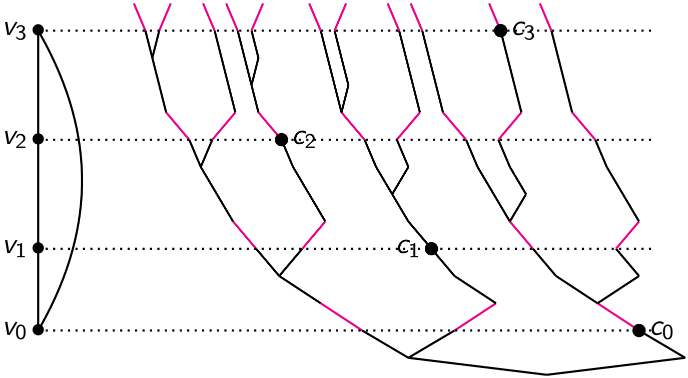
One can develop almost all the Ramsey theory one needs on strong triangle-free trees

Almost sufficient

One can develop almost all the Ramsey theory one needs on strong triangle-free trees

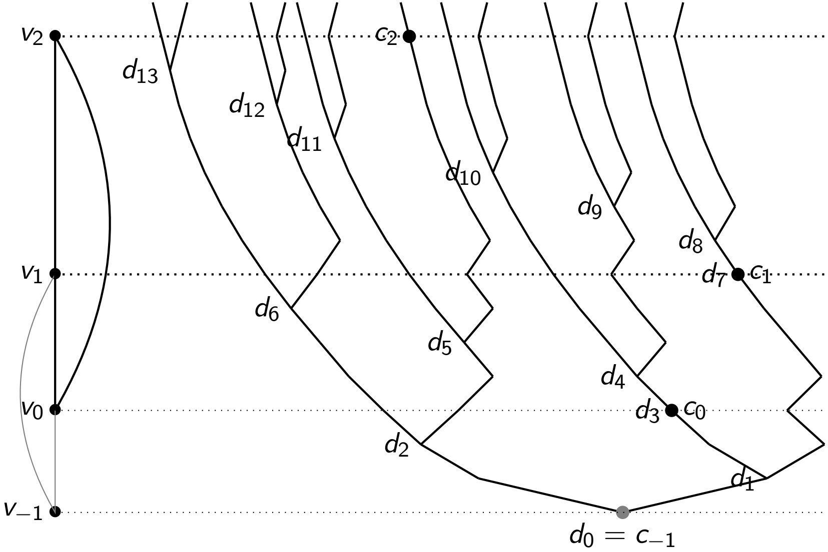
except for vertex colorings: there is a bad coloring of coding nodes.

Refined Approach: Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



Skew the levels of interest.

Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$



Strong Similarity Map

Let $k \geq 3$ be given and let $S, T \subseteq \mathbb{T}_k$ be meet-closed subsets. A bijection $f : S \to T$ is a strong similarity map if for all nodes $s, t, u, v \in S$, the following hold:

1. f preserves lexicographic order.
2. f preserves meets, and hence splitting nodes.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves coding nodes.
6. f preserves passing numbers at coding nodes.

Mutual Pre-a-Clique: A key concept

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level subset X of $\mathbb{T}_k$ of size at least two has a (mutual) pre-a-clique if $\exists \mathcal{I} \subseteq [\omega]^{a-2}$ such that, letting $i_* = \max(\mathcal{I})$ and $l_* = |c^k_{i_*}|$:

1. $l_* \leq l_X$, and there are exactly the same number of nodes in the level set $X \restriction l_*$ as in X;
2. The set $\{c^k_i : i \in \mathcal{I}\}$ codes a $(a-2)$-clique;
3. Each node in X^+ has passing number 1 at c^k_i, for each $i \in \mathcal{I}$.

The set $\{c^k_i : i \in \mathcal{I}\}$ witnesses that X has a pre-a-clique at l_*.

Mutual Pre-a-Clique: A key concept

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level subset X of $\mathbb{T}_k$ of size at least two has a (mutual) pre-a-clique if $\exists \mathcal{I} \subseteq [\omega]^{a-2}$ such that, letting $i_* = \max(\mathcal{I})$ and $l_* = |c^k_{i_*}|$:

1. $l_* \leq l_X$, and there are exactly the same number of nodes in the level set $X \restriction l_*$ as in X;
2. The set $\{c^k_i : i \in \mathcal{I}\}$ codes a $(a-2)$-clique;
3. Each node in X^+ has passing number 1 at c^k_i, for each $i \in \mathcal{I}$.

The set $\{c^k_i : i \in \mathcal{I}\}$ witnesses that X has a pre-a-clique at l_*.

The Point. Pre-a-cliques for $a \in [3, k]$ code entanglements that affect how nodes can extend.

Mutual Pre-a-Clique: A key concept

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level subset X of $\mathbb{T}_k$ of size at least two has a (mutual) pre-a-clique if $\exists \mathcal{I} \subseteq [\omega]^{a-2}$ such that, letting $i_* = \max(\mathcal{I})$ and $l_* = |c^k_{i_*}|$:

1. $l_* \leq l_X$, and there are exactly the same number of nodes in the level set $X \restriction l_*$ as in X;
2. The set $\{c^k_i : i \in \mathcal{I}\}$ codes a $(a-2)$-clique;
3. Each node in X^+ has passing number 1 at c^k_i, for each $i \in \mathcal{I}$.

The set $\{c^k_i : i \in \mathcal{I}\}$ witnesses that X has a pre-a-clique at l_*.

The Point. Pre-a-cliques for $a \in [3, k]$ code entanglements that affect how nodes can extend. These need to be witnessed by coding nodes *in* a subtree in order for things to work.

Stable Map

Let S and T be strongly similar subtrees of $\mathbb{T}_k$ with $M \leq \omega$ critical nodes. The strong similarity map $f : T \to S$ is stable if for each $m \in [1, M)$, the following holds:

Stable Map

Let S and T be strongly similar subtrees of $\mathbb{T}_k$ with $M \leq \omega$ critical nodes. The strong similarity map $f : T \to S$ is stable if for each $m \in [1, M)$, the following holds:

For each $a \in [3, k]$, a level subset $X \subseteq T \restriction |d^T_m|$ has a maximal new pre-a-clique in T in the interval $(|d^T_{m-1}|, |d^T_m|]$ if and only if $f[X]$ has a maximal new pre-a-clique in S in the interval $(|d^S_{m-1}|, |d^S_m|]$.

Stable Map

Let S and T be strongly similar subtrees of $\mathbb{T}_k$ with $M \le \omega$ critical nodes. The strong similarity map $f : T \to S$ is stable if for each $m \in [1, M)$, the following holds:

For each $a \in [3, k]$, a level subset $X \subseteq T \restriction |d^T_m|$ has a maximal new pre-a-clique in T in the interval $(|d^T_{m-1}|, |d^T_m|]$ if and only if $f[X]$ has a maximal new pre-a-clique in S in the interval $(|d^S_{m-1}|, |d^S_m|]$.

We say that S and T are stably isomorphic and write $S \cong T$.

The Space of Strong $\mathcal{H}_k$-Coding Trees: $(\mathcal{T}_k, \leq, r)$

$\mathcal{T}_k$ is the collection of all subtrees of $\mathbb{T}_k$ which are stably isomorphic to $\mathbb{T}_k$.

The Space of Strong $\mathcal{H}_k$-Coding Trees: $(\mathcal{T}_k, \leq, r)$

$\mathcal{T}_k$ is the collection of all subtrees of $\mathbb{T}_k$ which are stably isomorphic to $\mathbb{T}_k$.

$S \leq T$ iff S is a subtree of T

$r_n(T)$ is the first n levels of T.

The Space of Strong $\mathcal{H}_k$-Coding Trees: $(\mathcal{T}_k, \leq, r)$

$\mathcal{T}_k$ is the collection of all subtrees of $\mathbb{T}_k$ which are stably isomorphic to $\mathbb{T}_k$.

$S \leq T$ iff S is a subtree of T

$r_n(T)$ is the first n levels of T.

The space $\mathcal{T}_k$ is very near a topological Ramsey space.

A structural characterization of members of $\mathcal{T}_k$

A subtree T of $\mathbb{T}_k$ has the Witnessing Property (WP) if for each $a \in [3, k]$, each new pre-a-clique in T takes place in some interval in T of the form $(|d^T_{m_n - 1}|, |c^T_n|]$ and is witnessed by a set of coding nodes in T.

A structural characterization of members of $\mathcal{T}_k$

A subtree T of $\mathbb{T}_k$ has the Witnessing Property (WP) if for each $a \in [3, k]$, each new pre-a-clique in T takes place in some interval in T of the form $(|d^T_{m_n-1}|, |c^T_n|]$ and is witnessed by a set of coding nodes in T.

Lem. A tree $T \subseteq \mathbb{T}_k$ is a member of $\mathcal{T}_k$ iff T is strongly similar to $\mathbb{T}_k$ and has the Witnessing Property.

Extension Lemmas

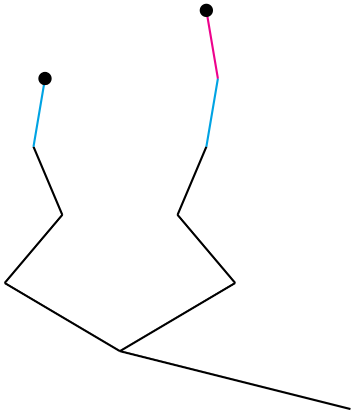
Not every finite subtree A of some $T \in \mathcal{T}_k$ can be extended within T as desired.

A series of extension lemmas shows that whenever A has the Witnessing Property and free level sets, then A is extendible as desired within T.

We call such finite trees valid in T.

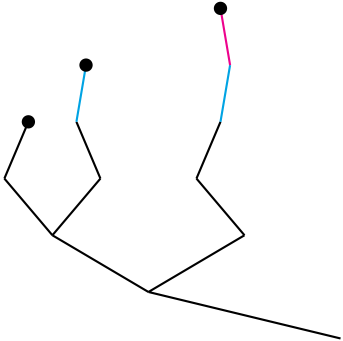
A subtree of $\mathbb{T}_3$ in which WP fails

It has a pre-3-clique not witnessed by a coding node.



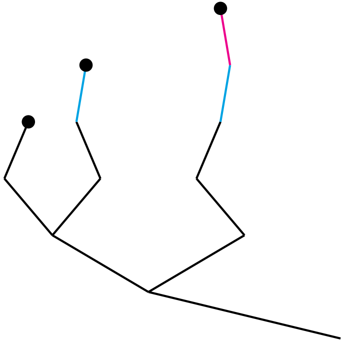
A subtree of $\mathbb{T}_3$ in which WP holds

Its pre-3-cliques are witnessed by a coding node.



A subtree of $\mathbb{T}_3$ in which WP holds

Its pre-3-cliques are witnessed by a coding node.



This gives the basic idea of WP.

Part II: A Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

Part II: A Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

Ideas:

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets.

Part II: A Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

Ideas:

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

Part II: A Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

Ideas:

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

(b) Then weave together to obtain an analogue of Milliken's Theorem.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.
Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.
Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.
Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.
Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.
Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.
Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.

Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.

Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.

Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP.

Assume moreover that $0^{(l_{\tilde{X}})} \in \tilde{X}$.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.

Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.

Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.

Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP.

Assume moreover that $0^{(l_{\tilde{X}})} \in \tilde{X}$.

Case (a). $\tilde{X}$ contains a splitting node.

Case (b). $\tilde{X}$ contains a coding node.

(a) Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and A a finite valid subtree of T with WP.
Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.
Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.
Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP.
Assume moreover that $0^{(l_{\tilde{X}})} \in \tilde{X}$.

Case (a). $\tilde{X}$ contains a splitting node.

Case (b). $\tilde{X}$ contains a coding node.

$$\mathrm{Ext}_T(A, \tilde{X}) = \{X \subseteq T : X \sqsupseteq \tilde{X} \text{ is a level set, } A \cup X \cong A \cup \tilde{X}, \\ \text{and } A \cup X \text{ is valid in } T\}.$$

(a) Ramsey Theorem for Level Set Colorings

Thm. Assume the previous set-up.
Given any coloring $h : \mathrm{Ext}_T(A, \tilde{X}) \to 2$, there is a strong coding tree $S \in [B, T]$ such that h is monochromatic on $\mathrm{Ext}_S(A, \tilde{X})$.
If $\tilde{X}$ has a coding node, then the strong coding tree S is, moreover, taken to be in $[r_{m_0-1}(B'), T]$, where m_0 is the integer for which there is a $B' \in r_{m_0}[B, T]$ with $\tilde{X} \subseteq \max(B')$.

(a) Ramsey Theorem for Level Set Colorings

Thm. Assume the previous set-up.
Given any coloring $h : \mathrm{Ext}_T(A, \tilde{X}) \to 2$, there is a strong coding tree $S \in [B, T]$ such that h is monochromatic on $\mathrm{Ext}_S(A, \tilde{X})$.
If $\tilde{X}$ has a coding node, then the strong coding tree S is, moreover, taken to be in $[r_{m_0-1}(B'), T]$, where m_0 is the integer for which there is a $B' \in r_{m_0}[B, T]$ with $\tilde{X} \subseteq \max(B')$.

We will go over the ideas of the proof later.

(b) Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.

(b) Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.
2. If d^A_{m+1} is a coding node, A has at most one new pre-clique in this interval.

(b) Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.
2. If d^A_{m+1} is a coding node, A has at most one new pre-clique in this interval.
3. If Y is a new pre-clique in this interval, then each proper subset of Y has a new pre-clique in some interval $(|d^A_j|, |d^A_{j+1}|]$, where $j < m$.

(b) Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.
2. If d^A_{m+1} is a coding node, A has at most one new pre-clique in this interval.
3. If Y is a new pre-clique in this interval, then each proper subset of Y has a new pre-clique in some interval $(|d^A_j|, |d^A_{j+1}|]$, where $j < m$.

Lem. If $A \subseteq \mathbb{T}_k$ has the Strict Witnessing Property and $B \cong A$, then B also has the Strict Witnessing Property.

(b) Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.
2. If d^A_{m+1} is a coding node, A has at most one new pre-clique in this interval.
3. If Y is a new pre-clique in this interval, then each proper subset of Y has a new pre-clique in some interval $(|d^A_j|, |d^A_{j+1}|]$, where $j < m$.

Lem. If $A \subseteq \mathbb{T}_k$ has the Strict Witnessing Property and $B \cong A$, then B also has the Strict Witnessing Property.

Any B stably isomorphic to A is a copy of A.

(b) Ramsey Theorem for Finite Trees with SWP

Thm. Let $T \in \mathcal{T}_k$ and A be a finite subtree of T with the Strict Witnessing Property. Let c be a coloring of all copies of A in T. Then there is a strong $\mathcal{H}_k$-coding tree $S \leq T$ in which all copies of A in S have the same color.

(b) Ramsey Theorem for Finite Trees with SWP

Thm. Let $T \in \mathcal{T}_k$ and A be a finite subtree of T with the Strict Witnessing Property. Let c be a coloring of all copies of A in T. Then there is a strong $\mathcal{H}_k$-coding tree $S \leq T$ in which all copies of A in S have the same color.

This is an analogue of Milliken's Theorem for strong coding trees.

Part III: Ramsey Theorem for Strictly Similar Finite Antichains

Ramsey Theorem for Strictly Similar Antichains

Thm. Let Z be a finite antichain of coding nodes in an incremental tree $T \in \mathcal{T}_k$, and suppose h colors of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an incremental strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.

Ramsey Theorem for Strictly Similar Antichains

Thm. Let Z be a finite antichain of coding nodes in an incremental tree $T \in \mathcal{T}_k$, and suppose h colors of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an incremental strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.

New Concepts: incremental new pre-cliques, strict similarity, envelopes to transform an antichain to a tree with SWP.

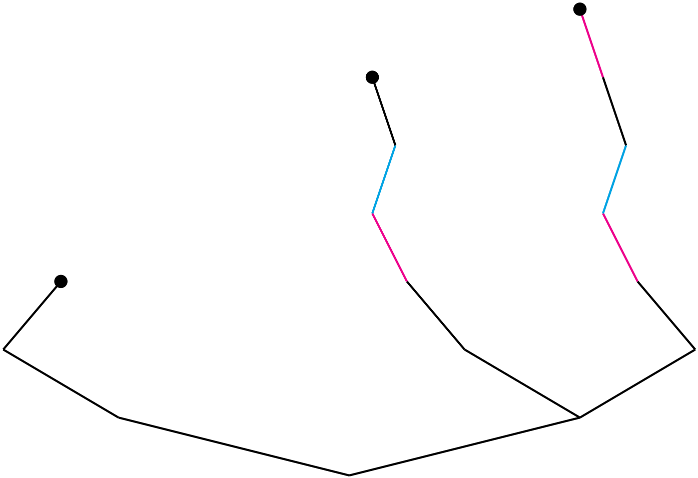
Some Examples of Strict Similarity Types for $k = 3$

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

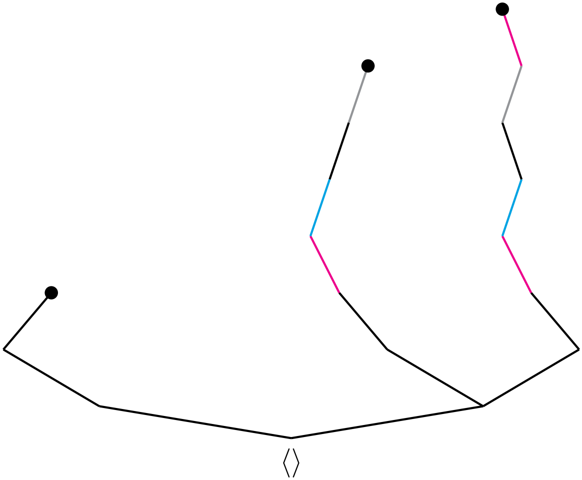
G a graph with three vertices and no edges

A tree A coding G - not WP but still a valid strict similarity type



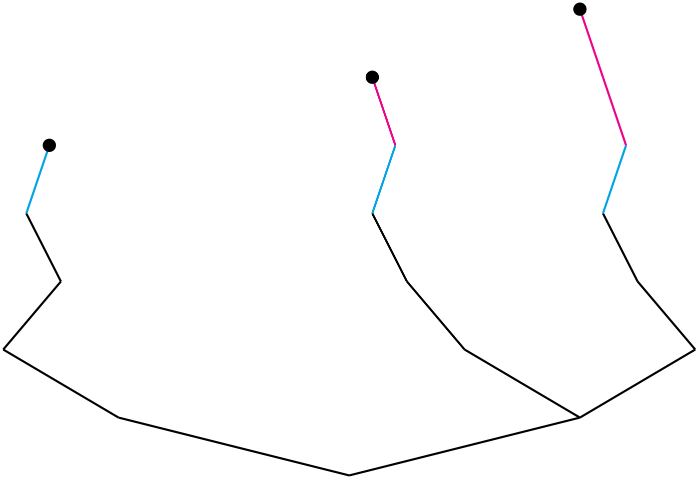
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



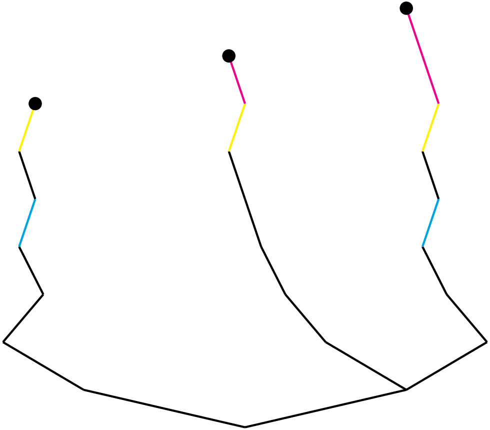
The tree C codes G

C is not strictly similar to A.

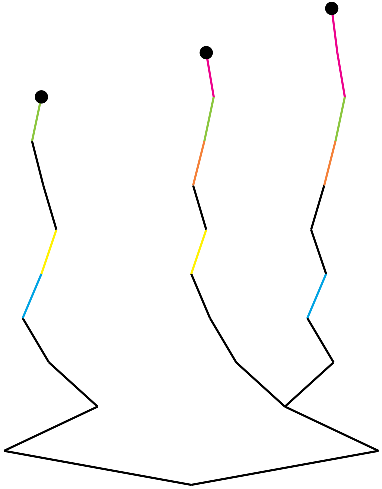


The tree D codes G

D is not strictly similar to either A or C.

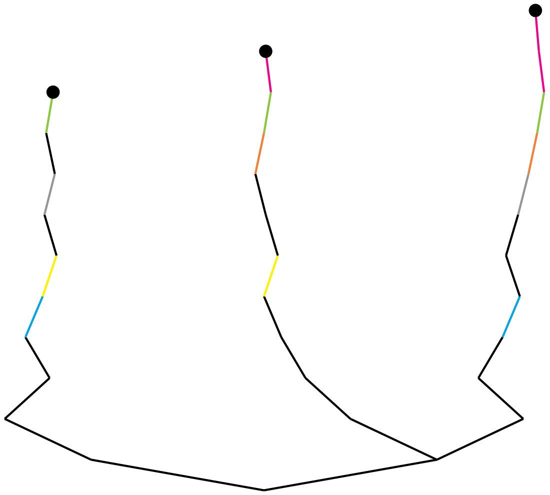


The tree E codes G and is not strictly similar to A - D



E is incremental. More on that later.

The tree F codes G and is strictly similar to E



F is also incremental.

Envelopes and Witnessing Coding Nodes

Envelopes add some neutral coding nodes to a finite tree to make it satisfy the Strict Witnessing Property.

Envelopes and Witnessing Coding Nodes

Envelopes add some neutral coding nodes to a finite tree to make it satisfy the Strict Witnessing Property.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Envelopes and Witnessing Coding Nodes

Envelopes add some neutral coding nodes to a finite tree to make it satisfy the Strict Witnessing Property.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

Envelopes and Witnessing Coding Nodes

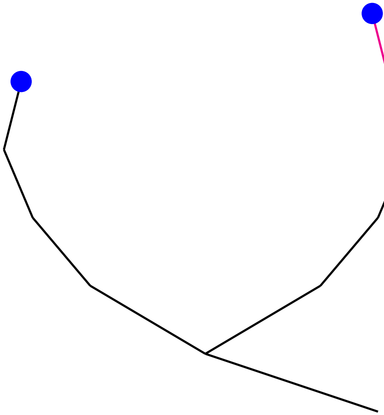
Envelopes add some neutral coding nodes to a finite tree to make it satisfy the Strict Witnessing Property.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

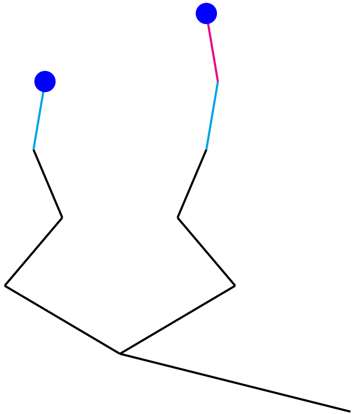
We now give some examples of envelopes.

H codes a non-edge



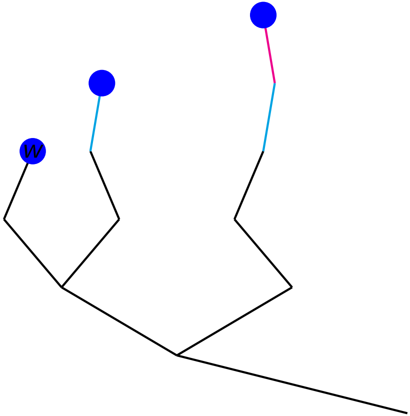
This satisfies the SWP, so H is its own envelope.

I codes a non-edge



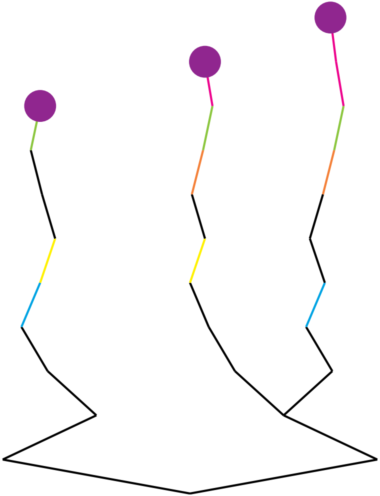
I does not satisfy the WP.

An Envelope $\mathbf{E}(I)$

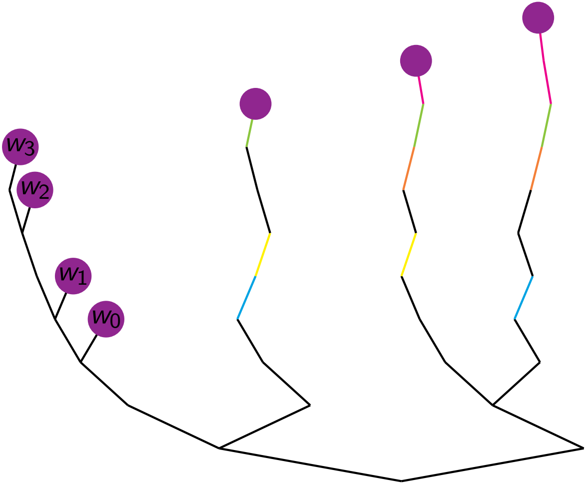


The witnessing coding node w is added to make an envelope.

The incremental tree E from before

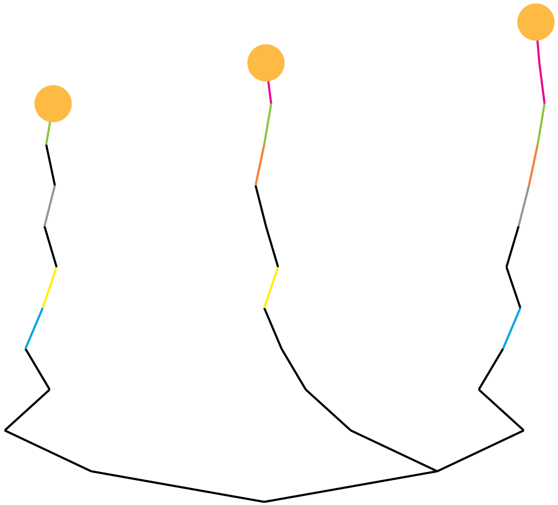


An envelope $\mathbf{E}(E)$

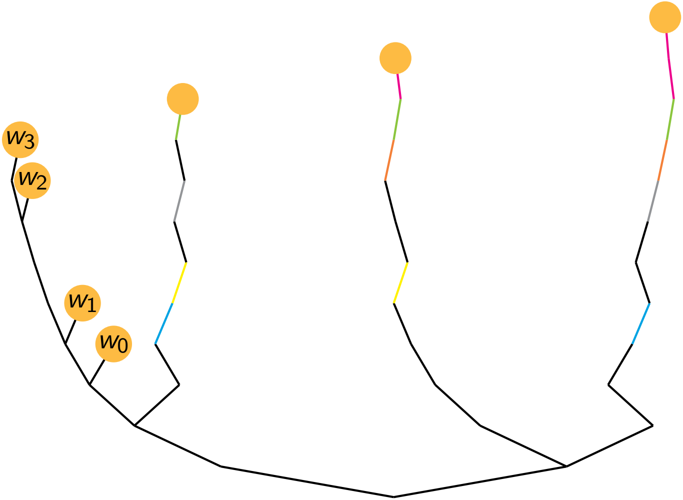


The witnessing coding nodes $w_1, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E



$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



The witnessing coding nodes $w_0, \dots, w_3$ make an envelope of F.

Part IV: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.
4. Apply the Ramsey Theorem from Part III, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.

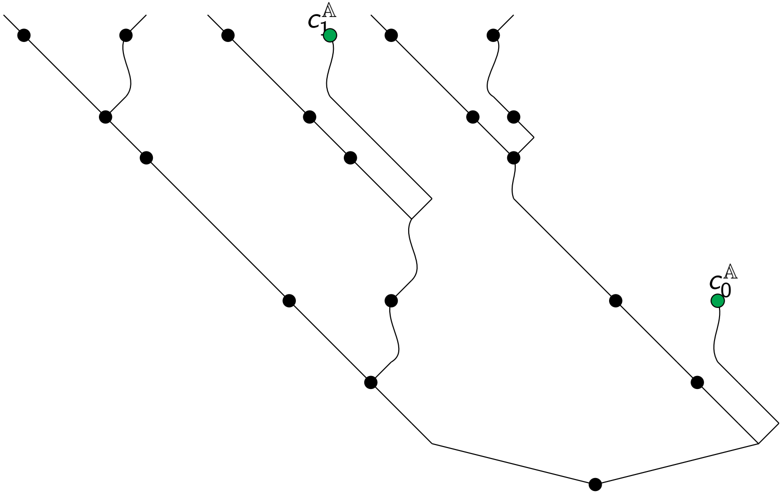
Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.
4. Apply the Ramsey Theorem from Part III, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take an antichain of coding nodes, $\mathbb{A}$ in S, which codes $\mathcal{H}_k$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_k$ coded by $\mathbb{A}$.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.
4. Apply the Ramsey Theorem from Part III, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take an antichain of coding nodes, $\mathbb{A}$ in S, which codes $\mathcal{H}_k$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_k$ coded by $\mathbb{A}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of (incremental) strict similarity types of antichains coding G.

An antichain $\mathbb{A}$ of coding nodes of S coding $\mathcal{H}_3$

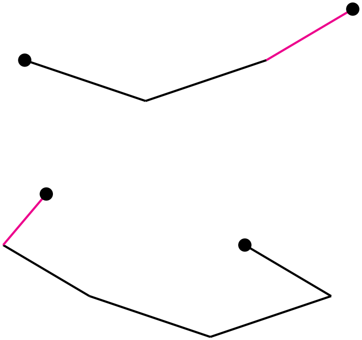


The tree minus the antichain of $c_n^{\mathbb{A}}$'s is isomorphic to $\mathbb{T}_3$.

Proving the lower bounds in general for big Ramsey degrees of $\mathcal{H}_k$ is a work in progress.

Big Ramsey degrees for edges and non-edges have been computed.

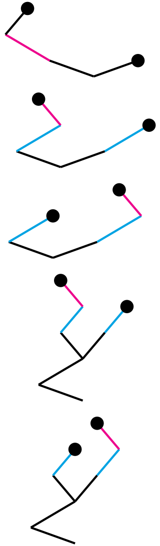
Edges have big Ramsey degree 2 in $\mathcal{H}_3$



These are their own envelopes.

$T(Edge, \mathcal{G}_3) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types in $\mathcal{H}_3$ (D.)



The beginning of a more general theory

The techniques developed for Henson graphs are very broad and likely to extend to a large class of Fraïssé structures with forbidden configurations.

The beginning of a more general theory

The techniques developed for Henson graphs are very broad and likely to extend to a large class of Fraïssé structures with forbidden configurations.

I am currently working to extend this research to big Ramsey degrees of the homogeneous partial order, homogeneous bowtie-free graph, and others.

Remarks

If time, we present some of the forcing argument for Halpern-Läuchli analogues.

Remarks

If time, we present some of the forcing argument for Halpern-Läuchli analogues.

Remark: Ideas from work on the Henson graph are yielding Ramsey theory for clopen colorings of a space of Rado graphs.

Remarks

If time, we present some of the forcing argument for Halpern-Läuchli analogues.

Remark: Ideas from work on the Henson graph are yielding Ramsey theory for clopen colorings of a space of Rado graphs.

This is the first result on infinite dimensional Ramsey theory of the Rado graphs.

Remarks

If time, we present some of the forcing argument for Halpern-Läuchli analogues.

Remark: Ideas from work on the Henson graph are yielding Ramsey theory for clopen colorings of a space of Rado graphs.

This is the first result on infinite dimensional Ramsey theory of the Rado graphs.

Hopefully it can be pushed to Borel colorings, or even property of Baire in some Ellentuck-style topology.

References

Dobrinen, *The Ramsey theory of the universal homogeneous triangle-free graph*, (2018) 65 pages (Submitted).

Dobrinen, *The Ramsey theory of Henson graphs*, (2019) 66 pages.

El-Zahar-Sauer, *The indivisibility of the homogeneous K_n-free graphs*, Jour. Combinatorial Th. (1989).

Halpern-Läuchli, *A partition theorem*, TAMS (1966).

Henson, *A family of countable homogeneous graphs*, Pacific Jour. Math. (1971).

Laflamme-Nguyen Van Thé-Sauer, *Partition properties of the dense local order and a colored version of Milliken's Theorem*, Combinatorica (2010).

Laflamme-Sauer-Vuksanovic, *Canonical partitions of universal structures*, Combinatorica (2006).

Larson, J. *Counting canonical partitions in the Random graph*, Combinatorica (2008).

References

Komjáth-Rödl, *Coloring of universal graphs*, Graphs and Combinatorics (1986).

Milliken, *A Ramsey theorem for trees*, Jour. Combinatorial Th., Ser. A (1979).

Nešetřil-Rödl, *Partitions of finite relational and set systems*, Jour. Combinatorial Th., Ser. A (1977).

Nešetřil-Rödl, *Ramsey classes of set systems*, Jour. Combinatorial Th., Ser. A (1983).

Nguyen Van Thé, *Big Ramsey degrees and divisibility in classes of ultrametric spaces*, Canadian Math. Bull. (2008).

Pouzet-Sauer, *Edge partitions of the Rado graph*, Combinatorica (1996).

Sauer, *Edge partitions of the countable triangle free homogeneous graph*, Discrete Math. (1998).

Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica (2006).

Zucker, *Big Ramsey degrees and topological dynamics*, Groups Geom. Dyn. (2018) 38 pp, (To appear).

II(a) HL - Case (i): level set X contains a splitting node

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \leq d$.

Fix κ large enough so that $\kappa \rightarrow (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new pre-cliques in T.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new pre-cliques in T.

$q \leq p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \geq l_p$, and

(i) $q(d) \supset p(d)$, and $q(i, \delta) \supset p(i, \delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) $\{q(i, \delta) : (i, \delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new pre-cliques above $\mathrm{ran}(p)$.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}$, a name for the levels in $\dot{b}_d$.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}$, a name for the levels in $\dot{b}_d$.

For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle$.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}$, a name for the levels in $\dot{b}_d$.

For $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \ldots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle$.

- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \dots, \theta_{\iota(2d-2)}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \dots, \theta_{\iota(2d-1)}).$$

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2)}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p(d), &\langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ &\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ &\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order.

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p(d), \langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ \langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ \langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and } (\forall i < d)\, \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and } (\forall i < d)\, \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf uses homogeneity of f.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and } (\forall i < d)\,\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf uses homogeneity of f.

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and } (\forall i < d)\,\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf uses homogeneity of f.

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

Pf uses 'sliding' idea.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and}\ (\forall i < d)\,\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf uses homogeneity of f.

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

Pf uses 'sliding' idea.

Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 3. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 3. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d\rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$.

Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 3. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d\rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$. Then for each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} =: t_i^*.$$

Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 3. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d\rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$. Then for each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} =: t_i^*.$$

The $t_0^*, \dots, t_d^*$ provide good starting nodes for constructing the tree homogeneous for the coloring on $\mathrm{Ext}_T(A, \tilde{X})$.

Case (i)

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Case (i)

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

Case (i)

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

(2) These forcings are not simply Cohen forcings; the partial orderings are stronger in order to guarantee that the new levels we obtain by forcing are extendible inside T to another strong coding tree.

Case (i)

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

(2) These forcings are not simply Cohen forcings; the partial orderings are stronger in order to guarantee that the new levels we obtain by forcing are extendible inside T to another strong coding tree.

(3) The assumption that $A \cup \tilde{X}$ satisfies the Witnessing Property is necessary.