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Generalized Weak Distributive Laws in Boolean Algebras
and Issues Related to a Problem of von Neumann
Regarding Measurable Algebras

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
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BY

Natasha Lynne Dobrinen

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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DOCTOR OF PHILOSOPHY

Karel Prikry, Advisor

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

KAREL PRIKRY

Name of Faculty Advisor

Karel Prikrý

Signature of Faculty Advisor

July 24, 2001

Date

GRADUATE SCHOOL

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“How precious to me are your thoughts, O God! How vast is the sum of them!”

Psalm 139:17

ABSTRACT

Von Neumann conjectured that every complete, c.c.c. Boolean algebra which satisfies the weak (ω, ω) -distributive law carries a strictly positive, σ -additive measure. Although consistent counterexamples have been obtained, whether von Neumann's conjecture is consistent with ZFC remains an open problem. In view of this, it is of interest to investigate distributive laws in complete, c.c.c. Boolean algebras. We construct complete embeddings of the Cohen algebra into several classic examples of complete, non-measurable, c.c.c. Boolean algebras, namely, the Galvin-Hajnal, Argyros, and atomless Gaifman algebras. We give game-theoretic characterizations of the (η, κ) -d.l. (for $\kappa^{<\eta} = \kappa$ or η), the weak (η, κ) -d.l. (for $\kappa^{<\eta} = \eta$), the $(\eta, < \lambda, \kappa)$ -d.l. (for $(\kappa^{<\lambda})^{<\eta} = \eta$), and the hyper-weak (η, κ) -d.l. (for $\kappa^{<\eta} = \eta$). For η regular, we use \Diamond_{η^+} to construct η^+ -Suslin algebras in which the related games are undetermined.

Chapter 1

INTRODUCTION

The present thesis concerns generalized distributive laws and connections with the well-known problem of John von Neumann, whether the countable chain condition and the weak (ω, ω) -distributive law characterize measurable algebras among Boolean σ -algebras [24]. Negative results using non-standard axioms in addition to ZFC have been obtained by Glówczyński [10], Jensen [18], Kelley [20], and Velickovic [28]. In addition to a consistent counterexample to von Neumann's problem, Kelley [20] obtained a different characterization of measurable algebras by strengthening the countable chain condition. These results point to the likelihood of the failure of von Neumann's proposed characterization of measurable algebras within ZFC. However, whether von Neumann's proposed characterization of measurable algebras among Boolean σ -algebras can be refuted within ZFC remains an open problem.

Von Neumann's problem has been widely investigated regarding chain conditions, while an analysis from the viewpoint of distributive laws has remained largely ignored. Because of their role in von Neumann's problem, we are interested in generalized distributive laws. In particular, we are interested in stronger and weaker forms of the weak (ω, ω) -distributive law and their connections with von Neumann's problem.

The first two sections of Chapter 2 provide basic definitions and the background of von Neumann's problem, including consistent counter-examples obtained by Glówczyński, Jensen, Kelley, and Velickovic. §2.3 concerns one of the primary constructions of Boolean algebras, namely, the construction of regular open algebras from partial orderings. In §2.4 we give characterizations of the (ω, ω) -d.l., the weak (ω, ω) -d.l., the countable chain condition, and related chain

conditions in terms of order properties of dense subsets of Boolean algebras. This yields characterizations of these Boolean algebraic properties in regular open algebras (i.e. complete Boolean algebras) in terms of their underlying partial orderings. We conclude Chapter 2 with two topological characterizations of the weak (ω, ω) -distributive law obtained by Kelley [20] and Balcar, Glówczyński and Jech [1] (see §2.5).

Chapter 3 concerns complete embeddings of the Cohen algebra into three different non-measurable Boolean algebras which satisfy the countable chain condition, namely the Argyros, Galvin-Hajnal, and atomless Gaifman algebras [3]. This implies that forcing by any of these three Boolean algebras yields an extension of a Cohen extension. As these three algebras satisfy the c.c.c. (and thus, half of von Neumann's conditions), it is of interest to find just how badly they fail to carry a strictly positive, σ -additive measure. Since the Cohen algebra does not satisfy any weak version of distributivity, it follows that these three Boolean algebras also do not satisfy any weak form of distributivity. This shows in a quantitative manner why these three Boolean algebras are not measurable and, in addition, show just how miserably the weak (ω, ω) -d.l. fails in each of them. These results point out that if one is to find a counter-example to von Neumann's proposed characterization of measurable algebras within ZFC, one must construct a Boolean algebra which has smaller elements than these three so that the weak (ω, ω) -d.l. holds, yet large enough that the countable chain condition still holds. Moreover, since it seems odd that the Cohen algebra embeds as a complete subalgebra into three of the classic examples of complete, atomless, c.c.c., non-measurable Boolean algebras, our results bring to light the following question: Does the Cohen algebra embed as a complete subalgebra into every complete, atomless, c.c.c., non-measurable Boolean algebra? In addition to these embeddings, we have found that there are non-atomless Gaifman algebras (see §3.4), and that even in this case, the hyper-weak (ω, ω) -d.l. (which is strictly

weaker than the weak (ω, ω) -d.l. (see §4.1)) still fails (see §3.6).

In Chapter 4 we obtain relationships between generalized distributive laws and infinitary games played between two players in a Boolean algebra. As von Neumann's problem has been studied almost entirely from the point of view of chain conditions (by strengthening the countable chain condition and leaving the weak (ω, ω) -d.l., Kelley found a characterization of measurable σ -algebras), it remains unknown whether leaving the countable chain condition and strengthening the weak (ω, ω) -d.l. could lead to a different characterization of measurable algebras. In §4.6, we show that the existence of a winning strategy for Player 2 in the game $\mathcal{G}_{\text{fin}}^\omega(\omega)$ is strictly stronger than the weak (ω, ω) -d.l., assuming \Diamond . This opens an alternative approach to von Neumann's problem: investigate whether the c.c.c. and the existence of a winning strategy for Player 2 in the game $\mathcal{G}_{\text{fin}}^\omega(\omega)$ characterize measurable algebras among Boolean σ -algebras.

Another point of interest in the connections between games and distributive laws is that each general distributive law is equivalent to a forcing property which says that functions in the extension model are bounded, in a way related to the particular distributive law, by functions in the ground model. Since it is often easier to prove the existence or non-existence of a winning strategy for a game than to show that a distributive law holds, it is of interest to find game-theoretic characterizations of distributive laws.

Jech pioneered relationships between distributive laws and games when he found a game-theoretic characterization of the (ω, ∞) -d.l. [14]. The background of this subject is covered in more detail in §4.1. In that section, we also give definitions of generalized distributive laws and implications between their differing strengths.

We classify the relative strengths of generalized distributive laws and the existence of a winning strategy for the first or second player in the related games. The distributive laws and their corresponding games are dealt with in order of

descending strength. §4.2 involves the (η, κ) -d.l. and the game $\mathcal{G}_1^\eta(\kappa)$; §4.3 involves the weak (η, κ) -d.l. and the game $\mathcal{G}_{\text{fin}}^\eta(\kappa)$; §4.4 involves the $(\eta, < \lambda, \kappa)$ -d.l. and the game $\mathcal{G}_{<\lambda}^\eta(\kappa)$; and §4.5 involves the hyper-weak (η, ω) -d.l. and the game $\mathcal{G}_{\omega-1}^\eta$. It turns out that for each generalized distributive law, there are certain cardinals for which that distributive law holds if and only if Player 1 does not have a winning strategy in the related game. Under GCH, these characterizations hold for many pairs and triples of cardinals.

For regular cardinals η , \Diamond_{η^+} implies that the existence of a winning strategy for Player 2 in a given game is strictly stronger than the distributive law related to that game. Moreover, the existence of a winning strategy for Player 2 in the game $\mathcal{G}_{\kappa-1}^\eta$ (where $\kappa = |\mathbf{B}|$), which is the easiest game of length η for Player 2 to win in \mathbf{B} , does not follow from the (η, ∞) -distributive law. These results, given in §4.6, point to the possible use of the existence of a winning strategy for Player 2 in one of these games to make more narrow the bounds on the possible area where von Neumann's proposed characterization of measurable algebras could be consistent with ZFC.

In §4.7 we show the relative strengths of the existence of a winning strategy for Player 2 and the non-existence of a winning strategy for Player 1 between the various games. This section concludes with diagrams summarizing the results of Chapter 4.

Chapter 2

PRELIMINARIES

2.1. DEFINITIONS

We begin with some basic definitions regarding Boolean algebras, distributive laws, and chain conditions. For details beyond this, we refer the reader to the *Handbook of Boolean Algebras* [21].

Definition 2.1.1. [21] A *Boolean algebra* is a structure $(\mathbf{B}, \vee, \wedge, -, \mathbf{0}, \mathbf{1})$ satisfying the following axioms: for all $x, y, z \in \mathbf{B}$,

- (1) $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity)
- (2) $x \vee y = y \vee x$ (commutativity)
- (3) $x \vee (x \wedge y) = x$ (absorption)
- (4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity)
- (5) $x \vee (-x) = \mathbf{1}$ (complementation)

and the five dual axioms, where \vee and \wedge are interchanged and (5) becomes $x \wedge (-x) = \mathbf{0}$.

Following the standard abuse of notation, we shall write simply \mathbf{B} when we mean $(\mathbf{B}, \vee, \wedge, -, \mathbf{0}, \mathbf{1})$.

By the Stone Representation Theorem, every Boolean algebra \mathbf{B} is isomorphic to the set algebra of clopen subsets of a totally disconnected, compact Hausdorff space, called the Stone space of \mathbf{B} .

We use the Greek letters $\zeta, \eta, \kappa, \lambda, \nu$ to denote cardinal numbers; $\alpha, \beta, \gamma, \delta$ to denote ordinal numbers; and μ to denote measures. For an ordinal α , ω_α denotes the $(1 + \alpha)$ -th infinite cardinal number. In particular, ω_0 denotes the least infinite cardinal, the countable infinity, which we shall usually write as ω instead of ω_0 . For a set S , $|S|$ denotes the cardinality of S . We shall not make a complete list

of the standard set-theoretic notation with which we shall avail ourselves in this thesis.

One defines the order \leq on \mathbf{B} by $x \leq y \leftrightarrow x \wedge y = x$. Infinite suprema and infima are defined using the partial order \leq in the following manner: for $M \subseteq \mathbf{B}$, $\bigvee M$ ($\bigwedge M$) is the least upper bound (greatest lower bound) of M in the partial order (\mathbf{B}, \leq) , if it exists. By de Morgan's laws, $\bigvee M$ exists for all $M \subseteq \mathbf{B}$ of cardinality less than κ if and only if $\bigwedge M$ exists for all $M \subseteq \mathbf{B}$ of cardinality less than κ .

Definition 2.1.2. [21] \mathbf{B} is κ -complete if for all $M \subseteq \mathbf{B}$ such that $|M| < \kappa$, $\bigvee M$ exists. We say that \mathbf{B} is a σ -algebra if it is ω_1 -complete. \mathbf{B} is complete if it is κ -complete for all cardinals κ .

Definition 2.1.3. [21] \mathbf{B} satisfies the (κ, λ) -distributive law $((\kappa, \lambda)$ -d.l.) if for each $|I| \leq \kappa$, $|J| \leq \lambda$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(2.1.1) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} a_{if(i)},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} a_{if(i)}$ for each $f: I \rightarrow J$ exist in \mathbf{B} . We say that \mathbf{B} is (κ, ∞) -distributive if it satisfies the (κ, λ) -d.l. for all cardinals λ and is *completely distributive* if it is (κ, ∞) -distributive for all cardinals κ . We say that the (κ, λ) -d.l. *fails everywhere* in \mathbf{B} if there exists a family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ($|I| \leq \kappa$, $|J| \leq \lambda$) such that $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = 1$ and $\bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} a_{if(i)} = 0$.

Definition 2.1.4. [21] \mathbf{B} satisfies the *weak* (κ, λ) -distributive law (weak (κ, λ) -d.l.) if for each $|I| \leq \kappa$, $|J| \leq \lambda$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(2.1.2) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow [J]^{<\omega}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij}$ for each $f: I \rightarrow [J]^{<\omega}$ exist in \mathbf{B} , where $[J]^{<\omega}$ denotes the set of all finite subsets

of J . \mathbf{B} is *weakly* (κ, ∞) -*distributive* if it satisfies the weak (κ, λ) -d.l. for all cardinals λ . We say that the weak (κ, λ) -distributive law *fails everywhere* in \mathbf{B} if there exists a family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ($|I| \leq \kappa$, $|J| \leq \lambda$) such that $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = 1$ and $\bigvee_{f: I \rightarrow [J]^{<\omega}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij} = 0$.

It is immediate that for all cardinals $\zeta \leq \kappa$, $\eta \leq \lambda$, (κ, λ) -distributivity implies (ζ, η) -distributivity, weak (κ, λ) -distributivity implies weak (ζ, η) -distributivity, and (κ, λ) -distributivity implies weak (κ, λ) -distributivity. Every $(\kappa, 2)$ -distributive Boolean algebra is (κ, κ) -distributive; moreover, every $(2^\kappa)^+$ -complete, $(\kappa, 2)$ -distributive Boolean algebra is $(\kappa, 2^\kappa)$ -distributive. Note that every Boolean algebra is (n, ∞) -distributive for all $n < \omega$. (See [21].)

Remark. For any set S the power set algebra $(\mathcal{P}(S), \cup, \cap, -, \emptyset, S)$ is completely distributive. Moreover, a complete Boolean algebra is completely distributive if and only if it is isomorphic to a power set algebra.

Definition 2.1.5. [21] \mathbf{B} satisfies the κ -*chain condition* (κ -c.c.) if $|X| < \kappa$ for each pairwise disjoint family X in \mathbf{B} . We say that \mathbf{B} satisfies the *countable chain condition* (c.c.c.) if it satisfies the ω_1 -chain condition.

Remark. Every Boolean algebra which is κ -complete and satisfies the κ -c.c. is complete. In particular, every σ -algebra which satisfies the c.c.c. is complete.

Definition 2.1.6. [6] A *finitely additive measure* on \mathbf{B} is a function $\mu : \mathbf{B} \rightarrow [0, \infty)$ such that $\mu(a \vee b) = \mu(a) + \mu(b)$ whenever a and b are disjoint elements of \mathbf{B} . A finitely additive measure μ on a σ -algebra \mathbf{B} is σ -*additive* if $\mu(\bigvee_{i < \omega} a_i) = \sum_{i < \omega} \mu(a_i)$ whenever $\{a_i : i < \omega\}$ is a pairwise disjoint subset of \mathbf{B} . A finitely additive measure μ on \mathbf{B} is *strictly positive* if for all $b \in \mathbf{B}$, $\mu(b) = 0 \leftrightarrow b = 0$.

Remark. Note that the range of a finitely additive measure μ has a maximum element, namely $\mu(1)$.

Definition 2.1.7. [6] A σ -algebra \mathbf{B} is a *measurable algebra* if there exists a strictly positive, σ -additive measure μ on \mathbf{B} .

Definition 2.1.8. [21] $a \in \mathbf{B}$ is an *atom* of \mathbf{B} if $0 < a$ but there is no $b \in \mathbf{B}$ satisfying $0 < b < a$. \mathbf{B} is *atomless* if it has no atoms and *atomic* if for each nonzero $b \in \mathbf{B}$ there is some atom a such that $a \leq b$.

Theorem 2.1.9. [21] If \mathbf{B} has a strictly positive, finitely additive measure, then \mathbf{B} satisfies the c.c.c. Moreover, every measurable algebra \mathbf{B} satisfies the weak (ω, ω) -d.l., and if \mathbf{B} is non-atomic, then the $(\omega, 2)$ -d.l. fails in \mathbf{B} ; if \mathbf{B} is atomless, then the $(\omega, 2)$ -d.l. fails everywhere in \mathbf{B} .

Remark. Another example of a Boolean algebra which is weakly (ω, ω) -distributive and not $(\omega, 2)$ -distributive but also not c.c.c. (hence, it cannot serve as a counterexample to von Neumann's proposed characterization) is the regular open algebra of the Sacks partial ordering \mathcal{S} (see paragraph preceding Definition 2.2.7).

Measurable algebras arise from the following standard construction. Given a probability measure space (X, \mathcal{A}, μ) , where \mathcal{A} is a σ -algebra of μ -measurable subsets of X , let $Z = \{S \in \mathcal{A} : \mu(S) = 0\}$. Then \mathcal{A}/Z is a measurable algebra. Moreover, every measurable algebra is isomorphic to one of the above form. This fact is a corollary of the Loomis-Sikorski Theorem, which says that for each σ -algebra \mathbf{B} there exists a σ -algebra of sets \mathbf{A} and a σ -complete epimorphism from \mathbf{A} onto \mathbf{B} (see [6] and [21]).

2.2. PARTIAL RESULTS ON VON NEUMANN'S PROBLEM

The question of von Neumann stated in the introduction (whether measurable algebras can be characterized as those σ -algebras which satisfy the countable chain condition and the weak (ω, ω) -distributive law) was motivated by his desire to obtain an algebraic characterization of measurable algebras; i.e. a charac-

terization in terms of purely Boolean algebraic properties. A number of partial results have been obtained, usually involving special set-theoretic assumptions; however, whether von Neumann's question can be solved within ZFC remains an open problem. We shall now discuss some of the work on von Neumann's problem.

In 1959, Kelley [20] gave a negative answer to von Neumann's question assuming that the Suslin Hypothesis is false and also obtained a different algebraic characterization of measurable algebras within ZFC, where a much stronger chain condition is substituted for the c.c.c. Moreover, he gave a topological characterization of those c.c.c. σ -algebras which satisfy the weak (ω, ω) -d.l. Balcar, Glówczyński and Jech [1] recently obtained a topological characterization of those complete Boolean algebras which satisfy the weak (ω, ω) -d.l. and the \bar{b} -c.c., where \bar{b} is the *bounding number* (see Definition 2.5.4). As a corollary of their result one obtains a different topological characterization of those c.c.c. σ -algebras which satisfy the weak (ω, ω) -d.l.

We now define Suslin algebras, which, provided they exist, serve as a counterexample to von Neumann's proposed characterization of measurable algebras.

Definition 2.2.1. [21] A *Suslin algebra* is an atomless, c.c.c. σ -algebra which satisfies the (ω, ω) -d.l.

Suslin's Hypothesis (SH) "There are no Suslin lines" is equivalent to the statement: "There are no Suslin algebras". Both SH and its negation are consistent with ZFC. In particular, the negation of SH (i.e. the existence of Suslin algebras) follows from the principle \diamond (see Definition 2.2.6), which in turn is a consequence of the Axiom of Constructibility, proved consistent with ZFC by Gödel (see [22]). Hence, the existence of Suslin algebras is consistent with ZFC. (Another application of \diamond to von Neumann's problem was given by Jensen [18] and will be discussed below.)

Kelley pointed out that Suslin algebras give a counterexample to von Neu-

mann's proposed characterization of measurable algebras: by definition, Suslin algebras are atomless, c.c.c., (ω, ω) -distributive σ -algebras; hence they are weakly (ω, ω) -distributive. Thus, it is immediate from Theorem 2.1.9 that Suslin algebras cannot be measurable algebras. However, SH follows from $\text{MA}(\omega_1)$ (see Definition 2.2.10), which is also consistent with ZFC. Thus, Kelley's example does not prove von Neumann's proposed characterization to be false in ZFC.

In order to introduce the principle \Diamond , we need the following definitions.

Definition 2.2.2. [22] Let κ be an infinite cardinal. We call a set $C \subseteq \kappa$ *closed unbounded* (c.u.b.) in κ if

- (1) For every sequence $\alpha_0 < \alpha_1 < \dots < \alpha_\xi < \dots$ ($\xi < \gamma$) of elements of C , of length $\gamma < \kappa$, we have $\lim_{\xi \rightarrow \gamma} \alpha_\xi \in C$ (closed);
- (2) For every $\alpha < \kappa$, there is a $\beta > \alpha$ such that $\beta \in C$ (unbounded).

Proposition 2.2.3. [22] *Let κ be an uncountable regular cardinal. The intersection of less than κ c.u.b. subsets of κ is c.u.b. in κ .*

Definition 2.2.4. [22] Let κ be an infinite cardinal. A set $S \subseteq \kappa$ is *stationary* in κ if $S \cap C \neq \emptyset$ for every c.u.b. subset C of κ .

Remark. It immediately follows from Proposition 2.2.3 that for any uncountable regular cardinal κ , if S is stationary in κ and C is c.u.b. in κ , then $S \cap C$ is stationary in κ . The property of being stationary is primarily of interest for uncountable regular cardinals.

Theorem 2.2.5 (Fodor). [22] *Let κ be an uncountable regular cardinal, S a stationary subset of κ , and $f : S \rightarrow \kappa$ such that for each nonzero $\gamma \in S$ ($f(\gamma) < \gamma$); then there is a stationary set $T \subseteq S$ such that f is constant on T .*

Remark. It is this property of stationary sets that explains the use of the term "stationary".

Now we shall introduce Jensen's principle \Diamond .

Definition 2.2.6. [22] \Diamond is the statement: There are sets $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ such that

$$(2.2.1) \quad \forall A \subseteq \omega_1 (\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\} \text{ is stationary}).$$

The sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ is called a \Diamond -sequence.

As noted above, \Diamond is consistent with ZFC.

Using \Diamond , Jensen [18] obtained a counterexample to von Neumann's proposed characterization of measurable algebras which differs from Kelley's counterexample of Suslin algebras. Jensen's counterexample is the algebra of regular open subsets (see §2.3) of a subset of the Sacks partial ordering of perfect trees which we shall now describe.

Definition 2.2.7. [18] Let \leq denote inclusion on $2^{<\omega}$, where $2^{<\omega}$ is the collection of all finite sequences of 0's and 1's. $(T, \leq) \subseteq (2^{<\omega}, \leq)$ is a *perfect tree* if $T \neq \emptyset$ and for every $u \in T$ there are $v, v' \in T$ such that $u \leq v, v'$ but neither $v \subseteq v'$ nor $v' \subseteq v$.

Let $(S, \subseteq) = (\{T \subseteq 2^{<\omega} : (T, \leq) \text{ is a perfect tree}\}, \subseteq)$, the *Sacks partial ordering* [27]. The regular open algebra of S , $\text{r.o.}(S)$, is weakly (ω, ω) -distributive and not $(\omega, 2)$ -distributive, but is not c.c.c. (hence, as remarked earlier, it cannot serve as a counterexample to von Neumann's proposed characterization of measurable algebras). To obtain a counterexample to von Neumann's proposed characterization of measurable algebras, Jensen used the following Fusion Lemma.

Lemma 2.2.8 (Fusion). [18] *If $\langle T_s : s \in 2^{<\omega} \rangle$ is a sequence of perfect trees such that*

- (1) $s \subseteq s' \rightarrow T_{s'} \subseteq T_s$;
- (2) *If $f \in 2^\omega$, then $\bigcap_{n < \omega} T_{f \upharpoonright n}$ contains only one branch;*
- (3) *If $f, f' \in 2^\omega$ and $f \neq f'$, then $\bigcap_{n < \omega} T_{f \upharpoonright n} \neq \bigcap_{n < \omega} T_{f' \upharpoonright n}$;*

then

$$(2.2.2) \quad T^* = \bigcap_{n < \omega} \bigcup_{s \in 2^n} T_s$$

is a perfect tree.

Using \Diamond and the Fusion Lemma, Jensen constructed a partial order $(\tilde{\mathcal{S}}, \leq) \subseteq (\mathcal{S}, \leq)$ such that $\text{r.o.}(\tilde{\mathcal{S}})$ is a c.c.c., weakly (ω, ω) -distributive σ -algebra but is not a measurable algebra; hence it is a counterexample to von Neumann's proposed characterization of measurable algebras. Moreover, the $(\omega, 2)$ -d.l. fails everywhere in $\text{r.o.}(\tilde{\mathcal{S}})$; thus, it differs from a Suslin algebra.

Next we shall discuss a third consistent negative result on von Neumann's problem obtained by Glówczyński [10]. As already pointed out, $\text{MA}(\omega_1)$ (and thus $\text{MA} + \neg\text{CH}$, where $\neg\text{CH}$ denotes the negation of CH) implies SH , which in turn implies $\neg\Diamond$, thus ruling out both Suslin algebras and Jensen's construction as counterexamples to von Neumann's problem. In light of this, the work of Glówczyński is of interest. He proved that the failure of von Neumann's proposed characterization is consistent with $\text{ZFC} + \text{MA} + \neg\text{CH}$ if the existence of a measurable cardinal is consistent with ZFC . In order to state the main result of [10] we shall now state MA and some relevant definitions.

Definition 2.2.9. [22] $G \subseteq \mathbf{P}$ is a *filter* in \mathbf{P} if

- (1) $\forall p, q \in G \exists r \in G$ such that $r \leq p$ and $r \leq q$;
- (2) $\forall p \in G \forall q \in \mathbf{P} (p \leq q \rightarrow q \in G)$.

A subset $D \subseteq \mathbf{P}$ is called *dense* in \mathbf{P} if for every $p \in \mathbf{P}$ there is some $d \in D$ such that $d \leq p$.

Definition 2.2.10. [22] $\text{MA}(\kappa)$ is the statement: Whenever \mathbf{P} is a non-empty c.c.c. partial order and \mathcal{D} is a family of $\leq \kappa$ -many dense subsets of \mathbf{P} , then there is a filter G in \mathbf{P} such that $\forall D \in \mathcal{D} (G \cap D \neq \emptyset)$. MA is the statement $\forall \kappa < 2^\omega (\text{MA}(\kappa))$.

Remark. $\text{MA}(\omega)$ is a theorem of ZFC and thus, CH implies MA; $\text{MA}(\omega_1)$, in turn, implies $\neg\text{CH}$ (and $\text{MA} + \neg\text{CH}$ clearly implies $\text{MA}(\omega_1)$). $\text{MA} + \neg\text{CH}$ is consistent with ZFC (see [22]).

Definition 2.2.11. [15] Let \mathbf{B}^+ denote $\mathbf{B} \setminus \{0\}$. We say that a subset $G \subseteq \mathbf{B}^+$ is a *filter* in \mathbf{B} if G is a filter in the partial order (\mathbf{B}^+, \leq) . $G \subseteq \mathbf{B}^+$ is a *principal filter* in \mathbf{B} if there is some $a \in \mathbf{B}^+$ such that $G = \{b \in \mathbf{B}^+ : b \geq a\}$. A filter $G \subseteq \mathbf{B}^+$ is an *ultrafilter* if for every $b \in \mathbf{B}$, either $b \in G$ or $\neg b \in G$. A filter $G \subseteq \mathbf{B}^+$ is κ -*complete* if $\bigwedge H \in G$ for each $H \subseteq G$ of size less than κ such that $\bigwedge H$ exists in \mathbf{B} .

Definition 2.2.12. [15] An uncountable cardinal κ is a *measurable cardinal* if there exists a κ -complete nonprincipal ultrafilter in the power set algebra $\mathcal{P}(\kappa)$.

Definition 2.2.13. [10] A *Maharam submeasure* on a σ -algebra \mathbf{B} is a function $\mu : \mathbf{B} \rightarrow [0, \infty)$ such that

- (1) μ is strictly positive;
- (2) If $a \leq b$, then $\mu(a) \leq \mu(b)$; (order-preserving)
- (3) $\mu(a \vee b) \leq \mu(a) + \mu(b)$; (sub-additive)
- (4) If $\bigwedge_{k \in \omega} \bigvee_{n \geq k} b_n \Delta b = 0$, then $\lim_{n \rightarrow \infty} \mu(b_n) = \mu(b)$; (continuous)

where $b_n \Delta b = (b_n - b) \vee (b - b_n)$, the symmetric difference of b_n and b .

Theorem 2.2.14. [10] *If $\text{Con}(\text{ZFC} + \text{there exists a measurable cardinal})$, then $\text{Con}(\text{ZFC} + \text{MA} + \neg\text{CH} + \text{there exists a weakly } (\omega, \infty)\text{-distributive, c.c.c., atomless } \sigma\text{-algebra without any Maharam submeasure})$.*

Every measurable algebra has a Maharam submeasure, since every strictly positive, σ -additive measure on a σ -algebra is a Maharam submeasure. Thus, if ZFC and “there exists a measurable cardinal” are consistent, then it is consistent with $\text{ZFC} + \text{MA} + \neg\text{CH}$ that there exists a weakly (ω, ω) -distributive, c.c.c., atomless σ -algebra which is not a measurable algebra.

Remark. Whether every σ -algebra with a Maharam submeasure is a measurable algebra remains an open problem (see [6]).

Velickovic [28] improved on Jensen's and Glówczyński's results by using the axiom $\text{CCC}(\mathcal{S})$, where \mathcal{S} is the Sacks partial ordering described above, which is consistent with $\text{ZFC} + \text{MA} + \neg\text{CH}$ and is strictly weaker than \diamond . To introduce $\text{CCC}(\mathcal{S})$ we need the following notion of a perfect poset.

Definition 2.2.15. [28] A collection $\mathcal{P} \subseteq \mathcal{S}$ of perfect trees is called a *perfect poset* provided that

- (1) $(2^{<\omega})_s \in \mathcal{P}$ for all $s \in 2^{<\omega}$, where $(2^{<\omega})_s = \{t \in 2^{<\omega} : s \subseteq t \text{ or } t \subseteq s\}$;
- (2) If $T, S \in \mathcal{P}$ then $T \vee S \in \mathcal{P}$, and if in addition $T \wedge S \neq \emptyset$ then $T \wedge S \in \mathcal{P}$, where $T \vee S = T \cup S$ and $T \wedge S$ is the largest perfect tree contained in $T \cap S$ or \emptyset if none exists.

Definition 2.2.16. [28] $\text{CCC}(\mathcal{S})$ is the statement: For every family \mathcal{D} of 2^ω -many dense subsets of \mathcal{S} , there is a c.c.c. perfect subposet $\mathcal{P} \subseteq \mathcal{S}$ such that $D \cap \mathcal{P}$ is dense in \mathcal{P} , for all $D \in \mathcal{D}$.

$\text{CCC}(\mathcal{S})$ follows from \diamond and is consistent with $\text{ZFC} + \text{MA} + \neg\text{CH}$ (and thus with SH, which in turn implies consistency with $\neg\diamond$); thus, $\text{CCC}(\mathcal{S})$ is strictly weaker than \diamond . Velickovic showed that $\text{CCC}(\mathcal{S})$ implies there is a weakly (ω, ∞) -distributive, countably generated, c.c.c. complete Boolean algebra which is not a measurable algebra. Thus, von Neumann's proposed characterization of measurable algebras among σ -algebras fails in $\text{ZFC} + \text{CCC}(\mathcal{S})$.

Motivated by von Neumann's question, Kelley obtained a different algebraic characterization of measurable algebras. He characterized measurable algebras as those σ -algebras which satisfy the weak (ω, ω) -d.l. and carry a strictly positive, finitely additive measure (see Theorem 2.2.19). Kelley defined the intersection number of a family of sets (or of a collection of elements of a Boolean algebra) and

used this notion to show that the property of having a strictly positive, finitely additive measure has an algebraic characterization.

Definition 2.2.17. [6] Let $S \subseteq \mathbf{B}$ be a non-empty set. For $s_0, \dots, s_n \in S$, not necessarily distinct, let

$$(2.2.3) \quad \alpha^*(s_0, \dots, s_n) = \frac{1}{n+1} \max\{|I| : I \subseteq n+1, \bigwedge_{i \in I} s_i \neq 0\}.$$

The *intersection number* of S is

$$(2.2.4) \quad \alpha(S) = \inf\{\alpha^*(s_0, \dots, s_n) : n \in \omega, s_0, \dots, s_n \in S\}.$$

Remark. Note that if $0 \in S$, then $\alpha(S) = 0$. Consequently, one usually works with collections of nonzero elements of a Boolean algebra.

Following the notation in [9], we write $\text{CUP}(S)$ for: S is the countable union of subsets of S each of which has positive intersection number. Gaifman showed that $\text{CUP}(\mathbf{B}^+)$ is strictly stronger than the c.c.c. (see §3.4 and [8]).

Kelley obtained the following algebraic characterization of the existence of a strictly positive, finitely additive measure on a Boolean algebra.

Theorem 2.2.18 (Kelley). [20] *For each Boolean algebra \mathbf{B} , the following are equivalent:*

- (1) *There exists a strictly positive, finitely additive measure on \mathbf{B} ;*
- (2) *$\text{CUP}(\mathbf{B}^+)$ holds.*

Theorem 2.2.19 (Kelley). [20] *For each σ -algebra \mathbf{B} , the following are equivalent:*

- (1) *\mathbf{B} is a measurable algebra;*
- (2) *\mathbf{B} carries a strictly positive, finitely additive measure and satisfies the weak (ω, ω) -d.l.;*
- (3) *$\text{CUP}(\mathbf{B}^+)$ holds and \mathbf{B} satisfies the weak (ω, ω) -d.l.*

(3) of Theorem 2.2.19 gives a possible algebraic characterization of measurable algebras among σ -algebras.

Fremlin notes in [6] that in the definition of intersection number, the fact that s_0, \dots, s_n are not required to be distinct does matter. He gives the following example: for $S = \{\{0\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $\alpha(S) = \frac{2}{5}$, not $\frac{1}{2}$. Galvin and Prikry [9] defined the *weak intersection number* of S , $\alpha_w(S)$, exactly as $\alpha(S)$, except that s_0, \dots, s_n are required to be distinct. They investigated the relationship between the two intersection numbers and proved an analogue of Kelley's Theorem. In order to state their results, let $CUPW(S)$ stand for: S is the union of countably many collections each of which has positive weak intersection number.

Theorem 2.2.20 (Galvin-Prikry). [9] *Suppose that S is a collection of sets, or elements of a Boolean algebra, such that for every X_1 and X_2 in S , if $X_1 \cap X_2 \neq \emptyset$, then there is some X in S such that $X \subseteq X_1 \cap X_2$. Then $CUP(S)$ and $CUPW(S)$ are equivalent.*

Recalling Stone's Representation Theorem, one immediately sees that the hypothesis of Theorem 2.2.20 is satisfied by \mathbf{B}^+ for every Boolean algebra \mathbf{B} . Thus, combining Kelley's Theorem 2.2.19 and Galvin-Prikry's Theorem 2.2.20, one has the following analogue of Kelley's characterization of measurable algebras.

Corollary 2.2.21 (Kelley-Galvin-Prikry). *For each σ -algebra \mathbf{B} , the following are equivalent:*

- (1) \mathbf{B} is a measurable algebra;
- (2) $CUPW(\mathbf{B}^+)$ holds and \mathbf{B} satisfies the weak (ω, ω) -d.l.

We shall now state a chain condition which is intermediate between $CUP(\mathbf{B}^+)$ and the c.c.c.

Definition 2.2.22. [6] \mathbf{B} satisfies the *σ -bounded chain condition* (σ -bounded c.c.) if $\mathbf{B}^+ = \bigcup_{n < \omega} X_n$ where, for each $n < \omega$, any pairwise disjoint subset of X_n has at most $n + 1$ members.

Clearly $\text{CUP}(\mathbf{B}^+)$ implies \mathbf{B} satisfies the σ -bounded c.c., which in turn implies \mathbf{B} satisfies the c.c.c. Gaifman [8] showed that $\text{CUP}(\mathbf{B}^+)$ is strictly stronger than the σ -bounded c.c.; thus, by Kelley's Theorem 2.2.18, the σ -bounded c.c. is strictly weaker than the existence of a strictly positive, finitely additive measure. Gaifman also pointed out that if the c.c.c. implies the σ -bounded c.c., then SH holds (see [13]). Later, Galvin and Hajnal constructed a Boolean algebra satisfying the c.c.c., but not the σ -bounded c.c. (see [3]). Hence, the σ -bounded c.c. is strictly stronger than the c.c.c. within ZFC. We will work with the Galvin-Hajnal Boolean algebra in §3.2 and the atomless Gaifman algebra in §3.5 where we will show that the Cohen algebra (the complete Boolean algebra in which the countable atomless Boolean algebra is a complete, dense subalgebra) embeds as a complete subalgebra into these algebras. It remains an open problem whether every σ -algebra which satisfies the σ -bounded c.c. and the weak (ω, ω) -d.l. is a measurable algebra (see [6]).

2.3. REGULAR OPEN ALGEBRAS

Two important methods for constructing Boolean algebras are as follows. First, for every Boolean algebra \mathbf{B} and ideal I , there is the factor algebra \mathbf{B}/I . Second, for every topological space X one has the algebra $\text{r.o.}(X)$ of regular open subsets of X . This construction is described in Definition 2.3.1. An important special case of this is when the topological space is a partial order (\mathbf{P}, \leq) with a certain topology which we shall call the *partial order topology* on (\mathbf{P}, \leq) (see Definition 2.3.3).

Definition 2.3.1. [21] Let X be a topological space. For $u \subseteq X$, let $\text{cl } u$ and $\text{int } u$ denote the closure and the interior of u , respectively. $u \subseteq X$ is *regular open* if $\text{int cl } u = u$. Set $\text{r.o.}(X) = \{u \subseteq X : u \text{ is regular open}\}$, the collection of all

regular open subsets of X . $\text{r.o.}(X)$ is called the *regular open algebra* of X .

Theorem 2.3.2. [21] $\text{r.o.}(X)$, with the distinguished elements $0 = \emptyset$, $1 = X$ and the operations $\vee, \wedge, -$ given by $u \vee v = \text{int cl}(u \cup v)$, $u \wedge v = u \cap v$, $-u = \text{int}(X \setminus u)$, is a Boolean algebra. Moreover, $\text{r.o.}(X)$ is a complete Boolean algebra. Infinite suprema and infima are obtained as follows: for each $M \subseteq \mathbf{B}$,

$$(2.3.1) \quad \bigvee M = \text{int cl}\left(\bigcup M\right) \quad \text{and} \quad \bigwedge M = \text{int cl}\left(\bigcap M\right).$$

Following the standard abuse of notation, we shall write simply $\text{r.o.}(X)$ when we mean $(\text{r.o.}(X), \vee, \wedge, -, \emptyset, X)$. Clearly the order \leq on $\text{r.o.}(X)$ is the ordinary set-theoretic inclusion.

A partial order (\mathbf{P}, \leq) can be given the following partial order topology.

Definition 2.3.3. [21] Let (\mathbf{P}, \leq) be a partial order. For $p \in \mathbf{P}$, let $u_p = \{q \in \mathbf{P} : q \leq p\}$. The set $\{u_p : p \in \mathbf{P}\}$ is the base of the *partial order topology* on (\mathbf{P}, \leq) .

For the remainder of this paper, each partial order shall be endowed with its partial order topology. By abuse of notation, we shall usually write \mathbf{P} when we mean (\mathbf{P}, \leq) .

Definition 2.3.4. [22] For $p, q \in \mathbf{P}$, p and q are *compatible* ($p \parallel q$) if there exists $r \in \mathbf{P}$ such that $r \leq p$ and $r \leq q$. p and q are *incompatible* ($p \perp q$) if they are not compatible.

Note that for any given partial order \mathbf{P} , $\forall p \in \mathbf{P}$, $\text{cl } u_p = \{q \in \mathbf{P} : q \parallel p\}$ and $\text{int cl } u_p = \{q \in \mathbf{P} : \forall r \leq q (r \parallel p)\}$.

Definition 2.3.5. [21] A *completion* of a partial order \mathbf{P} is a pair (e, \mathbf{B}) such that \mathbf{B} is a complete Boolean algebra, $e : \mathbf{P} \rightarrow \mathbf{B}^+$, and

- (1) $p \leq q$ in $\mathbf{P} \rightarrow e(p) \leq e(q)$ in \mathbf{B} ;
- (2) $p \perp q$ in $\mathbf{P} \leftrightarrow e(p) \wedge e(q) = 0$ in \mathbf{B} ;
- (3) $e[\mathbf{P}]$ is a dense subset of \mathbf{B}^+ .

Theorem 2.3.6. [21] *Every partial order \mathbf{P} has $(e, r.o.(\mathbf{P}))$ as a completion, where the mapping $e : \mathbf{P} \rightarrow r.o.(\mathbf{P})^+$ is given by $e(p) = \text{int cl } u_p$, for each $p \in \mathbf{P}$. Moreover, every completion of \mathbf{P} is isomorphic over \mathbf{P} to $r.o.(\mathbf{P})$; i.e. if (e', \mathbf{B}') is a completion of \mathbf{P} , then there is an isomorphism $h : \mathbf{B}' \rightarrow r.o.(\mathbf{P})$ such that $h \circ e' = e$.*

Thus, we shall call $(e, r.o.(\mathbf{P}))$ *the completion* of \mathbf{P} , where e is the *canonical mapping* from \mathbf{P} into $r.o.(\mathbf{P})^+$ given in Theorem 2.3.6. By abuse of notation, we shall often refer to $r.o.(\mathbf{P})$ as the completion of \mathbf{P} , where it is understood that the mapping from \mathbf{P} to $r.o.(\mathbf{P})^+$ is the canonical mapping e .

The following property is necessary and sufficient for \mathbf{P} to embed into $r.o.(\mathbf{P})$ as a dense subset of $r.o.(\mathbf{P})^+$ (see Proposition 2.3.9).

Definition 2.3.7. [21] \mathbf{P} is *separative* if for all $p, q \in \mathbf{P}$, $q \not\leq p$ implies there is some $r \in \mathbf{P}$ such that $r \leq q$ and $r \perp p$.

Lemma 2.3.8. [21] *If \mathbf{P} is a dense subset of \mathbf{B}^+ and $p, q \in \mathbf{P}$, then $p \perp q$ in $\mathbf{P} \leftrightarrow p \wedge q = 0$ in \mathbf{B} .*

Remark. It follows from Lemma 2.3.8 that every dense subset of \mathbf{B}^+ is separative; in particular, \mathbf{B}^+ is separative.

Proposition 2.3.9. [21] *For each partial order \mathbf{P} , the following are equivalent:*

- (1) \mathbf{P} is separative;
- (2) $\text{int cl } u_p = u_p$;
- (3) e is an isomorphism from \mathbf{P} onto $e[\mathbf{P}] \subseteq r.o.(\mathbf{P})^+$.

It follows from Proposition 2.3.9 and the preceding remark that \mathbf{P} embeds as a dense subset of $r.o.(\mathbf{P})^+$ if and only if \mathbf{P} is separative. Thus, by abuse of notation, we shall consider each separative partial order \mathbf{P} to be a dense subset of $r.o.(\mathbf{P})^+$.

For each Boolean algebra \mathbf{B} , the completion of (\mathbf{B}^+, \leq) , $\text{r.o.}(\mathbf{B}^+)$, is called the *completion* of \mathbf{B} . Note that \mathbf{B} is a dense subalgebra of $\text{r.o.}(\mathbf{B}^+)$; i.e. \mathbf{B} is a subalgebra of $\text{r.o.}(\mathbf{B}^+)$ and \mathbf{B}^+ is a dense subset of $(\text{r.o.}(\mathbf{B}^+))^+$. In fact, every complete Boolean algebra having \mathbf{B} as a dense subalgebra is isomorphic to $\text{r.o.}(\mathbf{B}^+)$, and thus is also called the completion of \mathbf{B} . Since every complete Boolean algebra is its own completion, it follows that the regular open algebras are (up to isomorphism) exactly the complete Boolean algebras. Furthermore, it follows from Theorem 2.3.6, Lemma 2.3.8, and Proposition 2.3.9 that the completion of \mathbf{B} is the regular open algebra of any dense subset of \mathbf{B}^+ . Since every dense subset of \mathbf{B}^+ is separative, it follows that every complete Boolean algebra is isomorphic to the regular open algebra of some separative partial order.

Remark. We shall use the fact that every complete Boolean algebra \mathbf{B} is isomorphic to the regular open algebra of some separative partial order \mathbf{P} to characterize algebraic properties of \mathbf{B} in terms of order properties of \mathbf{P} .

Next we shall introduce a class of regular open algebras which demonstrates that for each infinite regular cardinal κ , for all cardinals $\eta < \kappa$ (η, ∞) -distributivity does not imply $(\kappa, 2)$ -distributivity. To do so, we shall need the following Definition 2.3.10 and Proposition 2.3.11.

Definition 2.3.10. [21] Let κ be an infinite cardinal. \mathbf{P} is κ -closed if for every ordinal $\rho < \kappa$, each decreasing sequence $(p_\alpha)_{\alpha < \rho}$ in \mathbf{P} has a lower bound in \mathbf{P} ; i.e. there exists $q \in \mathbf{P}$ satisfying $q \leq p_\alpha$, for each $\alpha < \rho$. We say that \mathbf{P} is *countably closed* if it is ω_1 -closed.

Proposition 2.3.11. [21] If \mathbf{B}^+ contains a dense κ -closed subset, then \mathbf{B} is (η, ∞) -distributive for all $\eta < \kappa$.

Example 2.3.12. [21] Let κ be an infinite regular cardinal, I and J be arbitrary sets such that $|I| \geq \kappa$ and $|J| \geq 2$, and

$$(2.3.2) \quad \text{Fn}(I, J, \kappa) = \{p : p \text{ a function from a subset of } I \text{ into } J, |\text{dom } p| < \kappa\}.$$

For $p, q \in \text{Fn}(I, J, \kappa)$, let $q \leq p \leftrightarrow p \subseteq q$. $\text{Fn}(I, J, \kappa)$ is separative and κ -closed; thus, $\text{r.o.}(\text{Fn}(I, J, \kappa))$ is (η, ∞) -distributive for every $\eta < \kappa$, by Proposition 2.11. However, $\text{r.o.}(\text{Fn}(I, J, \kappa))$ is not $(\kappa, 2)$ -distributive.

2.4. CHARACTERIZATIONS OF SOME BOOLEAN ALGEBRAIC PROPERTIES OF $\text{r.o.}(\mathbf{P})$ IN TERMS OF ORDER PROPERTIES OF \mathbf{P}

As discussed in §2.3, every complete Boolean algebra, and hence every measurable algebra, is isomorphic to the regular open algebra of some separative partial order. In this section we characterize the chain conditions c.c.c., σ -bounded c.c., and $\text{CUP}(\mathbf{B}^+)$ in regular open algebras $\text{r.o.}(\mathbf{P})$ in terms of the order properties of \mathbf{P} . Then we characterize (ω, ω) -distributivity and weak (ω, ω) -distributivity in regular open algebras of separative partial orders in terms of order properties of the underlying partial order. We then use Kelley's Theorem 2.2.19 to obtain a characterization of those separative partial orders which give rise to regular open algebras that are measurable.

We start by giving definitions of some chain conditions in partial orderings and showing their equivalences to the analogous chain condition in the regular open algebra. Note that for the chain conditions, \mathbf{P} does not need to be separative.

Definition 2.4.1. [22] A partial order \mathbf{P} satisfies the κ -chain condition (κ -c.c.) if each pairwise incompatible subset of \mathbf{P} has cardinality $< \kappa$.

Proposition 2.4.2. A partial order \mathbf{P} satisfies the κ -c.c. iff $\text{r.o.}(\mathbf{P})$ satisfies the κ -c.c. as a Boolean algebra.

Proof: Let $e : \mathbf{P} \rightarrow \text{r.o.}(\mathbf{P})$ be the canonical mapping of \mathbf{P} into $\text{r.o.}(\mathbf{P})$. Suppose \mathbf{P} satisfies the κ -c.c. Let $\{b_i : i < \kappa\} \subseteq \mathbf{B}^+$. Since $e[\mathbf{P}]$ is a dense subset of $\text{r.o.}(\mathbf{P})^+$, for each b_i , choose a $p_i \in \mathbf{P}$ such that $e(p_i) \leq b_i$. $\{p_i : i < \kappa\}$ is

not pairwise incompatible, since \mathbf{P} satisfies the κ -c.c. Thus, there are $i, j < \kappa$ such that $p_i \parallel p_j$. By Theorem 2.3.5 (2), $e(p_i) \wedge e(p_j) \neq 0$ in $\text{r.o.}(\mathbf{P})$. Since $e(p_i) \wedge e(p_j) \leq b_i \wedge b_j$, $\{b_i : i < \kappa\}$ is not pairwise disjoint.

Conversely, suppose that $\text{r.o.}(\mathbf{P})$ satisfies the κ -c.c. Let $\{p_i : i < \kappa\} \subseteq \mathbf{P}$. Then $\{e(p_i) : i < \kappa\} \subseteq \text{r.o.}(\mathbf{P})$, so $\exists i, j < \kappa$ such that $e(p_i) \wedge e(p_j) \neq 0$. By Theorem 2.3.5 (2), $p_i \parallel p_j$; hence \mathbf{P} satisfies the κ -c.c.

□

Definition 2.4.3. [6] A partial order \mathbf{P} satisfies the σ -bounded chain condition (σ -bounded c.c.) if there exist subsets $S_i \subseteq \mathbf{P}$, $i < \omega$ such that

$$(2.4.1) \quad \mathbf{P} = \bigcup_{i < \omega} S_i$$

and for each $i < \omega$, if $X \subseteq S_i$ and X is pairwise incompatible, then $|X| \leq i + 1$.

Proposition 2.4.4. A partial order \mathbf{P} satisfies the σ -bounded c.c. iff $\text{r.o.}(\mathbf{P})$ satisfies the σ -bounded c.c.

Proof: The proof is similar to that of Proposition 2.4.2 and follows directly from Theorem 2.3.5 (2).

□

It follows from Proposition 2.3.5 (2) that for all $p_1, \dots, p_n \in \mathbf{P}$, $e(p_1) \wedge \dots \wedge e(p_n) \neq 0$ in $\text{r.o.}(\mathbf{P}) \iff p_1, \dots, p_n$ are compatible in \mathbf{P} ; i.e. if there is a $q \in \mathbf{P}$ such that $q \leq p_i$, for all $i \in \{1, \dots, n\}$. Thus, one can define the intersection number of a subset $S \subseteq \mathbf{P}$ directly within (\mathbf{P}, \leq) in the obvious way so that it agrees with the definition of intersection number $\alpha(e[S])$ in $\text{r.o.}(\mathbf{P})$. Continuing in this manner, $\text{CUP}(\mathbf{P})$ can be defined directly in \mathbf{P} and is equivalent to $\text{CUP}(e[\mathbf{P}])$ in $\text{r.o.}(\mathbf{P})$.

Proposition 2.4.5. $\text{CUP}(\mathbf{P})$ holds iff $\text{CUP}(\text{r.o.}(\mathbf{P})^+)$ holds.

Proof: Since $\text{CUP}(\mathbf{P})$ holds if and only if $\text{CUP}(e[\mathbf{P}])$ holds in $\text{r.o.}(\mathbf{P})$, it suffices to show that $\text{CUP}(e[\mathbf{P}])$ is equivalent to $\text{CUP}(\text{r.o.}(\mathbf{P}))$. Suppose $\text{CUP}(e[\mathbf{P}])$

holds. Let $S_i \subseteq e[\mathbf{P}]$, $i < \omega$, be such that $e[\mathbf{P}] = \bigcup_{i < \omega} S_i$ and $\forall i < \omega$, $\alpha(S_i) \neq 0$. For each $i < \omega$, let $X_i = \{b \in \text{r.o.}(\mathbf{P}) : \exists e(p) \in S_i (e(p) \leq b)\}$. Since $e[\mathbf{P}]$ is dense in $\text{r.o.}(\mathbf{P})$, $\bigcup_{i < \omega} X_i = \text{r.o.}(\mathbf{P})$. Moreover, since every element of X_i is greater than or equal to an element of S_i , $\alpha(X_i) \geq \alpha(S_i) > 0$. Hence, $\text{CUP}(\text{r.o.}(\mathbf{P})^+)$ holds.

Conversely, if X_i , $i < \omega$, are sets which establish $\text{CUP}(\text{r.o.}(\mathbf{P})^+)$, then for each $i < \omega$, let $S_i = \{b \in X_i : b \in e[\mathbf{P}]\}$. $S_i \subseteq X_i$ implies $\alpha(S_i) \geq \alpha(X_i) > 0$. Moreover, since $\bigcup_{i < \omega} X_i = \text{r.o.}(\mathbf{P}) \supseteq e[\mathbf{P}]$, it follows that $\bigcup_{i < \omega} S_i = e[\mathbf{P}]$. Hence, $\text{CUP}(e[\mathbf{P}])$ holds.

□

We now restrict our attention to separative partial orders. Recall that if \mathbf{P} is separative, then $e[\mathbf{P}]$ is isomorphic to \mathbf{P} , by Proposition 2.3.9. Hence, we shall abuse notation and refer to \mathbf{P} as a dense subset of $\text{r.o.}(\mathbf{P})$.

Definition 2.4.6. [21] $D \subseteq \mathbf{B}^+$ is a *pairwise disjoint family* in \mathbf{B} if D is pairwise disjoint. D is a *partition of unity* in \mathbf{B} if it is a maximal pairwise disjoint family in \mathbf{B} .

Proposition 2.4.7. *Let \mathbf{P} be a dense subset of \mathbf{B}^+ . For each $D \subseteq \mathbf{P}$, the following are equivalent:*

- (1) D is a partition of unity in \mathbf{B} ;
- (2) D is a maximal pairwise incompatible subset of \mathbf{P} .

Proof: Let $D \subseteq \mathbf{P}$ be a partition of unity in \mathbf{B} . By definition, D is pairwise disjoint, which is equivalent to D being pairwise incompatible, by Definition 2.3.5 (2). $\forall p \in \mathbf{P}$, $\exists d \in D$ such that $d \wedge p \neq 0$, since D is a partition of unity in \mathbf{B}^+ . Thus, $d \parallel p$ in \mathbf{P} , so D is a maximal incompatible subset of \mathbf{P} .

Conversely, if D is a maximal pairwise incompatible subset of \mathbf{P} , then D is pairwise disjoint in \mathbf{B}^+ . For each $b \in \mathbf{B}^+$, $\exists p \in \mathbf{P}$ such that $p \leq b$, since \mathbf{P} is

dense in \mathbf{B}^+ . $\exists d \in D$ such that $d \parallel p$, since D is maximal pairwise incompatible in \mathbf{P} . Hence, $0 < d \wedge p \leq d \wedge b$ in \mathbf{B}^+ .

□

Definition 2.4.8. [15] For D, D' partitions of unity in \mathbf{B} , D' is a *refinement* of D if for each $d' \in D'$ there exists a $d \in D$ such that $d' \leq d$.

Proposition 2.4.9. *If \mathbf{P} is a dense subset of \mathbf{B}^+ , then for each partition of unity D in \mathbf{B} there is a partition of unity D' in \mathbf{B} consisting of members of \mathbf{P} which is a refinement of D .*

Proof: Let D be a partition of unity in \mathbf{B} . Using Zorn's Lemma, for each $d \in D$ there is a maximal pairwise incompatible family of elements of \mathbf{P} below d , call this family \mathcal{P}_d . Then $\bigcup_{d \in D} \mathcal{P}_d$ is a refinement of D .

□

Remark. If \mathbf{P} is a separative partial order, then every maximal pairwise incompatible subset of \mathbf{P} is a partition of unity in $\text{r.o.}(\mathbf{P})$, by Proposition 2.4.7. Conversely, if D is a partition of unity in $\text{r.o.}(\mathbf{P})$, then there is a refinement of D consisting of members of \mathbf{P} , by Proposition 2.4.9.

We now give a characterization of the (ω, ω) -d.l. in c.c.c. σ -algebras in terms of dense subsets of \mathbf{B}^+ .

Theorem 2.4.10. *For each c.c.c. σ -algebra \mathbf{B} , the following are equivalent:*

- (1) \mathbf{B} satisfies the (ω, ω) -d.l.;
- (2) The (ω, ω) -d.l. holds for all countable families of countable partitions of unity in \mathbf{B} ;
- (3) For each countable family of countable partitions of unity $D_i \subseteq \mathbf{B}^+$, ($i < \omega$), there exists a countable partition of unity $D \subseteq \mathbf{B}^+$ which refines D_i , for all $i < \omega$;
- (4) For each dense $\mathbf{P} \subseteq \mathbf{B}^+$ and for every countable family of countable

maximal pairwise incompatible $D_i \subseteq \mathbf{P}$, ($i < \omega$), there exists a countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ which refines D_i , for all $i < \omega$;

- (5) *There exists a dense $\mathbf{P} \subseteq \mathbf{B}^+$ such that for every countable family of countable maximal pairwise incompatible $D_i \subseteq \mathbf{P}$, ($i < \omega$), there exists a countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ which refines D_i , for all $i < \omega$.*

Proof: The proof of (1) \iff (2) can be found in [21]. We shall show (2) \implies (3) \implies (4) \implies (5) \implies (2).

(2) \implies (3): Let $D_i = \{d_{ij} : j < \omega\}$, $i < \omega$, be partitions of unity in \mathbf{B} . Since the (ω, ω) -d.l. holds,

$$(2.4.2) \quad 1 = \bigwedge_{i < \omega} \bigvee_{j < \omega} d_{ij} = \bigvee_{f: \omega \rightarrow \omega} \bigwedge_{i < \omega} d_{i, f(i)}.$$

Let $D = \{\bigwedge_{i < \omega} d_{i, f(i)} : f \in \omega^\omega\} \setminus \{0\}$. Since the D_i are pairwise disjoint, D must be pairwise disjoint. That is, if $f \neq g$ are in ω^ω , then for some $i_* < \omega$, $f(i_*) \neq g(i_*)$; hence

$$(2.4.3) \quad \bigwedge_{i < \omega} d_{i, f(i)} \wedge \bigwedge_{i < \omega} d_{i, g(i)} \leq d_{i_*, f(i_*)} \wedge d_{i_*, g(i_*)} = 0.$$

D is a partition of unity, since

$$(2.4.4) \quad \bigvee D = \bigvee_{f: \omega \rightarrow \omega} \bigwedge_{i < \omega} d_{i, f(i)} = 1.$$

Moreover, D is countable, since \mathbf{B} satisfies the c.c.c.

(3) \implies (4): Let \mathbf{P} be a dense subset of \mathbf{B}^+ and let $D_i = \{d_{ij} : j < \omega\} \subseteq \mathbf{P}$, $i < \omega$, be maximal pairwise incompatible. By Proposition 2.4.7, each D_i is a partition of unity in \mathbf{B} , so (3) implies that there exists a countable partition of unity $D \subseteq \mathbf{B}^+$ which refines all the D_i . By Proposition 2.4.9, there is a partition of unity $D' \subseteq \mathbf{P}$ which refines D . By Proposition 2.4.7, D' is a maximal pairwise incompatible subset of \mathbf{P} . Moreover, D' is countable, since \mathbf{B} satisfies the c.c.c.

(4) \implies (5): \mathbf{B}^+ is a dense subset of \mathbf{B}^+ , so (4) implies that (5) holds for the dense subset $\mathbf{P} = \mathbf{B}^+$.

(5) \implies (2): Let \mathbf{P} be a dense subset of \mathbf{B}^+ for which (5) holds. Let $D_i = \{d_{ij} : j < \omega\} \subseteq \mathbf{B}^+$, $i < \omega$, be partitions of unity. $\forall i < \omega$, there exists a refinement $P_i \subseteq \mathbf{P}$ of D_i , by Proposition 2.4.9. (5) implies that there exists a countable maximal pairwise incompatible $P \subseteq \mathbf{P}$ which refines P_i , $\forall i < \omega$. Since $\forall i < \omega$ P_i refines D_i , P also simultaneously refines all D_i . $\forall p \in P \forall i < \omega$, $\exists j(i) < \omega$ such that $p \leq d_{i,j(i)}$. Let $f_p \in \omega^\omega$ be given by $f_p(i) = j(i)$. Then $p \leq \bigwedge_{i < \omega} d_{i,f_p(i)}$. Hence,

$$(2.4.5) \quad 1 = \bigvee P \leq \bigvee_{p \in P} \bigwedge_{i < \omega} d_{i,f_p(i)} \leq \bigvee_{f: \omega \rightarrow \omega} \bigwedge_{i < \omega} d_{i,f(i)}.$$

Thus, the (ω, ω) -d.l. holds for the partitions of unity D_i , $i < \omega$.

□

Theorem 2.4.10 yields the following characterization of the (ω, ω) -d.l. in a c.c.c. regular open algebra $\text{r.o.}(\mathbf{P})$ in terms of the order properties of \mathbf{P} .

Corollary 2.4.11. *For each separative, c.c.c. partial order \mathbf{P} , the following are equivalent:*

- (1) *$\text{r.o.}(\mathbf{P})$ satisfies the (ω, ω) -d.l.;*
- (2) *For every countable family of countable maximal pairwise incompatible subsets $D_i \subseteq \mathbf{P}$, ($i \in \omega$), there exists an countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ which refines D_i , for all $i \in \omega$.*

We now give a characterization of the weak (ω, ω) -d.l. in c.c.c. σ -algebras in terms of dense subsets of \mathbf{B}^+ .

Theorem 2.4.12. *For each c.c.c. σ -algebra \mathbf{B} , the following are equivalent:*

- (1) *\mathbf{B} satisfies the weak (ω, ω) -d.l.;*

- (2) \mathbf{B} satisfies the weak (ω, ω) -d.l. for all countable families of countable partitions of unity;
- (3) For every countable family of countable partitions of unity $D_i = \{d_{ij} : j \in \omega\} \subseteq \mathbf{B}^+$, $(i \in \omega)$, there exists an countable partition of unity $D \subseteq \mathbf{B}^+$ such that $\forall d \in D, \forall i \in \omega, \exists k \in \omega$ such that $d \leq \bigvee_{j \leq k} d_{ij}$;
- (4) For each dense $\mathbf{P} \subseteq \mathbf{B}^+$, for every countable family of countable maximal pairwise incompatible $D_i = \{p_{ij} : j \in \omega\} \subseteq \mathbf{P}$, $(i \in \omega)$, there exists a countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ such that $\forall p \in D, \forall i \in \omega$, p is compatible with only finitely many $p_{ij} \in D_i$;
- (5) There exists a dense $\mathbf{P} \subseteq \mathbf{B}^+$ such that for every countable family of countable maximal pairwise incompatible $D_i = \{p_{ij} : j \in \omega\} \subseteq \mathbf{P}$, $(i \in \omega)$, there exists a countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ such that $\forall p \in D, \forall i \in \omega$, p is compatible with only finitely many $p_{ij} \in D_i$.

The proof of Theorem 2.4.12 is similar to that of Theorem 2.4.10.

Theorem 2.4.12 yields the following characterization of the weak (ω, ω) -d.l. in the regular open algebra of a separative, c.c.c. partial order.

Corollary 2.4.13. *For each separative, c.c.c. partial order \mathbf{P} , the following are equivalent:*

- (1) $r.o.(\mathbf{P})$ satisfies the weak (ω, ω) -d.l.;
- (2) For every countable family of countable maximal incompatible $D_i = \{p_{ij} : j \in \omega\} \subseteq \mathbf{P}$, $(i \in \omega)$, there exists a countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ such that $\forall p \in D, \forall i \in \omega$, p is compatible with only finitely many $p_{ij} \in D_i$.

As discussed in §2.2, Kelley characterized measurable algebras as those σ -algebras \mathbf{B} for which the weak (ω, ω) -d.l. and $\text{CUP}(\mathbf{B}^+)$ hold. In order to state our corollary of Kelley's Theorem 2.2.19 for regular open algebras, we need the following characterization of $\text{CUP}(\mathbf{B}^+)$.

Lemma 2.4.14. *For each Boolean algebra \mathbf{B} , the following are equivalent:*

- (1) $\text{CUP}(\mathbf{B}^+)$ holds;
- (2) For each subset $A \subseteq \mathbf{B}^+$, $\text{CUP}(\mathbf{P})$ holds;
- (3) There exists a dense $\mathbf{P} \subseteq \mathbf{B}^+$ such that $\text{CUP}(\mathbf{P})$ holds.

Proof: (2) \implies (1) \implies (3) is trivial, since \mathbf{B}^+ is dense in itself.

(3) \implies (2): Suppose \mathbf{P} is dense in \mathbf{B}^+ and $\text{CUP}(\mathbf{P})$ holds. Let A be any subset of \mathbf{B}^+ . Let $\mathcal{S}_i, i < \omega$, be sets for which $\text{CUP}(\mathbf{P})$ holds. Let

$$(2.4.6) \quad \mathcal{T}_i = \{a \in A : \exists p \in \mathbf{P} (p \leq a)\}.$$

Since \mathbf{P} is dense in \mathbf{B}^+ , $\forall a \in A, \exists p \in \mathbf{P}$ for which $p \leq a$. Since (2.4.6) holds, $\exists i < \omega$ for which $p \in \mathcal{S}_i$. Hence, $a \in \mathcal{T}_i$. Thus,

$$(2.4.7) \quad \bigcup_{i < \omega} \mathcal{T}_i = A.$$

Moreover, since each element of \mathcal{T}_i is greater than or equal to some element of \mathcal{S}_i , $\alpha(\mathcal{T}_i) \geq \alpha(\mathcal{S}_i) > 0$. Therefore, $\text{CUP}(A)$ holds.

□

Note that for any partial order \mathbf{P} , $\text{CUP}(\mathbf{P})$ implies that \mathbf{P} satisfies the c.c.c. Thus, by Kelley's Theorem 2.2.19, Corollary 2.4.13 and Lemma 2.4.14 we obtain the following.

Corollary 2.4.15. *For each separative partial order \mathbf{P} , the following are equivalent:*

- (1) $r.o.(\mathbf{P})$ is a measurable algebra;
- (2) $\text{CUP}(\mathbf{P})$ holds, and for every countable family of countable maximal pairwise incompatible $D_i = \{p_{ij} : j \in \omega\} \subseteq \mathbf{P}, (i \in \omega)$, there exists an countable maximal pairwise incompatible $D \subseteq \mathbf{P}$ such that $\forall p \in D, \forall i \in \omega, p$ is compatible with only finitely many $p_{ij} \in D_i$.

2.5. TOPOLOGICAL CHARACTERIZATIONS OF THE WEAK (ω, ω) -DISTRIBUTIVE LAW IN C.C.C. σ -ALGEBRAS

Kelley obtained the following topological characterization of the weak (ω, ω) -d.l. in a c.c.c. Boolean σ -algebra in terms of the topology of its Stone space.

Theorem 2.5.1 (Kelley). [20] *For each c.c.c. σ -algebra \mathbf{B} , the following are equivalent:*

- (1) \mathbf{B} satisfies the weak (ω, ω) -d.l.;
- (2) Every subset of the Stone space X of \mathbf{B} which is of the first category is nowhere dense.

Balcar, Glówczyński and Jech [1] obtained a topological characterization of those complete Boolean algebras which satisfy the weak (ω, ω) -d.l. and the \underline{b} -c.c. in terms of the sequential topology on a Boolean algebra. As an immediate corollary of their Theorem 2.5.7, one obtains a topological characterization of the weak (ω, ω) -d.l. in c.c.c. σ -algebras. In order to state their results, we shall need the following definitions.

Definition 2.5.2. [1] Let (X, τ) be a topological space. The space X is

- (1) *sequential* if each $A \subseteq X$ which contains all limit points of τ -convergent sequences of elements of A is closed;
- (2) *Fréchet* if for every $A \subseteq X$, $\text{cl}_\tau(A) = \{x \in X : (\exists \langle x_n : n < \omega \rangle \subseteq A) x_n \xrightarrow[\tau]{} x\}$,

where $\xrightarrow[\tau]{} denotes convergence in the topology τ .$

Remark. Every first countable space is Fréchet and every Fréchet space is sequential, but the reverse implications do not hold (see [4]).

Definition 2.5.3. [1] Let \mathbf{B} be a σ -algebra. For a sequence $\langle b_n : n < \omega \rangle$ of elements of \mathbf{B} , we denote

$$(2.5.1) \quad \overline{\lim} b_n = \bigwedge_{k < \omega} \bigvee_{n \geq k} b_n \quad \text{and} \quad \underline{\lim} b_n = \bigvee_{k < \omega} \bigwedge_{n \geq k} b_n.$$

We say that a sequence $\langle b_n \rangle$ *algebraically converges* to an element $b \in \mathbf{B}$ ($b_n \longrightarrow b$) if $\overline{\lim} b_n = \underline{\lim} b_n = b$.

Remark. Note that $b_n \longrightarrow b$ if and only if $\bigwedge_{k < \omega} \bigvee_{n \geq k} (b \triangle b_n) = 0$.

The *sequential topology*, τ_s , can be described by the following closure operation: For $A \subseteq \mathbf{B}$, let $c(A) = \{x \in \mathbf{B} : x \text{ is the algebraic limit of a sequence } \{x_n\} \text{ of elements of } A\}$. $\text{cl}_{\tau_s}(A) = \bigcup_{\alpha < \omega_1} c^{(\alpha)}(A)$, where $c^{(\alpha+1)}(A) = c(c^{(\alpha)}(A))$ and for a limit ordinal λ , $c^{(\lambda)}(A) = \bigcup_{\beta < \lambda} c^{(\beta)}(A)$. τ_s is the largest topology on \mathbf{B} , with respect to inclusion among all topologies with the following property: if $b_n \longrightarrow b$ then $b_n \xrightarrow{\tau} b$. Note that (\mathbf{B}, τ_s) is Fréchet if and only if $\text{cl}_{\tau_s}(A) = c(A)$ for each $A \subseteq \mathbf{B}$. (See [1].)

Definition 2.5.4. [1] The *bounding number* is the least cardinal \underline{b} of a family \mathcal{F} of functions from ω to ω such that \mathcal{F} is unbounded with respect to eventual domination; i.e. for every $g : \omega \rightarrow \omega$ there is some $f \in \mathcal{F}$ such that $g(n) \leq f(n)$ for infinitely many n .

Remark. $\omega < \underline{b} \leq 2^\omega$, so the c.c.c. implies the \underline{b} -c.c. MA implies $\underline{b} = 2^\omega$. It is also consistent with ZFC that $\underline{b} = \omega_1$ and $2^\omega > \omega_1$.

Definition 2.5.5. [1] *Fréchet's diagonal condition* is the following: If $b_n \longrightarrow b$ as $n \rightarrow \infty$ and for each $n < \omega$, $b_{nk} \longrightarrow b_n$ as $k \rightarrow \infty$, then there exists a function $g : \omega \rightarrow \omega$ such that $b_{ng(n)} \longrightarrow b$ as $n \rightarrow \infty$.

Definition 2.5.6. [1] A matrix $\{a_{mn}\}_{m < \omega, n < \omega}$ is *increasing* if each row $\{a_{mn} : n < \omega\}$ is an increasing sequence with limit 1.

We now state a characterization of those complete Boolean algebras in which the weak (ω, ω) -d.l. and the \underline{b} -c.c. hold.

Theorem 2.5.7. [1] *For each complete Boolean algebra \mathbf{B} , the following are equivalent:*

- (1) *\mathbf{B} satisfies the weak (ω, ω) -d.l. and the \underline{b} -chain condition;*
- (2) *Fréchet's diagonal condition holds for every increasing matrix;*
- (3) *The sequential space (\mathbf{B}, τ_s) is Fréchet.*

Recalling our remark in §2.1 that each c.c.c. σ -algebra is complete, we extract from Theorem 2.5.7 the following characterization of the weak (ω, ω) -d.l. in c.c.c. σ -algebras.

Corollary 2.5.8. *For each c.c.c. σ -algebra \mathbf{B} , the following are equivalent:*

- (1) *The weak (ω, ω) -d.l. holds in \mathbf{B} ;*
- (2) *Fréchet's diagonal condition holds for every increasing matrix;*
- (3) *The sequential space (\mathbf{B}, τ_s) is Fréchet.*

(3) of Corollary 2.5.8 gives a topological characterization of the weak (ω, ω) -d.l. in c.c.c. σ -algebras.

Chapter 3

COMPLETE EMBEDDINGS OF THE COHEN ALGEBRA INTO THREE EXAMPLES OF C.C.C., NON-MEASURABLE BOOLEAN ALGEBRAS

3.1. PRELIMINARIES

In order to gain more insight into von Neumann's problem within ZFC, we investigate some examples of complete, c.c.c. Boolean algebras which are non-measurable. The goal is to find out whether these Boolean algebras sustain any weak form of distributivity. In the next few sections, we work with three classic examples of complete, c.c.c., non-measurable Boolean algebras constructed by Galvin and Hajnal, Argyros, and Gaifman. In particular, we are interested in whether or not the hyper-weak (ω, ω) -d.l. holds in these algebras, and if not, whether the Cohen algebra, the completion of the countable, atomless Boolean algebra, embeds as a complete subalgebra.

Galvin and Hajnal, Argyros, and Gaifman constructed Boolean algebras in order to establish strict implications between various chain conditions, including the countable chain condition, the σ -bounded chain condition, and $\text{CUP}(\mathbf{B}^+)$:

Definition 3.1.1. [21] \mathbf{B} satisfies the *countable chain condition* (c.c.c.) if for each pairwise disjoint subset $X \subseteq \mathbf{B}^+$, $|X| \leq \omega$.

Definition 3.1.2. [6] \mathbf{B} satisfies the *σ -bounded chain condition* (σ -bounded c.c.) if there exist subsets $X_n \subseteq \mathbf{B}^+$, $n < \omega$, such that

$$(3.1.1) \quad \mathbf{B}^+ = \bigcup_{n < \omega} X_n$$

where $\forall n < \omega$, each pairwise disjoint subset of X_n has cardinality $\leq n + 1$.

Definition 3.1.3. [9] $\text{CUP}(\mathbf{B}^+)$ holds if there exist subsets $X_n \subseteq \mathbf{B}^+$, $n < \omega$, such that

$$(3.1.2) \quad \mathbf{B}^+ = \bigcup_{n < \omega} X_n$$

and $\forall n < \omega$, $\alpha(X_n) > 0$, where α is Kelley's intersection number (see Definition 2.2.17).

Horn and Tarski asked the following question: Does every c.c.c. Boolean algebra satisfy the σ -bounded c.c.? [13] While trying to answer this question, Gaifman established that $\text{CUP}(\mathbf{B}^+)$ is strictly stronger than the σ -bounded c.c., since his example satisfies the σ -bounded c.c., but not $\text{CUP}(\mathbf{B}^+)$ [8]. Argyros also established that fact and, in addition, showed that under CH, Knaster's condition K_3 is strictly stronger than K_2 . Later, Galvin and Hajnal answered Horn and Tarski's question in the negative. They constructed a Boolean algebra which satisfies the c.c.c. but not the σ -bounded c.c. (See [3].)

In each of the Galvin-Hajnal, Argyros, and Gaifman algebras, $\text{CUP}(\mathbf{B}^+)$ fails; thus, by Kelley's Theorem 2.2.19, these Boolean algebras do not carry a strictly positive, finitely additive measure, and thus, are not measurable algebras. However, this is not the only reason measurability fails. In addition, not only do our results show that the weak (ω, ω) -d.l. fails in these algebras, but moreover, that the more general hyper-weak (ω, ω) -d.l. fails. Furthermore, in the Galvin-Hajnal, Argyros, and atomless Gaifman algebras, the Cohen algebra embeds as a complete subalgebra. In every Gaifman algebra, the hyper-weak (ω, ω) -d.l. fails.

Recall the definition of the weak (ω, ω) -d.l.

Definition 3.1.4. [21] \mathbf{B} satisfies the *weak (ω, ω) -distributive law* (weak (ω, ω) -d.l.) if for each $|I| \leq \omega$, $|J| \leq \omega$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(3.1.3) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow [J]^{<\omega}} \bigwedge_{i \in I} \bigvee \{a_{ij} : j \in f(i)\},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee \{a_{ij} : j \in f(i)\}$ for each $f : I \rightarrow [J]^{<\omega}$ exist in \mathbf{B} .

The hyper-weak (ω, ω) -distributive law is the following generalization of the weak (ω, ω) -distributive law.

Definition 3.1.5. [26] \mathbf{B} satisfies the *hyper-weak (ω, ω) -distributive law* (hyper-weak (ω, ω) -d.l.) if for each $|I| \leq \omega$, $|J| = \omega$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(3.1.4) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} \bigvee \{a_{ij} : j \in J \setminus \{f(i)\}\},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee \{a_{ij} : j \in J \setminus \{f(i)\}\}$ for each $f : I \rightarrow J$ exist in \mathbf{B} .

The hyper-weak (ω, ω) -d.l. is an extremely weak form of distributivity. On the right hand side of (3.1.4), for each $i < \omega$, instead of taking the supremum over finitely many members of each set $\{a_{ij} : j < \omega\}$ as one does in the weak (ω, ω) -d.l., we now omit one element of $\{a_{ij} : j < \omega\}$ and take the supremum of the rest. The weak (ω, ω) -d.l. trivially implies the hyper-weak (ω, ω) -d.l. Moreover, the hyper-weak (ω, ω) -d.l. is strictly weaker than the weak (ω, ω) -d.l.: Laver forcing satisfies the hyper-weak (ω, ω) -d.l., but not the weak (ω, ω) -d.l.

Theorem 3.1.6. *The hyper-weak (ω, ω) -d.l. fails everywhere in the Cohen algebra.*

Proof: Let $\mathbf{B} = \text{r.o.}(\text{Clop}(\omega^\omega))$, the Cohen algebra. Let $a_{ij} = \{f \in \omega^\omega : f(i) = j\}$ and let $W_i = \{a_{ij} : j < \omega\}$. Then

$$(3.1.5) \quad \bigwedge_{i < \omega} \bigvee_{j < \omega} a_{ij} = 1 > 0 = \bigvee_{f: \omega \rightarrow \omega} \bigwedge_{i < \omega} \bigvee \{a_{ij} : j \in \omega \setminus \{f(i)\}\}.$$

Thus, the hyper-weak (ω, ω) -d.l. fails everywhere in $\text{Clop}(\omega^\omega)$; hence also in \mathbf{B} .

□

By Theorem 3.1.6, if the Cohen algebra embeds into a complete Boolean algebra \mathbf{B} as a complete subalgebra, then the hyper-weak (ω, ω) -d.l. fails in \mathbf{B} . Moreover, since the Cohen algebra satisfies the σ -bounded c.c., a complete embedding of the Cohen algebra into a complete Boolean algebra \mathbf{B} implies that there is a complete subalgebra of \mathbf{B} which satisfies the σ -bounded c.c., but is still non-measurable. This is especially intriguing in the case of the Galvin-Hajnal algebra, since it does not satisfy the σ -bounded c.c. The complete embedding of the Cohen algebra is the strongest form of non-distributivity which a Boolean algebra can possess.

Each of our embeddings of the Cohen algebra will use the following notions and lemmas.

Definition 3.1.7. [21] A subalgebra \mathbf{A} of a Boolean algebra \mathbf{B} is a *regular subalgebra* of \mathbf{B} if for each $M \subseteq \mathbf{A}$ such that $\bigvee^{\mathbf{A}} M$ exists in \mathbf{A} , $\bigvee^{\mathbf{B}} M$ exists in \mathbf{B} and $\bigvee^{\mathbf{A}} M = \bigvee^{\mathbf{B}} M$.

The following Lemma 3.1.8 is useful for showing that a subalgebra is regular.

Lemma 3.1.8. Let \mathbf{B} be a Boolean algebra, \mathbf{P} a dense subset of \mathbf{B}^+ , and \mathbf{A} a subalgebra of \mathbf{B} . \mathbf{A} is a regular subalgebra of \mathbf{B} iff $\forall p \in \mathbf{P} \exists a_p \in \mathbf{A}^+$ such that whenever $a \in \mathbf{A}$ and $a_p \wedge a \neq 0$, then $p \wedge a \neq 0$.

Proof: Suppose that $\forall p \in \mathbf{P} \exists a_p \in \mathbf{A}^+$ such that whenever $a \in \mathbf{A}$ and $a_p \wedge a \neq 0$, then $p \wedge a \neq 0$. To show that \mathbf{A} is a regular subalgebra of \mathbf{B} , it suffices to show that whenever $M \subseteq \mathbf{A}$ and $\bigvee^{\mathbf{A}} M = 1$, then $\bigvee^{\mathbf{B}} M = 1$. Let $M \subseteq \mathbf{A}$ be such that $\bigvee^{\mathbf{A}} M = 1$. Given $p \in \mathbf{P}$, $a_p \in \mathbf{A}^+$ implies there exists some $a \in M$ such that $a_p \wedge a \neq 0$. Thus, $p \wedge a \neq 0$. Hence, $\bigvee^{\mathbf{B}} M = 1$.

Conversely, suppose $\exists p \in \mathbf{P}$ such that $\forall a \in \mathbf{A}^+ \exists b_a \in \mathbf{A}$ such that $a \wedge b_a \neq 0$ but $p \wedge b_a = 0$. Then $\bigvee_{a \in \mathbf{A}}^{\mathbf{A}} b_a = 1$, but $\bigvee_{a \in \mathbf{A}}^{\mathbf{B}} b_a \neq 1$, since p is disjoint with every element of $\{b_a : a \in \mathbf{A}\}$. Thus, \mathbf{A} is not a regular subalgebra of \mathbf{B} .

□

Recall the following definitions.

Definition 3.1.9. [21] A subalgebra \mathbf{A} of a Boolean algebra \mathbf{B} is a *complete subalgebra* of \mathbf{B} if for each subset $M \subseteq \mathbf{A}$ such that $\bigvee^{\mathbf{B}} M$ exists, $\bigvee^{\mathbf{A}} M$ exists and $\bigvee^{\mathbf{B}} M = \bigvee^{\mathbf{A}} M$.

Definition 3.1.10. [21] A monomorphism $f : \mathbf{A} \rightarrow \mathbf{B}$ is *complete* if for each $M \subseteq \mathbf{A}$ for which $\bigvee^{\mathbf{A}} M$ exists, $f(\bigvee^{\mathbf{A}} M) = \bigvee^{\mathbf{B}} \{f(b) : b \in M\}$.

The following lemma is a natural consequence of the Sikorski Extension Theorem [21].

Lemma 3.1.11. [21] *If \mathbf{B} is a complete Boolean algebra and \mathbf{A} is a regular subalgebra of \mathbf{B} , then there is a complete monomorphism from $\text{r.o.}(\mathbf{A})$ into \mathbf{B} ; that is, $\text{r.o.}(\mathbf{A})$ embeds into \mathbf{B} as a complete subalgebra of \mathbf{B} .*

Lemma 3.1.8 combined with Lemma 3.1.11 give useful conditions for embedding the Cohen algebra as a complete subalgebra of a complete Boolean algebra \mathbf{B} . For the Galvin-Hajnal, Argyros, and atomless Gaifman algebras, the method employed in this chapter for completely embedding the Cohen algebra is the following: Choose countably many independent elements $\{c_i : i < \omega\} \subseteq \mathbf{B}$ in such a way that the subalgebra \mathbf{C} generated by $\{c_i : i < \omega\}$ satisfies the conditions of Lemma 3.1.8. Then \mathbf{C} is an atomless, regular subalgebra of \mathbf{B} . Since \mathbf{B} is complete, it follows from Lemma 3.1.11 that the completion of \mathbf{C} , namely $\text{r.o.}(\mathbf{C})$, embeds as a complete subalgebra of \mathbf{B} . Since there is only one atomless, countable Boolean algebra (see [21]), $\text{r.o.}(\mathbf{C})$ is isomorphic to the Cohen algebra. Hence, the Cohen algebra embeds as a complete subalgebra of \mathbf{B} .

3.2. A COMPLETE EMBEDDING OF THE COHEN ALGEBRA INTO THE GALVIN-HAJNAL ALGEBRA

Galvin and Hajnal constructed a partial ordering \mathbf{P} which satisfies the c.c.c. but not the σ -bounded c.c. To do this, they used the following family of sets.

Lemma 3.2.1. [3] *There is a family of sets $\{S_\alpha : \alpha < 2^\omega\}$ with the following four properties:*

- (S1) $\forall \alpha < 2^\omega \quad S_\alpha \subseteq \alpha;$
- (S2) $\forall \alpha < 2^\omega \quad [S_\alpha]^2 \subseteq \bigcup_{\gamma < 2^\omega} \{\{\beta, \gamma\} : \beta \in S_\gamma\};$
- (S3) $\forall \alpha < 2^\omega \quad \text{type}(S_\alpha) \leq \omega;$
- (S4) *If $S \subseteq 2^\omega$, $[S]^2 \subseteq \bigcup_{\gamma < 2^\omega} \{\{\beta, \gamma\} : \beta \in S_\gamma\}$, and $\text{type}(S) \leq \omega$, then $\exists \alpha < 2^\omega$ such that $S = S_\alpha$.*

The following lemmas will be used extensively.

Lemma 3.2.2. *Suppose $\eta, \kappa < 2^\omega$ and $|S_\eta| = \omega$. Then there exist $\kappa < \alpha < \beta < 2^\omega$ such that $S_\alpha \cup S_\beta = S_\eta$ and $S_\alpha \cap S_\beta = \emptyset$.*

Proof: Let $\eta, \kappa < 2^\omega$ be given. $\forall S \subseteq S_\eta$, $[S]^2 \subseteq [S_\eta]^2 \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$, by (S2). Thus, (S4) implies $\exists \alpha < 2^\omega$ such that $S_\alpha = S$. Let $I \subseteq 2^\omega$ be such that $\forall S \subseteq S_\eta \exists$ a unique $\alpha \in I$ such that $S = S_\alpha$. $|I| = 2^\omega$ since $|S_\eta| = \omega$.

Let $J = \{\alpha \in I : \alpha \leq \kappa\}$ and let $K = \{\alpha \in I : \alpha > \kappa\}$. For each $\gamma \in J$ let $\beta_\gamma \in I$ be the unique ordinal in I such that $S_\gamma \cup S_{\beta_\gamma} = S_\eta$ and $S_\gamma \cap S_{\beta_\gamma} = \emptyset$. Then for each $\alpha \in K \setminus \{\beta_\gamma : \gamma \in J\}$, which is uncountable, there is some $\beta \in K \setminus \{\beta_\gamma : \gamma \in J\}$ such that $S_\alpha \cup S_\beta = S_\eta$ and $S_\alpha \cap S_\beta = \emptyset$.

□

Lemma 3.2.3. *If $\zeta \in S_\eta$, then $\exists \kappa > \eta$ for which $S_\kappa = \{\zeta, \eta\}$. Given $\zeta < 2^\omega$,*

there is a sequence $\zeta < \alpha_0 < \alpha_1 < \alpha_2 < \dots$ such that

$$\begin{aligned}
 S_{\alpha_0} &= \{\zeta\} \\
 S_{\alpha_1} &= \{\zeta, \alpha_0\} \\
 S_{\alpha_2} &= \{\zeta, \alpha_0, \alpha_1\} \\
 &\vdots \\
 S_{\alpha_{i+1}} &= \{\zeta, \alpha_0, \dots, \alpha_i\} \\
 &\vdots
 \end{aligned}
 \tag{3.2.1}$$

and $\forall X \subseteq \bigcup_{i < \omega} S_{\alpha_i} = \{\zeta, \alpha_0, \alpha_1, \alpha_2, \dots\}$, $\exists \lambda > \sup(\alpha_i : i < \omega)$ such that $S_\lambda = X$.

Proof: The proof makes heavy use of properties (S1) and (S4). Suppose $\zeta \in S_\eta$. Then $[\{\zeta, \eta\}]^2 = \{\{\zeta, \eta\}\} \subseteq \{\{\beta, \eta\} : \beta \in S_\eta\} \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$. By (S4), $\exists \kappa < 2^\omega$ for which $S_\kappa = \{\zeta, \eta\}$. By (S1), $\kappa > \eta$.

Given $\zeta < 2^\omega$, $[\{\zeta\}]^2 = \emptyset$; so $\exists \alpha_0 < 2^\omega$ for which $S_{\alpha_0} = \{\zeta\}$, by (S4). (S1) implies $\alpha_0 > \zeta$, since $\{\zeta\} = S_{\alpha_0} \subseteq \alpha_0$. $[\{\zeta, \alpha_0\}]^2 = \{\{\zeta, \alpha_0\}\} = \{\{\beta, \alpha_0\} : \beta \in S_{\alpha_0}\} \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$; so (S4) implies $\exists \alpha_1 < 2^\omega$ such that $S_{\alpha_1} = \{\zeta, \alpha_0\}$. (S1) implies $\alpha_1 > \alpha_0$.

Given $\zeta < \alpha_0 < \alpha_1 < \dots < \alpha_n$ where for each $j < n$ $S_{\alpha_{j+1}} = \{\zeta, \alpha_0, \dots, \alpha_j\}$, the set $[\{\zeta, \alpha_0, \dots, \alpha_n\}]^2 = \bigcup_{j \leq n} \{\{\beta, \alpha_j\} : \beta \in S_{\alpha_j}\} \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$. Thus, (S1) and (S4) imply there is some $\alpha_{n+1} > \alpha_n$ such that $S_{\alpha_{n+1}} = \{\zeta, \alpha_0, \dots, \alpha_n\}$. By induction, there is a sequence $\alpha_0 < \alpha_1 < \dots < \alpha_j < \alpha_{j+1} < \dots$ such that for each $j < \omega$, $S_{\alpha_{j+1}} = \{\zeta, \alpha_0, \dots, \alpha_j\}$.

Suppose $X \subseteq \bigcup_{i < \omega} S_{\alpha_i}$. Then $[X]^2 \subseteq [\{\zeta, \alpha_0, \alpha_1, \dots\}]^2 = \bigcup_{j < \omega} \{\{\beta, \alpha_j\} : \beta \in S_{\alpha_j}\} \subseteq \bigcup_{\alpha < 2^\omega} \{\{\beta, \alpha\} : \beta \in S_\alpha\}$. Thus, by (S1) and (S4), there is a $\lambda < 2^\omega$ such that $S_\lambda = X$ and $\lambda > \alpha_j$ for all $j < \omega$.

□

Now we are ready to describe the Galvin-Hajnal partial ordering **P**.

Construction of \mathbf{P} : Well-order 2^ω and let Y denote 2^{2^ω} , the set of all functions from 2^ω into 2. $\forall \alpha < 2^\omega$ let

$$(3.2.2) \quad V_\alpha = \{f \in Y : \forall \beta \in S_\alpha (f(\beta) = 0), \text{ and } f(\alpha) = 1\}.$$

Take the collection of all non-empty intersections of finitely many V_α 's. Well-order these sets and let them be denoted by U_ξ , $\xi < 2^\omega$. Let

$$(3.2.3) \quad (\mathbf{P}, \leq) = (\{U_\xi : \xi < 2^\omega\}, \subseteq).$$

For each $\xi < 2^\omega$, let $F(\xi)$ be the finite subset of 2^ω such that

$$(3.2.4) \quad U_\xi = \bigcap_{\alpha \in F(\xi)} V_\alpha.$$

The following lemma gives a useful characterization of non-empty intersections of finitely many V_α 's and their complements in Y .

Lemma 3.2.4. *Let V_α^c denote $Y \setminus V_\alpha$, the set-theoretic complement of V_α in Y .*

$(\bigcap_{i=1}^m V_{\alpha_i}) \cap (\bigcap_{j=1}^n V_{\beta_j}^c) \neq \emptyset$ iff the following two conditions hold:

- (1) $(\bigcup_{i=1}^m S_{\alpha_i}) \cap \{\alpha_1, \dots, \alpha_m\} = \emptyset$;
- (2) $\{\alpha_1, \dots, \alpha_m\} \cap \{\beta_1, \dots, \beta_n\} = \emptyset$.

Proof: Suppose $(\bigcap_{i=1}^m V_{\alpha_i}) \cap (\bigcap_{j=1}^n V_{\beta_j}^c) \neq \emptyset$. Then $\bigcap_{i=1}^m V_{\alpha_i} \neq \emptyset$, so (1) must hold. If (2) fails, then $\exists 1 \leq k \leq m, \exists 1 \leq l \leq n$ such that $\alpha_k = \beta_l$. This implies

$$(3.2.5) \quad \left(\bigcap_{i=1}^m V_{\alpha_i}\right) \cap \left(\bigcap_{j=1}^n V_{\beta_j}^c\right) \subseteq V_{\alpha_k} \cap V_{\beta_l}^c = \emptyset.$$

Contradiction. Thus, (2) holds.

Now suppose $(\bigcap_{i=1}^m V_{\alpha_i}) \cap (\bigcap_{j=1}^n V_{\beta_j}^c) = \emptyset$. Then $\bigcap_{i=1}^m V_{\alpha_i} \subseteq (\bigcap_{j=1}^n V_{\beta_j}^c)^c = \bigcup_{j=1}^n V_{\beta_j}$. Suppose (1) holds. Then $\bigcap_{i=1}^m V_{\alpha_i} \neq \emptyset$. If (2) also holds, then $\exists f \in \bigcap_{i=1}^m V_{\alpha_i}$ such that $f(\beta_1) = \dots = f(\beta_n) = 0$. But such an f is clearly not in $\bigcup_{j=1}^n V_{\beta_j}$. Contradiction. Thus, either (1) or (2) fails.

□

We now use Lemmas 3.2.1, 3 and 4 to show that \mathbf{P} is separative.

Proposition 3.2.5. (\mathbf{P}, \subseteq) is separative.

Proof: Suppose $U_\xi, U_\chi \in \mathbf{P}$ and $U_\xi \not\subseteq U_\chi$. Then

$$(3.2.6) \quad \emptyset \neq U_\xi \cap U_\chi^c = \bigcap_{\eta \in F(\xi)} V_\eta \cap \left(\bigcap_{\zeta \in F(\chi)} V_\zeta \right)^c = \bigcup_{\zeta \in F(\chi)} \left(\bigcap_{\eta \in F(\xi)} V_\eta \cap V_\zeta^c \right).$$

Thus, for at least one $\zeta \in F(\chi)$, $(\bigcap_{\eta \in F(\xi)} V_\eta) \cap V_\zeta^c \neq \emptyset$. Fix such a ζ . By Lemma 3.2.4, $(\bigcup_{\eta \in F(\xi)} S_\eta) \cap F(\xi) = \emptyset$ and $\zeta \notin F(\xi)$.

By Lemma 3.2.3, there is a sequence of ordinals α_{ij} of type $\omega \cdot \omega$ such that $\forall i, i', j, j' < \omega$,

$$(3.2.7) \quad \alpha_{ij} < \alpha_{i'j'} \iff (i < i') \text{ or } (i = i' \text{ and } j < j')$$

$$(3.2.8) \quad \forall 0 < j < \omega \quad S_{\alpha_{ij}} = \{\zeta, \alpha_{ik} : k < j\}.$$

and

$$(3.2.9) \quad S_{\alpha_{i+1,0}} = \{\zeta, \alpha_{ij} : j < \omega\}.$$

$\forall 0 < i < \omega$, let $T_i = S_{\alpha_{i0}} \setminus \{\zeta\}$. Then the T_i 's are pairwise disjoint, and $\forall i < \omega$, $\text{type}(T_i) = \omega$. Let $\mathcal{I} = \{0 < i < \omega : T_i \not\subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha\}$. By (S3), $\text{type}(\bigcup_{\alpha \in F(\xi)} S_\alpha) \leq \omega \cdot |F(\xi)| < \omega \cdot \omega = \text{type}(\bigcup_{0 < i < \omega} T_i)$, so \mathcal{I} is infinite. Let $\mathcal{I}' = \{i \in \mathcal{I} : S_{\alpha_{i0}} \cap F(\xi) = \emptyset\}$. $|F(\xi)| < \omega$, and $\zeta \notin F(\xi)$; so \mathcal{I}' is infinite. Choose some $i \in \mathcal{I}'$ and $j < \omega$ for which $\alpha_{ij} \notin \bigcup_{\alpha \in F(\xi)} S_\alpha$. $S_{\alpha_{ij}} \cap F(\xi) = \emptyset$ and $\alpha_{ij} \notin \bigcup_{\alpha \in F(\xi)} S_\alpha$ imply $V_{\alpha_{ij}} \cap U_\xi \neq \emptyset$, by Lemma 3.2.4. Further, $V_{\alpha_{ij}} \cap V_\zeta = \emptyset$, since $\zeta \in S_{\alpha_{ij}}$. Let $q = V_{\alpha_{ij}} \cap U_\xi$. Then $q \leq U_\xi$ and $q \cap U_\chi \subseteq q \cap V_\zeta = \emptyset$. Thus \mathbf{P} is separative.

□

Galvin and Hajnal showed that \mathbf{P} satisfies the c.c.c. but not the σ -bounded c.c. (see [3]). Thus, by Propositions 2.4.2 and 2.4.4, $\text{r.o.}(\mathbf{P})$ is a complete, c.c.c. Boolean algebra in which the σ -bounded c.c. fails.

Throughout this section, we use the following notation. Let $e : \mathbf{P} \rightarrow \text{r.o.}(\mathbf{P})$ denote the canonical embedding of \mathbf{P} into $\text{r.o.}(\mathbf{P})$. We will frequently write $q \leq p$ to denote $q \subseteq p$ as subsets of Y .

The following fact will be used later in Proposition 3.2.10.

Fact 3.2.6. *If $\{p_i : i < \omega\} \subseteq \mathbf{P}$ is infinite, then $\bigwedge_{i < \omega} e(p_i) = \mathbf{0}$ in $\text{r.o.}(\mathbf{P})$.*

Proof: Clearly $\bigwedge_{i < \omega} e(p_i) \subseteq \bigcap_{i < \omega} e(p_i)$. $\forall i < \omega$ let $\xi_i < 2^\omega$ be such that $p_i = U_{\xi_i}$. If $q = U_\zeta \in \bigcap_{i < \omega} e(p_i)$, then $\forall i < \omega$, $q \leq p_i$. Thus, $F(\zeta) \supseteq \bigcup_{i < \omega} F(\xi_i)$, which is infinite since the p_i are distinct. But $F(\zeta)$ is finite. Contradiction. Thus, $\forall q \in \mathbf{P}$ $q \notin \bigcap_{i < \omega} e(p_i)$; so $\bigcap_{i < \omega} e(p_i) = \emptyset$. Hence, $\bigwedge_{i < \omega} e(p_i) = \emptyset = \mathbf{0}$ in $\text{r.o.}(\mathbf{P})$. \square

We now construct a countable, atomless, regular subalgebra \mathbf{C} of $\text{r.o.}(\mathbf{P})$.

Construction of \mathbf{C} : By Lemma 3.2.3, there is a sequence of type $\omega \cdot \omega$,

$$\begin{aligned} \alpha(0, 0) &< \alpha(0, 1) < \cdots < \lambda(0) < \alpha(1, 0) \\ &< \alpha(1, 1) < \cdots < \lambda(1) < \alpha(2, 0) < \alpha(2, 1) < \cdots \end{aligned}$$

such that $\forall i < \omega$, $0 < j < \omega$

$$\begin{aligned} (3.2.10) \quad S_{\alpha(i, j)} &= \{\alpha(i, k) : k < j\}; \\ S_{\lambda(i)} &= \{\alpha(i, j) : 0 < j < \omega\}. \end{aligned}$$

Note that the set $\{S_{\lambda(i)} : i < \omega\}$ is pairwise disjoint. Recall that $\forall i < \omega$, $0 < j < \omega$,

$$(3.2.11) \quad V_{\alpha(i, j)} = \{f \in Y : \forall k < j \ f(\alpha(i, k)) = 0, \text{ and } f(\alpha(i, j)) = 1\},$$

and

$$(3.2.12) \quad V_{\lambda(i)} = \{f \in Y : \forall 0 < j < \omega \ f(\alpha(i, j)) = 0, \text{ and } f(\lambda(i)) = 1\}.$$

We will use the set $\{V_{\alpha(i, j)} : i < \omega, 0 < j < \omega\}$ to construct a subalgebra of $\text{r.o.}(\mathbf{P})$ which is countable and atomless. The elements $V_{\lambda(i)}$, $i < \omega$ will be used later to ensure that the generators of \mathbf{C} , defined below, are independent.

$\forall i < \omega$, $0 < j < \omega$, let $u(i, j)$ denote $e(V_{\alpha(i, j)})$ and let $v(i)$ denote $e(V_{\lambda(i)})$. For each $i < \omega$, let

$$(3.2.13) \quad a_i = \bigvee_{0 < j < \omega} u(i, j)$$

in $\text{r.o.}(\mathbf{P})$. Let \mathbf{C} be the subalgebra of $\text{r.o.}(\mathbf{P})$ generated by $\{a_i : i < \omega\}$. Then \mathbf{C} is countable. We will show that \mathbf{C} is an atomless, regular subalgebra of $\text{r.o.}(\mathbf{P})$. In order to do so, the following facts will be useful.

Fact 3.2.7. $\forall i < \omega$, $a_i = \bigvee_{0 < j < \omega} u(i, j) = \bigcup_{0 < j < \omega} u(i, j)$.

Proof: First, note that

$$(3.2.14) \quad \begin{aligned} a_i &= \bigvee_{0 < j < \omega} u(i, j) \\ &= \text{int cl} \left(\bigcup_{0 < j < \omega} u(i, j) \right) \\ &= \{r \in \mathbf{P} : \forall q \leq r \exists 0 < j < \omega (q \cap V_{\alpha(i, j)} \neq \emptyset)\}. \end{aligned}$$

Since $\bigcup_{0 < j < \omega} u(i, j)$ is an open set in the order topology on \mathbf{P} , it is immediate that $a_i = \text{int cl} (\bigcup_{0 < j < \omega} u(i, j)) \supseteq \bigcup_{0 < j < \omega} u(i, j)$. To show inclusion, it suffices to show that $\forall p \in \mathbf{P}$, whenever $p \notin \bigcup_{0 < j < \omega} u(i, j)$, then $\exists q \leq p$ such that $\forall 0 < j < \omega$, $q \cap V_{\alpha(i, j)} = \emptyset$, since this will imply $p \notin \bigvee_{0 < j < \omega} u(i, j)$.

Suppose $U_\xi \in \mathbf{P}$ and $U_\xi \notin \bigcup_{0 < j < \omega} u(i, j)$. Then $\forall 0 < j < \omega$, $U_\xi \not\leq V_{\alpha(i, j)}$; so $\{\alpha(i, j) : 0 < j < \omega\} \cap F(\xi) = \emptyset$. Recall that $S_{\lambda(i)} = \{\alpha(i, j) : 0 < j < \omega\}$. By Lemma 3.2.2, $\exists \alpha < \beta < 2^\omega$ such that $\alpha > \sup(F(\xi) \cup \{\lambda_i\})$ and $S_\alpha \cup S_\beta = S_{\lambda(i)}$.

$$(3.2.15) \quad \{\alpha, \beta\} \cap \left(S_\alpha \cup S_\beta \cup \bigcup_{\eta \in F(\xi)} S_\eta \right) = \emptyset,$$

since $\alpha, \beta > \sup(F(\xi) \cup \{\lambda(i)\})$, and

$$(3.2.16) \quad F(\xi) \cap \left(S_\alpha \cup S_\beta \cup \bigcup_{\eta \in F(\xi)} S_\eta \right) = F(\xi) \cap S_{\lambda(i)} = \emptyset,$$

since we've already shown that $F(\xi) \cap S_{\lambda(i)} = \emptyset$, and $F(\xi) \cap (\bigcup_{\eta \in F(\xi)} S_\eta)$ must be empty, since $U_\xi \neq \emptyset$. Thus, Lemma 3.2.4 implies

$$(3.2.17) \quad V_\alpha \cap V_\beta \cap U_\xi \neq \emptyset.$$

Let $q = V_\alpha \cap V_\beta \cap U_\xi$. Then $q \leq U_\xi$. For each $0 < j < \omega$, $\forall f \in V_{\alpha(i,j)}$, $f(\alpha(i,j)) = 1$; whereas $\forall f \in q$, $f(\alpha(i,j)) = 0$. Thus, $\forall 0 < j < \omega$, $q \cap V_{\alpha(i,j)} = \emptyset$. Therefore, $U_\xi \notin \bigvee_{0 < j < \omega} u(i,j)$, by (3.2.14).

□

Fact 3.2.8. $\forall \xi < 2^\omega$, if $\bigcup_{\alpha \in F(\xi)} S_\alpha \supseteq S_{\lambda(i)}$, then $e(U_\xi) \leq -a_i$. In particular, $\forall i < \omega$, $v(i) \leq -a_i$.

Proof: If $\bigcup_{\alpha \in F(\xi)} S_\alpha \supseteq S_{\lambda(i)}$, then for each $0 < j < \omega$, $\forall f \in U_\xi$ $f(\alpha(i,j)) = 0$, whereas $\forall f \in V_{\alpha(i,j)}$, $f(\alpha(i,j)) = 1$. Thus, $\forall 0 < j < \omega$, $e(U_\xi) \wedge u(i,j) = 0$. Hence, $e(U_\xi) \wedge a_i = 0$.

□

Fact 3.2.9. For finite sets $I, J \subseteq \omega$, $\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} -a_j = 0$ iff $I \cap J \neq \emptyset$. That is, the generators of \mathbf{C} are independent.

Proof: Let $I, J \subseteq \omega$ be finite sets. Clearly, if $I \cap J \neq \emptyset$, then $\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} -a_j = 0$. On the other hand, if $I \cap J = \emptyset$, then Fact 3.2.8 and Lemma 3.2.4 imply

$$(3.18) \quad \bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} -a_j \geq \bigwedge_{i \in I} u(i,1) \wedge \bigwedge_{j \in J} v(j) \neq 0.$$

This follows from the fact that, for $r = \bigcap_{i \in I} V_{\alpha(i,1)} \cap \bigcap_{j \in J} V_{\lambda(j)}$, $r \neq \emptyset$, since

$$(3.2.19) \quad \left(\bigcup_{i \in I} S_{\alpha(i,1)} \cup \bigcup_{j \in J} S_{\lambda(j)} \right) \cap \left(\{ \alpha(i,1) : i \in I \} \cup \{ \lambda(j) : j \in J \} \right) = \emptyset,$$

and $e(r) \leq \bigwedge_{i \in I} u(i,1) \wedge \bigwedge_{j \in J} v(j)$.

□

By Fact 3.2.9, \mathbf{C} is atomless, since its generators are independent.

The next Proposition will aid us in constructing the elements a_p in Lemma 3.1.8.

Proposition 3.2.10. *Given $p \in \mathbf{P}$, there are at most finitely many $i < \omega$ for which either $e(p) \wedge a_i = \mathbf{0}$ or $e(p) \wedge -a_i = \mathbf{0}$.*

Proof: Let $U_\xi \in \mathbf{P}$ and let $J \subseteq \omega$ be given by $j \in J \leftrightarrow e(U_\xi) \wedge a_j = \mathbf{0}$. For each $j \in J$,

$$(3.2.20) \quad \mathbf{0} = e(U_\xi) \wedge a_j = e(U_\xi) \wedge \left(\bigvee_{0 < k < \omega} u(j, k) \right) = \bigvee_{0 < k < \omega} (e(U_\xi) \wedge u(j, k)).$$

Thus, $\forall j \in J, \forall 0 < k < \omega, e(U_\xi) \cap u(j, k) = \emptyset$, which in turn, implies $U_\xi \cap V_{\alpha(j, k)} = \emptyset$.

We claim that if $e(U_\xi) \wedge a_j = \mathbf{0}$, then either $\alpha(j, 0) \in F(\xi)$ or else $\{\alpha(j, k) : 0 < k < \omega\} \subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha$. $U_\xi \cap U_{\alpha(j, 1)} = \emptyset$ implies either $\alpha(j, 0) \in F(\xi)$ or else $\alpha(j, 1) \in \bigcup_{\alpha \in F(\xi)} S_\alpha$. Suppose $\alpha(j, 0) \notin F(\xi)$. Then $\alpha(j, 1) \in \bigcup_{\alpha \in F(\xi)} S_\alpha$. Now let $1 < k < \omega$ and assume $\{\alpha(j, l) : 0 < l < k\} \subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha$. $U_\xi \cap V_{\alpha(j, k)} = \emptyset$ implies $\forall f \in U_\xi, f(\alpha(j, 0)) = 1$ or ... or $f(\alpha(j, k-1)) = 1$ or $f(\alpha(j, k)) = 0$. Since $\{\alpha(j, l) : 0 < l < k\} \subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha$ and $\alpha(j, 0) \notin F(\xi)$, $\forall f \in U_\xi, f(\alpha(j, k))$ must equal 0. Thus, $\{\alpha(j, l) : 0 < l < k+1\} \subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha$. By induction, $\{\alpha(j, k) : 0 < k < \omega\} \subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha$.

$|F(\xi)| < \omega$ implies that $\alpha(j, 0) \in F(\xi)$ for at most finitely many $j < \omega$; by (S3), $\text{type}(\bigcup_{\alpha \in F(\xi)} S_\alpha) < \omega \cdot \omega$, which implies $\{\alpha(j, k) : 0 < k < \omega\} \subseteq \bigcup_{\alpha \in F(\xi)} S_\alpha$ for at most finitely many $j < \omega$. Thus, $|J| < \omega$.

Now let I be the subset of ω defined by $i \in I \leftrightarrow e(U_\xi) \wedge -a_i = \mathbf{0}$. $\forall i \in I, e(U_\xi) \leq a_i$, so using Fact 3.2.7, we have

$$(3.2.21) \quad e(U_\xi) \leq \bigwedge_{i \in I} a_i \leq \bigcap_{i \in I} a_i = \bigcap_{i \in I} \bigcup_{0 < k < \omega} u(i, k) = \bigcup_{g: I \rightarrow \omega \setminus \{0\}} \bigcap_{i \in I} u(i, g(i)).$$

If $|I| = \omega$, then it follows from Fact 3.2.6 that $\forall g : I \rightarrow \omega \setminus \{0\}, \bigcap_{i \in I} u(i, g(i)) = \emptyset$. Hence, $e(U_\xi) = \mathbf{0}$, contradicting $U_\xi \neq \emptyset$. Thus, $|I| < \omega$.

□

Now we use Proposition 3.2.10 to show that \mathbf{C} satisfies the conditions of Lemma 3.1.8.

Proposition 3.2.11. $\forall p \in \mathbf{P} \exists c_p \in \mathbf{C}^+$ such that $\forall c \in \mathbf{C}$, if $c_p \wedge c \neq \mathbf{0}$, then $e(p) \wedge c \neq \mathbf{0}$.

Proof: Let $p = U_\xi \in \mathbf{P}$. By Proposition 3.2.10, let $I, J \subseteq \omega$ be the finite disjoint sets given by $i \in I \leftrightarrow e(U_\xi) \wedge -a_i = \mathbf{0}$ and $j \in J \leftrightarrow e(U_\xi) \wedge a_j = \mathbf{0}$. If $I \cup J = \emptyset$, then let $c_p = \mathbf{1}$. If $I \cup J \neq \emptyset$, then let $c_p = \bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} -a_j$. Let $c = \bigwedge_{k \in K} a_k \wedge \bigwedge_{l \in L} -a_l \in \mathbf{C}$ be any element of \mathbf{C} such that $c_p \wedge c \neq \mathbf{0}$. By Fact 3.2.8, $I \cap L = J \cap K = \emptyset$.

Since $J \cap K = \emptyset$, $\forall k \in K \ e(U_\xi) \wedge a_k \neq \mathbf{0}$. This implies $\exists 0 < j_k < \omega$ such that $e(U_\xi) \cap u(k, j_k) \neq \mathbf{0}$, so $U_\xi \cap V_{\alpha(k, j_k)} \neq \emptyset$. Thus, by Lemma 3.2.4,

$$(3.2.22) \quad \left(F(\xi) \cup \{\alpha(k, j_k) : k \in K\} \right) \cap \left(\bigcup_{\alpha \in F(\xi)} S_\alpha \cup \{\alpha(k, j) : k \in K, j < j_k\} \right) = \emptyset.$$

Since $I \cap L = \emptyset$, $\forall l \in L \ e(U_\xi) \not\leq a_l$. Thus, $\forall l \in L \ \forall 0 < j < \omega$, $e(U_\xi) \not\leq u(l, j)$; so $\forall l \in L \ \forall 0 < j < \omega$, $U_\xi \not\leq V_{\alpha(l, j)}$, which implies $\alpha(l, j) \notin F(\xi)$. Therefore,

$$(3.2.23) \quad F(\xi) \cap \{\alpha(l, j) : l \in L, 0 < j < \omega\} = \emptyset.$$

Using Lemma 3.2.2, $\forall l \in L$ choose $\beta_l > \alpha_l > \sup(F(\xi) \cup \{\lambda(i) : i < \omega\} \cup \{\beta_i : i < l\})$ such that $S_{\alpha_l} \cup S_{\beta_l} = S_{\lambda(l)}$. Note that $e(V_{\alpha_l} \cap V_{\beta_l}) \wedge a_l = \mathbf{0}$, by Fact 3.2.8. Let

$$(3.2.24) \quad r = \bigcap_{k \in K} V_{\alpha(k, j_k)} \cap \bigcap_{l \in L} (V_{\alpha_l} \cap V_{\beta_l}).$$

$r \neq \emptyset$, by Lemma 3.2.4. $e(r) \leq c$, since $\forall k \in K \ u(k, j_k) \leq a_k$ and $\forall l \in L \ e(V_{\alpha_l} \cap V_{\beta_l}) \leq -a_l$. Furthermore, $r \cap U_\xi \neq \emptyset$, by Lemma 3.2.4, since (3.2.22) and (3.2.23) imply $\left(F(\xi) \cup \{\alpha(k, j_k) : k \in K\} \cup \{\alpha_l, \beta_l : l \in L\} \right) \cap \left(\bigcup_{\alpha \in F(\xi)} S_\alpha \cup \bigcup_{k \in K} \{\alpha(k, j) : j < j_k\} \cup \bigcup_{l \in L} S_{\lambda(l)} \right) = \emptyset$. Hence, $\mathbf{0} < e(r \cap U_\xi) \leq c$. Thus, $e(U_\xi) \wedge c \neq \mathbf{0}$.

□

It follows from Lemma 3.1.8 and Proposition 3.2.11 that \mathbf{C} is a regular subalgebra of $\text{r.o.}(\mathbf{P})$. Thus, by Lemma 3.1.11, $\text{r.o.}(\mathbf{C})$ embeds into $\text{r.o.}(\mathbf{P})$ as a complete subalgebra. This concludes the proof of the complete embedding of $\text{r.o.}(\mathbf{C})$ into $\text{r.o.}(\mathbf{P})$.

3.3. A COMPLETE EMBEDDING OF THE COHEN ALGEBRA INTO THE ARGYROS ALGEBRA

Argyros constructed the tree $T \subseteq [\omega]^2$ as follows [3]: Let $\{S_{nm} : n < \omega, 1 \leq m \leq 3^n\}$ be a family of sets such that $\forall n, m < \omega, S_{nm} \in [\omega]^3$ and $S_{nm} \cap S_{n'm'} = \emptyset$ whenever $\langle n, m \rangle \neq \langle n', m' \rangle$. For each $n < \omega$, let

$$(3.3.1) \quad \text{Lev}(n) = \bigcup_{1 \leq m \leq 3^n} [S_{nm}]^2.$$

For each $n < \omega$, index the elements of $\text{Lev}(n)$ so that $\text{Lev}(n) = \{s_{nj} : 1 \leq j \leq 3^{n+1}\}$. The tree $\langle T, \prec \rangle$ is defined as follows:

$$(3.3.2) \quad T = \bigcup_{n < \omega} \text{Lev}(n)$$

with the partial ordering \prec on T given by

$$(3.3.3) \quad \begin{aligned} s \prec t &\iff \exists n, j < \omega \text{ such that } s = s_{nj} \in \text{Lev}(n), t \in \text{Lev}(n+1), \\ &\text{and } t \in [S_{n+1,j}]^2. \end{aligned}$$

The partial ordering \mathbf{P} is constructed using three basic types of elements. For $X, Y \in [\omega]^{<\omega}$, let

$$(3.3.4) \quad B_X = \{f \in 2^\omega : \forall x \in X \ f(x) = 1\}$$

and let

$$(3.3.5) \quad \overline{B}_Y = \{f \in 2^\omega : \forall y \in Y \ f(y) = 0\}.$$

Let $\tilde{\Sigma}$ be the set of all finite and infinite branches of T , where all branches are assumed to start at $\text{Lev}(0)$. There are 2^ω many branches in T , so index them: $\tilde{\Sigma} = \{\Sigma(i) : i < 2^\omega\}$. For $s = \{k, l\} \in T$, let

$$(3.3.6) \quad \begin{aligned} K_s &= (B_{\{k\}} \cap \overline{B}_{\{l\}}) \cup (B_{\{l\}} \cap \overline{B}_{\{k\}}) \\ &= \{f \in 2^\omega : (f(k) = 1 \text{ and } f(l) = 0) \text{ or } (f(k) = 0 \text{ and } f(l) = 1)\}. \end{aligned}$$

In other words, K_s is the set of all functions in 2^ω which are non-constant on s . For $\Sigma(i) \in \tilde{\Sigma}$, let

$$(3.3.7) \quad A_i = \bigcap_{s \in \Sigma(i)} K_s.$$

That is, A_i is the set of all functions in 2^ω which are non-constant on every element $s \in \Sigma(i)$.

The *Argyros partial ordering* $\langle \mathbf{P}, \subseteq \rangle$ is the collection of all finite, non-empty intersections of elements of the three forms B_X, \overline{B}_Y , and A_i :

$$(3.3.8) \quad \mathbf{P} = \left\{ B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i : X, Y \in [\omega]^{<\omega} \text{ and } I \in [2^\omega]^{<\omega} \right\} \setminus \{\emptyset\}.$$

Definition 3.3.1. Let $s, t \in T$. s and t are *siblings* if they have the same immediate predecessor; that is, if $\exists u \in T$ such that s and t are both immediate successors of u . $s, t, u \in T$ are *triplets* if they all have the same immediate predecessor.

Remark. If s and t are siblings, then there exist unique $m, n \in \omega$ such that $s, t \in [S_{mn}]^2$ and $K_s \cap K_t \neq \emptyset$. If s, t, u are triplets, then there exist unique $m, n \in \omega$ such that $\{s, t, u\} = [S_{mn}]^2$, $s = t \Delta u$ (the set-theoretic difference of t and u in ω), and $K_s \cap K_t \cap K_u = \emptyset$.

Note. The elements of \mathbf{P} are not uniquely represented by the form $B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i$. For instance, if $s = \{k, l\} \in \Sigma(i)$, then

$$(3.3.9) \quad B_{\{k\}} \cap A_i = B_{\{k\}} \cap \overline{B}_{\{l\}} \cap A_i = \overline{B}_{\{l\}} \cap A_i.$$

We shall hold to the following convention: given $S \subseteq T$ and $X, Y \in [\omega]^{<\omega}$, the representation $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ of a subset of 2^ω is said to be in the *normal form* if and only if $X \cap Y = \emptyset$, and $\forall s \in S, s \cap (X \cup Y) = \emptyset$.

It follows that if $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ is the normal form representation of $B_U \cap \overline{B}_V \cap \bigcap_{i \in I} A_i \neq \emptyset$, then $X \supseteq U, Y \supseteq V, X \cap Y = \emptyset, S \subseteq \bigcup_{i \in I} \Sigma(i)$, and S contains no triplets. It is not hard to see that for each element $p \in \mathbf{P}$ there is a unique normal form representation of p .

We now give conditions for testing whether or not a given $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s \subseteq 2^\omega$ is empty.

Lemma 3.3.2. *Let $p = B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ be a subset of 2^ω , not necessarily in the normal form. Then $p \neq \emptyset$ iff the following four conditions hold:*

- (L1) $X \cap Y = \emptyset$;
- (L2) $\forall s \in S, s \not\subseteq X$ and $s \not\subseteq Y$;
- (L3) S does not contain any triplets.
- (L4) If $s, t \in S$ are siblings, then either $(s \Delta t) \cap X = \emptyset$ or $(s \Delta t) \cap Y = \emptyset$

In particular, if $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ is in normal form, then $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s \neq \emptyset$.

Proof: Let p denote $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$, not necessarily in the normal form. If (L1) fails, then $p \subseteq B_X \cap \overline{B}_Y = \emptyset$. Suppose (L2) fails. Then $\exists s \in S$ such that either $s \subseteq X$ or $s \subseteq Y$. If $s \subseteq X$, then $p \subseteq B_s \cap K_s = \emptyset$. If $s \subseteq Y$, then $p \subseteq \overline{B}_s \cap K_s = \emptyset$. If (L3) fails, then there are triplets $s, t, u \in S$; so $p \subseteq K_s \cap K_t \cap K_u = \emptyset$. If (L4) fails, then $\exists s, t$ siblings in S for which $(s \Delta t) \cap X \neq \emptyset$ and $(s \Delta t) \cap Y \neq \emptyset$. Let r denote the mutual sibling of s and t . Note that $r = s \Delta t$. Let $k, l \in \omega$ for which $\{k, l\} = r, k \in r \cap X$, and $l \in r \cap Y$. Then $p \subseteq K_s \cap K_t \cap B_{\{k\}} \cap \overline{B}_{\{l\}} \subseteq K_s \cap K_t \cap K_r = \emptyset$, since r, s, t are triplets. Hence, if one of (L1)-(L4) fail, then $p = \emptyset$.

Now suppose $p = B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ and (L1)-(L4) hold. By (L3), S contains no triplets, so we can divide S into two disjoint subsets as follows: Let $\Theta = \{s \in S : s \text{ has a sibling in } S\}$ and let $\Gamma = \{s \in S : s \text{ has no sibling in } S\}$.

We shall show that $p \neq \emptyset$. $B_X \cap \overline{B}_Y \neq \emptyset$, since (L1) implies $X \cap Y = \emptyset$. Let $s = \{k, l\} \in \Gamma$. Since s has no siblings in S , we only need to check how s interacts with X and Y . By (L2), $s \not\subseteq X$ and $s \not\subseteq Y$. We have 3 cases. If $k \in X$ then $l \notin X$, since $s \not\subseteq X$; let $h_s = B_X \cap \overline{B}_{Y \cup \{l\}}$. If $k \in Y$, then $l \notin Y$, since $s \not\subseteq Y$; let $h_s = B_{X \cup \{l\}} \cap \overline{B}_Y$. If $k \notin X \cup Y$, then it does not matter where l lies; for $l \in X$ let $h_s = B_X \cap \overline{B}_{Y \cup \{k\}}$ and for $l \notin X$ let $h_s = B_{X \cup \{k\}} \cap \overline{B}_{Y \cup \{l\}}$. In each of these three cases, $h_s \neq \emptyset$. Moreover, $B_X \cap \overline{B}_Y \cap \bigcap_{s \in \Gamma} h_s \neq \emptyset$, since all $s \in \Gamma$ are disjoint.

For each pair of siblings $s = \{k, l\}$, $t = \{k, m\} \in \Theta$ we only need to check how $s \cup t$ interacts with $X \cup Y$. If $k \in X$, then $l, m \notin X$, since $s, t \not\subseteq X$ by (L2); so let $h_{s,t} = B_X \cap \overline{B}_{Y \cup \{l, m\}}$. Similarly, if $k \in Y$, then $l, m \notin Y$ by (L2); so let $h_{s,t} = B_{X \cup \{l, m\}} \cap \overline{B}_Y$. Now suppose $k \notin X \cup Y$. If $l \in X$, then $(s \Delta t) \cap X \neq \emptyset$, so (L4) implies $(s \Delta t) \cap Y = \emptyset$. Hence, $m \notin Y$; so let $h_{s,t} = B_{X \cup \{m\}} \cap \overline{B}_{Y \cup \{k\}}$. Similarly, if $l \in Y$, then $m \notin X$; so let $h_{s,t} = B_{X \cup \{k\}} \cap \overline{B}_{Y \cup \{m\}}$. If $l \notin X \cup Y$, then it does not matter where m lies; if $m \in X$, let $h_{s,t} = B_{X \cap \{l\}} \cap \overline{B}_{Y \cup \{k\}}$ and if $m \notin X$, let $h_{s,t} = B_{X \cup \{k\}} \cap \overline{B}_{Y \cup \{m, l\}}$. In each of these cases, $h_{s,t} \neq \emptyset$. Let $\tilde{\Theta} = \{(s, t) : s, t \in \Theta \text{ \& } s, t \text{ are siblings}\}$. $\forall (s, t) \neq (u, v) \in \tilde{\Theta}, (s \cup t) \cap (u \cup v) = \emptyset$. Thus, $\bigcap_{(s, t) \in \tilde{\Theta}} h_{s,t} \neq \emptyset$.

The set of siblings of elements of Γ is disjoint from the set of siblings of elements of Θ . Hence, $p \supseteq B_X \cap \overline{B}_Y \cap \bigcap_{s \in \Gamma} h_s \cap \bigcap_{(s, t) \in \tilde{\Theta}} h_{s,t} \neq \emptyset$.

Note that if $p = B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ in the normal form, then (L1)-(L4) are satisfied. Hence, $p \neq \emptyset$.

□

Proposition 3.3.3. *P is separative.*

Suppose $p, q \in \mathbf{P}$ and $q \not\subseteq p$. Let $B_X \cap \overline{B}_Y \cap \bigcap_{r \in R} K_r$ and $B_U \cap \overline{B}_V \cap \bigcap_{s \in S} K_s$ be the normal forms of p and q , respectively. $q \subseteq p$ if and only if $X \subseteq U$, $Y \subseteq V$, and $\forall r \in R$ either $r \in S$ or else r is split by U and V . Hence, $q \not\subseteq p$ implies either (1) $X \not\subseteq U$, or (2) $Y \not\subseteq V$, or (3) $\exists r \in R$ such that $r \notin S$ and either $r \cap U = \emptyset$ or $r \cap V = \emptyset$.

Case 1: Suppose $U \not\supseteq X$. Choose some $k \in X \setminus U$ and let $w = q \cap \overline{B}_{\{k\}} = B_U \cap \overline{B}_{V \cup \{k\}} \cap \bigcap_{s \in S} K_s$. Note that $w \cap p \subseteq \overline{B}_{\{k\}} \cap B_{\{k\}} = \emptyset$. We will show that $w \neq \emptyset$. If $k \in V$, then $w = q$ and we are done, so assume $k \notin V$. Since $q \neq \emptyset$, Lemma 3.3.2 implies (L1)-(L4) hold for $B_U \cap \overline{B}_V \cap \bigcap_{s \in S} K_s$. There are four possible subcases.

subcase (i): $\forall s \in S, k \notin s$. Then $B_U \cap \overline{B}_{V \cup \{k\}} \cap \bigcap_{s \in S} K_s$ is the normal form for w .

subcase (ii): There exists exactly one $t \in S$ such that $k \in t$, and t has no siblings in S . Suppose $t = \{k, l\}$. $t \cap (U \cup V) = \emptyset$, since q is given in the normal form. Hence, $B_{U \cup \{l\}} \cap \overline{B}_{V \cup \{k\}} \cap \bigcap_{s \in S \setminus \{t\}} K_s$ is the normal form for w . This is because $(U \cup \{l\}) \cap (V \cup \{k\}) = \emptyset$ (since $k \notin U$ and $l \notin V$) and $(U \cup V \cup \{k, l\}) \cap (S \setminus \{t\}) = \emptyset$.

subcase (iii): There is exactly one $t \in S$ containing k , and t has one sibling $t' \in S$ which does not contain k . Suppose $t = \{k, l\}$ and $t' = \{l, m\}$. $B_{U \cup \{l\}} \cap \overline{B}_{V \cup \{k, m\}} \cap \bigcap_{s \in S \setminus \{t, t'\}} K_s$ is the normal form for w : $t, t' \in S \implies (t \cup t') \cap (U \cup V) = \emptyset$, and $\forall s \in S \setminus \{t, t'\}, s \cap (U \cup V \cup \{k, l, m\}) = \emptyset$, (since $q = B_U \cap \overline{B}_V \cap \bigcap_{s \in S} K_s$ is in normal form).

subcase (iv): There are two siblings in S which both contain k , say $t = \{k, l\}$ and $t' = \{k, m\}$. In this case, $B_{U \cup \{l, m\}} \cap \overline{B}_{V \cup \{k\}} \cap \bigcap_{s \in S \setminus \{t, t'\}} K_s$ is the normal form for w . This follows from the facts that $t, t' \in S \implies (U \cup V) \cap (t \cup t') = \emptyset$, and $(U \cup V) \cap (\bigcup S) = \emptyset \implies (U \cup V \cup t \cup t') \cap (\bigcup (S \setminus \{t, t'\})) = \emptyset$.

In each of these subcases, w has a normal form. Hence, by Lemma 3.3.2, $w \neq \emptyset$.

Case 2: $V \not\supseteq Y$. Let $k \in Y \setminus V$ and let $w = q \cap B_{\{k\}}$. Analogously to Case 1, $w \neq \emptyset$ and $w \cap p = \emptyset$.

Case 3: $\exists r \in R \setminus S$ such that r is not split by U and V (i.e. either $r \cap U = \emptyset$ or $r \cap V = \emptyset$). Suppose $r = \{k, l\}$. We have three possible subcases.

subcase (i): $\exists s, t \in S$ such that r, s, t are triplets. Then $p \cap q = \emptyset$ and we are done.

subcase (ii): $r = \{k, l\}$ and there is exactly one $t = \{k, m\}$ a sibling of r in S . $k, m \notin U \cup V$, since q in normal form implies $t \cap (U \cup V) = \emptyset$. If $r \cap U = \emptyset$, let $w = q \cap \overline{B}_r$. Then $B_{U \cup \{m\}} \cap \overline{B}_{V \cup \{k, l\}} \cap \bigcap_{s \in S \setminus \{t\}} K_s$ is the normal form for w ; moreover, $w \cap p \subseteq \overline{B}_r \cap K_r = \emptyset$. If $r \cap V = \emptyset$, let $w = q \cap B_r$. Then $B_{U \cup \{k, l\}} \cap \overline{B}_{V \cup \{m\}} \cap \bigcap_{s \in S \setminus \{t\}} K_s$ is the normal form for w ; and $w \cap p = \emptyset$.

subcase (iii): r has no siblings in S . Then $\forall s \in S, s \cap r = \emptyset$. We know that either $r \cap U = \emptyset$ or $r \cap V = \emptyset$. If $r \cap U = \emptyset$, let $w = q \cap \overline{B}_r$. Then $B_U \cap \overline{B}_{V \cup r} \cap \bigcap_{s \in S} K_s$ is the normal form for w ; and $w \cap p = \emptyset$. If $r \cap V = \emptyset$, let $w = q \cap B_r$. Then $B_{U \cup r} \cap \overline{B}_V \cap \bigcap_{s \in S} K_s$ is the normal form for w ; and $w \cap p = \emptyset$.

Hence, in each case, we found the normal form for some $w \subseteq q$ for which $w \cap p = \emptyset$. By Lemma 3.3.2, $\emptyset \neq w \in \mathbf{P}$. Therefore, \mathbf{P} is separative.

□

Argyros showed that \mathbf{P} satisfies the σ -bounded chain condition, but $\text{CUP}(\mathbf{P})$ fails (see [3]). Hence, $\text{r.o.}(\mathbf{P})$ satisfies the σ -bounded c.c., but is not a measurable algebra. We shall show that the Cohen algebra embeds into $\text{r.o.}(\mathbf{P})$ as a complete subalgebra. To do so, we first construct a countable, atomless subalgebra \mathbf{C} of $\text{r.o.}(\mathbf{P})$ and show that \mathbf{C} is a regular subalgebra of $\text{r.o.}(\mathbf{P})$.

Construction of \mathbf{C} : Choose an infinite branch of T and call it β . $\forall n < \omega$, let t_n be the unique element of T such that $\{t_n\} = \beta \cap \text{Lev}(n)$. $\forall n < \omega$, choose one $s_{n+1} \in T$ such that $s_{n+1} \neq t_{n+1}$ and s_{n+1} is an immediate successor of t_n . Note that $s_{n+1} \in \text{Lev}(n+1)$, $s_{n+1} \notin \beta$, and s_{n+1} and t_{n+1} are siblings. $\forall 0 < n < \omega$, let β_n be an infinite branch in T which contains s_n . For $0 < m < n$, $\beta_m \cap \beta_n = \{t_0, \dots, t_m\}$. Without loss of generality, we can re-index the elements of $\bar{\Sigma}$ so that $\forall n < \omega$ $\beta_n = \Sigma(n)$, where $\beta_0 = \beta = \Sigma(0)$.

Let

$$(3.3.10) \quad \mathcal{T} = \{t \in T : \exists s \in \bigcup_{n < \omega} \Sigma(n) \text{ such that } t \text{ and } s \text{ are siblings}\},$$

and let

$$(3.3.11) \quad \mathcal{S} = \bigcup \mathcal{T} = \{k < \omega : \exists l < \omega \text{ such that } \{k, l\} \in \mathcal{T}\}.$$

Let

$$(3.3.12) \quad \mathcal{C} = \{B_X \cap \bar{B}_Y \cap \bigcap_{i \in I} A_{\Sigma(i)} \in \mathbf{P} : X, Y \in [\mathcal{S}]^{<\omega}, I \in [2^\omega]^{<\omega}, \\ \text{and } \forall i \in I (\Sigma(i) \subseteq \mathcal{T}) \setminus \{\emptyset\}\}.$$

Let $e : \mathbf{P} \rightarrow \text{r.o.}(\mathbf{P})$ be the canonical embedding of \mathbf{P} into its completion and let

$$(3.3.13) \quad \mathbf{C} = \langle \{e(p) : p \in \mathcal{C}\} \rangle,$$

the subalgebra of $\text{r.o.}(\mathbf{P})$ generated by the set $\{e(p) : p \in \mathcal{C}\}$. Note that $|\mathbf{C}| = \omega$, since $|\mathcal{S}| = \omega$ and there are only countably many finite and infinite branches in \mathcal{T} .

We shall now give two easy, but useful facts.

Fact 3.3.4. $\forall x \in \omega$, $-e(B_{\{x\}}) = e(\bar{B}_{\{x\}})$.

Proof: $\bar{B}_{\{x\}} \cap B_{\{x\}} = \emptyset$, so $e(\bar{B}_{\{x\}}) \wedge e(B_{\{x\}}) = \mathbf{0}$ in $\text{r.o.}(\mathbf{P})$. Since $\bar{B}_{\{x\}} \cup B_{\{x\}} = 2^\omega$, $\forall p \in \mathbf{P}$ either $p \cap B_{\{x\}} \neq \emptyset$ or else $p \cap \bar{B}_{\{x\}} \neq \emptyset$, so $e(p) \wedge (e(\bar{B}_{\{x\}}) \vee e(B_{\{x\}})) \neq \mathbf{0}$. Therefore, $e(B_{\{x\}}) \vee e(\bar{B}_{\{x\}}) = \mathbf{1}$. Thus, $-e(B_{\{x\}}) = e(\bar{B}_{\{x\}})$.

□

Fact 3.3.5. $\forall p, q \in \mathbf{P}, e(p) \wedge e(q) = e(p \cap q)$.

Proof: Let $p, q \in \mathbf{P}$. Clearly, $e(p \cap q) \leq e(p)$ and $e(p \cap q) \leq e(q)$, so $e(p \cap q) \leq e(p) \wedge e(q)$. Suppose $r \in \mathbf{P}$ and $e(r) \leq e(p) \wedge e(q)$. Then $e(r) \leq e(p)$ and $e(r) \leq e(q)$, so $r \subseteq p$ and $r \subseteq q$. Thus, $r \subseteq p \cap q$, which implies that $e(r) \leq e(p \cap q)$. Since this holds for all r such that $e(r) \leq e(p) \wedge e(q)$, $e(p) \wedge e(q) \leq e(p \cap q)$.

□

Let

$$\begin{aligned} \mathbf{D} &= \{e(p) : p \in \mathcal{C}\} \\ (3.3.14) \quad &= \{e(B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i) : X, Y \in [\mathcal{S}]^{<\omega}, I \in [2^\omega]^{<\omega}, \\ &\text{and } \forall i \in I (\Sigma(i) \subseteq \mathcal{T})\} \setminus \{0\}. \end{aligned}$$

Proposition 3.3.6. \mathbf{D} is dense in \mathbf{C}^+ .

Proof: First of all, every element of \mathbf{C}^+ is a finite disjunction of elements of the form

$$(3.3.15) \quad \bigwedge_{i \in I} e(B_{X_i} \cap \overline{B}_{Y_i} \cap \bigcap_{m \in M_i} A_m) \wedge \bigwedge_{j \in J} -e(B_{X_j} \cap \overline{B}_{Y_j} \cap \bigcap_{n \in N_j} A_n),$$

where I, J are finite, disjoint index sets, $\forall k \in I \cup J (X_k, Y_k \in [\mathcal{S}]^{<\omega})$, $\forall i \in I (M_i \in [2^\omega]^{<\omega})$, $\forall m \in M_i (\Sigma(m) \subseteq \mathcal{T})$, $\forall j \in J (N_j \in [2^\omega]^{<\omega})$, and $\forall n \in N_j (\Sigma(n) \subseteq \mathcal{T})$.

By Fact 3.3.5,

$$(3.3.16) \quad \bigwedge_{i \in I} e(B_{X_i} \cap \overline{B}_{Y_i} \cap \bigcap_{m \in M_i} A_m) = e(B_X \cap \overline{B}_Y \cap \bigcap_{k \in K} A_k),$$

where $X = \bigcup_{i \in I} X_i$, $Y = \bigcup_{i \in I} Y_i$, and $K = \bigcup_{i \in I} M_i$. For each $j \in J$, Fact 3.3.5 implies

$$(3.3.17) \quad e(B_{X_j} \cap \overline{B}_{Y_j} \cap \bigcap_{n \in N_j} A_n) = e(B_{X_j}) \wedge e(\overline{B}_{Y_j}) \wedge \bigwedge_{n \in N_j} e(A_n),$$

so

(3.3.18)

$$\begin{aligned}
 -e(B_{X_j} \cap \overline{B}_{Y_j} \cap \bigcap_{n \in N_j} A_n) &= -e(B_{X_j}) \vee -e(\overline{B}_{Y_j}) \vee \left(\bigvee_{n \in N_j} -e(A_n) \right) \\
 &= -\left(\bigwedge_{x \in X_j} e(B_{\{x\}}) \right) \vee -\left(\bigwedge_{y \in Y_j} e(\overline{B}_{\{y\}}) \right) \vee \bigvee_{n \in N_j} -e(A_n) \\
 &= \bigvee_{x \in X_j} -e(B_{\{x\}}) \vee \bigvee_{y \in Y_j} -e(\overline{B}_{\{y\}}) \vee \bigvee_{n \in N_j} -e(A_n) \\
 &= \bigvee_{x \in X_j} e(\overline{B}_{\{x\}}) \vee \bigvee_{y \in Y_j} e(B_{\{y\}}) \vee \bigvee_{n \in N_j} -e(A_n),
 \end{aligned}$$

by Facts 3.3.4 and 3.3.5. Combining (3.3.15), (3.3.16) and (3.3.18), we see that every element of \mathbf{C}^+ is a finite disjunction of elements of the form

$$(3.3.19) \quad e(B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i) \wedge \bigwedge_{j \in J} -e(A_j),$$

where $X, Y \in [S]^{<\omega}$; $I, J \in [2^\omega]^{<\omega}$; and $\forall k \in I \cup J, \Sigma(k) \subseteq \mathcal{T}$.

Let $c = e(B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i) \wedge (\bigwedge_{j \in J} -e(A_j)) \in \mathbf{C}^+$. $e[\mathbf{P}]$ is dense in $\text{r.o.}(\mathbf{P})$, so $\exists p \in \mathbf{P}$ such that $e(p) \leq c$. $\forall j \in J, e(p) \wedge e(A_j) = \mathbf{0} \implies p \cap A_j = \emptyset \implies p \subseteq A_j^c = \bigcup_{s \in \Sigma(j)} (B_s \cup \overline{B}_s)$. Let $f \in p \subseteq 2^\omega$. Then $f \in B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i$ and $\forall j \in J, f \in \bigcup_{s \in \Sigma(j)} (B_s \cup \overline{B}_s)$. For each $j \in J$ choose an $s_j \in \Sigma(j)$ for which $f \in (B_{s_j} \cup \overline{B}_{s_j})$. Let $j \in J'$ if $f \in B_{s_j}$ and let $j \in J''$ if $f \in \overline{B}_{s_j}$. Let

$$(3.3.20) \quad q = B_{X \cup (\bigcup \{s_j : j \in J'\})} \cap \overline{B}_{Y \cup (\bigcup \{s_j : j \in J''\})} \cap \bigcap_{i \in I} A_i.$$

$f \in q$, so $q \neq \emptyset$. $X \cup (\bigcup \{s_j : j \in J'\}) \cup Y \cup (\bigcup \{s_j : j \in J''\}) \subseteq S$ and $\bigcup_{i \in I} \Sigma(i) \subseteq \mathcal{T} \implies q \in \mathcal{C}$; so $e(q) \in \mathbf{D}$. Clearly $\forall j \in J', e(B_{s_j}) \leq -e(A_j)$ and $\forall j \in J'', e(\overline{B}_{s_j}) \leq -e(A_j)$. Hence, $e(q) \leq c$. Therefore, \mathbf{D} is dense in \mathbf{C}^+ .

□

Proposition 3.3.7. \mathbf{C} is atomless.

Proof: By Proposition 3.3.6, \mathbf{D} is dense in \mathbf{C}^+ , so it suffices to show that \mathbf{D} is atomless. Let $c = e(B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i) \in \mathbf{D}$, where $p = B_X \cap \overline{B}_Y \cap \bigcap_{i \in I} A_i$

is in the normal form. Choose some $z \in S$ with the properties that $z \notin X \cup Y$, z is not in any element of $\bigcup_{i \in I} \Sigma(i)$, and z is not in any sibling of any element of $\bigcup_{i \in I} \Sigma(i)$. Let $d = c \wedge e(B_{\{z\}})$. $d \neq 0$ by the way z was chosen (by Lemma 3.3.2).

Now, $-e(B_{\{z\}}) \wedge c = e(\overline{B}_{\{z\}}) \wedge c \neq 0$, since $z \notin (X \cup Y \cup \bigcup_{i \in I} (\bigcup \Sigma(i)))$ and p is in the normal form. However, $-e(B_{\{z\}}) \wedge d = e(\overline{B}_{\{z\}}) \wedge d \leq e(\overline{B}_{\{z\}} \cap B_{\{z\}}) = 0$. Thus, $0 \neq -e(B_{\{z\}}) \wedge c \leq c \setminus d$. Hence, $d < c$. Therefore, \mathbf{D} is atomless.

□

Proposition 3.3.8. $\forall p \in \mathbf{P}, \exists c_p \in \mathbf{C}^+$ such that $\forall c \in \mathbf{C}$, if $c_p \wedge c \neq 0$, then $e(p) \wedge c \neq 0$.

Proof: Let $p = B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s \in \mathbf{P}$ in normal form. Let $X' = X \cap S$, $Y' = Y \cap S$, and let $S' = S \cap \mathcal{T}$. Let $q = B_{X'} \cap \overline{B}_{Y'} \cap \bigcap_{s' \in S'} K_{s'}$ and let $c_p = e(q)$. $c_p \in \mathbf{C}$, since $X', Y' \in [S]^{<\omega}$, $S' \subseteq \mathcal{T}$, and S' is a finite union of branches in \mathcal{T} since S is a finite union of branches in T . Note that $c_p \geq e(p) \neq 0$ and that $B_{X'} \cap \overline{B}_{Y'} \cap \bigcap_{s' \in S'} K_{s'}$ is the normal form of q . Since \mathbf{D} is dense in \mathbf{C}^+ , it suffices to show that $\forall d \in \mathbf{D}, c_p \wedge d \neq 0 \implies e(p) \wedge d \neq 0$.

Let $d \in \mathbf{D}$ and let $r \in \mathcal{C}$ be such that $d = e(r)$. Let $B_U \cap \overline{B}_V \cap \bigcap_{w \in W} K_w$ be the normal form for r . Suppose $c_p \wedge d \neq 0$. Then

$$\begin{aligned}
 q \cap r &= B_{X'} \cap \overline{B}_{Y'} \cap \bigcap_{s' \in S'} K_{s'} \cap B_U \cap \overline{B}_V \cap \bigcap_{w \in W} K_w \\
 (3.3.21) \quad &= B_{X' \cup U} \cap \overline{B}_{Y' \cup V} \cap \bigcap_{t \in S' \cup W} K_t \\
 &\neq \emptyset.
 \end{aligned}$$

By Lemma 3.3.2,

$$(3.3.22) \quad (X' \cup U) \cap (Y' \cup V) = \emptyset;$$

$$(3.3.23) \quad \forall s \in S' \cup W, \quad s \not\subseteq X' \cup U \quad \text{and} \quad s \not\subseteq Y' \cup V;$$

(3.3.24) $S' \cup W$ has no triplets;

(3.3.25)

s, t are siblings in $S' \cup W \implies ((s\Delta t) \cap (X' \cup U) = \emptyset \text{ or } (s\Delta t) \cap (Y' \cup V) = \emptyset)$.

We will show that $e(p) \wedge d \neq 0$.

$$\begin{aligned}
 (3.3.26) \quad p \cap r &= B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s \cap B_U \cap \overline{B}_V \cap \bigcap_{w \in W} K_w \\
 &= B_{((X \setminus X') \cup X' \cup U)} \cap \overline{B}_{((Y \setminus Y') \cup Y' \cup V)} \cap \bigcap_{s \in (S \setminus S') \cup S' \cup W} K_s.
 \end{aligned}$$

Note: $(X \setminus X') \cap S = (Y \setminus Y') \cap S = \emptyset$; $X' \cup U, Y' \cup V \subseteq S$; $(S \setminus S') \cap T = \emptyset$; and $S' \cup W \subseteq T$. Since T contains all siblings of all elements of itself, if $a = B_{X \setminus X'} \cap \overline{B}_{Y \setminus Y'} \cap \bigcap_{s \in S \setminus S'} K_s \neq \emptyset$ and $b = B_{X' \cup U} \cap \overline{B}_{Y' \cup V} \cap \bigcap_{s \in S' \cup W} K_s \neq \emptyset$, then $a \cap b \neq \emptyset$, since $a \cap b$ satisfies the properties of Lemma 3.3.2.

We claim that (L1)-(L4) hold for the last expression of (3.3.26). (L1): $X \cap Y = \emptyset$ since p is in normal form; $U \cap V = \emptyset$ since r is in normal form; $U, V \subseteq S \implies (U \cup V) \cap ((X \setminus X') \cup (Y \setminus Y')) = \emptyset$; and $X' \cap V = Y' \cap U = \emptyset$ by (3.3.22). Hence, $(X \cup U) \cap (Y \cup V) = \emptyset$.

(L2): Let $s \in S \cup W$. Either $s \in S' \cup W$ or else $s \in S \setminus S'$. For $s \in S' \cup W$, $s \not\subseteq X' \cup U$ and $s \not\subseteq Y' \cup V$ by (3.3.23). Moreover, $s \cap ((X \setminus X') \cup (Y \setminus Y')) = \emptyset$, since $s \subseteq S$. For $s \in S \setminus S'$, Lemma 3.3.2 implies $s \not\subseteq X$ and $s \not\subseteq Y$, since $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ is the normal form of p . Moreover, $s \cap (X' \cup Y' \cup U \cup V) = \emptyset$, since $s \cap S = \emptyset$. Hence, $\forall s \in S \cup W$, $s \not\subseteq X \cup U$ and $s \not\subseteq Y \cup V$.

(L3): $S' \cup W$ contains no triplets, by (3.3.24). $S \setminus S'$ contains no triplets, since q is in normal form. $S \setminus S' \subseteq T \setminus T$ and $S' \cup W \subseteq T$, so $S \setminus S'$ and $S' \cup W$ have no common siblings. Hence, $S \cup W$ contains no triplets.

(L4): Suppose s, t are siblings in $S \cup W$. Either $s, t \in S \setminus S'$ or else $s, t \in S' \cup W$. Suppose $s, t \in S \setminus S'$. Then $(s \cup t) \cap S = \emptyset$, so $(s \Delta t) \cap (X' \cup Y' \cup U \cup V) = \emptyset$. Further, $B_X \cap \overline{B}_Y \cap \bigcap_{s \in S} K_s$ is the normal form of p , so (L4) of Lemma 3.3.2 implies either

$(s\Delta t) \cap X = \emptyset$ or else $(s\Delta t) \cap Y = \emptyset$. Thus, either $(s\Delta t) \cap (X \cup U) = \emptyset$ or else $(s\Delta t) \cap (Y \cup V) = \emptyset$.

Otherwise, $s, t \in S' \cup W \subseteq \mathcal{T}$. Then $(s \cup t) \cap ((X \setminus X') \cup (Y \setminus Y')) = \emptyset$. By (3.3.25), either $(s\Delta t) \cap (X' \cup U) = \emptyset$ or else $(s\Delta t) \cap (Y' \cup V) = \emptyset$. Thus, either $(s\Delta t) \cap (X \cup U) = \emptyset$ or else $(s\Delta t) \cap (Y \cup V) = \emptyset$.

Thus, $p \cap r$ satisfies (L1)-(L4), so Lemma 3.3.2 implies $p \cap r \neq \emptyset$. Thus, $e(p) \wedge d = e(p \cap r) \neq \mathbf{0}$.

□

By Proposition 3.3.8, Lemma 3.1.8, and Lemma 3.1.11, $\text{r.o.}(\mathbf{C})$ is a complete subalgebra of \mathbf{B} . This completes our construction of a complete embedding of the Cohen algebra into the Argyros algebra.

3.4. INTRODUCTION TO THE GAIFMAN ALGEBRA

Gaifman constructed the following Boolean algebra in which the σ -bounded c.c. holds, but $\text{CUP}(\mathbf{B}^+)$ fails. First, one starts with the clopen subsets of $2^{(0,1)}$, which we shall denote as $\text{Clop}(2^{(0,1)})$. For $X, Y \in [(0, 1)]^{<\omega}$, let

$$(3.4.1) \quad B_X = \{f \in 2^{(0,1)} : \forall x \in X \ f(x) = 1\}$$

and let

$$(3.4.2) \quad \overline{B}_Y = \{f \in 2^{(0,1)} : \forall y \in Y \ f(y) = 0\}.$$

Note that the set

$$(3.4.3) \quad \{B_X \cap \overline{B}_Y : X, Y \in [(0, 1)]^{<\omega} \text{ and } X \cap Y = \emptyset\}$$

is dense in $\text{Clop}(2^{(0,1)})$.

Let $\{T_i : 2 \leq i < \omega\}$ be an enumeration of the open subintervals of $(0, 1)$ with rational endpoints. For each $2 \leq i < \omega$, choose i^2 -many disjoint, open

subintervals of T_i , and label them $T_{i1}, T_{i2}, \dots, T_{ii^2}$. Note that $\bigcup_{1 \leq j \leq i^2} T_{ij} \subseteq T_i$. $T_i \setminus \bigcup_{1 \leq j \leq i^2} T_{ij} \neq \emptyset$; so let $T_{i0} = (0, 1) \setminus \bigcup_{1 \leq j \leq i^2} T_{ij}$. Then $\{T_{ij} : 0 \leq j \leq i^2\}$ is a partition of $(0, 1)$. Define

$$(3.4.4) \quad I = \{B_X \cap \overline{B}_Y \in \text{Clop}(2^{(0,1)}) : \exists 2 \leq i < \omega \exists J \subseteq \{1, \dots, i^2\} \text{ such that} \\ |J| \geq i \text{ and } \forall j \in J \ X \cap T_{ij} \neq \emptyset\}.$$

That is, I consists of those elements of $\text{Clop}(2^{(0,1)})$ of the form $B_X \cap \overline{B}_Y$ such that for some $2 \leq i < \omega$, X intersects at least i -many of the open intervals T_{i1}, \dots, T_{ii^2} . Let \mathcal{I} be the ideal generated by I in $\text{Clop}(2^{(0,1)})$. The *Gaifman algebra* is the quotient algebra

$$(3.4.5) \quad \mathbf{B} = \text{Clop}(2^{(0,1)}) / \mathcal{I}.$$

We shall denote the elements of \mathbf{B} by $[b]$, where $b \in \text{Clop}(2^{(0,1)})$.

Theorem 3.4.1 (Gaifman). [8] *\mathbf{B} satisfies the σ -bounded c.c., but does not satisfy $\text{CUP}(\mathbf{B}^+)$, and thus, is not a measurable algebra.*

A Boolean algebra constructed in the manner of Gaifman is not necessarily atomless. To show this, we need the following Corollary 3.4.3 and Lemma 3.4.4. Lemma 3.4.2 is a generalization obtained by Prikry of a lemma (Corollary 3.4.3) used by Gaifman in his proof that \mathbf{B} satisfies the σ -bounded c.c., but not $\text{CUP}(\mathbf{B}^+)$.

Lemma 3.4.2 (Prikry). [25] *Let $\tilde{\mathcal{T}}$ be a family of finite subsets of $(0, 1)$ closed under finite supersets. Let $\tilde{\mathcal{G}} = \{B_X : X \in \tilde{\mathcal{T}}\}$ and let $\tilde{\mathcal{I}}$ be the ideal in $\text{Clop}(2^{(0,1)})$ generated by $\tilde{\mathcal{G}}$. Then for $X, Y \in \tilde{\mathcal{T}}$ with $X \cap Y = \emptyset$, $B_X \cap \overline{B}_Y \in \tilde{\mathcal{I}}$ iff $B_X \in \tilde{\mathcal{G}}$.*

Proof: Clearly if $B_X \in \tilde{\mathcal{G}}$, then $B_X \cap \overline{B}_Y \in \tilde{\mathcal{I}}$, since $B_X \cap \overline{B}_Y \subseteq B_X \in \tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}}$ is an ideal.

Claim: For any non-empty $B_X \cap \overline{B}_Y \in \text{Clop}(2^{(0,1)})$, $X \notin \tilde{\mathcal{T}} \implies (\forall n < \omega \ \forall j \leq n \ \forall Z_j \in \tilde{\mathcal{T}} \ (B_X \cap \overline{B}_Y \not\subseteq \bigcup_{j \leq n} B_{Z_j}))$.

Suppose $\emptyset \neq B_X \cap \overline{B}_Y$ and $X \notin \tilde{\mathcal{T}}$. Let Z_j be some elements of $\tilde{\mathcal{T}}$, for $j \leq n < \omega$. If $X \supseteq Z_j$ for any $j \leq n$, then $X \in \tilde{\mathcal{T}}$; so $\forall j \leq n$, $Z_j \setminus X \neq \emptyset$. For each $j \leq n$, choose a $z_j \in Z_j \setminus X$. Let $Z = \{z_j : j \leq n\}$. Now $\emptyset \neq B_X \cap \overline{B}_Y \cap \overline{B}_Z \subseteq B_X$, since $Z \cap X = \emptyset$. But $\overline{B}_Z \cap (\bigcup_{j \leq n} B_{Z_j}) = \emptyset$, so $B_X \cap \overline{B}_Y \cap \overline{B}_Z \not\subseteq \bigcup_{j \leq n} B_{Z_j}$. Therefore, $B_X \cap \overline{B}_Y \not\subseteq \bigcup_{j \leq n} B_{Z_j}$.

Now, suppose $B_X \cap \overline{B}_Y \in \tilde{\mathcal{I}}$. Then there are finitely many elements of the form $B_{Z_j} \in \tilde{\mathcal{I}}$ such that $B_X \cap \overline{B}_Y \subseteq \bigcup_{j \leq n} B_{Z_j}$, since $\tilde{\mathcal{I}}$ is the ideal generated by B_Z 's with $Z \in \tilde{\mathcal{T}}$. By the Claim, $X \in \tilde{\mathcal{T}}$; hence, $B_X \in \tilde{\mathcal{G}}$.

□

Corollary 3.4.3 (Gaifman). [8] *For each $B_X \cap \overline{B}_Y \in \text{Clop}(2^{(0,1)})$ with $X \cap Y = \emptyset$, $B_X \cap \overline{B}_Y \in \mathcal{I}$ iff $B_X \in \mathcal{I}$.*

Proof: Let \mathcal{T} denote the set of all finite subsets of $(0, 1)$ which intersect at least i -many T_{ij} 's for some $i < \omega$. \mathcal{T} is closed under finite supersets, and Gaifman's ideal \mathcal{I} is the ideal generated by $\{B_X : X \in \mathcal{T}\}$.

□

Let

$$(3.4.6) \quad \mathcal{D} = \{B_X \cap \overline{B}_Y \in \text{Clop}(2^{(0,1)}) : X \cap Y = \emptyset\}.$$

\mathcal{D} is dense in $\text{Clop}(2^{(0,1)})^+$. Thus, by Corollary 3.4.3, the set

$$(3.4.7) \quad \mathbf{D} = \{[B_X \cap \overline{B}_Y] : B_X \cap \overline{B}_Y \in \mathcal{D} \text{ and } B_X \notin \mathcal{I}\}$$

is dense in \mathbf{B}^+ .

The following Lemma 3.4.4 is useful for testing whether a given element of \mathbf{B} is an atom. In Example 3.4.5, we will use this lemma to show that atoms exist in some Gaifman algebras.

Lemma 3.4.4. *An element $[B_X \cap \overline{B}_Y] \in \mathbf{D}$ is an atom in \mathbf{B} iff for each $x \in (0, 1) \setminus (X \cup Y)$, $B_X \cap B_{\{x\}} \in \mathcal{I}$.*

Proof: Suppose $\exists x \in (0, 1) \setminus (X \cup Y)$ such that $B_X \cap B_{\{x\}} \notin \mathcal{I}$. Then by Corollary 3.4.3, $B_X \cap B_{\{x\}} \cap \overline{B}_Y \notin \mathcal{I}$. Moreover, $[B_{X \cup \{x\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$, since $x \notin X$ implies $[0] < [B_X \cap \overline{B}_{Y \cup \{x\}}] \leq [B_X \cap \overline{B}_Y] \wedge -[B_{X \cup \{x\}} \cap \overline{B}_Y]$. Thus, $[B_X \cap \overline{B}_Y]$ is not an atom.

Now suppose $\forall x \in (0, 1) \setminus (X \cup Y)$, $B_X \cap B_{\{x\}} \in \mathcal{I}$. Let $[B_U \cap \overline{B}_V] \in \mathbf{D}$ and suppose that $[B_U \cap \overline{B}_V] \wedge [B_X \cap \overline{B}_Y] \neq [0]$. We will show that $[B_U \cap \overline{B}_V] \wedge [B_X \cap \overline{B}_Y] = [B_X \cap \overline{B}_Y]$.

$[B_U \cap \overline{B}_V] \wedge [B_X \cap \overline{B}_Y] \neq [0]$ implies $(U \cup X) \cap (V \cup Y) = \emptyset$ and $U \cup X \notin \mathcal{T}$. $\forall v \in V \cap Y$, $\overline{B}_{\{v\}} \supseteq \overline{B}_Y$; so

$$(3.4.8) \quad \bigwedge_{v \in V \cap Y} [\overline{B}_{\{v\}}] \geq [\overline{B}_Y].$$

$\forall v \in V \setminus Y$, $B_X \cap B_{\{v\}} \in \mathcal{I}$, since $v \notin X$; so $[B_X] \wedge [B_{\{v\}}] = [0]$, which implies $[B_X] \leq -[B_{\{v\}}]$. Hence,

$$(3.4.9) \quad \bigwedge_{v \in V \setminus Y} [\overline{B}_{\{v\}}] = \bigwedge_{v \in V \setminus Y} -[B_{\{v\}}] \geq [B_X].$$

Thus, by (3.4.8) and (3.4.9),

$$(3.4.10) \quad [\overline{B}_V] = \bigwedge_{v \in V} [\overline{B}_{\{v\}}] \geq [B_X \cap \overline{B}_Y].$$

Furthermore, $[B_U \cap \overline{B}_V] \wedge [B_X \cap \overline{B}_Y] \neq [0]$ implies $B_U \cap B_X \notin \mathcal{I}$, by Corollary 3.4.3. If $u \in U \setminus X$, then by our assumption, $B_X \cap B_{\{u\}} \in \mathcal{I}$, contradictory to $B_X \cap B_U \notin \mathcal{I}$. Thus, it follows that $U \subseteq X$. Hence,

$$(3.4.11) \quad [B_U] \geq [B_X].$$

By (3.4.10) and (3.4.11),

$$(3.4.12) \quad [B_U \cap \overline{B}_V] = [B_U] \wedge [\overline{B}_V] \geq [B_X \cap \overline{B}_Y].$$

Hence, $[B_U \cap \overline{B}_V] \wedge [B_X \cap \overline{B}_Y] = [B_X \cap \overline{B}_Y]$, so $[B_X \cap \overline{B}_Y]$ is an atom.

□

Depending on the intervals T_{ij} used in the construction, there may be atoms in a Gaifman algebra. For instance, we construct the following atom.

Example 3.4.5 (Construction of an atom). Let $T_2 = (0, 1)$, $T_3 = (0, \frac{31}{32})$, and $T_4 = (0, \frac{59}{64})$, and choose the following T_{ij} 's in these T_i 's:

$$(3.4.13) \quad \{T_{2,1}, \dots, T_{2,4}\} = \left\{ \left(0, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{7}{8}\right), \left(\frac{7}{8}, \frac{15}{16}\right), \left(\frac{15}{16}, 1\right) \right\}$$

$$(3.4.14) \quad \{T_{3,1}, \dots, T_{3,9}\} = \left\{ \left(0, \frac{1}{16}\right), \left(\frac{1}{16}, \frac{3}{16}\right), \left(\frac{3}{16}, \frac{5}{16}\right), \left(\frac{5}{16}, \frac{7}{16}\right), \left(\frac{7}{16}, \frac{9}{16}\right), \right. \\ \left. \left(\frac{9}{16}, \frac{11}{16}\right), \left(\frac{11}{16}, \frac{13}{16}\right), \left(\frac{13}{16}, \frac{29}{32}\right), \left(\frac{29}{32}, \frac{31}{32}\right) \right\}$$

$$(3.4.15) \quad \{T_{4,1}, \dots, T_{4,16}\} = \left\{ \left(0, \frac{1}{32}\right), \left(\frac{1}{32}, \frac{3}{32}\right), \left(\frac{3}{32}, \frac{5}{32}\right), \left(\frac{5}{32}, \frac{7}{32}\right), \right. \\ \left(\frac{7}{32}, \frac{9}{32}\right), \left(\frac{9}{32}, \frac{11}{32}\right), \left(\frac{11}{32}, \frac{13}{32}\right), \left(\frac{13}{32}, \frac{15}{32}\right), \\ \left(\frac{15}{32}, \frac{17}{32}\right), \left(\frac{17}{32}, \frac{19}{32}\right), \left(\frac{19}{32}, \frac{21}{32}\right), \left(\frac{21}{32}, \frac{23}{32}\right), \\ \left(\frac{23}{32}, \frac{25}{32}\right), \left(\frac{25}{32}, \frac{27}{32}\right), \left(\frac{27}{32}, \frac{57}{64}\right), \left(\frac{57}{64}, \frac{59}{64}\right) \right\}.$$

Let

$$(3.4.16) \quad X = \left\{ \frac{1}{32}, \frac{3}{32}, \frac{5}{32}, \frac{10}{32}, \frac{14}{32}, \frac{18}{32} \right\}.$$

$[B_X]$ is an atom in \mathbf{B} : For any $x \in (0, 1) \setminus X$, $B_{X \cup \{x\}} \in \mathcal{I}$, since x must lie in at least one of the $T_{2,j}$'s, $T_{3,j}$'s, or $T_{4,j}$'s which X does not intersect. Thus, by Lemma 3.4.4, $[B_X]$ is an atom. Note: $[B_X]$ is an atom no matter how the rest of the T_{ij} 's are picked for $i > 4$.

□

Remark. In order to produce an atom $[B_X \cap \overline{B}_Y]$, the set X must consist of elements which lie on the boundary of some T_{ij} with $2 \leq i \leq |X| + 1$. For if $\exists x \in X \cap \text{int}(\bigcap_{2 \leq i \leq |X|+1} T_{i,j_i})$ for some j_i 's, then for all $z \in \text{int}(\bigcap_{2 \leq i \leq |X|+1} T_{i,j_i})$, $[0] < [B_{X \cup \{z\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$. Hence, there are at most countably many atoms in any Gaifman algebra.

3.5. THE COHEN ALGEBRA COMPLETELY EMBEDS INTO EACH ATOMLESS GAIFMAN ALGEBRA

In this section, we work with atomless Gaifman algebras. Atomless Gaifman algebras do exist. For instance, if the T_{ij} 's are nested so that $(i < k, 0 \leq j \leq i^2, 1 \leq l \leq k^2, \text{ and } T_{ij} \cap T_{kl} \neq \emptyset) \implies T_{kl} \subseteq T_{ij}$, then the resulting Gaifman algebra is atomless.

Let the T_i 's and T_{ij} 's be chosen according to the Gaifman construction given in §3.4 in such a way that the Gaifman algebra \mathbf{B} is atomless. Let

$$(3.5.1) \quad E = \{x \in (0, 1) : \exists 2 \leq i < \omega (x \in T_{i0} \setminus \text{int}(T_{i0}))\}.$$

E is the set of all endpoints of the intervals T_{ij} , $2 \leq i < \omega$, $1 \leq j \leq i^2$. The following Lemma 3.5.1 is useful for constructing an atomless subalgebra of \mathbf{B} .

Lemma 3.5.1. *If $[0] < [B_X \cap \overline{B}_Y]$ and F is a finite subset of $(0, 1)$, then $\exists z \in (0, 1) \setminus F$ such that $[0] < [B_{X \cup \{z\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$.*

Proof: Since $[0] < [B_X \cap \overline{B}_Y]$ and \mathbf{B} is atomless, $\exists [B_U \cap \overline{B}_V] \in \mathbf{D}$, (the dense subset of \mathbf{B}^+ defined in (3.4.7)), such that $[B_U \cap \overline{B}_V] < [B_X \cap \overline{B}_Y]$.

Claim: $\exists z \in (0, 1) \setminus (X \cup Y)$ for which $[0] < [B_{X \cup \{z\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$. Since $[0] < [B_X \cap \overline{B}_Y]$ and \mathbf{B} is atomless, Lemma 3.4.4 implies that $\exists x \in (0, 1) \setminus (X \cup Y)$ such that $B_{X \cup \{x\}} \notin \mathcal{I}$. $x \notin X \implies [B_{X \cup \{x\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$. $B_{X \cup \{x\}} \notin \mathcal{I}$

and $(X \cup \{x\}) \cap Y = \emptyset \implies [B_{X \cup \{x\}} \cap \overline{B}_Y] > [0]$, by Corollary 3.4.3. Hence, the Claim holds.

Let F be a finite subset of $(0, 1)$. By the above Claim, there is a $z_0 \in (0, 1) \setminus (X \cup Y)$ such that $[0] < [B_{X \cup \{z_0\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$. In general, given $z_0 \in (0, 1) \setminus (X \cup Y), \dots, z_n \in (0, 1) \setminus (X \cup Y \cup \{z_0, \dots, z_{n-1}\})$ such that $[0] < [B_{X \cup Y \cup \{z_0, \dots, z_n\}} \cap \overline{B}_Y] < [B_{X \cup \{z_0, \dots, z_{n-1}\}} \cap \overline{B}_Y]$, the Claim implies there is some $z_{n+1} \in (0, 1) \setminus (X \cup Y \cup \{z_0, \dots, z_n\})$ for which $[0] < [B_{X \cup \{z_0, \dots, z_{n+1}\}} \cap \overline{B}_Y] < [B_{X \cup \{z_0, \dots, z_n\}} \cap \overline{B}_Y]$. So we have a strictly decreasing sequence of non-zero elements, each of which is strictly below $[B_X \cap \overline{B}_Y]$. Take $N \geq |F|$. Since the z_n 's are distinct, there is some $n \leq N$ for which $z_n \notin F \cup X \cup Y$. Then $[0] < [B_{X \cup \{z_n\}} \cap \overline{B}_Y]$, since $[0] < [B_{X \cup \{z_i : i \leq N\}} \cap \overline{B}_Y]$; and $[B_{X \cup \{z_n\}} \cap \overline{B}_Y] < [B_X \cap \overline{B}_Y]$, since $z_n \notin X$.

□

We now construct a countable, atomless subalgebra \mathbf{C} of \mathbf{B} and show that \mathbf{C} is a regular subalgebra of \mathbf{B} . Our construction uses two types of sets which will ensure that \mathbf{C} is a regular subalgebra of \mathbf{B} : F_i 's which take care of elements of E , and X_i 's which take care of elements which lie outside of E . We start by constructing the F_i 's by induction.

Let $E_2 = T_{20} \setminus \text{int}(T_{20})$. Let

$$(3.5.2) \quad F_2 = E_2.$$

Let $E_3 = T_{30} \setminus (\text{int}(T_{30}) \cup F_2)$. $\forall F \subseteq F_2$ for which $[B_F] > [0]$, by Lemma 3.5.1 choose one $x_F \in (0, 1) \setminus F_2$ such that $[B_{F \cup \{x_F\}}] > [0]$. Let

$$(3.5.3) \quad F_3 = E_3 \cup \{x_F : F \subseteq F_2, [B_F] > [0]\}.$$

Suppose that for all $3 \leq i \leq n$, we have constructed sets F_i with the properties that $F_i \cap (\bigcup_{2 \leq k < i} F_k) = \emptyset$, and whenever $F \subseteq \bigcup_{2 \leq k < i} F_k$ and $[B_F] > [0]$, then $\exists x_F \in F_i$ such that $[B_{F \cup \{x_F\}}] > [0]$. Let $E_{n+1} = T_{n+1,0} \setminus (\text{int}(T_{n+1,0}) \cup$

$\bigcup_{2 \leq i \leq n} F_i$). $\forall F \subseteq \bigcup_{2 \leq i \leq n} F_i$ for which $[B_F] > [0]$, by Lemma 3.5.1 choose an $x_F \in (0, 1) \setminus (\bigcup_{2 \leq i \leq n} F_i)$ such that $[B_{F \cup \{x_F\}}] > [0]$. Let

$$(3.5.4) \quad F_{n+1} = E_{n+1} \cup \{x_F : F \subseteq \bigcup_{2 \leq i \leq n} F_i, [B_F] > [0]\}.$$

Note. The sets F_i , $2 \leq j < i < \omega$, have the following properties:

- (F1) $F_i \neq \emptyset$;
- (F2) $E \subseteq \bigcup_{2 \leq i < \omega} F_i$;
- (F3) $F_i \cap F_j = \emptyset$;
- (F4) $\forall F \subseteq \bigcup_{2 \leq k < i} F_k$ such that $[B_F] > [0]$, there is an $x_F \in F_i$ for which $[B_{F \cup \{x_F\}}] > [0]$.

Now we construct the X_i 's. Let

$$(3.5.5) \quad J_2 = \left\{ s = \langle s(2), s(3) \rangle \in \{0, 1, 2, 3, 4\} \times \{0, 1, \dots, 9\} : \text{int}(T_{2,s(2)} \cap T_{3,s(3)}) \neq \emptyset \right\}.$$

$J_2 \neq \emptyset$, since $\{T_{2j} : 0 \leq j \leq 4\}$ and $\{T_{3j} : 0 \leq j \leq 9\}$ are finite partitions of $(0, 1)$, where each piece of each partition is either an open interval, or else a finite union of closed intervals. For each $s \in J_2$, choose some $x_s \in \text{int}(T_{2,s(2)} \cap T_{3,s(3)}) \setminus (\bigcup_{2 \leq i < \omega} F_i)$. Such an x_s exists, since $\text{int}(T_{2,s(2)} \cap T_{3,s(3)}) \neq \emptyset$ and $\bigcup_{2 \leq i < \omega} F_i$ is countable. Let

$$(3.5.6) \quad X_2 = \{x_s : s \in J_2\}.$$

In general, for $i \geq 3$, let

$$(3.5.7) \quad J_i = \left\{ s = \langle s(2), \dots, s(i+1) \rangle \in \prod_{k=2}^{i+1} (k^2 + 1) : \text{int}\left(\bigcap_{k=2}^{i+1} T_{k,s(k)}\right) \neq \emptyset \right\}.$$

$J_i \neq \emptyset$, since for each $2 \leq k \leq i+1$, $\{T_{kj} : 0 \leq j \leq k^2\}$ is a partition of $(0, 1)$ where each T_{kj} is either an open interval or else a finite union of closed intervals. For each $s \in J_i$, choose some $x_s \in \text{int}(\bigcap_{2 \leq k \leq i+1} T_{k,s(k)}) \setminus (\bigcup_{2 \leq k < \omega} F_k \cup \bigcup_{2 \leq k < i} X_k)$. Let

$$(3.5.8) \quad X_i = \{x_s : s \in J_i\}.$$

Note. The sets $X_i \cup F_i$ have the following properties:

$$(XF1) \quad (\bigcup_{2 \leq i < \omega} X_i) \cap (\bigcup_{2 \leq k < \omega} F_k) = \emptyset;$$

$$(XF2) \quad \forall 2 \leq i < j < \omega, (X_i \cup F_i) \cap (X_j \cup F_j) = \emptyset.$$

For each $2 \leq i < \omega$, let

$$(3.5.9) \quad c_i = \bigvee_{x \in X_i} [B_{\{x\}}] \vee \bigvee_{f \in F_i} [B_{\{f\}}].$$

Let

$$(3.5.10) \quad \mathbf{C} = \langle \{c_i : 2 \leq i < \omega\} \rangle,$$

the subalgebra of \mathbf{B} generated by $\{c_i : 2 \leq i < \omega\}$.

\mathbf{C} is certainly countable, since \mathbf{C} is generated by a countable subset of \mathbf{B} . We will show that \mathbf{C} is an atomless, regular subalgebra of \mathbf{B} . It will then follow by Lemma 3.1.11 that $\text{r.o.}(\mathbf{C})$ is a complete subalgebra of $\text{r.o.}(\mathbf{B})$.

Proposition 3.5.2. *\mathbf{C} is countable and atomless.*

Proof: To show \mathbf{C} is atomless, we shall show that its generators are independent. Suppose K, L are finite, disjoint subsets of $\{2, 3, 4, \dots\}$. We claim that

$$(3.5.11) \quad \bigwedge_{k \in K} a_k \wedge \bigwedge_{l \in L} -a_l \neq [0].$$

By definition,

$$(3.5.12) \quad \bigwedge_{k \in K} a_k \wedge \bigwedge_{l \in L} -a_l = \bigwedge_{k \in K} \left(\bigvee_{x \in X_k} [B_{\{x\}}] \vee \bigvee_{f \in F_k} [B_{\{f\}}] \right) \wedge [\overline{B}_{\bigcup_{l \in L} (X_l \cup F_l)}].$$

By (XF2), $(\bigcup_{k \in K} X_k \cup F_k) \cap (\bigcup_{l \in L} (X_l \cup F_l)) = \emptyset$. Hence, by Corollary 3.4.3, it suffices to show that $\exists \langle u_k : k \in K \rangle \in \prod_{k \in K} (X_k \cup F_k)$ for which $B_{\{u_k : k \in K\}} \notin \mathcal{I}$.

Let $n = |K| + 1$. Order the elements of K as $k_2 < k_3 < \dots < k_n$. Choose one $x_2 \in X_{k_2}$. $\forall 2 \leq i \leq k_n + 1$ let $s(i) \in i^2 + 1$ be such that $x_2 \in T_{i, s(i)}$. $x_2 \notin E \implies x_2 \in \text{int}(\bigcap_{2 \leq i \leq k_n + 1} T_{i, s(i)})$. Hence, $\forall 2 \leq m \leq n$, $\langle s(2), s(3), \dots, s(k_m + 1) \rangle \in$

J_{k_m} ; so choose $x_m \in X_m \cap \text{int}(\bigcap_{2 \leq i \leq k_m+1} T_{i,s(i)})$. Let $X = \{x_m : 2 \leq m \leq n\}$. $\forall 2 \leq i \leq n-1$, $k_i \geq i$, so $\{x_2, x_{i-1}, \dots, x_n\}$ all lie in $T_{i,s(i)}$. $|\{x_3, \dots, x_{i-2}\}| = i-4$. Therefore, X lies in at most $(i-3)$ -many T_{ij} 's. For $i \geq n$, X lies in less than i -many T_{ij} 's, since $|X| \leq |K| < n$. Thus, $B_X \notin \mathcal{I}$.

□

Remark. By (F1), since each $F_i \neq \emptyset$, it is also possible to do the above argument using a sequence $\langle u_k : k \in K \rangle \in \prod_{k \in K} F_k$.

Next, we show that **C** satisfies the conditions of Lemma 3.1.8. It will then follow that **C** is a regular subalgebra of **B**.

Proposition 3.5.3. *For each $d \in \mathbf{D}$, there exists a $c_d \in \mathbf{C}^+$ such that whenever $c \in \mathbf{C}$ and $c \wedge c_d \neq [0]$, then $c \wedge d \neq [0]$.*

Proof: Let $d = [B_X \cap \overline{B}_Y] \in \mathbf{D}$. Let $N \geq |X \cup Y| + 1$ be such that $(X \cup Y) \cap (\bigcup_{N < i < \omega} (X_i \cup F_i)) = \emptyset$. Let

$$(3.5.13) \quad I = \{2 \leq i \leq N : X \cap (X_i \cup F_i) \neq \emptyset\}$$

and

$$(3.5.14) \quad J = \{2 \leq i \leq N : X \cap (X_i \cup F_i) = \emptyset\}.$$

Let

$$(3.5.15) \quad c_d = \bigwedge_{i \in I} c_i \wedge \bigwedge_{j \in J} -c_j.$$

Suppose $c = \bigwedge_{k \in K} c_k \wedge \bigwedge_{l \in L} -c_l \in \mathbf{C}^+$ and $c_d \wedge c > [0]$, (where $K, L \in [\omega \setminus 2]^{<\omega}$). Then

$$(3.5.16) \quad [0] < c_d \wedge c = \bigwedge_{i \in I \cup K} c_i \wedge \bigwedge_{j \in J \cup L} -c_j,$$

so $(I \cup K) \cap (J \cup L) = \emptyset$. Let $K' = K \cap (N + 1)$ and $K'' = K \setminus K'$. $K' \cap J = \emptyset$ and $K' \subseteq I \cup J \implies K' \subseteq I$.

Note that

$$\begin{aligned}
 (3.5.17) \quad c \wedge d &= \bigwedge_{k \in K} c_k \wedge \bigwedge_{l \in L} -c_l \wedge [B_X \cap \overline{B}_Y] \\
 &= \bigwedge_{k \in K} \left(\bigvee_{u \in X_k \cup F_k} [B_{\{u\}}] \right) \wedge [B_X \cap \overline{B}_{Y \cup (\bigcup_{l \in L} X_l \cup F_l)}].
 \end{aligned}$$

By Corollary 3.4.3, it suffices to find a sequence $\langle u_k : k \in K \rangle \in \prod_{k \in K} (X_k \cup F_k)$ for which (a) $[B_{\{u_k : k \in K\} \cup X}] > [0]$ and (b) $(\{u_k : k \in K\} \cup X) \cap (Y \cup \bigcup_{l \in L} (X_l \cup F_l)) = \emptyset$. There are two cases.

Case 1: $X \subseteq E$. (This includes the possibility that $X = \emptyset$.) By (F2), $E \subseteq \bigcup_{2 \leq i < \omega} F_i$, so $X \subseteq \bigcup_{2 \leq i \leq N} F_i$. $\forall k \in K'$ choose some $f_k \in F_k \cap X$. Then $X \cup \{f_k : k \in K'\} = X$, so it suffices to show (a) and (b) for some sequence $\langle f_k : k \in K'' \rangle \in \prod_{k \in K''} (X_k \cup F_k)$. List the elements of K'' in order: $N + 1 \leq k_0 < k_1 < \dots < k_M$. By (F4), $X \subseteq \bigcup_{2 \leq i \leq N} F_i$ and $[B_X] > [0] \implies \exists f_{k_0} \in F_{k_0}$ such that $[B_{X \cup \{f_{k_0}\}}] > [0]$. By (F4), $X \cup \{f_{k_0}\} \subseteq \bigcup_{2 \leq i \leq k_0} F_i$ and $[B_{X \cup \{f_{k_0}\}}] > [0] \implies \exists f_{k_1} \in F_{k_1}$ such that $[B_{X \cup \{f_{k_0}, f_{k_1}\}}] > [0]$. In general, given $f_{k_i} \in F_{k_i}$, $0 \leq i \leq m$ such that $[B_{X \cup \{f_{k_0}, \dots, f_{k_m}\}}] > [0]$ and $X \cup \{f_{k_0}, \dots, f_{k_m}\} \subseteq \bigcup_{2 \leq i \leq k_m} F_i$, (F4) implies $\exists f_{k_{m+1}} \in F_{k_{m+1}}$ such that $[B_{X \cup \{f_{k_0}, \dots, f_{k_m}, f_{k_{m+1}}\}}] > [0]$. By induction on m , we get a sequence $\langle f_{k_0}, \dots, f_{k_M} \rangle \in \prod_{0 \leq i \leq M} F_{k_i}$ for which $[B_{X \cup \{f_{k_0}, \dots, f_{k_M}\}}] > [0]$. Thus, (a) holds for the sequence $\langle f_k : k \in K'' \rangle$.

$\{f_k : k \in K''\} \cap (Y \cup (\bigcup_{l \in L} X_l \cup F_l)) = \emptyset$, since $Y \cap (\bigcup_{N < i < \omega} (X_i \cup F_i)) = \emptyset$ and $K \cap L = \emptyset$. $X \cap Y = \emptyset$, since $d > [0]$. $X \subseteq \bigcup_{i \in I} F_i$ and $I \cap L = \emptyset \implies X \cap (\bigcup_{l \in L} X_l \cup F_l) = \emptyset$. Thus, (b) holds for the sequence $\langle f_k : k \in K'' \rangle$.

Case 2: $X \setminus E \neq \emptyset$. $\forall k \in K'$ choose some $u_k \in (X_k \cup F_k) \cap X$. (This is possible, since $K' \subseteq I$.) $X \cup \{u_k : k \in K'\} = X$, so it suffices to show (a) and (b) for some sequence $\langle u_k : k \in K'' \rangle \in \prod_{k \in K''} (X_k \cup F_k)$. Fix an element $x \in X \setminus E$. Order the elements of K'' as $N + 1 \leq k_0 < \dots < k_M$. Let $s \in$

$\prod_{2 \leq i \leq k_M+1} (i^2 + 1)$ be such that $x \in \text{int}(\bigcap_{2 \leq i \leq k_M+1} T_{i,s(i)})$. $\forall 0 \leq m \leq M$, choose an $x_{k_m} \in X_{k_m} \cap \text{int}(\bigcap_{2 \leq i \leq k_m+1} T_{i,s(i)})$.

For $2 \leq i \leq N+2$, $X \cup \{x_k : k \in K''\}$ intersects exactly the same T_{ij} 's as X , since $\{x_k : k \in K''\} \subseteq \text{int}(\bigcap_{2 \leq n \leq k_0+1} T_{n,s(n)})$. For $N+3 \leq i \leq k_M$, $\{x_k : k \in K'', k \geq i-1\}$ all lie in $T_{i,s(i)}$, as does x . Hence, $X \cup \{x_k : k \in K''\}$ intersects at most $|X| + |\{x_k : k \in K'', N < k \leq i-2\}| \leq i-2-N+N < i$ -many T_{ij} 's. For $i > k_M$, $|X \cup \{x_k : k \in K''\}| \leq k_M < i$, so $X \cup \{x_k : k \in K''\}$ intersects less than i -many T_{ij} 's. Thus, $X \cup \{x_k : x \in K\}$ intersects less than i -many T_{ij} 's, for all $2 \leq i < \omega$. Hence, (a) holds. (b) follows from the same reasoning as in Case 1. Thus, $c \wedge d \neq [0]$.

□

Proposition 3.5.3 states that \mathbf{C} satisfies the conditions of Lemma 3.1.8. Hence, \mathbf{C} is a regular subalgebra of \mathbf{B} . It follows from Lemma 3.1.11 that $\text{r.o.}(\mathbf{C})$ embeds as a complete subalgebra of $\text{r.o.}(\mathbf{B})$. That is, the Cohen algebra embeds as a complete subalgebra into the completion of each atomless Gaifman algebra.

3.6. THE HYPER-WEAK (ω, ω) -DISTRIBUTIVE LAW FAILS IN EACH GAIFMAN ALGEBRA

We now return to general Gaifman algebras. As we saw in §3.4, a Gaifman algebra may be constructed so that it has atoms. Since the Cohen algebra is atomless, it cannot embed as a complete subalgebra of a non-atomless Gaifman algebra. However, even if there are atoms, the hyper-weak (ω, ω) -d.l. still fails for every Gaifman algebra.

Theorem 3.6.1. *The hyper-weak (ω, ω) -d.l. fails in \mathbf{B} .*

Proof: We will construct a relative subalgebra of \mathbf{B} in which the hyper-weak (ω, ω) -d.l. fails everywhere. This will give the failure of the hyper-weak (ω, ω) -d.l.

in \mathbf{B} .

The set

(3.6.1)

$$E = \{z \in (0, 1) : \exists 2 \leq i < \omega \exists 1 \leq j \leq i^2 \text{ such that } z \text{ is an endpoint of } T_{ij}\}$$

is countable; so choose an $r \in (0, 1) \setminus E$. For each $2 \leq i < \omega$, let $j(i) \in \{0, 1, \dots, i^2\}$ be the integer such that $r \in T_{i, j(i)}$. Define the following sets:

$$(3.6.2) \quad \text{Clop}(2^{(0,1)}) \upharpoonright B_{\{r\}} = \{b \in \text{Clop}(2^{(0,1)}) : b \subseteq B_{\{r\}}\};$$

$$(3.6.3) \quad \mathcal{I}' = \mathcal{I} \cap (\text{Clop}(2^{(0,1)}) \upharpoonright B_{\{r\}}) = \{b \in \mathcal{I} : b \subseteq B_{\{r\}}\};$$

$$(3.6.4) \quad \mathbf{A} = (\text{Clop}(2^{(0,1)}) \upharpoonright B_{\{r\}}) / \mathcal{I}'.$$

We will show that the hyper-weak (ω, ω) -d.l. fails everywhere in \mathbf{A} . Later, we will show that \mathbf{A} is isomorphic to a relative subalgebra of \mathbf{B} .

To prove the failure of hyper-weak (ω, ω) -d.l., the following dense subset $\mathbf{D}_{\mathbf{A}}$ will be useful. First, note that the set

$$(3.6.5) \quad \mathcal{D}_r = \{B_X \cap \overline{B}_Y : X, Y \in [(0, 1)]^{<\omega}, r \in X, \text{ and } X \cap Y = \emptyset\}$$

is dense in $(\text{Clop}(2^{(0,1)}) \upharpoonright B_{\{r\}})^+$. Thus, the set

$$(3.6.6) \quad \mathbf{D}_{\mathbf{A}} = \{[B_X \cap \overline{B}_Y]_{\mathbf{A}} : B_X \cap \overline{B}_Y \in \mathcal{D}_r\} \setminus \{[0]_{\mathbf{A}}\}$$

is dense in \mathbf{A} , where $[b]_{\mathbf{A}}$ denotes the equivalence class of b under the equivalence relation $a \sim_{\mathbf{A}} b \iff a, b \in \text{Clop}(2^{(0,1)}) \upharpoonright B_{\{r\}} \text{ and } a \Delta b \in \mathcal{I}'$. By $[b]_{\mathbf{B}}$, we will denote the equivalence class of b under the equivalence relation $a \sim_{\mathbf{B}} b \iff a, b \in \text{Clop}(2^{(0,1)}) \text{ and } a \Delta b \in \mathcal{I}$.

Although not necessary for the proof of the failure of the hyper-weak (ω, ω) -d.l. in \mathbf{B} , it is interesting, and necessary for the proof that the hyper-weak (ω, ω) -d.l. fails everywhere in \mathbf{A} , that \mathbf{A} is atomless.

Proposition 3.6.2. *A is atomless.*

Proof: Suppose $[B_X \cap \overline{B}_Y]_{\mathbf{A}} \in \mathbf{D}_{\mathbf{A}}$. Then $[B_X \cap \overline{B}_Y]_{\mathbf{A}} > [0]_{\mathbf{A}}$, so $B_X \cap \overline{B}_Y \notin \mathcal{I}'$. Let $m = |X|$. $r \in \text{int}(\bigcap_{2 \leq i \leq m+1} T_{i,j(i)})$ implies $\text{int}(\bigcap_{2 \leq i \leq m+1} T_{i,j(i)})$ contains an open interval; choose some $z \in \text{int}(\bigcap_{2 \leq i \leq m+1} T_{i,j(i)}) \setminus (X \cup Y)$. We claim that $[0]_{\mathbf{A}} < [B_{X \cup \{z\}} \cap \overline{B}_Y]_{\mathbf{A}} < [B_X \cap \overline{B}_Y]_{\mathbf{A}}$.

For all $2 \leq i \leq m+1$, the set $X \cup \{z\}$ intersects exactly the same T_{ij} 's as X . $|X \cup \{z\}| = m+1$, so for all $i > m+1$, $X \cup \{z\}$ does not intersect i -many T_{ij} 's. Hence $B_{X \cup \{z\}} \notin \mathcal{I}'$. Since $z \notin Y$, $[B_{X \cup \{z\}} \cap \overline{B}_Y]_{\mathbf{A}} > [0]_{\mathbf{A}}$. Moreover, $z \notin X \implies [B_{X \cup \{z\}} \cap \overline{B}_Y]_{\mathbf{A}} < [B_X \cap \overline{B}_Y]_{\mathbf{A}}$. Hence, $[B_X \cap \overline{B}_Y]_{\mathbf{A}}$ is not an atom. \square

Next we show that the hyper-weak (ω, ω) -d.l. fails everywhere in \mathbf{A} .

Proposition 3.6.3. *The hyper-weak (ω, ω) -d.l. fails everywhere in \mathbf{A} .*

Proof: Recall that for each $2 \leq i < \omega$, $j(i)$ is the integer in $\{0, 1, \dots, i^2\}$ such that $r \in \text{int}(T_{i,j(i)})$. Hence, $\text{int}(\bigcap_{2 \leq k \leq i} T_{k,j(k)}) \neq \emptyset$. Choose a $z_2 \in \text{int}(T_{2,j(2)}) \setminus \{r\}$. Choose a $z_3 \in \text{int}(T_{2,j(2)} \cap T_{3,j(3)}) \setminus \{r, z_2\}$. In general, for $i \geq 3$, choose a $z_i \in \text{int}(\bigcap_{2 \leq k \leq i} T_{k,j(k)}) \setminus \{r, z_2, \dots, z_{i-1}\}$. Thus, we have a sequence z_2, z_3, \dots such that

$$(3.6.7) \quad \forall 2 \leq i < \omega \quad z_i \in \text{int}\left(\bigcap_{2 \leq k \leq i} T_{k,j(k)}\right) \setminus \{r, z_2, \dots, z_{i-1}\}.$$

Construction of the partitions: Let

$$(3.6.8) \quad \begin{aligned} a_{00} &= (B_{\{r\}} \cap \overline{B}_{\{z_2\}}) \cup B_{\{r, z_2, z_3\}} \\ a_{01} &= B_{\{r, z_2, z_4\}} \cap \overline{B}_{\{z_3\}} \\ a_{02} &= B_{\{r, z_2, z_5\}} \cap \overline{B}_{\{z_3, z_4\}} \\ &\vdots \\ a_{0n} &= B_{\{r, z_2, z_{n+3}\}} \cap \overline{B}_{\{z_3, z_4, \dots, z_{n+2}\}} \\ &\vdots \end{aligned}$$

and let

$$(3.6.9) \quad W_0 = \{[a_{0n}]_{\mathbf{A}} : n < \omega\}.$$

Let

$$(3.6.10) \quad \begin{aligned} a_{10} &= (B_{\{r\}} \cap \overline{B}_{\{z_3\}}) \cup B_{\{r, z_3, z_4\}} \\ a_{11} &= B_{\{r, z_3, z_5\}} \cap \overline{B}_{\{z_4\}} \\ a_{12} &= B_{\{r, z_3, z_6\}} \cap \overline{B}_{\{z_4, z_5\}} \\ &\vdots \\ a_{1n} &= B_{\{r, z_3, z_{n+4}\}} \cap \overline{B}_{\{z_4, z_5, \dots, z_{n+3}\}} \\ &\vdots \end{aligned}$$

and let

$$(3.6.11) \quad W_1 = \{[a_{1n}]_{\mathbf{A}} : n < \omega\}.$$

In general, for $m < \omega$, let

$$(3.6.12) \quad \begin{aligned} a_{m0} &= (B_{\{r\}} \cap \overline{B}_{\{z_{m+2}\}}) \cup B_{\{r, z_{m+2}, z_{m+3}\}} \\ a_{m1} &= B_{\{r, z_{m+2}, z_{m+4}\}} \cap \overline{B}_{\{z_{m+3}\}} \\ a_{m2} &= B_{\{r, z_{m+2}, z_{m+5}\}} \cap \overline{B}_{\{z_{m+3}, z_{m+4}\}} \\ &\vdots \\ a_{mn} &= B_{\{r, z_{m+2}, z_{m+n+3}\}} \cap \overline{B}_{\{z_{m+3}, z_{m+4}, \dots, z_{m+n+2}\}} \\ &\vdots \end{aligned}$$

and let

$$(3.6.13) \quad W_m = \{[a_{mn}]_{\mathbf{A}} : n < \omega\}.$$

Claim 1: $\forall m < \omega$, W_m is a partition of unity in \mathbf{A} .

Let $m < \omega$ be given and suppose $n < n' < \omega$. If $n = 0$, then

(3.6.14)

$$\begin{aligned}
 a_{mn} \wedge a_{mn'} &= a_{m0} \wedge a_{mn'} \\
 &= (B_{\{r\}} \cap \overline{B}_{\{z_{m+2}\}} \cup B_{\{r, z_{m+2}, z_{m+3}\}}) \cap (B_{\{r, z_{m+2}, z_{m+n'+3}\}} \\
 &\quad \cap \overline{B}_{\{z_{m+3}, z_{m+4}, \dots, z_{m+n'+2}\}}) \\
 &\subseteq (\overline{B}_{\{z_{m+2}\}} \cap B_{\{z_{m+2}\}}) \cup (B_{\{z_{m+3}\}} \cap \overline{B}_{\{z_{m+3}\}}) \\
 &= \emptyset.
 \end{aligned}$$

If $n \neq 0$, then

$$\begin{aligned}
 a_{mn} \wedge a_{mn'} &\subseteq B_{\{r, z_{m+2}, z_{m+n+3}\}} \cap \overline{B}_{\{z_{m+3}, z_{m+4}, \dots, z_{m+n'+2}\}} \\
 (3.6.15) \quad &\subseteq B_{\{z_{m+n+3}\}} \cap \overline{B}_{\{z_{m+n+3}\}} \\
 &= \emptyset
 \end{aligned}$$

since $n < n'$ implies $m+3 \leq m+n+3 \leq m+n'+2$. In either case, $[a_{mn}]_{\mathbf{A}} \wedge [a_{mn'}]_{\mathbf{A}} = [0]_{\mathbf{A}}$. Hence, W_m is pairwise disjoint in \mathbf{A} .

Next, we will show that $\bigvee_{n < \omega} [a_{mn}]_{\mathbf{A}} = [1]_{\mathbf{A}}$. Let $[B_X \cap \overline{B}_Y]_{\mathbf{A}} \in \mathbf{D}_{\mathbf{A}}$ and let $b = B_X \cap \overline{B}_Y$. (Note that $r \in X$ and $B_X \notin \mathcal{I}'$.) If $z_{m+2} \notin X$, then

$$(3.6.16) \quad b \cap a_{m0} \supseteq B_X \cap \overline{B}_{Y \cup \{z_{m+2}\}} \notin \mathcal{I}'$$

by Corollary 3.4.3, since $B_X \notin \mathcal{I}'$. Thus, $[b]_{\mathbf{A}} \wedge [a_{m0}]_{\mathbf{A}} \neq [0]_{\mathbf{A}}$.

If $z_{m+2} \in X$, then we have two cases. If $\exists n < \omega$ such that $z_{m+n+3} \in X$, then let n be the least element of ω such that $z_{m+n+3} \in X$.

$$\begin{aligned}
 (3.6.17) \quad b \cap a_{mn} &\supseteq B_X \cap \overline{B}_Y \cap B_{\{z_{m+2}, z_{m+n+3}\}} \cap \overline{B}_{\{z_{m+3}, \dots, z_{m+n+2}\}} \\
 &= B_X \cap \overline{B}_{Y \cup \{z_{m+3}, \dots, z_{m+n+2}\}} \notin \mathcal{I}'
 \end{aligned}$$

by Corollary 3.4.3, since $B_X \notin \mathcal{I}'$. Thus, $[b]_{\mathbf{A}} \wedge [a_{mn}]_{\mathbf{A}} \neq [0]_{\mathbf{A}}$. Otherwise, $z_{m+2} \in X$ and $\forall n < \omega$, $z_{m+n+3} \notin X$. In this case, let $n < \omega$ be such that

$m+n+3 \geq |X|+1$ and $z_{m+n+3} \notin Y$. (This is possible since Y is finite.) Then,

$$(3.6.18) \quad \begin{aligned} b \cap a_{mn} &= B_X \cap \overline{B}_Y \cap B_{\{z_{m+2}, z_{m+n+3}\}} \cap \overline{B}_{\{z_{m+3}, \dots, z_{m+n+2}\}} \\ &= B_{X \cup \{z_{m+n+3}\}} \cap \overline{B}_{Y \cup \{z_{m+3}, \dots, z_{m+n+2}\}} \notin \mathcal{I}' \end{aligned}$$

by Corollary 3.4.3, since z_{m+n+3} and r are both contained in $T_{i,j(i)}$ for all $2 \leq i \leq |X|+1$. Thus, $[b]_{\mathbf{A}} \wedge [a_{mn}]_{\mathbf{A}} \neq [0]_{\mathbf{A}}$.

Thus, for each $[b]_{\mathbf{A}} \in \mathbf{D}_{\mathbf{A}}$, $\exists n < \omega$ such that $[b]_{\mathbf{A}} \wedge [a_{mn}]_{\mathbf{A}} \neq [0]_{\mathbf{A}}$. Hence, $\bigvee_{n < \omega} [a_{mn}]_{\mathbf{A}} = [1]_{\mathbf{A}}$. This concludes the proof of Claim 1.

To show that the hyper-weak (ω, ω) -d.l. fails in \mathbf{A} , we will use the following Lemma 3.6.4.

Lemma 3.6.4. *Let \mathcal{B} be a Boolean algebra and \mathcal{D} a dense subset of \mathcal{B}^+ . If there exist partitions of unity $W_m = \{a_{mn} : n < \omega\} \subseteq \mathcal{B}^+$, $m < \omega$, such that for each $d \in \mathcal{D}$, $\exists m < \omega$ such that $\forall n < \omega$ $d \wedge a_{mn} \neq 0$, then the hyper-weak (ω, ω) -d.l. fails everywhere in \mathcal{B} .*

Proof: Suppose $W_m = \{a_{mn} : n < \omega\} \subseteq \mathcal{B}^+$, $m < \omega$, are a family of partitions of unity with the property that for each $d \in \mathcal{D}$, $\exists m < \omega$ such that $\forall n < \omega$, $d \wedge a_{mn} \neq 0$. Since each W_m is a partition of unity,

$$(3.6.19) \quad \bigwedge_{m < \omega} \bigvee_{n < \omega} a_{mn} = 1.$$

We will show that for each $f : \omega \rightarrow \omega$

$$(3.6.20) \quad \bigwedge_{m < \omega} \bigvee_{n \neq f(m)} a_{mn} = 0.$$

Let $f : \omega \rightarrow \omega$ be given. Let $d \in \mathcal{D}$ and let \bar{m} be an element of ω such that $\forall n < \omega$, $d \wedge a_{\bar{m},n} \neq 0$. Then in particular, $d \wedge a_{\bar{m},f(\bar{m})} \neq 0$. Thus, $d \not\leq \bigvee_{n \neq f(\bar{m})} a_{\bar{m}n}$, since $W_{\bar{m}}$ is a partition of unity in \mathcal{B} . Hence,

$$(3.6.21) \quad d \not\leq \bigwedge_{m < \omega} \bigvee_{n \neq f(m)} a_{mn}.$$

Since this is true for all d in the dense set \mathcal{D} ,

$$(3.6.22) \quad \bigwedge_{m < \omega} \bigvee_{n \neq f(m)} a_{mn} = 0.$$

Since $f : \omega \rightarrow \omega$ was arbitrary, we have

$$(3.6.23) \quad \bigvee_{f: \omega \rightarrow \omega} \bigwedge_{m < \omega} \bigvee_{n \neq f(m)} a_{mn} = 0.$$

(3.6.19) and (3.6.23) show that the hyper-weak (ω, ω) -d.l. fails everywhere in \mathcal{B} .

□

Claim 2: The hyper-weak (ω, ω) -d.l. fails everywhere in \mathcal{A} .

By Lemma 3.6.4, it suffices to show that for each $[d]_{\mathcal{A}} \in \mathcal{D}_{\mathcal{A}}$, there is an $m < \omega$ such that $\forall n < \omega$, $[d]_{\mathcal{A}} \wedge [a_{mn}]_{\mathcal{A}} \neq [0]_{\mathcal{A}}$.

Let $[d]_{\mathcal{A}} = [B_X \cap \overline{B}_Y]_{\mathcal{A}} \in \mathcal{D}_{\mathcal{A}}$ be given. Choose $m < \omega$ large enough so that $m + 2 \geq |X| + 2$ and

$$(3.6.24) \quad (X \cup Y) \cap \{z_{m+2}, z_{m+3}, z_{m+4}, \dots\} = \emptyset.$$

For $n = 0$,

$$(3.6.25) \quad d \cap a_{m0} \supseteq B_{X \cup \{z_{m+2}, z_{m+3}\}} \cap \overline{B}_Y.$$

$\forall 2 \leq i \leq m + 2$, $r, z_{m+2}, z_{m+3} \in T_{i,j(i)}$; so $X \cup \{z_{m+2}, z_{m+3}\}$ intersects less than i -many T_{ij} 's. For $i = m + 3$, $r, z_{m+3} \in T_{m+3,j(m+3)}$; so $X \cup \{z_{m+2}, z_{m+3}\}$ intersects the same T_{ij} 's as $X \cup \{z_{m+3}\}$, which is less than $|X| + 2$ -many T_{ij} 's, where $|X| + 2 < m + 3$. $\forall i > m + 3$, $X \cup \{z_{m+2}, z_{m+3}\}$ intersects at most $|X| + 2$ -many T_{ij} 's, where $|X| + 2 < m + 3 < i$. Thus, $B_{X \cup \{z_{m+2}, z_{m+3}\}} \notin \mathcal{I}'$. By Corollary 3.4.3, $B_{X \cup \{z_{m+2}, z_{m+3}\}} \cap \overline{B}_Y \notin \mathcal{I}'$; so $d \cap a_{m0} \notin \mathcal{I}'$. Thus, $[d]_{\mathcal{A}} \wedge [a_{m0}]_{\mathcal{A}} \neq [0]_{\mathcal{A}}$.

In general, for $1 \leq n < \omega$,

$$(3.6.26) \quad d \cap a_{mn} = B_{X \cup \{z_{m+2}, z_{m+n+3}\}} \cap \overline{B}_{Y \cup \{z_{m+3}, \dots, z_{m+n+2}\}}.$$

As before, for each $2 \leq i \leq |X| + 2$, $r, z_{m+2}, z_{m+n+3} \in T_{i,j(i)}$; and $r, z_{m+n+3} \in T_{m+3,j(m+3)}$. Thus, $B_{X \cup \{z_{m+2}, z_{m+n+3}\}} \notin \mathcal{I}'$; so Corollary 3.4.3 implies that $B_{X \cup \{z_{m+2}, z_{m+n+3}\}} \cap \overline{B}_{Y \cup \{z_{m+3}\}} \notin \mathcal{I}'$. Thus, $d \cap a_{mn} \notin \mathcal{I}'$, from which it follows that $[d]_{\mathbf{A}} \wedge [a_{mn}]_{\mathbf{A}} \neq [0]_{\mathbf{A}}$.

Therefore, by Lemma 3.6.4, the hyper-weak (ω, ω) -d.l. fails everywhere in \mathbf{A} . This concludes the proof of Proposition 3.6.3.

□

Proposition 3.6.5. *\mathbf{A} is isomorphic to $\mathbf{B} \upharpoonright [B_{\{r\}}]_{\mathbf{B}}$.*

Proof: Let \mathbf{C} denote $\mathbf{B} \upharpoonright [B_{\{r\}}]_{\mathbf{B}}$. Define $a \sim_{\mathbf{C}} b \iff [a]_{\mathbf{B}}, [b]_{\mathbf{B}} \leq [B_{\{r\}}]_{\mathbf{B}}$ and $a \triangle b \in \mathcal{I}$. Note that for $b \subseteq B_{\{r\}}$, $[b]_{\mathbf{C}} = [b]_{\mathbf{B}}$.

Let $\phi : \mathbf{A} \rightarrow \mathbf{C}$ be defined by

$$(3.6.27) \quad \phi([b]_{\mathbf{A}}) = [b]_{\mathbf{C}},$$

for each $[b]_{\mathbf{A}} \in \mathbf{A}$. We shall show that ϕ is an isomorphism.

First, ϕ is well-defined: Suppose $[a]_{\mathbf{A}}, [b]_{\mathbf{A}} \in \mathbf{A}$ and $a \sim_{\mathbf{A}} b$. Then $a \triangle b \in \mathcal{I}' \subseteq \mathcal{I}$, so $a \sim_{\mathbf{B}} b$. Since $a, b \subseteq B_{\{r\}}$, $[a]_{\mathbf{B}} = [a]_{\mathbf{C}}$ and $[b]_{\mathbf{B}} = [b]_{\mathbf{C}}$. Thus, $[a]_{\mathbf{C}} = [b]_{\mathbf{C}}$.

$$(3.6.28) \quad \phi([a]_{\mathbf{A}} \vee [b]_{\mathbf{A}}) = \phi([a \cup b]_{\mathbf{A}}) = [a \cup b]_{\mathbf{C}} = [a]_{\mathbf{C}} \vee [b]_{\mathbf{C}} = \phi([a]_{\mathbf{A}}) \vee \phi([b]_{\mathbf{A}});$$

so ϕ preserves \vee . Let b^c denote the complement of b in $\text{Clop}(2^{(0,1)})$. Then

$$(3.6.29) \quad \begin{aligned} \phi(-[b]_{\mathbf{A}}) &= \phi([B_{\{r\}} \cap b^c]_{\mathbf{A}}) = [B_{\{r\}} \cap b^c]_{\mathbf{C}} = [B_{\{r\}} \cap b^c]_{\mathbf{B}} \\ &= [B_{\{r\}}]_{\mathbf{B}} \wedge [b^c]_{\mathbf{B}} = [B_{\{r\}}]_{\mathbf{B}} \wedge -[b]_{\mathbf{B}} = -[b]_{\mathbf{C}} = -\phi([b]_{\mathbf{C}}). \end{aligned}$$

Thus, ϕ preserves $-$. Thus, ϕ is a homomorphism.

Suppose $[a]_{\mathbf{A}} \neq [b]_{\mathbf{A}}$. Then $a \triangle b \notin \mathcal{I}'$. Since $a, b \subseteq B_{\{r\}}$, $a \triangle b \notin \mathcal{I}'$ implies $a \triangle b \notin \mathcal{I}$. Thus, $[a]_{\mathbf{B}} \neq [b]_{\mathbf{B}}$. Since $a, b \subseteq B_{\{r\}}$, we have $[a]_{\mathbf{C}} = [a]_{\mathbf{B}} \neq [b]_{\mathbf{B}} = [b]_{\mathbf{C}}$. Thus, $\phi([a]_{\mathbf{A}}) \neq \phi([b]_{\mathbf{A}})$, so ϕ is 1-1.

Suppose $[b]_{\mathbf{C}} \in \mathbf{C}$. Then $[b]_{\mathbf{C}} \leq [B_{\{r\}}]_{\mathbf{C}}$, so there is a $d \in \mathcal{I}$ such that $b \subseteq B_{\{r\}} \vee d$. Let $a = b \cap B_{\{r\}}$. $a \sim_{\mathbf{B}} b$, since $a \triangle b \subseteq d \setminus B_{\{r\}} \in \mathcal{I}$. Moreover, $[a]_{\mathbf{A}}$

is a well-defined element of \mathbf{A} , since $a \subseteq B_{\{r\}}$. Thus, $\phi([a]_{\mathbf{A}}) = [a]_{\mathbf{C}} = [a]_{\mathbf{B}} = [b]_{\mathbf{B}} = [b]_{\mathbf{C}}$. Hence, ϕ is onto. Thus, ϕ is an isomorphism.

□

Since the hyper-weak (ω, ω) -d.l. fails completely in \mathbf{A} and \mathbf{A} is isomorphic to the relative subalgebra \mathbf{C} of \mathbf{B} , the hyper-weak (ω, ω) -d.l. fails in \mathbf{B} . Furthermore, the hyper-weak (ω, ω) -d.l. fails everywhere on the relative subalgebra $\mathbf{B} \upharpoonright [B_{\{r\}}]_{\mathbf{B}}$.

3.7. OPEN PROBLEMS

We conclude this chapter with some open problems. In §§3.2, 3.3, and 3.5, we showed that the Cohen algebra embeds as a complete subalgebra into the Galvin-Hajnal, Argyros, and atomless Gaifman algebras. The first natural question is the following.

- (1) Is there a complete, atomless, c.c.c., non-measurable Boolean algebra in which the Cohen algebra does not embed as a complete subalgebra?

We have already pointed out that the Laver forcing \mathbf{L} satisfies the hyper-weak (ω, ω) -d.l. but not the weak (ω, ω) -d.l.; so $\text{r.o.}(\mathbf{L})$ is a complete, atomless, non-measurable algebra in which the Cohen algebra does not embed. However, $\text{r.o.}(\mathbf{L})$ does not satisfy the c.c.c. We conjecture that the answer to (1) is “yes”. If so, we ask whether there is a complete, atomless, c.c.c. Boolean algebra in which the hyper-weak (ω, ω) -d.l. fails, but the Cohen algebra does not embed as a complete subalgebra. Precisely, show that

- (2) The failure of the hyper-weak (ω, ω) -d.l. in a complete, atomless, c.c.c. Boolean algebra does not imply that the Cohen algebra can be embedded as a complete subalgebra.

or the stronger version,

- (3) There is a complete, atomless, c.c.c. Boolean algebra in which the hyper-weak (ω, ω) -d.l. fails everywhere, but the Cohen algebra does not embed as a complete subalgebra.

Chapter 4

GENERAL DISTRIBUTIVE LAWS AND RELATED GAMES

4.1. INTRODUCTION TO GENERAL DISTRIBUTIVE LAWS AND RELATED GAMES

Infinite games between two players arise naturally in the study of Boolean algebras. Foreman, Gray, Jech, Kamburelis, Shelah, and Vojtáš, among others, have investigated relationships between games and Boolean algebraic notions such as σ -closed dense subsets, proper forcing, and distributive laws. We focus on this latter property and its connections to games.

Each general distributive law is equivalent to a forcing property which says that functions in the extension model are bounded, in a way related to the particular distributive law, by functions in the ground model. Since it is often easier to prove the existence or non-existence of a winning strategy for a game than to show that a distributive law holds, it is of interest to find game-theoretic characterizations of distributive laws.

Games have a particularly desirable connection with von Neumann's problem. As von Neumann's problem has been studied almost entirely from the point of view of chain conditions, it remains unknown whether the countable chain condition along with some stronger form of the weak (ω, ω) -d.l. characterizes measurable algebras among Boolean σ -algebras. In §4.6, we show that the existence of a winning strategy for Player 2 in the game $\mathcal{G}_{\text{fin}}^\omega(\omega)$ (see Definition 4.3.3) is strictly stronger than the weak (ω, ω) -d.l., assuming \diamond . This opens an alternative approach to von Neumann's problem: investigate whether or not the c.c.c. and the existence of a winning strategy for Player 2 in the game $\mathcal{G}_{\text{fin}}^\omega(\omega)$ characterize

measurable algebras among Boolean σ -algebras.

Jech pioneered the field of games related to distributive laws in 1979 with his paper, *A game theoretic property of Boolean algebras* [14]. In it, he gave a game-theoretic characterization of (ω, ∞) -distributivity in complete Boolean algebras (see Theorem 4.1.11). A few years later, he obtained a game-theoretic characterization of (ω, κ) -distributivity in complete Boolean algebras [16] (see Theorem 4.1.14). This spurred related research in the early 1980's by Foreman, Gray, Vojtáš, and others. In particular, Foreman [5] obtained a game-theoretic characterization of the (η, ∞) -distributive law in complete Boolean algebras for all successor cardinals η .

In regard to weak distributivity, Jech found a relationship between the weak (ω, κ) -d.l. and a game he invented called $\mathcal{G}_{\text{fn}}(\kappa)$. Namely, he showed that, in a complete Boolean algebra, if the weak (ω, κ) -d.l. fails, then Player 1 has a winning strategy in $\mathcal{G}_{\text{fn}}(\kappa)$ [16] (see Definition 4.3.3 and Theorem 4.3.4). Jech asked whether the converse holds. This remained unanswered for a decade until Kamburelis showed that the converse does not hold; in fact, he showed that Player 1 does not have a winning strategy in the game $\mathcal{G}_{\text{fn}}(\kappa)$ played in a Boolean algebra \mathbf{B} if and only if the weak (ω, κ) -distributive law holds in \mathbf{B} and $V^{\mathbf{B}} \models ([\kappa]^\omega \cap V \text{ is stationary})$ [19] (see Theorem 4.3.6).

Except for Kamburelis' result, research in the area of games and distributive laws was sparse between 1984 and 1998. Then, in 1999, using a game similar to Foreman's game, Fuchino, Mildenerger, Shelah, and Vojtáš obtained a game-theoretic characterization of the (η, ∞) -distributive law in complete Boolean algebras for all cardinals η [7].

In view of these results, it is natural to ask: For which generalized distributive laws and for which cardinals, if any, can one obtain game-theoretic characterizations? For each generalized version of distributivity which we have investigated, there are certain pairs and triples of cardinals for which we have obtained game-

theoretic characterizations. These are given in §§4.2 - 4.5.

Now let us turn to the definitions and implications between various distributive laws. The following distributive laws are given in order of decreasing strength. For instance, the (η, κ) -d.l. implies the weak (η, κ) -d.l., which in turn implies the $(\eta, < \lambda, \kappa)$ -d.l., and so on.

Definition 4.1.1. [21] **B** satisfies the (η, κ) -distributive law $((\eta, \kappa)$ -d.l.) if for each $|I| \leq \eta$, $|J| \leq \kappa$, and family $(b_{ij})_{i \in I, j \in J}$ of elements of **B**,

$$(4.1.1) \quad \bigwedge_{i \in I} \bigvee_{j \in J} b_{ij} = \bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} b_{if(i)},$$

provided that $\bigvee_{j \in J} b_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij}$, and $\bigwedge_{i \in I} b_{if(i)}$ for each $f: I \rightarrow J$ exist in **B**. We say that **B** is (η, ∞) -distributive if it satisfies the (η, κ) -d.l. for all cardinals κ .

Definition 4.1.2. [21] **B** satisfies the weak (κ, λ) -distributive law (weak (κ, λ) -d.l.) if for each $|I| \leq \kappa$, $|J| \leq \lambda$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of **B**,

$$(4.1.2) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow [J]^{<\omega}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij}$ for each $f: I \rightarrow [J]^{<\omega}$ exist in **B**. We say that **B** is weakly (κ, ∞) -distributive if it satisfies the weak (κ, λ) -d.l. for all cardinals λ .

Definition 4.1.3. [17] For $2 \leq \lambda \leq \kappa$, **B** satisfies the $(\eta, < \lambda, \kappa)$ -distributive law $((\eta, < \lambda, \kappa)$ -d.l.) if for each $|I| \leq \eta$, $|J| \leq \kappa$, and family $(b_{ij})_{i \in I, j \in J}$ of elements of **B**,

$$(4.1.3) \quad \bigwedge_{i \in I} \bigvee_{j \in J} b_{ij} = \bigvee_{f: I \rightarrow [J]^{<\lambda}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} b_{ij},$$

provided that $\bigvee_{j \in J} b_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in f(i)} b_{ij}$ for each $f: I \rightarrow [J]^{<\lambda}$ exist in **B**. We say that **B** is $(\eta, < \lambda, \infty)$ -distributive if it satisfies the $(\eta, < \lambda, \kappa)$ -d.l. for all cardinals κ .

Definition 4.1.4. [26] For $\kappa \geq \omega$, \mathbf{B} satisfies the *super-weak* (η, κ) -*distributive law* (super-weak (η, κ) -d.l.) if for each $|I| \leq \eta$, $\omega \leq |J| \leq \kappa$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(4.1.4) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{\substack{f: I \rightarrow \mathcal{P}(J) \\ \forall i \in I, |J \setminus f(i)| = |J|}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in f(i)} a_{ij}$ for each $f : I \rightarrow \mathcal{P}(J)$ such that $\forall i \in I (|J \setminus f(i)| = |J|)$ exist in \mathbf{B} . We say that \mathbf{B} is *super-weakly* (η, ∞) -*distributive* if it satisfies the super-weak (η, κ) -d.l. for all cardinals $\kappa \geq \omega$.

Definition 4.1.5. [26] For $\kappa \geq \omega$, \mathbf{B} satisfies the *hyper-weak* (η, κ) -*distributive law* (hyper-weak (η, κ) -d.l.) if for each $|I| \leq \eta$, $\omega \leq |J| \leq \kappa$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(4.1.5) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} \bigvee_{j \in J \setminus \{f(i)\}} a_{ij},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in J \setminus \{f(i)\}} a_{ij}$ for each $f : I \rightarrow J$ exist in \mathbf{B} . We say that \mathbf{B} is *hyper-weakly* (η, ∞) -*distributive* if it satisfies the hyper-weak (η, κ) -d.l. for all cardinals $\kappa \geq \omega$.

Remark. The (η, κ) -d.l. is the same as the $(\eta, < 2, \kappa)$ -d.l., and the weak (η, κ) -d.l. is the same as the $(\eta, < \omega, \kappa)$ -d.l.

Theorem 4.1.6. Let $\eta, \kappa, \lambda, \nu$ be cardinals.

- (1) $\forall \eta, \kappa$, the (η, κ) -d.l. implies the weak (η, κ) -d.l.
- (2) $\forall \eta, \kappa$ and $\omega \leq \lambda$, the weak (η, κ) -d.l. implies the $(\eta, < \lambda, \kappa)$ -d.l.
- (3) $\forall \eta, \forall \omega \leq \lambda \leq \kappa$, the $(\eta, < \lambda, \kappa)$ -d.l. implies the super-weak (η, κ) -d.l.
- (4) $\forall \eta$ and $\forall \kappa \geq \omega$, the super-weak (η, κ) -d.l. implies the hyper-weak (η, κ) -d.l.

The proofs of (1) and (2) are found in [21]. The proofs of (3) and (4) follow naturally from the definitions.

Theorem 4.1.7. *For all cardinals $\eta_0 \leq \eta_1$, $\kappa_0 \leq \kappa_1$, $2 \leq \lambda_0 \leq \lambda_1$, and cardinals η, κ , in a complete Boolean algebra, the following implications hold:*

- (1) (η_1, κ_1) -distributivity $\implies (\eta_0, \kappa_0)$ -distributivity.
- (2) weak (η_1, κ_1) -distributivity \implies weak (η_0, κ_0) -distributivity.
- (3) For $\lambda_0 \leq \kappa_1$, $(\eta_1, < \lambda_0, \kappa_1)$ -distributivity $\implies (\eta_0, < \lambda_1, \kappa_0)$ -distributivity.
- (4) For $\kappa \geq \omega$, super-weak (η, κ) -distributivity \iff hyper-weak (η, κ) -distributivity.
- (5) For $\kappa_0 \geq \omega$, hyper-weak (η_1, κ_0) -distributivity \implies hyper-weak (η_0, κ_1) -distributivity.
- (6) hyper-weak (η, ω) -distributivity \iff hyper-weak (η, ∞) -distributivity.

(1)-(3) follow naturally from the definitions and can be found in [21]. (4)-(6) seem odd at first glance. It is easy to see that super-weak (η, κ) -distributivity implies hyper-weak (η, κ) -distributivity, but it does not seem obvious that the converse should hold. Likewise, it follows naturally from the definition that the hyper-weak (η_1, κ_1) -d.l. implies the hyper-weak (η_0, κ_0) -d.l., but the fact that the hyper-weak (η, κ_0) -d.l. implies the hyper-weak (η, κ_1) -d.l. is an anomaly of hyper-weak distributivity. In fact (5) and (6) are properties of hyper-weak distributivity which do not hold for other forms of distributivity. After some thought, though, (5) and (6) are not so surprising: roughly speaking, since partitions of larger cardinality are generally composed of smaller pieces, leaving out one piece of a partition of large cardinality should generally yield a larger supremum than the supremum of all but one piece of a partition of smaller cardinality. This will be made precise in the proof of Theorem 4.1.7, below. First, we need the following definition and lemma.

Definition 4.1.8. [21] For any cardinal κ , a collection $\{b_i : i < \kappa\} \subseteq \mathbf{B}$ is a

quasi-partition of \mathbf{B} if $\{b_i : i < \kappa\}$ is pairwise disjoint, and

$$(4.1.6) \quad \bigvee_{i < \kappa} b_i = 1.$$

Every partition of unity is a quasi-partition, but not the reverse: a quasi-partition may include the $\mathbf{0}$ element repeatedly.

Lemma 4.1.9. *For a κ -complete Boolean algebra \mathbf{B} and for cardinals η, κ with $\kappa \geq \omega$, the super-weak (η, κ) -d.l. holds in \mathbf{B} iff it holds for all collections of quasi-partitions $\{b_{ij} : j < \kappa\}$, $(i < \eta)$, of \mathbf{B} . Similarly, the hyper-weak (η, κ) -d.l. holds in \mathbf{B} iff it holds for all collections of quasi-partitions $\{b_{ij} : j < \kappa\}$, $(i < \eta)$, of \mathbf{B} .*

Proof: Certainly if the super-weak (η, κ) -d.l. holds, then it holds for all collections of quasi-partitions. Conversely, suppose that the super-weak (η, κ) -d.l. fails for some collection $\{b_{ij} : j < \kappa\}$, $(i < \eta)$. Then there are $b, c \in \mathbf{B}$ such that $\forall f : \eta \rightarrow \mathcal{T}$, where $\mathcal{T} = \{T \subseteq \kappa : |\kappa \setminus T| = \kappa\}$, $\bigwedge_{i < \eta} \bigvee_{j < \kappa} b_{ij} = b > c \geq \bigwedge_{i < \eta} \bigvee_{j \in f(i)} b_{ij}$.

For each $i < \eta$, construct a quasi-partition from $\{b_{ij} : j < \kappa\}$ as follows: Let

$$(4.1.7) \quad \begin{aligned} a_{i0} &= b_{i0} \wedge b \\ a_{i1} &= (b_{i1} \setminus a_{i0}) \wedge b \\ a_{i2} &= (b_{i2} \setminus (a_{i0} \vee a_{i1})) \wedge b \\ &\vdots \\ a_{ij} &= (b_{ij} \setminus (\bigvee_{k < j} a_{ik})) \wedge b \\ &\vdots \end{aligned}$$

for each $j < \kappa$, and let $a_{i\kappa} = 1 \setminus b$. Then $\{a_{ij} : j \leq \kappa\}$ is a quasi-partition, since $\{a_{ij} : j \leq \kappa\}$ is certainly pairwise disjoint, and

$$(4.1.8) \quad \bigvee_{j \leq \kappa} a_{ij} = (1 \setminus b) \vee \bigvee_{j < \kappa} a_{ij} = (1 \setminus b) \vee b = 1.$$

We will show that the super-weak (η, κ) -d.l. fails for the quasi-partitions $\{a_{ij} : j \leq \kappa\}$, $(i < \eta)$. Let $\mathcal{S} = \{S \subseteq \kappa + 1 : |\kappa + 1 \setminus S| = \kappa\}$ and let $f : \eta \rightarrow \mathcal{S}$ be given.

$$(4.1.9) \quad \bigwedge_{i < \eta} \bigvee_{j \in f(i)} a_{ij} \leq \bigwedge_{i < \eta} \bigvee_{j \in f(i)} (b_{ij} \vee a_{i\kappa}),$$

since $\forall i < \kappa$, $a_{ij} \leq b_{ij}$. Recall $\mathcal{T} = \{T \subseteq \kappa : |\kappa \setminus T| = \kappa\}$. Define $g : \eta \rightarrow \mathcal{T}$ by $g(i) = f(i) \cap \kappa$, $\forall i < \eta$. Then,

$$(4.1.10) \quad \begin{aligned} \bigwedge_{i < \eta} \bigvee_{j \in f(i)} a_{ij} &\leq \bigwedge_{i < \eta} \bigvee_{j \in g(i)} (b_{ij} \vee a_{i\kappa}) \\ &= \bigwedge_{i < \eta} \bigvee_{j \in g(i)} (b_{ij} \vee (1 \setminus b)) \\ &\leq c \vee (1 \setminus b) \end{aligned}$$

Since for each $f : \eta \rightarrow \mathcal{S}$ there is a $g : \eta \rightarrow \mathcal{T}$ such that (4.1.10) holds,

$$(4.1.11) \quad \begin{aligned} \bigvee_{f: \eta \rightarrow \mathcal{S}} \bigwedge_{i < \eta} \bigvee_{j \in f(i)} a_{ij} &\leq \bigvee_{g: \eta \rightarrow \mathcal{T}} \bigwedge_{i < \eta} \bigvee_{j \in g(i)} (a_{ij} \vee a_{i\kappa}) \\ &\leq c \vee (1 \setminus b) \\ &< 1, \end{aligned}$$

since $c < b$. However, $\bigwedge_{i < \eta} \bigvee_{j < \kappa} a_{ij} = 1$. Thus, the super-weak (η, κ) -d.l. fails for the collection of quasi-partitions $\{a_{ij} : j < \kappa\}$, $(i < \eta)$.

The proof for the hyper-weak (η, κ) -d.l. is analogous.

□

Now we are ready to prove (4)-(6) of Theorem 4.1.7.

Proof of Theorem 4.1.7: (4). It follows naturally from the definitions that the super-weak (η, κ) -d.l. implies the hyper-weak (η, κ) -d.l. To show the converse, suppose that the hyper-weak (η, κ) -d.l. holds. Let $\{a_{ijk} : j, k < \kappa\}$, $(i < \eta)$, be quasi-partitions of \mathbf{B} . (We use triple subscripts for ease of notation later.) Let $\{b_{ij} : j < \kappa\}$, $(i < \eta)$, be quasi-partitions of \mathbf{B} given by $b_{ij} = \bigvee_{k < \kappa} a_{ijk}$, $\forall j < \kappa$.

Let $\mathcal{S} = \{S \subseteq \kappa \times \kappa : |(\kappa \times \kappa) \setminus S| = \kappa\}$. Let $f : \eta \rightarrow \kappa$ be given. Define $g : \eta \rightarrow \mathcal{S}$ by

$$(4.1.12) \quad g(i) = \{(j, k) \in \kappa \times \kappa : j \neq f(i)\}.$$

Then

$$(4.1.13) \quad \bigwedge_{i < \eta} \bigvee_{(j, k) \in g(i)} a_{ijk} = \bigwedge_{i < \eta} \bigvee \{b_{ij} : j < \kappa, j \neq f(i)\}.$$

Since for each $f : \eta \rightarrow \kappa$ we can find a $g : \eta \rightarrow \mathcal{S}$ such that (4.1.13) holds,

$$(4.1.14) \quad \bigvee_{g: \eta \rightarrow \mathcal{S}} \bigwedge_{i < \eta} \bigvee_{(j, k) \in g(i)} a_{ijk} \geq \bigvee_{f: \eta \rightarrow \kappa} \bigwedge_{i < \eta} \bigvee \{b_{ij} : j < \kappa, j \neq f(i)\} = 1,$$

since the hyper-weak (η, κ) -d.l. holds in \mathbf{B} . Thus, the super-weak (η, κ) -d.l. holds in \mathbf{B} .

(5). Suppose $\eta_0 \leq \eta_1$ and $\kappa_0 \leq \kappa_1$. It follows immediately from the definition that the hyper-weak (η_1, κ) -d.l. implies the hyper-weak (η_0, κ) -d.l. We will show that the hyper-weak (η, κ_0) -d.l. implies the hyper-weak (η, κ_1) -d.l.

Suppose the hyper-weak (η, κ_0) -d.l. holds. By Lemma 4.1.9, it suffices to show that the hyper-weak (η, κ_1) -d.l. holds for all families of quasi-partitions of \mathbf{B} . Let $\{b_{ij} : j < \kappa_1\}$, $(i < \eta)$, be a family of quasi-partitions. For each $i < \eta$, for each $0 < j < \kappa_0$, let $a_{ij} = b_{ij}$. Let $a_{i0} = b_{i0} \vee \bigvee_{\kappa_0 \leq j < \kappa_1} b_{ij}$. Then $\{a_{ij} : j < \kappa_0\}$, $(i < \eta)$, are quasi-partitions of \mathbf{B} .

Given $f : \eta \rightarrow \kappa_0$, let $g : \eta \rightarrow \kappa_1$ be given by

$$(4.1.15) \quad g(i) = f(i).$$

Then

$$(4.1.16) \quad \bigwedge_{i < \eta} \bigvee_{j < \kappa_1, j \neq g(i)} b_{ij} \geq \bigwedge_{i < \eta} \bigvee_{j < \kappa_0, j \neq f(i)} a_{ij}.$$

Since for each $f : \eta \rightarrow \kappa_0$ we can construct a g such that (4.1.16) holds,

$$(4.1.17) \quad \bigvee_{g:\eta \rightarrow \kappa_1} \bigwedge_{i < \eta} \bigvee_{j < \kappa_1, j \neq g(i)} b_{ij} \geq \bigvee_{f:\eta \rightarrow \kappa_0} \bigwedge_{i < \eta} \bigvee_{j < \kappa_0, j \neq f(i)} a_{ij} \geq 1,$$

since the hyper-weak (η, κ_0) -d.l. holds. Thus, the hyper-weak (η, κ_1) -d.l. holds.

(6). This follows from (5). Let \mathbf{B} be a Boolean algebra and let $\kappa \geq \omega$. If the hyper-weak (η, ω) -d.l. holds in \mathbf{B} , then by (5), the hyper-weak (η, κ) -d.l. holds. Letting κ range over all cardinals greater than or equal to ω , we find that the hyper-weak (η, ∞) -d.l. holds. By Definition 4.1.5, the hyper-weak (η, ∞) -d.l. implies the hyper-weak (η, ω) -d.l.

□

Games between two players arise naturally in the study of distributive laws in Boolean algebras. Jech obtained a game-theoretic characterization of the (ω, ∞) -distributive law using the following game \mathcal{G} .

Definition 4.1.10. [14] The game \mathcal{G} is played by two players in a complete Boolean algebra \mathbf{B} as follows: Player 1 begins the game by choosing some $a_0 \in \mathbf{B}^+$ (where $\mathbf{B}^+ = \mathbf{B} \setminus \{0\}$); then Player 2 chooses some $b_0 \in \mathbf{B}^+$ such that $b_0 \leq a_0$. The two players take turns choosing elements $a_n, b_n \in \mathbf{B}^+$ to form a descending sequence

$$(4.1.18) \quad a_0 \geq b_0 \geq a_1 \geq b_1 \geq \cdots \geq a_n \geq b_n \geq \cdots$$

Player 1 *wins the play* (4.1.18) iff the sequence (4.1.18) has no lower bound in \mathbf{B}^+ .

Theorem 4.1.11 (Jech). [14] *For \mathbf{B} complete, the (ω, ∞) -distributive law holds in \mathbf{B} iff Player 1 does not have a winning strategy in \mathcal{G} .*

Foreman obtained a game-theoretic characterization of the (η, ∞) -distributive law for successor cardinals η [5]. His game is played like Jech's \mathcal{G} , but played in

η -many rounds, where P2 chooses first at limit ordinals. Vojtáš made an improvement on Foreman's result, but still did not obtain a complete characterization of (η, ∞) -distributivity [30]. Over a decade later, for a partial ordering \mathbf{P} , Fuchino, Mildenberger, Shelah, and Vojtáš defined the game $G(\mathbf{P}, \eta)$, played in \mathbf{P} just like Foreman's game, except that Player 1 plays first at each limit ordinal. Using $G(\mathbf{P}, \eta)$, they obtained a game-theoretic characterization of the (η, ∞) -d.l. for all cardinals η .

Theorem 4.1.12 (Fuchino, Mildenberger, Shelah, and Vojtáš). [7] *For \mathbf{P} separative, for all cardinals η , the (η, ∞) -distributive law holds in $r.o.(\mathbf{P})$ iff Player 1 does not have a winning strategy in $G(\mathbf{P}, \eta)$.*

Interested in the second parameter of the (ω, κ) -distributive law, Jech defined the following game.

Definition 4.1.13. [16] The game $\mathcal{G}_1(\kappa)$ is played by two players in a complete Boolean algebra as follows. At the beginning of the game, Player 1 chooses some $a \in \mathbf{B}^+$ which is then fixed throughout the ω -many rounds. On the n -th round, Player 1 chooses $W_n \subseteq \mathbf{B}^+$, a partition of a such that $|W_n| \leq \kappa$; then Player 2 chooses some $b_n \in W_n$. Player 1 *wins the play*

$$(4.1.19) \quad \langle a, W_0, b_0, W_1, b_1, \dots, W_n, b_n, \dots \rangle$$

iff

$$(4.1.20) \quad \bigwedge_{n < \omega} b_n = 0.$$

Jech obtained the following game-theoretic characterization of the (ω, κ) -d.l.

Theorem 4.1.14. [16] *For \mathbf{B} complete, for all cardinals κ , the (ω, κ) -d.l. holds in \mathbf{B} iff Player 1 does not have a winning strategy in $\mathcal{G}_1(\kappa)$.*

In the proceeding sections, we will work with variations of Jech's game $\mathcal{G}_1(\kappa)$. In §4.2, we present the game $\mathcal{G}_1^\eta(\kappa)$ which generalizes $\mathcal{G}_1(\kappa)$ to plays of uncountable length. We show that, for cardinals η, κ such that $\kappa^{<\eta} = \eta$ or $\kappa^{<\eta} = \kappa$, the

(η, κ) -distributive law is characterized by the non-existence of a winning strategy for Player 1 in $\mathcal{G}_1^\eta(\kappa)$. It immediately follows that the (η, ∞) -distributive law holds in \mathbf{B} iff Player 1 does not have a winning strategy in $\mathcal{G}_1^\eta(\infty)$. This yields a characterization of (η, ∞) -distributivity different than the characterization obtained by Fuchino, Mildenberger, Shelah and Vojtáš [7]. Moreover, it implies that the existence of a winning strategy for Player 1 in the games $G(\mathbf{B}^+, \eta)$ and $\mathcal{G}_1^\eta(\infty)$ are equivalent.

In §4.3, we present the game $\mathcal{G}_{\text{fin}}^\eta(\kappa)$, a generalization of Jech's game $\mathcal{G}_{\text{fin}}(\kappa)$ to plays of uncountable length. This game is played similarly to $\mathcal{G}_1^\eta(\kappa)$, except that now Player 2 chooses finitely-many pieces from each of Player 1's partitions. We show that, for cardinals η, κ such that $\kappa^{<\eta} = \eta$, the weak (η, κ) -distributive law holds iff Player 1 does not have a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$. §4.4 and §4.5 consider generalizations of §4.3 to the $(\eta, < \lambda, \kappa)$ -d.l. and the hyper-weak (η, κ) -d.l., respectively.

In §4.6, for regular cardinals $\eta \geq \omega$, we generalize a result of Jech [16] and use \Diamond_{η^+} to construct an η^+ -Suslin algebra in which the games $\mathcal{G}_1^\eta(\kappa)$ (for κ), $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ (for $\kappa \geq \omega$), and $\mathcal{G}_{<\lambda}^\eta(\kappa)$ (for $2 \leq \lambda \leq \min(\eta, \kappa)$) are undetermined, and an η^+ -Suslin algebra in which the game $\mathcal{G}_{\kappa-1}^\eta$ (for $\omega \leq \kappa \leq \eta$) is undetermined. This result is surprising, as it implies the the existence of a winning strategy for Player 2 in each of the games does not follow from any of the distributive laws. §4.7 summarizes relationships between winning strategies for the two players in the various games and their corresponding distributive laws.

4.2. THE (η, κ) -DISTRIBUTIVE LAW AND THE GAME $\mathcal{G}_1^\eta(\kappa)$

Recall the (η, κ) -distributive law.

Definition 4.2.1. [21] \mathbf{B} satisfies the (η, κ) -distributive law $((\eta, \kappa)$ -d.l.) if for

each $|I| \leq \eta$, $|J| \leq \kappa$, and family $(b_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(4.2.1) \quad \bigwedge_{i \in I} \bigvee_{j \in J} b_{ij} = \bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} b_{if(i)},$$

provided that $\bigvee_{j \in J} b_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij}$, and $\bigwedge_{i \in I} b_{if(i)}$ for each $f: I \rightarrow J$ exist in \mathbf{B} . We say that \mathbf{B} is (η, ∞) -distributive if it satisfies the (η, κ) -d.l. for all cardinals κ .

Jech showed that the (η, κ) -d.l. is equivalent to the following forcing property.

Proposition 4.2.2. [17] *If \mathbf{B} is complete, then \mathbf{B} is (η, κ) -distributive iff every function from η to κ in the generic extension $V[G]$ belongs to the ground model V . \mathbf{B} is (η, ∞) -distributive iff every function $f: \eta \rightarrow V$ in $V[G]$ is in V .*

The following generalizes Jech's game $\mathcal{G}_1(\kappa)$ to plays of uncountable length .

Definition 4.2.3. Let κ be any cardinal and η be any infinite cardinal. The game $\mathcal{G}_1^\eta(\kappa)$ is played between two players in an η^+ -complete Boolean algebra \mathbf{B} as follows: At the beginning of the game, Player 1 (P1) chooses some $a \in \mathbf{B}^+$ which is fixed throughout the η -many rounds. For $\alpha < \eta$, the α -th round is played as follows: P1 chooses a partition W_α of a such that $|W_\alpha| \leq \kappa$; then Player 2 (P2) chooses some $b_\alpha \in W_\alpha$. In this manner, the two players construct a sequence of length η

$$(4.2.2) \quad \langle a, W_0, b_0, W_1, b_1, \dots, W_\alpha, b_\alpha, \dots : \alpha < \eta \rangle$$

called a *play* of the game. P1 *wins the play* (4.2.2) if and only if

$$(4.2.3) \quad \bigwedge_{\alpha < \eta} b_\alpha = \mathbf{0}.$$

A *strategy for P1* is a function $\sigma: \{\mathbf{0}\} \cup (\mathbf{B}^+)^{<\eta} \rightarrow [\mathbf{B}^+]^{\leq \kappa}$ such that $\sigma(\mathbf{0}) = \{a\}$ and for each $\langle b_\alpha : \alpha < \beta \rangle \in (\mathbf{B}^+)^{<\eta}$, $\sigma(\langle b_\alpha : \alpha < \beta \rangle)$ is a partition of a . σ is a *winning strategy* if P1 wins every time P1 follows σ .

A *strategy for P2* is a function $\tau : ([\mathbf{B}^+]^{\leq \kappa})^{< \eta} \rightarrow \mathbf{B}^+$ such that for each $\langle W_\alpha : \alpha \leq \beta \rangle \in ([\mathbf{B}^+]^{\leq \kappa})^{< \eta}$, $\tau(\langle W_\alpha : \alpha \leq \beta \rangle) \in W_\beta$. τ is a *winning strategy* if whenever P2 plays by τ , P2 wins.

$\mathcal{G}_1^\eta = \mathcal{G}_1^\eta(\infty)$ is the game played as above, except now P1 can choose partitions of any size. We note that Jech's game $\mathcal{G}_1(\kappa)$ is the same as $\mathcal{G}_1^\omega(\kappa)$ in our notation.

Remark. \mathcal{G}_1^η can be played in a partial ordering \mathbf{P} in the natural way: At the beginning of the game, P1 chooses some fixed $p \in \mathbf{P}$. On the α -th round, P1 chooses a maximal incompatible subset $M_\alpha \subseteq \mathbf{P}$ below p ; then P2 chooses one element $p_\alpha \in M_\alpha$. This constructs the sequence

$$(4.2.4) \quad \langle p, M_0, p_0, M_1, p_1, \dots, M_\alpha, p_\alpha, \dots : \alpha < \eta \rangle.$$

P1 wins the play (4.2.4) if and only if $\forall q \in \mathbf{P}, \exists \alpha < \eta$ such that $q \not\leq p_\alpha$. If \mathbf{P} is separative, then the existence of a winning strategy for P1 or P2 is invariant between \mathbf{P} and $\text{r.o.}(\mathbf{P})$.

Note. One can easily show that if P2 has a winning strategy in $\mathcal{G}_1^\eta(\kappa)$, then the (η, κ) -d.l. holds. It then follows that if P2 has a winning strategy in \mathcal{G}_1^η , then the (η, ∞) -d.l. holds.

Jech obtained the following characterization of the (ω, κ) -d.l.

Theorem 4.2.4 (Jech). [16] *If \mathbf{B} is complete, then the (ω, κ) -d.l. holds in \mathbf{B} iff P1 does not have a winning strategy in $\mathcal{G}_1^\omega(\kappa)$.*

The “if” direction easily generalizes to all pairs of cardinals η, κ .

Theorem 4.2.5. *If \mathbf{B} is η^+ -complete and the (η, κ) -d.l. fails in \mathbf{B} , then P1 has a winning strategy in $\mathcal{G}_1^\eta(\kappa)$.*

Proof: Follows from Theorem 4.4.4 with $\lambda = 2$.

□

We have obtained the following partial converse to Theorem 4.2.5.

Theorem 4.2.6. *If \mathbf{B} is η^+ -complete and $P1$ has a winning strategy in $\mathcal{G}_1^\eta(\kappa)$, then*

- (1) *the $(\kappa^{<\eta}, \kappa)$ -d.l. fails;*
- (2) *the $(\eta, \kappa^{<\eta})$ -d.l. fails.*

Proof: (1) follows from Theorem 4.4.5 with $\lambda = 2$. The proof of (2) uses ideas from Jech's proof of Theorem 4.2.4, the main difference being that here, limit ordinals must be treated with care, as in Case 1, below. Let σ be a winning strategy for $P1$ in $\mathcal{G}_1^\eta(\kappa)$. Let $\{a\} = \sigma(\emptyset)$ and $P_0 = \sigma(\langle \rangle)$. For each $x_0 \in P_0$, let $W_1(\langle x_0 \rangle) = \{x_0 \wedge z : z \in \sigma(\langle x_0 \rangle)\}$. Let $P_1 = \bigcup \{W_1(\langle x_0 \rangle) : x_0 \in \sigma(\langle \rangle)\}$. For each $x_1 \in \sigma(\langle x_0 \rangle)$, let $W_2(\langle x_0, x_1 \rangle) = \{x_0 \wedge x_1 \wedge z : z \in \sigma(\langle x_0, x_1 \rangle)\}$. Let $P_2 = \bigcup \{W_2(\langle x_0, x_1 \rangle) : x_0 \in \sigma(\langle \rangle), x_1 \in \sigma(\langle x_0 \rangle)\}$. In general, given $\alpha < \eta$ and a sequence $\langle x_\beta : \beta < \alpha \rangle \in (\mathbf{B}^+)^{<\alpha}$ such that $\forall \beta < \alpha \ x_\beta \in \sigma(\langle x_\gamma : \gamma < \beta \rangle)$, let

$$(4.2.5) \quad W_\alpha(\langle x_\beta : \beta < \alpha \rangle) = \{(\bigwedge_{\beta < \alpha} x_\beta) \wedge z : z \in \sigma(\langle x_\beta : \beta < \alpha \rangle)\},$$

and let

$$(4.2.6) \quad P_\alpha = \bigcup \{W_\alpha(\langle x_\beta : \beta < \alpha \rangle) : \forall \beta < \alpha, x_\beta \in \sigma(\langle x_\gamma : \gamma < \beta \rangle)\}.$$

Note that

- (a) $\forall \alpha < \eta$, $P_\alpha \subseteq \mathbf{B}^+$ is pairwise disjoint and $|P_\alpha| \leq \kappa^{\alpha+1} \leq \kappa^{<\eta}$;
- (b) If $\beta < \alpha < \eta$, then $\forall x \in P_\alpha \ \exists y \in P_\beta$ such that $y \geq x$;
- (c) If $\alpha < \eta$ and $\forall \beta < \alpha, x_\beta \in \sigma(\langle x_\gamma : \gamma < \beta \rangle)$, then $\bigvee W_\alpha(\langle x_\beta : \beta < \alpha \rangle) = \bigwedge_{\beta < \alpha} x_\beta$;
- (d) $\forall \alpha < \eta$, $\bigvee P_{\alpha+1} = \bigvee P_\alpha$;
- (e) If
 - (i) $\beta \leq \gamma < \eta$;
 - (ii) $\langle x_\zeta : \zeta < \beta \rangle, \langle y_\zeta : \zeta < \gamma \rangle$ are sequences such that $\forall \zeta < \beta$
 $x_\zeta \in \sigma(\langle x_\theta : \theta < \zeta \rangle)$ and $y_\zeta \in \sigma(\langle y_\theta : \theta < \zeta \rangle)$; and
 - (iii) $\exists \delta \leq \beta$ such that $x_\delta \neq y_\delta$;

then $\bigwedge_{\zeta \leq \beta} x_\zeta \wedge \bigwedge_{\zeta \leq \gamma} y_\zeta = 0$.

(a)-(d) are clear. For (e), if $\delta < \alpha$ is least such that $x_\delta \neq y_\delta$, then $x_\delta, y_\delta \in \sigma(\langle x_\gamma : \gamma < \delta \rangle)$, a partition of a , so $x_\delta \wedge y_\delta = 0$.

Claim: The $(\eta, \kappa^{<\eta})$ -d.l. fails for the collection P_α , $(\alpha < \eta)$. By (a), we can index the elements of P_α using the index set $\kappa^{<\eta}$ so that $P_\alpha = \{b_{\alpha, \gamma} : \gamma < \kappa^{<\eta}\}$, allowing repetitions.

Case 1: $\exists \alpha < \eta$ for which $\bigvee P_\alpha < a$. Let α be least such that $\bigvee P_\alpha < a$. (d) implies α is a limit ordinal. We will show that the $(\alpha, \kappa^{<\eta})$ -d.l. fails for the partitions P_β , $(\beta < \alpha)$, of a . Since $\bigwedge_{\beta < \alpha} \bigvee P_\beta = a$, it suffices to show that

$$(4.2.7) \quad \bigvee_{f: \alpha \rightarrow \kappa^{<\eta}} \bigwedge_{\beta < \alpha} b_{\beta, f(\beta)} \leq \bigvee P_\alpha.$$

Let $f : \alpha \rightarrow \kappa^{<\eta}$ be given. $\forall \beta < \alpha$, $b_{\beta, f(\beta)} \in P_\beta$; so there is a sequence $\langle x_\zeta^\beta : \zeta \leq \beta \rangle$ such that $\forall \zeta \leq \beta$, $x_\zeta^\beta \in \sigma(\langle x_\gamma^\beta : \gamma < \zeta \rangle)$, and $b_{\beta, f(\beta)} = \bigwedge_{\zeta \leq \beta} x_\zeta^\beta$.

Suppose $\forall \zeta \leq \beta < \gamma < \alpha$, $x_\zeta^\beta = x_\zeta^\gamma$. $\forall \beta < \alpha$, let x_β denote x_β^β . Then $\forall \beta < \alpha$, $x_\beta \in \sigma(\langle x_\zeta : \zeta < \beta \rangle)$, so (c) implies $\bigwedge_{\beta < \alpha} x_\beta = \bigvee W_\alpha(\langle x_\beta : \beta < \alpha \rangle)$. Therefore,

$$(4.2.8) \quad \bigwedge_{\beta < \alpha} b_{\beta, f(\beta)} = \bigwedge_{\beta < \alpha} \bigwedge_{\zeta \leq \beta} x_\zeta^\beta = \bigwedge_{\beta < \alpha} x_\beta = \bigvee W_\alpha(\langle x_\beta : \beta < \alpha \rangle) \leq \bigvee P_\alpha.$$

Otherwise, $\exists \delta \leq \beta < \gamma < \alpha$ such that $x_\delta^\beta \neq x_\delta^\gamma$. In this case, (e) implies

$$(4.2.9) \quad \bigwedge_{\zeta < \alpha} b_{\zeta, f(\zeta)} \leq b_{\beta, f(\beta)} \wedge b_{\gamma, f(\gamma)} = \bigwedge_{\zeta \leq \beta} x_\zeta^\beta \wedge \bigwedge_{\zeta \leq \gamma} x_\zeta^\gamma = 0.$$

Since (4.2.8) and (4.2.9) hold for all $f : \alpha \rightarrow \kappa^{<\eta}$, (4.2.7) holds. Therefore, the $(\alpha, \kappa^{<\eta})$ -d.l. fails.

Case 2: For each $\alpha < \eta$, $\bigvee P_\alpha = a$. Then (b) implies $\forall \beta < \alpha < \eta$, P_α is a refinement of P_β . Let $f : \eta \rightarrow \kappa^{<\eta}$ be given. We will show that $\bigwedge_{\alpha < \eta} b_{\alpha, f(\alpha)} = 0$. If $\exists \alpha < \eta$ such that $b_{\alpha, f(\alpha)} = 0$, then we are done; so assume $\forall \alpha < \eta$ $b_{\alpha, f(\alpha)} \neq 0$.

Suppose $\exists \beta < \alpha < \eta$ such that $b_{\beta, f(\beta)} \not\geq b_{\alpha, f(\alpha)}$. Then, $b_{\beta, f(\beta)} \wedge b_{\alpha, f(\alpha)} = \mathbf{0}$, since P_α is a refinement of P_β . Hence, $\bigwedge_{\alpha < \eta} b_{\alpha, f(\alpha)} = \mathbf{0}$.

Otherwise, $\forall \beta < \alpha < \eta$, $b_{\beta, f(\beta)} \geq b_{\alpha, f(\alpha)} > \mathbf{0}$. Again, $\forall \beta < \alpha$ let $\langle x_\zeta^\beta : \zeta < \beta \rangle$ be the sequence such that $\forall \zeta \leq \beta$, $x_\zeta^\beta \in \sigma(\langle x_\gamma^\beta : \gamma < \zeta \rangle)$, and $b_{\beta, f(\beta)} = \bigwedge_{\zeta \leq \beta} x_\zeta^\beta$.

We claim that

$$(4.2.10) \quad \forall \zeta \leq \beta < \alpha < \eta, \quad x_\zeta^\beta = x_\zeta^\alpha.$$

If (4.2.10) fails, then $\exists \delta, \beta, \alpha$ with $\delta \leq \beta < \alpha < \eta$ such that $x_\delta^\beta \neq x_\delta^\alpha$. Then $b_{\beta, f(\beta)} \wedge b_{\alpha, f(\alpha)} = \mathbf{0}$, by (e). But this implies $b_{\alpha, f(\alpha)} = \mathbf{0}$, since $b_{\alpha, f(\alpha)} \leq b_{\beta, f(\beta)}$. Contradiction. Thus, (4.2.10) holds.

$\forall \alpha < \eta$ let x_α denote x_α^α . Then

$$(4.2.11) \quad \langle a, \sigma(\langle \rangle), x_0, \sigma(\langle x_0 \rangle), x_1, \dots, \sigma(\langle x_\beta : \beta < \alpha \rangle), x_\alpha, \dots : \alpha < \eta \rangle$$

is a play in $\mathcal{G}_1^\eta(\kappa)$ in which P1 follows σ . So,

$$(4.2.12) \quad \bigwedge_{\alpha < \eta} b_{\alpha, f(\alpha)} = \bigwedge_{\alpha < \eta} \bigwedge_{\zeta \leq \alpha} x_\zeta^\alpha = \bigwedge_{\alpha < \eta} x_\alpha = \mathbf{0}.$$

Since $f : \eta \rightarrow \kappa^{<\eta}$ was arbitrary,

$$(4.2.13) \quad \bigvee_{f: \eta \rightarrow \kappa^{<\eta}} \bigwedge_{\alpha < \eta} b_{\alpha, f(\alpha)} = \mathbf{0} < a.$$

Thus, the $(\eta, \kappa^{<\eta})$ -d.l. fails for the partitions P_α , $(\alpha < \eta)$, of a .

□

Whether the full converse of Theorem 4.2.6 holds is unknown. However, for certain pairs of cardinals η, κ we have the following:

Corollary 4.2.7. *If \mathbf{B} is η^+ -complete and $\kappa^{<\eta} = \eta$ or $\kappa^{<\eta} = \kappa$, then the (η, κ) -d.l. holds in \mathbf{B} iff P1 does not have a winning strategy in $\mathcal{G}_1^\eta(\kappa)$.*

From Corollary 4.2.7, we obtain a characterization of the (η, ∞) -d.l. which differs from the that of Theorem 4.1.12 obtained by Fuchino, Mildenberger, Shelah, and Vojtáš [7].

Corollary 4.2.8. *If \mathbf{B} is η^+ -complete, then the (η, ∞) -d.l. holds iff $P1$ does not have a winning strategy in \mathcal{G}_1^η .*

Corollary 4.2.8 implies that the existence of a winning strategy for $P1$ in \mathcal{G}_1^η is equivalent to the existence of a winning strategy for $P1$ in the game $G(\mathbf{B}^+, \eta)$.

Assuming GCH, Corollary 4.2.7 gives a characterization of the (η, κ) -d.l. for all $\kappa < \eta$ and for all $\kappa \geq \eta$ with $\text{cf}(\kappa) \geq \eta$.

Corollary 4.2.9. (GCH) *If \mathbf{B} is η^+ -complete, then $\forall \kappa < \eta$ and $\forall \kappa$ with $\text{cf}(\kappa) \geq \eta$, the (η, κ) -d.l. holds iff $P1$ does not have a winning strategy in $\mathcal{G}_1^\eta(\kappa)$.*

4.3. THE WEAK (η, κ) -DISTRIBUTIVE LAW AND THE GAME $\mathcal{G}_{\text{fin}}^\eta(\kappa)$

In this section we present connections between the weak (η, κ) -distributive law and the game $\mathcal{G}_{\text{fin}}^\eta(\kappa)$. Recall the following definition.

Definition 4.3.1. [21] \mathbf{B} satisfies the *weak (η, κ) -distributive law* if for each $|I| \leq \eta$, $|J| \leq \kappa$, and family $(b_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(4.3.1) \quad \bigwedge_{i \in I} \bigvee_{j \in J} b_{ij} = \bigvee_{f: I \rightarrow [J]^{<\omega}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} b_{ij},$$

provided that $\bigvee_{j \in J} b_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in f(i)} b_{ij}$ for each $f: I \rightarrow [J]^{<\omega}$ exist in \mathbf{B} . We say that \mathbf{B} is *weakly (η, ∞) -distributive* if it satisfies the weak (η, κ) -d.l. for all cardinals κ .

The weak (η, κ) -d.l. has the following forcing property.

Proposition 4.3.2. *If \mathbf{B} is complete, then \mathbf{B} is weakly (η, κ) -distributive iff for every function $g: \eta \rightarrow \kappa$ in $V[G]$ there is some function $f: \eta \rightarrow [\kappa]^{<\omega}$ in V such that $\forall \alpha < \eta$, $g(\alpha) \in f(\alpha)$. \mathbf{B} is weakly (η, ∞) -distributive iff for each function*

$g : \eta \rightarrow V$ in $V[G]$, there is a function $f : \eta \rightarrow [V]^{<\omega}$ in V such that $\forall \alpha < \eta$, $g(\alpha) \in f(\alpha)$.

Proof: Follows from Proposition 4.4.2 with $\lambda = \omega$.

□

We now introduce the game $\mathcal{G}_{\text{fin}}^\eta(\kappa)$, which generalizes Jech's game $\mathcal{G}_{\text{fin}}(\kappa)$ (in [16]) to plays of uncountable length. $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ is played just like $\mathcal{G}_1^\eta(\kappa)$ except now P2 can choose finitely many pieces from each of P1's partitions.

Definition 4.3.3. Let η and $\kappa \geq \omega$ be cardinals. The game $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ is played between two players in an η^+ -complete Boolean algebra \mathbf{B} as follows: At the beginning of the game, P1 chooses some $a \in \mathbf{B}^+$ to be fixed throughout the η -many rounds. For $\alpha < \eta$, the α -th round is played as follows: P1 chooses a partition W_α of a such that $|W_\alpha| \leq \kappa$; then P2 chooses a finite subset $F_\alpha \in [W_\alpha]^{<\omega}$. In this manner, the two players construct a sequence of length η

$$(4.3.2) \quad \langle a, W_0, F_0, W_1, F_1, \dots, W_\alpha, F_\alpha, \dots : \alpha < \eta \rangle$$

called a *play* of the game. P1 *wins the play* (4.3.2) if and only if

$$(4.3.3) \quad \bigwedge_{\alpha < \eta} \bigvee F_\alpha = \mathbf{0}.$$

A *strategy for P1* is a function $\sigma : \{\mathbf{0}\} \cup ([\mathbf{B}^+]^{<\omega})^{<\eta} \rightarrow [\mathbf{B}^+]^{\leq \kappa}$ such that $\sigma(\mathbf{0}) = a \in \mathbf{B}^+$ and for each $\langle F_\alpha : \alpha < \beta \rangle \in ([\mathbf{B}^+]^{<\omega})^{<\eta}$, $\sigma(\langle F_\alpha : \alpha < \beta \rangle)$ is a partition of a . σ is a *winning strategy* if P1 wins every time P1 follows σ .

A *strategy for P2* is a function $\tau : ([\mathbf{B}^+]^{\leq \kappa})^{<\eta} \rightarrow [\mathbf{B}^+]^{<\omega}$ such that for each $\langle W_\alpha : \alpha \leq \beta \rangle \in ([\mathbf{B}^+]^{\leq \kappa})^{<\eta}$, $\tau(\langle W_\alpha : \alpha \leq \beta \rangle) \in [W_\beta]^{<\omega}$. τ is a *winning strategy* if whenever P2 follows τ , P2 wins.

$\mathcal{G}_{\text{fin}}^\eta = \mathcal{G}_{\text{fin}}^\eta(\infty)$ is the game played as above, except now P1 can choose partitions of any size.

Jech's game $\mathcal{G}_{\text{fin}}(\kappa)$ is the same as our game $\mathcal{G}_{\text{fin}}^\omega(\kappa)$.

Remark. $\mathcal{G}_{\text{fin}}^\eta$ can be played in a partial ordering in the natural way. If \mathbf{P} is a separative partial ordering, then the existence of a winning strategy for P1 or P2 in $\mathcal{G}_{\text{fin}}^\eta$ is invariant between \mathbf{P} and $\text{r.o.}(\mathbf{P})$.

Note. In an η^+ -complete Boolean algebra, the following hold: If P2 has a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ ($\mathcal{G}_{\text{fin}}^\eta$), then the weak (η, κ) -d.l. (weak (η, ∞) -d.l.) holds. A winning strategy for P1 in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ ($\mathcal{G}_{\text{fin}}^\eta$) is a winning strategy in $\mathcal{G}_1^\eta(\kappa)$ (\mathcal{G}_1^η). Conversely, a winning strategy for P2 in $\mathcal{G}_1^\eta(\kappa)$ (\mathcal{G}_1^η) is a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ ($\mathcal{G}_{\text{fin}}^\eta$).

Jech showed the following.

Theorem 4.3.4 (Jech). [16] *If \mathbf{B} is complete and the weak (ω, κ) -d.l. fails in \mathbf{B} , then P1 has a winning strategy in $\mathcal{G}_{\text{fin}}(\kappa)$.*

Jech's Theorem 4.3.4 naturally generalizes to games of uncountable length, where we only require \mathbf{B} to be η^+ -complete.

Theorem 4.3.5. *If \mathbf{B} is η^+ -complete and the weak (η, κ) -d.l. fails in \mathbf{B} , then P1 has a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$.*

Proof: Follows from Theorem 4.4.4 with $\lambda = \omega$.

□

Having obtained Theorem 4.3.4, Jech asked whether the converse holds. Kamburelis showed that, in general, it does not, by obtaining the following.

Theorem 4.3.6 (Kamburelis). [19] *P1 does not have a winning strategy in $\mathcal{G}_{\text{fin}}(\kappa)$ played in a complete Boolean algebra \mathbf{B} iff the weak (ω, κ) -d.l. holds in \mathbf{B} and $V^{\mathbf{B}} \models [\kappa]^\omega \cap V$ is stationary.*

Remark. For the special case when $\kappa = \omega$, Kamburelis also showed that weak (ω, ω) -distributivity is equivalent to the non-existence of a winning strategy for P1 in $\mathcal{G}_{\text{fin}}(\omega)$.

It seems likely that one should find a similar characterization of the non-existence of a winning strategy for Player 1 in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ in terms of stationary sets. Currently, we do not know whether that is the case. However, for certain combinations of cardinals, we do have the following partial converse to Theorem 4.3.5.

Theorem 4.3.7. *If \mathbf{B} is η^+ -complete and $P1$ has a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$, then the weak $(\kappa^{<\eta}, \kappa)$ -d.l. fails.*

Proof: Follows from Theorem 4.4.5 with $\lambda = \omega$.

□

Observe that in Theorem 4.3.7, the first coordinate in the weak distributive law is $\kappa^{<\eta}$, not necessarily η . However, in certain cases, we obtain a purely game-theoretic characterization of the weak (η, κ) -d.l.

Corollary 4.3.8. *Let \mathbf{B} be η^+ -complete. $\forall \kappa, \eta$ such that $\kappa^{<\eta} = \eta$, the weak (η, κ) -d.l. holds in \mathbf{B} iff $P1$ does not have a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$.*

Corollary 4.3.9. (GCH) *Let \mathbf{B} be η^+ -complete. If (a) $\kappa < \eta$, or (b) $\kappa = \eta$ and η is regular, then the weak (η, κ) -d.l. holds iff $P1$ does not have a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$.*

Proof: GCH and (a) imply $\kappa^{<\eta} = \eta$; GCH and (b) imply $\kappa^{<\eta} = \eta^{<\eta} = \eta$. The result follows from Corollary 4.3.8. □

4.4. THE $(\eta, < \lambda, \kappa)$ -DISTRIBUTIVE LAW AND THE GAME $\mathcal{G}_{<\lambda}^\eta(\kappa)$

Three-parameter distributivity is the natural generalization of weak distributivity and is defined as follows.

Definition 4.4.1. [17] \mathbf{B} satisfies the $(\eta, < \lambda, \kappa)$ -distributive law $((\eta, < \lambda, \kappa)$ -

d.l.) if for each $|I| \leq \eta$, $|J| \leq \kappa$, and family $(b_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(4.4.1) \quad \bigwedge_{i \in I} \bigvee_{j \in J} b_{ij} = \bigvee_{f: I \rightarrow [J]^{<\lambda}} \bigwedge_{i \in I} \bigvee_{j \in f(i)} b_{ij},$$

provided that $\bigvee_{j \in J} b_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} b_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in f(i)} b_{ij}$ for each $f: I \rightarrow [J]^{<\lambda}$ exist in \mathbf{B} . We say that \mathbf{B} is $(\eta, < \lambda, \infty)$ -distributive if it satisfies the $(\eta, < \lambda, \kappa)$ -d.l. for all cardinals κ .

Recall that the (η, κ) -d.l. is the same as the $(\eta, < 2, \kappa)$ -d.l., and the weak (η, κ) -d.l. is the same as the $(\eta, < \omega, \kappa)$ -d.l.

$(\eta, < \lambda, \kappa)$ -distributivity is equivalent to the following forcing property.

Proposition 4.4.2. *If \mathbf{B} is complete, then \mathbf{B} is $(\eta, < \lambda, \kappa)$ -distributive iff for every function $g: \eta \rightarrow \kappa$ in $V[G]$ there is a function $f: \eta \rightarrow [\kappa]^{<\lambda}$ in V such that $\forall \alpha < \eta$, $g(\alpha) \in f(\alpha)$. \mathbf{B} is $(\eta, < \lambda, \infty)$ -distributive iff for every function $g: \eta \rightarrow V$ in $V[G]$ there is some function $f: \eta \rightarrow [V]^{<\lambda}$ such that $\forall \alpha < \eta$, $g(\alpha) \in f(\alpha)$.*

Proof: Let G be a generic filter in \mathbf{B}^+ and let g be a Boolean-valued name. Let

$$(4.4.2) \quad b = \|g \text{ is a function from } \eta \text{ to } \kappa\|,$$

and $\forall \alpha < \eta$, $\beta < \kappa$, let $b_{\alpha\beta} = \|g(\alpha) = \beta\| \wedge b$. $\forall \alpha < \eta$, let $W_\alpha = \{b_{\alpha\beta} : \beta < \kappa\}$. Note that each W_α is a partition of b . For each $h: \eta \rightarrow [\kappa]^{<\lambda}$, let $b_h = \bigwedge_{\alpha < \eta} \bigvee_{\beta \in h(\alpha)} b_{\alpha\beta}$. For each h such that $b_h \neq 0$, we will show $\exists f: \eta \rightarrow [\kappa]^{<\lambda}$ in V such that $b_h \Vdash \forall \alpha < \eta (g(\alpha) \in f(\alpha))$.

Fix $h: \eta \rightarrow [\kappa]^{<\lambda}$ such that $b_h \neq 0$. For $\alpha < \eta$, define $f(\alpha) = \{\beta < \kappa : b_h \wedge b_{\alpha\beta} \neq 0\}$. f is in V , since f is defined using b_h and $b_{\alpha\beta}$ which are all in \mathbf{B} . $\forall \beta < \kappa$, $b_h \wedge b_{\alpha\beta} \neq 0$ implies $\beta \in h(\alpha)$; so $f(\alpha) \subseteq h(\alpha) \in [\kappa]^{<\lambda}$. Thus, f is a function in V and $f: \eta \rightarrow [\kappa]^{<\lambda}$. Let $b_f = \bigwedge_{\alpha < \eta} \bigvee_{\beta \in f(\alpha)} b_{\alpha\beta}$. Note that

$$(4.4.3) \quad b_f \Vdash (g: \eta \rightarrow \kappa \text{ is a function and } \forall \alpha < \eta, g(\alpha) \in f(\alpha)).$$

$\forall \alpha < \eta,$

$$\begin{aligned}
 b_h &= b_h \wedge \bigvee_{\beta \in h(\alpha)} b_{\alpha\beta} \\
 &= \bigvee_{\beta \in h(\alpha)} (b_h \wedge b_{\alpha\beta}) \\
 (4.4.4) \quad &= \bigvee \{b_h \wedge b_{\alpha\beta} : \beta \in h(\alpha) \text{ and } b_h \wedge b_{\alpha\beta} \neq \mathbf{0}\} \\
 &\leq \bigvee \{b_h \wedge b_{\alpha\beta} : \beta \in f(\alpha)\} \\
 &\leq \bigvee_{\beta \in f(\alpha)} b_{\alpha\beta}.
 \end{aligned}$$

Since (4.4.4) holds for all $\alpha < \eta$,

$$(4.4.5) \quad b_h \leq \bigwedge_{\alpha < \eta} \bigvee_{\beta \in f(\alpha)} b_{\alpha,\beta} = b_f.$$

By (4.4.3) and (4.4.5), $\forall h : \eta \rightarrow [\kappa]^{<\lambda}$ such that $b_h \neq \mathbf{0}$, $b_h \Vdash (g : \eta \rightarrow \kappa \text{ is a function and } \forall \alpha < \eta, g(\alpha) \in f(\alpha))$. $(\eta, < \lambda, \kappa)$ -distributivity implies

$$(4.4.6) \quad b = \bigvee_{h: \eta \rightarrow [\kappa]^{<\lambda}} b_h = \bigvee \{b_h : h : \eta \rightarrow [\kappa]^{<\lambda} \text{ and } b_h \neq \mathbf{0}\}.$$

Thus, $b \Vdash (g : \eta \rightarrow \kappa \text{ is a function and } \forall \alpha < \eta, g(\alpha) \in f(\alpha))$. Hence, if g is a function from η to κ in $V[G]$, then $\exists f : \eta \rightarrow [\kappa]^{<\lambda}$ in V such that $\forall \alpha < \eta, g(\alpha) \in f(\alpha)$ in $V[G]$.

To prove the converse, suppose that the $(\eta, < \lambda, \kappa)$ -d.l. fails in \mathbf{B} . We will show that there exist a generic $G \subseteq \mathbf{B}^+$ and a function $f : \eta \rightarrow \kappa$ in G such that for all $h : \eta \rightarrow [\kappa]^{<\lambda}$ in V , $\exists \alpha < \eta (f(\alpha) \notin h(\alpha))$.

The failure of the $(\eta, < \lambda, \kappa)$ -d.l. implies that there exist $b \in \mathbf{B}^+$ and $\{b_{\alpha\beta} : \beta < \kappa\}$, $(\alpha < \eta)$, quasi-partitions of b for which

$$(4.4.7) \quad \bigwedge_{\alpha < \eta} \bigvee_{\beta < \kappa} b_{\alpha\beta} = b > \mathbf{0} = \bigvee_{g: \eta \rightarrow [\kappa]^{<\lambda}} \bigwedge_{\alpha < \eta} \bigvee_{\beta \in g(\alpha)} b_{\alpha\beta}.$$

Let \mathbf{f} be a \mathbf{B} -valued name for a function such that $\|\mathbf{f}(\check{\alpha}) = \check{\beta}\| = b \wedge b_{\alpha\beta} = b_{\alpha\beta}$. Note that for each $\alpha < \eta$, $b = \bigvee_{\beta < \kappa} b_{\alpha\beta} \Vdash (\mathbf{f} \text{ is a function from } \eta \text{ to } \kappa)$.

Claim: $b \Vdash \forall \check{h} : \check{\eta} \rightarrow [\check{\kappa}]^{<\check{\lambda}} \text{ in } V(\exists \check{\alpha} < \check{\eta}(\mathbf{f}(\check{\alpha}) \notin \check{h}(\check{\alpha})))$.

Let $h : \eta \rightarrow [\kappa]^{<\lambda}$ be a function in V . Let $p \leq b$. Since the $(\eta, < \lambda, \kappa)$ -d.l. fails, $\exists \alpha < \eta$ for which $p \not\leq \bigvee_{\beta \in h(\alpha)} b_{\alpha\beta}$. Hence, $\exists \gamma \in \kappa \setminus h(\alpha)$ for which $p \wedge b_{\alpha\gamma} \neq 0$. It follows that $p \wedge b_{\alpha\gamma} \Vdash \mathbf{f}(\check{\alpha}) \notin \check{h}(\check{\alpha})$. Thus, $p \nVdash \mathbf{f}(\check{\alpha}) \in \check{h}(\check{\alpha})$. Since this holds for all $p \leq b$, $b \Vdash \mathbf{f}(\check{\alpha}) \notin \check{h}(\check{\alpha})$.

□

The following generalization of the game $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ corresponds naturally to the $(\eta, < \lambda, \kappa)$ -d.l.

Definition 4.4.3. Let η, κ be infinite cardinals and λ be a cardinal such that $2 \leq \lambda \leq \kappa$. The game $\mathcal{G}_{<\lambda}^\eta(\kappa)$ is played between two players in a $\max(\eta^+, \lambda)$ -complete Boolean algebra \mathbf{B} as follows: At the beginning of the game, P1 chooses some $a \in \mathbf{B}^+$. For $\alpha < \eta$, the α -th round is played as follows: P1 chooses a partition W_α of a such that $|W_\alpha| \leq \kappa$; then P2 chooses some $F_\alpha \in [W_\alpha]^{<\lambda}$. In this manner, the two players construct a sequence of length η

$$(4.4.8) \quad \langle a, W_0, F_0, W_1, F_1, \dots, W_\alpha, F_\alpha, \dots : \alpha < \eta \rangle$$

called a *play* of the game. P1 *wins the play* (4.4.8) iff

$$(4.4.9) \quad \bigwedge_{\alpha < \eta} \bigvee F_\alpha = 0.$$

A *strategy for P1* is a function $\sigma : \{0\} \cup ([\mathbf{B}^+]^{<\lambda})^{<\eta} \rightarrow [\mathbf{B}^+]^{\leq \kappa}$ such that $\sigma(0) = \{a\}$ and for each $\langle F_\alpha : \alpha < \beta \rangle \in ([\mathbf{B}^+]^{<\lambda})^{<\eta}$, $\sigma(\langle F_\alpha : \alpha < \beta \rangle)$ is a partition of a . σ is a *winning strategy* if P1 wins every time P1 follows σ .

A *strategy for P2* is a function $\tau : ([\mathbf{B}^+]^{\leq \kappa})^{<\eta} \rightarrow [\mathbf{B}^+]^{<\lambda}$ such that for each $\langle W_\alpha : \alpha \leq \beta \rangle \in ([\mathbf{B}^+]^{\leq \kappa})^{<\eta}$, $\tau(\langle W_\alpha : \alpha \leq \beta \rangle) \in [W_\beta]^{<\lambda}$. τ is a *winning strategy* if whenever P2 follows τ , P2 wins.

$\mathcal{G}_{<\lambda}^\eta = \mathcal{G}_{<\lambda}^\eta(\infty)$ is the game played just as $\mathcal{G}_{<\lambda}^\eta(\kappa)$, except now P1 can choose partitions of any size.

Note that $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ is the same as $\mathcal{G}_{<\omega}^\eta(\kappa)$, Jech's game $\mathcal{G}_{\text{fin}}(\kappa)$ (in [16]) is the same as $\mathcal{G}_{\text{fin}}^\omega(\kappa)$, and $\mathcal{G}_{<2}^\eta(\kappa)$ is the same as $\mathcal{G}_1^\eta(\kappa)$.

Remark. $\mathcal{G}_{<\lambda}^\eta$ can be played in a partial ordering in the natural way. As before, if \mathbf{P} is separative, the existence of a winning strategy for P1 or P2 in $\mathcal{G}_{<\lambda}^\eta$ is invariant between \mathbf{P} and $\text{r.o.}(\mathbf{P})$.

Note. In a $\max(\eta^+, \lambda)$ -complete Boolean algebra, the following hold: For $\eta_0 \leq \eta_1$, $\kappa_0 \leq \kappa_1$, and $2 \leq \lambda_0 \leq \lambda_1$, a winning strategy for P1 in $\mathcal{G}_{<\lambda_1}^{\eta_0}(\kappa_0)$ is a winning strategy for P1 in $\mathcal{G}_{<\lambda_0}^{\eta_1}(\kappa_1)$; conversely, a winning strategy for P2 in $\mathcal{G}_{<\lambda_0}^{\eta_1}(\kappa_1)$ is a winning strategy for P2 in $\mathcal{G}_{<\lambda_1}^{\eta_0}(\kappa_0)$. If P2 has a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$ ($\mathcal{G}_{<\lambda}^\eta$), then the $(\eta, < \lambda, \kappa)$ -d.l. ($(\eta, < \lambda, \infty)$ -d.l.) holds.

Theorems 4.2.5, 4.2.6 (1), 4.3.5, and 4.3.7 are consequences of the following two theorems.

Theorem 4.4.4. *If \mathbf{B} is $\max(\eta^+, \lambda)$ -complete, $\lambda \leq \kappa$, and the $(\eta, < \lambda, \kappa)$ -d.l. fails, then P1 has a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$.*

Proof: If the $(\eta, < \lambda, \kappa)$ -d.l. fails, then there is a family $W_\alpha = \{b_{\alpha\beta} : \beta < \kappa\} \subseteq \mathbf{B}^+$, $(\alpha < \eta)$, such that

$$(4.4.10) \quad \bigvee_{f: \eta \rightarrow [\kappa]^{<\lambda}} \bigwedge_{\alpha < \eta} \bigvee_{\beta \in f(\alpha)} b_{\alpha,\beta} = \mathbf{0} < \bigwedge_{\alpha < \eta} \bigvee_{\beta < \kappa} b_{\alpha,\beta}.$$

Let $a = \bigwedge_{\alpha < \eta} \bigvee_{\beta < \kappa} b_{\alpha\beta}$. A winning strategy for P1 is given as follows: P1 chooses a at the beginning of the game, and $\forall \alpha < \eta$, P1 plays W_α .

□

Theorem 4.4.5. *If \mathbf{B} is $\max(\eta^+, \lambda)$ -complete, $\lambda \leq \kappa$, and P1 has a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$, then the $((\kappa^{<\lambda})^{<\eta}, < \lambda, \kappa)$ -d.l. fails in \mathbf{B} .*

Proof: Suppose σ is a winning strategy for P1. Let $\{a\} = \sigma(0)$ and $W_{\langle \rangle} = \sigma(\langle \rangle)$. Index the elements of $[W_{\langle \rangle}]^{<\lambda}$ using $s(0) \in \kappa^{<\lambda}$ so that $[W_{\langle \rangle}]^{<\lambda} = \{F_{\langle s(0) \rangle} : s(0) \in \kappa^{<\lambda}\}$. Let $W_{\langle s(0) \rangle} = \sigma(\langle F_{\langle s(0) \rangle} \rangle)$ be the partition of a which P1 chooses according to σ if P2 has just chosen $F_{\langle s(0) \rangle} \in [W_{\langle \rangle}]^{<\lambda}$. In general, given $\alpha < \eta$, $s \in (\kappa^{<\lambda})^\alpha$, and W_s , index the elements of $[W_s]^{<\lambda}$ using $s(\alpha) \in \kappa^{<\lambda}$ so that

$$(4.4.11) \quad [W_s]^{<\lambda} = \{F_{s \smallfrown s(\alpha)} : s(\alpha) \in \kappa^{<\lambda}\}.$$

Let

$$(4.4.12) \quad W_{s \smallfrown s(\alpha)} = \sigma(\langle F_{s \restriction 1}, F_{s \restriction 2}, \dots, F_s, F_{s \smallfrown s(\alpha)} \rangle)$$

be the partition of a which P1 chooses according to σ if P2 has just chosen $F_{s \smallfrown s(\alpha)} \in [W_s]^{<\lambda}$. For limit ordinals $\gamma < \eta$, let

$$(4.4.13) \quad W_{\langle s(\alpha) : \alpha < \gamma \rangle} = \sigma(\langle F_{s(\alpha+1)} : \alpha < \gamma \rangle).$$

Note that $\{W_s : s \in (\kappa^{<\lambda})^{<\eta}\}$ is a listing of all the possible choices for P1 under σ , and $\{F_s : s \in (\kappa^{<\lambda})^{<\eta} \text{ and } \text{dom}(s) \text{ is a successor ordinal}\}$ is a listing of all the possible choices for P2 when P1 follows σ .

Claim: The $((\kappa^{<\lambda})^{<\eta}, < \lambda, \kappa)$ -d.l. fails for the partitions W_s , $(s \in (\kappa^{<\lambda})^{<\eta})$, of a . For each $s \in (\kappa^{<\lambda})^{<\eta}$, use κ to index the elements of W_s so that $W_s = \{b_{s,j} : j < \kappa\}$. It suffices to show that for every $f : (\kappa^{<\lambda})^{<\eta} \rightarrow [\kappa]^{<\lambda}$,

$$(4.4.14) \quad \bigwedge_{s \in (\kappa^{<\lambda})^{<\eta}} \bigvee_{j \in f(s)} b_{s,j} = 0.$$

Let $f : (\kappa^{<\lambda})^{<\eta} \rightarrow [\kappa]^{<\lambda}$ be given. To show (4.4.14), we construct a sequence $t \in (\kappa^{<\lambda})^\eta$ with the following two properties:

$$(a) \quad \forall \alpha < \eta, F_{t \restriction (\alpha+1)} = \{b_{t \restriction \alpha, j} : j \in f(t \restriction \alpha)\};$$

(b) $\langle a \rangle \frown \langle W_{t \upharpoonright \alpha}, F_{t \upharpoonright (\alpha+1)} : \alpha < \eta \rangle$ is a play of $\mathcal{G}_{<\lambda}^\eta(\kappa)$ in which P1 follows σ .

Let $t(0)$ be the unique element of $\kappa^{<\lambda}$ for which $F_{\langle t(0) \rangle} = \{b_{\langle \cdot \rangle, j} : j \in f(\langle \cdot \rangle)\}$. In general, for $\alpha < \eta$ and given $\langle t(\beta) : \beta < \alpha \rangle$, let $t(\alpha)$ be the unique element of $\kappa^{<\lambda}$ for which

$$(4.4.15) \quad F_{\langle t(\beta) : \beta < \alpha \rangle \frown t(\alpha)} = \{b_{\langle t(\beta) : \beta < \alpha \rangle, j} : j \in f(\langle t(\beta) : \beta < \alpha \rangle)\}.$$

Let

$$(4.4.16) \quad t = \langle t(\alpha) : \alpha < \eta \rangle \in (\kappa^{<\lambda})^\eta.$$

t satisfies (a), by (4.4.15); and t satisfies (b) by (4.4.11), (4.4.12), (4.4.13), and (4.4.15). (a) implies that $\forall \alpha < \eta, \bigvee_{j \in f(t \upharpoonright \alpha)} b_{s, j} = \bigvee F_{t \upharpoonright (\alpha+1)}$. Since $\forall \alpha < \eta, t \upharpoonright \alpha \in (\kappa^{<\lambda})^{<\eta}$, it follows that

$$(4.4.17) \quad \bigwedge_{s \in (\kappa^{<\lambda})^{<\eta}} \bigvee_{j \in f(s)} b_{s, j} \leq \bigwedge_{\alpha < \eta} \bigvee F_{t \upharpoonright (\alpha+1)}.$$

Since (b) holds and σ is a winning strategy for P1,

$$(4.4.18) \quad \bigwedge_{\alpha < \eta} \bigvee F_{t \upharpoonright (\alpha+1)} = 0.$$

Thus, (4.4.14) holds. Since f was arbitrary, the $((\kappa^{<\lambda})^{<\eta}, < \lambda, \kappa)$ -d.l. fails for the partitions $W_s, (s \in (\kappa^{<\lambda})^{<\eta})$, of a .

□

Corollary 4.4.6. *If \mathbf{B} is $\max(\eta^+, \lambda)$ -complete and $(\kappa^{<\lambda})^{<\eta} = \eta$, then the $(\eta, < \lambda, \kappa)$ -d.l. holds in \mathbf{B} iff P1 does not have a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$.*

Corollary 4.4.7. (GCH) *Suppose \mathbf{B} is $\max(\eta^+, \lambda)$ -complete, $\lambda \leq \kappa$, and either (a) $\kappa^+ \leq \eta$, or (b) $\kappa = \eta$ and η is regular. Then the $(\eta, < \lambda, \kappa)$ -d.l. holds in \mathbf{B} iff P1 does not have a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$.*

4.5. THE HYPER-WEAK (η, κ) -DISTRIBUTIVE LAW AND THE GAME $\mathcal{G}_{\kappa-1}^\eta$

The hyper-weak (η, κ) -distributive is a generalization of the $(\eta, < \lambda, \kappa)$ -distributive law, for any $\omega \leq \lambda \leq \kappa$. Recall the definition of the hyper-weak (η, κ) -distributive law.

Definition 4.5.1. [26] For $\kappa \geq \omega$, \mathbf{B} satisfies the *hyper-weak (η, κ) -distributive law* (hyper-weak (η, κ) -d.l.) if for each $|I| \leq \eta$, $\omega \leq |J| \leq \kappa$, and family $(a_{ij})_{i \in I, j \in J}$ of elements of \mathbf{B} ,

$$(4.5.1) \quad \bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \rightarrow J} \bigwedge_{i \in I} \bigvee_{j \in J \setminus \{f(i)\}} a_{ij},$$

provided that $\bigvee_{j \in J} a_{ij}$ for each $i \in I$, $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij}$, and $\bigwedge_{i \in I} \bigvee_{j \in J \setminus \{f(i)\}} a_{ij}$ for each $f: I \rightarrow J$ exist in \mathbf{B} . We say that \mathbf{B} is *hyper-weakly (η, ∞) -distributive* if it satisfies the hyper-weak (η, κ) -d.l. for all cardinals $\kappa \geq \omega$.

As was shown in Theorem 4.1.7 (5) and (6), in a complete Boolean algebra, for $\eta_0 \leq \eta_1$ and $\omega \leq \kappa_0 \leq \kappa_1$, hyper-weak (η_1, κ_0) -distributivity implies hyper-weak (η_0, κ_1) -distributivity. Moreover, the hyper-weak (η, ω) -d.l. holds iff the hyper-weak (η, κ) -d.l. holds (with $\kappa \geq \omega$) iff the hyper-weak (η, ∞) -d.l. holds.

Hyper-weak (η, κ) -distributivity has the following forcing property.

Proposition 4.5.2. *For \mathbf{B} complete and $\kappa \geq \omega$, \mathbf{B} satisfies the hyper-weak (η, κ) -d.l., iff for every function $g: \eta \rightarrow \kappa$ in $V[G]$ there is a function $f: \eta \rightarrow \kappa$ in V such that $\forall \alpha < \eta$, $g(\alpha) \neq f(\alpha)$.*

Proof: Suppose the hyper-weak (η, κ) -d.l. holds. Let G be a generic filter in \mathbf{B}^+ and let g be a Boolean-valued name. Let

$$(4.5.2) \quad b = \|\|g \text{ is a function from } \eta \text{ to } \kappa\|,$$

and $\forall \alpha < \eta$, $\forall \beta < \kappa$, let

$$(4.5.3) \quad b_{\alpha\beta} = \|\|g(\alpha) = \beta\| \wedge b.$$

$\forall \alpha < \eta$, let $W_\alpha = \{b_{\alpha\beta} : \beta < \kappa\}$. Note that each W_α is a partition of b . For each $h : \eta \rightarrow \kappa$, let

$$(4.5.4) \quad b_h = \bigwedge_{\alpha < \eta} \bigvee_{\beta \in \kappa \setminus \{h(\alpha)\}} b_{\alpha\beta}.$$

For each h such that $b_h \neq 0$, we will show there is an $f : \eta \rightarrow \kappa$ in V such that $b_h \Vdash \forall \alpha < \eta, g(\alpha) \neq f(\alpha)$.

Fix $h : \eta \rightarrow \kappa$ such that $b_h \neq 0$. For $\alpha < \eta$, define

$$(4.5.5) \quad \tilde{f}(\alpha) = \{\beta < \kappa : b_h \wedge b_{\alpha\beta} = 0\}$$

and let

$$(4.5.6) \quad f(\alpha) = \text{the least } \beta \in \tilde{f}(\alpha).$$

f is a well-defined function from η to κ , since $b_h \wedge b_{\alpha, h(\alpha)} = 0$. Moreover, f is in V , since b_h and all $b_{\alpha\beta}$ are in \mathbf{B} . Thus, $b \Vdash (f \text{ is a function in } V \text{ and } f : \eta \rightarrow \kappa)$.

Let $b_f = \bigwedge_{\alpha < \eta} \bigvee_{\beta \in \kappa \setminus \{f(\alpha)\}} b_{\alpha\beta}$. Note that

$$(4.5.7)$$

$b_f \Vdash (g : \eta \rightarrow \kappa \text{ is a function, } f : \eta \rightarrow \kappa \text{ is a function in } V,$

and $\forall \alpha < \eta, g(\alpha) \neq f(\alpha))$.

$\forall \alpha < \eta,$

$$\begin{aligned}
 b_h &= b_h \wedge \bigvee_{\beta \in \kappa \setminus \{h(\alpha)\}} b_{\alpha\beta} \\
 &= \bigvee_{\beta \in \kappa \setminus \{h(\alpha)\}} (b_h \wedge b_{\alpha\beta}) \\
 (4.5.8) \quad &\leq \bigvee \{b_h \wedge b_{\alpha\beta} : \beta < \kappa, \text{ and } b_h \wedge b_{\alpha\beta} \neq 0\} \\
 &\leq \bigvee \{b_h \wedge b_{\alpha\beta} : \beta < \kappa, \text{ and } \beta \neq f(\alpha)\} \\
 &\leq \bigvee_{\beta \in \kappa \setminus \{f(\alpha)\}} b_{\alpha\beta}.
 \end{aligned}$$

So,

$$(4.5.9) \quad b_h \leq \bigwedge_{\alpha < \eta} \bigvee_{\beta \in \kappa \setminus \{f(\alpha)\}} b_{\alpha, \beta} = b_f.$$

Thus, $b_h \Vdash (g : \eta \rightarrow \kappa \text{ is a function, and } \exists f : \eta \rightarrow \kappa (f \text{ is a function in } V \text{ and } \forall \alpha < \eta, g(\alpha) \neq f(\alpha)))$. This holds for each $h : \eta \rightarrow \kappa$ such that $b_h \neq 0$. Hyper-weak (η, κ) -distributivity implies

$$(4.5.10) \quad b = \bigvee_{h : \eta \rightarrow \kappa} b_h = \bigvee \{b_h : h : \eta \rightarrow \kappa \text{ and } b_h \neq 0\},$$

so $b \Vdash \exists f : \eta \rightarrow \kappa (f \in V \text{ and } \forall \alpha < \eta (g(\alpha) \neq f(\alpha)))$. $b \in G$ implies $\exists f : \eta \rightarrow \kappa$ in V such that $\forall \alpha < \eta, g(\alpha) \neq f(\alpha)$ in $V[G]$.

The proof of the converse is similar to that of Proposition 4.4.2.

□

We now introduce a game which generalizes $\mathcal{G}_{<\lambda}^\eta(\kappa)$ and corresponds closely to the hyper-weak (η, κ) -d.l.

Definition 4.5.3. Let η, κ be cardinals, with $\kappa \geq \omega$. The game $\mathcal{G}_{\kappa-1}^\eta$ is played by two players in a $\max(\eta^+, \kappa^+)$ -complete Boolean algebra as follows: First, P1 chooses some $a \in \mathbf{B}^+$. For $\alpha < \eta$, the α -th round is played as follows: P1 chooses a partition $W_\alpha = \{a_{\alpha j} : j < \kappa\}$ of a of cardinality κ . Then P2 chooses all but one of the members of W_α ; i.e. P2 chooses $E_\alpha = \{a_{\alpha j} : j < \kappa \text{ and } j \neq k_\alpha\} \subseteq W_\alpha$, for some $k_\alpha < \kappa$. Let $b_\alpha = \bigvee E_\alpha$. P1 wins the play

$$(4.5.11) \quad \langle a, W_0, E_0, W_1, E_1, \dots, W_\alpha, E_\alpha, \dots : \alpha < \eta \rangle$$

if and only if

$$(4.5.12) \quad \bigwedge_{\alpha < \eta} b_\alpha = 0.$$

A *strategy for P1* is a function $\sigma : \{\mathbf{0}\} \cup ([\mathbf{B}^+]^\kappa)^{<\eta} \rightarrow [\mathbf{B}^+]^\kappa$ such that $\sigma(\mathbf{0}) = \{a\}$ and for each $\langle E_\beta : \beta < \alpha \rangle \in ([\mathbf{B}^+]^\kappa)^{<\eta}$, $\sigma(\langle E_\beta : \beta < \alpha \rangle)$ is a partition of a of size κ . σ is a *winning strategy* if P1 wins every time P1 follows σ .

A *strategy for P2* is a function $\tau : ([\mathbf{B}^+]^\kappa)^{<\eta} \rightarrow \mathbf{B}^+$ such that for each $\langle W_\beta : \beta \leq \alpha \rangle \in ([\mathbf{B}^+]^\kappa)^{<\eta}$, $\tau(\langle W_\beta : \beta \leq \alpha \rangle) \subseteq W_\alpha$ such that $|W_\alpha \setminus \tau(\langle W_\beta : \beta \leq \alpha \rangle)| = 1$. τ is a *winning strategy* if whenever P2 follows τ , P2 wins.

$\mathcal{G}_{\infty-1}^\eta$ is the game played just as $\mathcal{G}_{<\lambda}^\eta(\kappa)$, except now P1 can choose infinite partitions of any size.

Note that we restrict partitions to being exactly size κ in the game $\mathcal{G}_{\kappa-1}^\eta$. If we do not, then P1 will always choose partitions of size ω , since this maximizes P1's chances for winning.

Remark. $\mathcal{G}_{\infty-1}^\eta$ can be played in a partial ordering in the natural way. As before, if \mathbf{P} is separative, the existence of a winning strategy for P1 or P2 in $\mathcal{G}_{<\lambda}^\eta$ is invariant between \mathbf{P} and $\text{r.o.}(\mathbf{P})$.

The following implications are similar to those for $\mathcal{G}_{<\lambda}^\eta(\kappa)$.

Note. In a $\max(\eta^+, \kappa^+)$ -complete Boolean algebra, the following hold for $\eta_0 \leq \eta_1$ and $\kappa_0 \leq \kappa_1$. If P1 has a winning strategy in $\mathcal{G}_{\kappa_1-1}^{\eta_0}$, then P1 has a winning strategy in $\mathcal{G}_{\kappa_0-1}^{\eta_1}$. If P2 has a winning strategy in $\mathcal{G}_{\kappa_0-1}^{\eta_1}$, then P2 has a winning strategy in $\mathcal{G}_{\kappa_1-1}^{\eta_0}$. If P2 has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$ ($\mathcal{G}_{\infty-1}^\eta$), then the hyper-weak (η, κ) -d.l. (hyper-weak (η, ∞) -d.l.) holds.

Theorem 4.5.4. For \mathbf{B} $\max(\eta^+, \kappa^+)$ -complete, if the hyper-weak (η, κ) -d.l. fails, then P1 has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.

Proof: If the hyper-weak (η, κ) -d.l. fails, then there is a family $W_\alpha = \{b_{\alpha\beta} : \beta < \kappa\} \subseteq \mathbf{B}^+$, $(\alpha < \eta)$, such that

$$(4.5.13) \quad \bigvee_{f:\eta \rightarrow \kappa} \bigwedge_{\alpha < \eta} \bigvee_{\beta \in \kappa \setminus \{f(\alpha)\}} b_{\alpha\beta} = \mathbf{0} < \bigwedge_{\alpha < \eta} \bigvee_{\beta < \kappa} b_{\alpha\beta}.$$

Let $a = \bigwedge_{\alpha < \eta} \bigvee_{\beta < \kappa} b_{\alpha\beta}$. A winning strategy for P1 is given as follows: P1 chooses a at the beginning of the game, and $\forall \alpha < \eta$, P1 plays W_α .

□

Theorem 4.5.5. *For \mathbf{B} $\max(\eta^+, \kappa^+)$ -complete, if P1 has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$, then the hyper-weak $(\kappa^{<\eta}, \kappa)$ -d.l. fails.*

Proof: The proof is similar to the proof of Theorem 4.4.5. Suppose σ is a winning strategy for P1. Let $\{a\} = \sigma(\mathbf{0})$ and $W_{\langle \rangle} = \sigma(\langle \rangle)$. Index the elements of the set of subsets of $W_{\langle \rangle}$ consisting of all but one element of $W_{\langle \rangle}$ using $s(0) \in \kappa$ so that $\{S \subseteq W_{\langle \rangle} : |W_{\langle \rangle} \setminus S| = 1\} = \{E_{\langle s(0) \rangle} : s(0) \in \kappa\}$. Let $W_{\langle s(0) \rangle} = \sigma(\langle E_{\langle s(0) \rangle} \rangle)$ be the partition of a which P1 chooses according to σ if P2 has just chosen $E_{\langle s(0) \rangle}$. In general, given $\alpha < \eta$, $s \in \kappa^\alpha$, and W_s , index the subsets $E_{s \smallfrown s(\alpha)} \subseteq W_s$ consisting of all but one element of W_s using $s(\alpha) \in \kappa$ so that

$$(4.5.14) \quad \{S \subseteq W_s : |W_s \setminus S| = 1\} = \{E_{s \smallfrown s(\alpha)} : s(\alpha) \in \kappa\}.$$

Let

$$(4.5.15) \quad W_{s \smallfrown s(\alpha)} = \sigma(\langle E_{s \upharpoonright 1}, E_{s \upharpoonright 2}, \dots, E_s, E_{s \smallfrown s(\alpha)} \rangle)$$

be the partition of a which P1 chooses according to σ if P2 has just chosen $E_{s \smallfrown s(\alpha)}$. For limit ordinals $\lambda < \eta$, let

$$(4.5.16) \quad W_{\langle s(\alpha) : \alpha < \lambda \rangle} = \sigma(\langle E_{s \upharpoonright (\alpha+1)} : \alpha < \lambda \rangle).$$

Note that $\{W_s : s \in \kappa^{<\eta}\}$ is a listing of all the possible choices for P1 under σ , and $\{E_s : s \in \kappa^{<\eta} \text{ and } \text{dom}(s) \text{ is a successor ordinal}\}$ is a listing of all the possible choices for P2 when P1 follows σ .

Claim: The hyper-weak $(\kappa^{<\eta}, \kappa)$ -d.l. fails for the partitions W_s , ($s \in \kappa^{<\eta}$), of a .

For each $s \in \kappa^{<\eta}$, use κ to index the elements of W_s so that $W_s = \{b_{s,j} : j < \kappa\}$. It suffices to show that for every $f : \kappa^{<\eta} \rightarrow \kappa$,

$$(4.5.17) \quad \bigwedge_{s \in \kappa^{<\eta}} \bigvee_{j \neq f(s)} b_{s,j} = 0.$$

Let $f : \kappa^{<\eta} \rightarrow \kappa$ be given. To show (4.5.17), we construct a sequence $t \in \kappa^\eta$ with the following two properties:

- (a) $\forall \alpha < \eta, E_{t \upharpoonright (\alpha+1)} = \{b_{t \upharpoonright \alpha, j} : j \neq f(t \upharpoonright \alpha)\}$;
- (b) $\langle a \rangle \frown \langle W_{t \upharpoonright \alpha}, E_{t \upharpoonright (\alpha+1)} : \alpha < \eta \rangle$ is a play of $\mathcal{G}_{\kappa-1}^\eta$ in which P1 follows σ .

Let $t(0)$ be the unique element of κ for which

$$(4.5.18) \quad E_{\langle t(0) \rangle} = \{b_{\langle \rangle, j} : j \neq f(\langle \rangle)\}.$$

In general, for $\alpha < \eta$ and given $\langle t(\beta) : \beta < \alpha \rangle$, let $t(\alpha)$ be the unique element of κ for which

$$(4.5.19) \quad E_{\langle t(\beta) : \beta < \alpha \rangle \frown t(\alpha)} = \{b_{\langle t(\beta) : \beta < \alpha \rangle, j} : j \neq f(\langle t(\beta) : \beta < \alpha \rangle)\}.$$

Let

$$(4.5.20) \quad t = \langle t(\alpha) : \alpha < \eta \rangle \in \kappa^\eta.$$

It follows from (4.5.14), (4.5.15), (4.4.16), (4.5.19), and (4.5.20) that t satisfies properties (a) and (b). Thus,

$$(4.5.21) \quad \bigwedge_{s \in \kappa^{<\eta}} \bigvee_{j \neq f(s)} b_{s,j} \leq \bigwedge_{\alpha < \eta} \bigvee E_{t \upharpoonright (\alpha+1)} = 0,$$

since property (a) implies the inequality and property (b) implies the equality in (4.5.21). Thus, (4.5.17) holds. Since f was arbitrary,

$$(4.5.22) \quad \bigvee_{f: \eta \rightarrow \kappa} \bigwedge_{s \in \kappa^{<\eta}} \bigvee_{j \neq f(s)} b_{s,j} = 0 < a = \bigwedge_{s \in \kappa^{<\eta}} \bigvee_{j < \kappa} b_{s,j}.$$

Hence, the hyper-weak $(\kappa^{<\eta}, \kappa)$ -d.l. fails for the partitions W_s , ($s \in \kappa^{<\eta}$), of a .

□

Corollary 4.5.6. *If \mathbf{B} is $\max(\eta^+, \kappa^+)$ -complete and $\kappa^{<\eta} = \eta$, then the hyper-weak (η, κ) -d.l. holds in \mathbf{B} iff P1 does not have a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.*

Corollary 4.5.7. (GCH) *Suppose \mathbf{B} is $\max(\eta^+, \kappa^+)$ -complete and either (a) $\kappa^+ \leq \eta$, or (b) $\kappa = \eta$ and η is regular. Then the hyper-weak (η, κ) -d.l. holds in \mathbf{B} iff P1 does not have a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.*

4.6. UNDETERMINED GAMES

In each of the games $\mathcal{G}_1^\eta(\kappa)$, $\mathcal{G}_{\text{fin}}^\eta(\kappa)$, $\mathcal{G}_{<\lambda}^\eta(\kappa)$, and $\mathcal{G}_{\kappa-1}^\eta$, it is not immediately clear whether or not the existence of a winning strategy for P2 is equivalent to the non-existence of a winning strategy for P1. Jech showed that they are not equivalent for $\mathcal{G}_1^\omega(\kappa)$ by constructing a Suslin algebra in which this game is undetermined [16]. The following examples generalize Jech's result. Assuming η is regular and \Diamond_{η^+} holds, we will show that the existence of a winning strategy for P2 and the non-existence of a winning strategy for P1 are not equivalent for $\mathcal{G}_1^\eta(\kappa)$, $\mathcal{G}_{\text{fin}}^\eta(\kappa)$, $\mathcal{G}_{<\lambda}^\eta(\kappa)$ when $\lambda \leq \min(\eta, \kappa)$, and for $\mathcal{G}_{\kappa-1}^\eta$ when $\omega \leq \kappa \leq \eta$.

We will construct a Suslin algebra in which the first three games are undetermined. Then, using similar ideas, we will construct a Suslin algebra in which $\mathcal{G}_{\kappa-1}^\eta$ (for $\omega \leq \kappa \leq \eta$) is undetermined. Note that by Theorem 4.7.1 in the proceeding section, the results of Example 4.6.1 for $\kappa \leq \eta$ follow from Example 4.6.2. However, Example 4.6.1 is necessary for the case when $\kappa \geq \eta$. Moreover, the two examples differ in construction and make clear exactly how we ensure that P2 does not have a winning strategy in each related game.

Example 4.6.1. *If η is regular, $2 \leq \lambda \leq \min(\kappa, \eta)$, and \Diamond_{η^+} holds, then there exists an η^+ -Suslin algebra in which the game $\mathcal{G}_{<\lambda}^\eta(\kappa)$ is undetermined.*

We construct a Suslin tree $(T, \preceq) = (\eta^+, \preceq)$ such that neither player has a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$ played in r.o. (T^*) , where $(T^*, \preceq^*) = (T, \succ)$. In light

of the natural correspondence between partitions of $\text{r.o.}(T^*)$ and partitions of levels of T , we need only work with partitions of levels of T . Since T will satisfy the η^+ -c.c. and P1 must choose partitions of levels of T of cardinality $\leq \kappa$, we need only consider partitions of levels of T of size $\leq \mu = \min(\eta, \kappa)$. Every partition of some level of T into $\leq \mu$ -many pieces will be an element of the set $([\eta^+]^{\leq \eta})^{\leq \mu}$. \Diamond_{η^+} implies $|([\eta^+]^{\leq \eta})^{\leq \mu}| = \eta^+$, so we can index the elements of $([\eta^+]^{\leq \eta})^{\leq \mu}$ by

$$(4.6.1) \quad ([\eta^+]^{\leq \eta})^{\leq \mu} = \{W_\alpha : \alpha < \eta^+\},$$

where each

$$(4.6.2) \quad W_\alpha = \langle F_{\alpha, \beta} : \beta < \mu \rangle,$$

and each $F_{\alpha, \beta} \in [\eta^+]^{\leq \eta}$. It will be useful later that $\forall \alpha < \eta^+$, the elements of W_α are ordered.

Once T is constructed, the collection $\{W_\alpha : \alpha < \eta^+\}$ will include all partitions of levels of T . Thus, every strategy for P2 will be coded as one of the functions $f : (\eta^+)^{< \eta} \rightarrow \mu^{< \lambda}$. If P2 is playing by a strategy f and P1 has played the sequence $\langle W_{\gamma(i)} : i \leq \delta \rangle$, then P2 chooses $\{F_{\gamma(\delta), \beta} : \beta \in f(\langle \gamma(i) : i \leq \delta \rangle)\} \subseteq W_{\gamma(\delta)}$. $\eta \geq \mu \geq \lambda$ and \Diamond_{η^+} imply $|(\eta^+)^{< \eta} \times \mu^{< \lambda}| = \eta^+$, so let $\phi : (\eta^+)^{< \eta} \times \mu^{< \lambda} \rightarrow \eta^+$ be a bijection. For each $\alpha < \eta^+$, we will construct $\text{Lev}(\alpha) = \eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ and a partition \mathcal{P}_α of $\text{Lev}(\alpha)$. The set $\{\mathcal{P}_\alpha : \alpha < \eta^+\}$ will be used later to show that P2 does not have a winning strategy.

Construction of (T, \leq) and $\{\mathcal{P}_\alpha : \alpha < \eta^+\}$: Let $\langle A_\alpha : \alpha < \eta^+ \rangle$ be a \Diamond_{η^+} -sequence. Let $\text{Lev}(0) = \eta$. Let $\mathcal{P}_0 = \langle P_{0, \beta} : \beta < \mu \rangle$ be a partition of $\text{Lev}(0)$ into μ -many non-empty subsets.

Let $\alpha < \eta$ and suppose $\text{Lev}(\alpha)$ and \mathcal{P}_α have been constructed. Let $s : \eta \times \mu \rightarrow \eta \cdot (\alpha + 2) \setminus \eta \cdot (\alpha + 1)$ be a bijection. $\forall \gamma < \eta$ let $S_\gamma = \{s(\gamma, \beta) : \beta < \mu\}$. For each $\gamma < \eta$, let the immediate successors of $\eta \cdot \alpha + \gamma \in \text{Lev}(\alpha)$ be the elements of S_γ . Note that each element of $\text{Lev}(\alpha)$ has exactly μ -many immediate successors.

This constructs $\text{Lev}(\alpha + 1)$. For $\beta < \mu$, let $P_{\alpha+1,\beta} = \{s(\gamma, \beta) : \gamma < \eta\}$, and let $\mathcal{P}_{\alpha+1} = \langle P_{\alpha+1,\beta} : \beta < \mu \rangle$. $\mathcal{P}_{\alpha+1}$ is a partition of $\text{Lev}(\alpha + 1)$ into μ -many disjoint subsets, each of size η .

Now consider a limit ordinal $\alpha < \eta^+$ and suppose that $\forall \beta < \alpha$, $\text{Lev}(\beta)$ and \mathcal{P}_β have been constructed. Let $T_\alpha = \bigcup_{\beta < \alpha} \text{Lev}(\beta)$. Let (a) be the statement, " $T_\alpha = \alpha$, $\{\mathcal{P}_\beta : \beta < \alpha\} \subseteq \{W_\beta : \beta < \alpha\}$, and $\phi''(\alpha^{<\eta} \times \mu^{<\lambda}) = \alpha$." If (a) does not hold, then for each $t \in T_\alpha$, pick one α -branch $B_t \subseteq T_\alpha$ which contains t and put one element of $\eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ on top of B_t at level α . Do this in such a way that $\text{Lev}(\alpha) = \eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$.

Suppose (a) holds. Let (b) be the statement, " A_α is a maximal antichain in T_α ", and let (c) be the statement, " $A_\alpha = \phi[f] \cap \alpha$ for some $f : (\eta^+)^{<\eta} \rightarrow \mu^{<\lambda}$." Let $t \in T_\alpha$. If (b) holds, then $\exists u \in A_\alpha$ such that $t \preceq u$ or $t \succ u$; let $r \in T_\alpha$ such that $r \succ t, u$. If (b) does not hold, then let $r = t$. If (c) does not hold, then pick $B_t \subseteq T_\alpha$ an α -branch containing r and put one element of $\eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ above B_t at level α . Otherwise, (c) holds. Let $\delta = \text{ht}(r)$ and let $g : \delta + 2 \rightarrow \eta^+$ be the function such that $\forall \gamma \leq \delta + 1$, $W_{g(\gamma)} = \mathcal{P}_\gamma$. If P1 plays the sequence $\langle \mathcal{P}_\gamma : \gamma \leq \delta + 1 \rangle$ and P2 follows f , then on the $\delta + 1$ -st round, P2 chooses

(4.6.3)

$$\{F_{g(\delta+1),\beta} : \beta \in f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)\} = \{P_{\delta+1,\beta} : \beta \in f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)\}.$$

Let $\zeta < \mu$ be such that $\zeta \notin f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)$. By construction, there exists exactly one successor of r in $P_{\delta+1,\zeta}$, say s . Choose an α -branch $B_t \subseteq T_\alpha$ which contains s and put one element of $\eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ above B_t at level α .

Do this for each $t \in T_\alpha$ in such a manner that $\text{Lev}(\alpha) = \eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$. Now that $\text{Lev}(\alpha)$ has been constructed, choose a partition $\mathcal{P}_\alpha = \langle P_{\alpha,\beta} : \beta < \mu \rangle$ of $\text{Lev}(\alpha)$ into μ -many non-empty, disjoint subsets.

Let

$$(4.6.4) \quad T = \bigcup_{\alpha < \eta^+} \text{Lev}(\alpha).$$

This concludes the construction of (T, \preceq) .

Let

$$\begin{aligned}
 C_L &= \{\alpha < \eta^+ : \alpha \text{ is a limit ordinal}\} \\
 C_T &= \{\alpha < \eta^+ : T_\alpha = \alpha\} \\
 C_P &= \{\alpha < \eta^+ : \{\mathcal{P}_\gamma : \gamma < \alpha\} \subseteq \{W_\gamma : \gamma < \alpha\}\} \\
 C_\phi &= \{\alpha < \eta^+ : \phi''(\alpha^{<\eta} \times \mu^{<\lambda}) = \alpha\}.
 \end{aligned}
 \tag{4.6.5}$$

By the usual arguments, C_L, C_T, C_P , and C_ϕ are c.u.b. subsets of η^+ . (Regularity of η is necessary to ensure that C_ϕ is c.u.b.)

T is an η^+ -Suslin tree, since the \Diamond_{η^+} -chain kills all maximal antichains at some level below η^+ . r.o. (T^*) satisfies the (η, ∞) -d.l., so P1 does not have a winning strategy in $\mathcal{G}_1^\eta(\kappa)$, by Theorem 4.2.6 (2). Since $\mathcal{G}_1^\eta(\kappa)$ is easier for P1 to win than $\mathcal{G}_{<\lambda}^\eta(\kappa)$, P1 does not have a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$ (see Theorem 4.7.1 in the next section).

Claim: P2 does not have a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$ played in r.o. (T^*) .

Let $f : (\eta^+)^{<\eta} \rightarrow \mu^{<\lambda}$ be a strategy for P2. The set

$$S = C_L \cap C_T \cap C_P \cap C_\phi \cap \{\beta < \eta^+ : \phi(f) \cap \beta = A_\beta\}.
 \tag{4.6.6}$$

is stationary, since $\langle A_\alpha : \alpha < \eta^+ \rangle$ is a \Diamond_{η^+} -sequence. Let $\alpha \in S$. Let $p \in \text{Lev}(\alpha)$, $t \in T_\alpha$ such that p is placed above B_t in the construction of $\text{Lev}(\alpha)$, $\delta = \text{ht}(t)$, and $s \in B_t \cap \text{Lev}(\delta + 1)$. Let $g : \delta + 2 \rightarrow \eta^+$ be the function such that $\forall \beta \leq \delta + 1$, $W_{g(\beta)} = \mathcal{P}_\beta$. Statements (a) and (c) hold for α , so the construction of $\text{Lev}(\alpha)$ ensures that $s \in P_{\delta+1, \zeta}$ for some $\zeta \notin f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)$. Thus, s does not get chosen by f when P1 plays the sequence $\langle \mathcal{P}_\beta : \beta < \eta \rangle$; that is,

$$s \notin \bigcup \{P_{\delta+1, \theta} : \theta \in f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)\}.
 \tag{4.6.7}$$

$p \succ s$ implies there is no α -branch in $\bigcap_{\beta < \alpha} \bigcup \{P_{\beta, \theta} : \theta \in f(\langle g(\gamma) : \gamma \leq \beta \rangle)\}$ below p . Since this holds for all $p \in \text{Lev}(\alpha)$, f is not a winning strategy for P2.

□

Remark. Since $\mathcal{G}_1^\eta(\kappa) = \mathcal{G}_{<2}^\eta(\kappa)$ and $2 \leq \min(\eta, \kappa)$, and since $\mathcal{G}_{\text{fin}}^\eta(\kappa) = \mathcal{G}_{<\omega}^\eta(\kappa)$ and it only makes sense to play $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ when $\omega \leq \min(\eta, \kappa)$, Example 4.6.1 also serves as an example where these games are undetermined.

We now construct a Suslin algebra in which the game $\mathcal{G}_{\kappa-1}^\eta$ is undetermined for $\omega \leq \kappa \leq \eta$. The non-existence of a winning strategy for P2 in $\mathcal{G}_{\kappa-1}^\eta$ played on the Suslin algebra will imply the non-existence of a winning strategy for P2 in all of the other games, when $\lambda \leq \eta \leq \kappa$, (see Theorem 4.7.1).

Example 4.6.2. *If η is regular, $\omega \leq \kappa \leq \eta$, and \Diamond_{η^+} holds, then there exists an η^+ -Suslin algebra in which the game $\mathcal{G}_{\kappa-1}^\eta$ is undetermined.*

We construct a Suslin tree $(T, \preceq) = (\eta^+, \preceq)$ such that neither player has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$ played in r.o. (T^*) , where $(T^*, \preceq^*) = (T, \succ)$. Every partition of some level of T into κ -many pieces will be an element of the set $([\eta^+]^{\leq \eta})^{\leq \kappa}$. \Diamond_{η^+} implies $|([\eta^+]^{\leq \eta})^{\leq \kappa}| = \eta^+$, so we can index the elements of $([\eta^+]^{\leq \eta})^{\leq \kappa}$ by

$$(4.6.8) \quad [\eta^+]^\kappa = \{W_\alpha : \alpha < \eta^+\} = \{\langle F_{\alpha,\beta} : \beta < \kappa \rangle : \alpha < \eta^+\},$$

where each $F_{\alpha,\beta} \in [\eta^+]^{\leq \eta}$. Note that the elements of the W_α are ordered.

The collection $\{W_\alpha : \alpha < \eta^+\}$ will include all partitions of levels of T . Every strategy for P2 is encoded in one of the functions $f : (\eta^+)^{<\eta} \rightarrow \kappa$: When P2 plays by f and P1 has played the sequence $\langle W_{\alpha(i)} : i \leq \gamma \rangle$, then P2 chooses $\{F_{\alpha(\gamma),\beta} : \beta < \kappa, \beta \neq f(\langle \alpha(i) : i \leq \gamma \rangle)\}$. That is, P2 chooses every element of $W_{\alpha(\gamma)}$ except for $F_{\alpha(\gamma),f(\langle \alpha(i) : i \leq \gamma \rangle)}$. \Diamond_{η^+} implies $|(\eta^+)^{<\eta} \times \kappa| = \eta^+$, so let $\phi : (\eta^+)^{<\eta} \times \kappa \rightarrow \eta^+$ be a bijection. For each $\alpha < \eta^+$, we will construct $\text{Lev}(\alpha) = \eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ and a partition \mathcal{P}_α of $\text{Lev}(\alpha)$. $\{\mathcal{P}_\alpha : \alpha < \eta^+\}$ will be used later to show that P2 does not have a winning strategy.

Construction of (T, \preceq) and $\{\mathcal{P}_\alpha : \alpha < \eta^+\}$: Let $\langle A_\alpha : \alpha < \eta^+ \rangle$ be a \Diamond_{η^+} -sequence. Let $\text{Lev}(0) = \eta$. Let $\mathcal{P}_0 = \langle P_{0,\beta} : \beta < \kappa \rangle$ be a partition of $\text{Lev}(0)$ into κ -many non-empty subsets.

Let $\alpha < \eta$ and suppose that $\text{Lev}(\alpha)$ and \mathcal{P}_α have been constructed. Let $s : \eta \times \kappa \rightarrow \eta \cdot (\alpha + 2) \setminus \eta \cdot (\alpha + 1)$ be a bijection. $\forall \gamma < \eta$ let $S_\gamma = \{s(\gamma, \beta) : \beta < \kappa\}$. For each $\gamma < \eta$, let the immediate successors of $\eta \cdot \alpha + \gamma \in \text{Lev}(\alpha)$ be the elements of S_γ . For $\beta < \kappa$, let $P_{\alpha+1, \beta} = \{s(\gamma, \beta) : \gamma < \eta\}$, and let $\mathcal{P}_{\alpha+1} = \langle P_{\alpha+1, \beta} : \beta < \kappa \rangle$. $\mathcal{P}_{\alpha+1}$ is a partition of $\text{Lev}(\alpha + 1)$ into κ -many disjoint subsets, each of size η .

Now consider a limit ordinal $\alpha < \eta^+$ and suppose that $\forall \beta < \alpha$, $\text{Lev}(\beta)$ and \mathcal{P}_β have been constructed. Let $T_\alpha = \bigcup_{\beta < \alpha} \text{Lev}(\beta)$. Let (a) be the statement, " $T_\alpha = \alpha$, $\{\mathcal{P}_\beta : \beta < \alpha\} \subseteq \{W_\beta : \beta < \alpha\}$, and $\phi''(\alpha^{<\eta} \times \kappa) = \alpha$." If (a) does not hold, then for each $t \in T_\alpha$, pick one α -branch $B_t \subseteq T_\alpha$ which contains t and put one element of $\eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ on top of B_t at level α , in such a way that $\text{Lev}(\alpha) = \eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$.

Suppose (a) holds. Let (b) be the statement, " A_α is a maximal antichain in T_α ", and let (c) be the statement, " $A_\alpha = \phi[f] \cap \alpha$ for some $f : (\eta^+)^{<\eta} \rightarrow \kappa$." Let $t \in T_\alpha$. If (b) holds, then $\exists u \in A_\alpha$ such that $t \preceq u$ or $t \succ u$; let $r \in T_\alpha$ such that $r \succ t, u$. If (b) does not hold, then let $r = t$. If (c) does not hold, then pick $B_t \subseteq T_\alpha$ an α -branch containing r and put one element of $\eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ above B_t at level α . Otherwise, (c) holds. Let $\delta = \text{ht}(r)$ and let $g : \delta + 2 \rightarrow \eta^+$ be the function such that $\forall \gamma \leq \delta + 1$, $W_{g(\gamma)} = \mathcal{P}_\gamma$. If P1 plays the sequence $\langle \mathcal{P}_\gamma : \gamma \leq \delta + 1 \rangle$ and P2 follows f , then on the $\delta + 1$ -st round, P2 chooses

$$(4.6.9) \quad \{F_{g(\delta+1), \beta} : \beta < \kappa, \beta \neq f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)\}$$

Let $\zeta < \kappa$ be such that $\zeta \neq f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)$. By construction, there exists exactly one successor of r in $P_{\delta+1, \zeta}$, say s . Choose an α -branch $B_t \subseteq T_\alpha$ which contains s and put one element of $\eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$ above B_t at level α .

Do this for each $t \in T_\alpha$ in such a manner that $\text{Lev}(\alpha) = \eta \cdot (\alpha + 1) \setminus \eta \cdot \alpha$. Now that $\text{Lev}(\alpha)$ has been constructed, choose a partition $\mathcal{P}_\alpha = \langle P_{\alpha, \beta} : \beta < \kappa \rangle$ of $\text{Lev}(\alpha)$ into κ -many non-empty, disjoint subsets.

Let

$$(4.6.10) \quad T = \bigcup_{\alpha < \eta^+} \text{Lev}(\alpha).$$

Let

$$(4.6.11) \quad \begin{aligned} C_L &= \{\alpha < \eta^+ : \alpha \text{ is a limit ordinal}\} \\ C_T &= \{\alpha < \eta^+ : T_\alpha = \alpha\} \\ C_P &= \{\alpha < \eta^+ : \{\mathcal{P}_\gamma : \gamma < \alpha\} \subseteq \{W_\gamma : \gamma < \alpha\}\} \\ C_\phi &= \{\alpha < \eta^+ : \phi''(\alpha^{<\eta} \times \kappa) = \alpha\}. \end{aligned}$$

By the usual arguments, T is an η^+ -Suslin tree and C_L, C_T, C_P , and C_ϕ are c.u.b. subsets of η^+ . $\text{r.o.}(T^*)$ satisfies the (η, ∞) -d.l., so P1 does not have a winning strategy in $\mathcal{G}_1^\eta(\kappa)$. By Theorem 4.7.1, P1 does not have a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.

Claim: P2 does not have a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$ played in $\text{r.o.}(T^*)$.

Let $f : (\eta^+)^{<\eta} \rightarrow \kappa$ be a strategy for P2, and let

$$(4.6.12) \quad S = C_L \cap C_T \cap C_P \cap C_\phi \cap \{\beta < \eta^+ : \phi(f) \cap \beta = A_\beta\}.$$

By \Diamond_{η^+} , S is stationary, so choose some $\alpha \in S$. Let $p \in \text{Lev}(\alpha)$, $t \in T_\alpha$ such that p is placed above B_t in the construction of $\text{Lev}(\alpha)$, $\delta = \text{ht}(t)$, and let s be the unique element of $B_t \cap \text{Lev}(\delta + 1)$ chosen in the construction of $\text{Lev}(\alpha)$. Let $g : \delta + 2 \rightarrow \eta^+$ be the function such that $\forall \beta \leq \delta + 1$, $W_{g(\beta)} = \mathcal{P}_\beta$. Statements (a) and (c) hold for α , so the construction of $\text{Lev}(\alpha)$ ensures that $s \in P_{\delta+1, \zeta}$, where $\zeta = f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)$. Thus, s does not get chosen by f when P1 plays the sequence $\langle \mathcal{P}_\beta : \beta < \eta \rangle$; i.e.,

$$(4.6.13) \quad s \notin \bigcup \{P_{\delta+1, \theta} : \theta \in \kappa, \theta \neq f(\langle g(\gamma) : \gamma \leq \delta + 1 \rangle)\}.$$

$p \succ s$ implies there is no α -branch in $\bigcap_{\beta < \alpha} \bigcup \{P_{\beta, \theta} : \theta < \kappa, \theta \neq f(\langle g(\gamma) : \gamma \leq \beta \rangle)\}$ below p . Since this holds for all $p \in \text{Lev}(\alpha)$, f is not a winning strategy for P2 in $\text{r.o.}(T^*)$.

□

4.7. RELATIONSHIPS BETWEEN THE VARIOUS GAMES AND DISTRIBUTIVE LAWS

In this section, we provide implications for the existences of winning strategies for P1 and P2 between the various games. Moreover, we tie together the material in the previous six sections with diagrams relating the various distributive laws and the existences of winning strategies for the two players.

Theorem 4.7.1. *Let the following statements be given, assuming that $\omega \leq \lambda \leq \kappa$:*

- (A1) *P1 has a winning strategy in $\mathcal{G}_1^\eta(\kappa)$.*
- (B1) *P1 has a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$.*
- (C1) *P1 has a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$.*
- (D1) *P1 has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.*
- (A2) *P2 has a winning strategy in $\mathcal{G}_1^\eta(\kappa)$.*
- (B2) *P2 has a winning strategy in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$.*
- (C2) *P2 has a winning strategy in $\mathcal{G}_{<\lambda}^\eta(\kappa)$.*
- (D2) *P2 has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.*

Whenever the games are defined in \mathbf{B} , the following implications hold: (D1) \implies (C1) \implies (B1) \implies (A1); and (A2) \implies (B2) \implies (C2) \implies (D2).

Proof: (D1) \implies (C1): Let σ be a winning strategy for P1 in $\mathcal{G}_{\kappa-1}^\eta$.

Using σ , define a strategy τ for P1 in $\mathcal{G}_{<\lambda}^\eta(\kappa)$ as follows. Let

$$(4.7.1) \quad \tau(0) = \sigma(0) = \{a\}$$

and let

$$(4.7.2) \quad \tau(\langle \rangle) = \sigma(\langle \rangle) = W_0,$$

a partition of a of cardinality κ . Given that P2 plays $F_0 \in [W_0]^{<\lambda}$, choose one element $b_0 \in W_0 \setminus F_0$ and let

$$(4.7.3) \quad E_0 = W_0 \setminus \{b_0\}.$$

Let

$$(4.7.4) \quad \tau(\langle F_0 \rangle) = \sigma(\langle E_0 \rangle).$$

In general, given a sequence $\langle a, W_0, F_0, W_1, F_1, \dots, W_\alpha, F_\alpha \rangle$ in which, so far, P1 has played by τ , and where $F_\alpha \in [W_\alpha]^{<\lambda}$, choose one element $b_\alpha \in W_\alpha \setminus F_\alpha$ and let

$$(4.7.5) \quad E_\alpha = W_\alpha \setminus \{b_\alpha\}.$$

Then let

$$(4.7.6) \quad \tau(\langle F_0, F_1, \dots, F_\alpha \rangle) = \sigma(\langle E_0, E_1, \dots, E_\alpha \rangle).$$

Note that for each $\alpha < \eta$, $\bigvee F_\alpha \leq \bigvee E_\alpha$. Thus,

$$(4.7.7) \quad \bigwedge_{\alpha < \eta} \bigvee F_\alpha \leq \bigwedge_{\alpha < \eta} \bigvee E_\alpha = 0,$$

since σ is a winning strategy for P1 in $\mathcal{G}_{\kappa-1}^\eta$. Hence, τ is a winning strategy for P1 in $\mathcal{G}_{<\lambda}^\eta(\kappa)$.

(C1) \implies (B1): A winning strategy for P1 in $\mathcal{G}_{<\lambda}^\eta(\kappa)$ is also a winning strategy for P1 in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$, because when P2 chooses finitely many pieces from each partition, it qualifies as choosing less than λ many pieces.

(B1) \implies (A1): A winning strategy for P1 in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ is also a winning strategy on $\mathcal{G}_1^\eta(\kappa)$, since P2 choosing one piece from each partition is a special case of P2 choosing finitely many pieces from each partition.

(A2) \implies (B2): A winning strategy τ for P2 in $\mathcal{G}_1^\eta(\kappa)$ is also a winning strategy for P2 in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$: when P2 plays by τ , P2 chooses one element from each partition, which qualifies as choosing finitely many elements from each partition.

(B2) \implies (C2): A winning strategy τ for P2 in $\mathcal{G}_{\text{fin}}^\eta(\kappa)$ is also a winning strategy for P2 in $\mathcal{G}_{<\lambda}^\eta(\kappa)$: by following τ , P2 can win by choosing finitely many pieces, which qualifies as choosing less than λ many pieces.

(C2) \implies (D2): Suppose P2 has a winning strategy σ in $\mathcal{G}_{<\lambda}^\eta(\kappa)$. Then a winning strategy τ for P2 in $\mathcal{G}_{\kappa-1}^\eta$ can be obtained as follows. Given a and W_0 , a partition of a , let $F_0 = \sigma(\langle W_0 \rangle)$. Choose a $b_0 \in W_0 \setminus F_0$ and let $\tau(\langle W_0 \rangle) = E_0 = W_0 \setminus \{b_0\}$. In general, given a sequence $\langle a, W_0, E_0, W_1, E_1, \dots, W_\alpha \rangle$ and sets $F_\beta = \sigma(\langle W_0, \dots, W_\beta \rangle)$ for $\beta \leq \alpha$, choose some $b_\alpha \in W_\alpha \setminus F_\alpha$. Then let

$$(4.7.8) \quad \tau(\langle W_0, \dots, W_\beta \rangle) = E_\alpha = W_\alpha \setminus \{b_\alpha\}.$$

Since for each $\alpha < \eta$, $E_\alpha \supseteq F_\alpha$, and since σ is a winning strategy for P2 in $\mathcal{G}_{<\lambda}^\eta(\kappa)$,

$$(4.7.9) \quad \bigwedge_{\alpha < \eta} \bigvee E_\alpha \geq \bigwedge_{\alpha < \eta} \bigvee F_\alpha > 0.$$

Thus, τ is a winning strategy for P2 in $\mathcal{G}_{\kappa-1}^\eta$.

□

We sum up the implications from §§4.1 - 4.6 and Theorem 4.7.1 between the various games and distributive laws for varying cardinals.

For cardinals $\eta_0 \leq \eta \leq \eta_1$, $\kappa_0 \leq \kappa \leq \kappa_1$, $\omega \leq \lambda_0 \leq \lambda \leq \lambda_1$, and $\lambda_1 \leq \kappa_0$, the following diagrams commute (whenever the properties on both sides of the arrow are defined).

4.7.2. Implications between distributive laws.

$$\begin{array}{ccc}
 (\eta, \kappa) - \text{d.l.} & \implies & (\eta_0, \kappa_0) - \text{d.l.} \\
 \Downarrow & & \Downarrow \\
 \text{weak } (\eta, \kappa) - \text{d.l.} & \implies & \text{weak } (\eta_0, \kappa_0) - \text{d.l.} \\
 \Downarrow & & \Downarrow \\
 (\eta, < \lambda, \kappa) - \text{d.l.} & \implies & (\eta_0, < \lambda_1, \kappa_0) - \text{d.l.} \\
 \Downarrow & & \Downarrow \\
 \text{hyper-weak } (\eta, \kappa) - \text{d.l.} & \implies & \text{hyper-weak } (\eta_0, \kappa_1) - \text{d.l.}
 \end{array}$$

4.7.3. Implications for the existence of a winning strategy for P1.

$$\begin{array}{ccc}
 \mathcal{G}_1^\eta(\kappa) & \Leftarrow & \mathcal{G}_1^{\eta_0}(\kappa_0) \\
 \uparrow & & \uparrow \\
 \mathcal{G}_{\text{fin}}^\eta(\kappa) & \Leftarrow & \mathcal{G}_{\text{fin}}^{\eta_0}(\kappa_0) \\
 \uparrow & & \uparrow \\
 \mathcal{G}_{<\lambda}^\eta(\kappa) & \Leftarrow & \mathcal{G}_{<\lambda_1}^{\eta_0}(\kappa_0) \\
 \uparrow & & \uparrow \\
 \mathcal{G}_{\kappa-1}^\eta & \Leftarrow & \mathcal{G}_{\kappa_1-1}^{\eta_0}
 \end{array}$$

4.7.4. Implications for the existence of a winning strategy for P2.

$$\begin{array}{ccc}
 \mathcal{G}_1^\eta(\kappa) & \Rightarrow & \mathcal{G}_1^{\eta_0}(\kappa_0) \\
 \Downarrow & & \Downarrow \\
 \mathcal{G}_{\text{fin}}^\eta(\kappa) & \Rightarrow & \mathcal{G}_{\text{fin}}^{\eta_0}(\kappa_0) \\
 \Downarrow & & \Downarrow \\
 \mathcal{G}_{<\lambda}^\eta(\kappa) & \Rightarrow & \mathcal{G}_{<\lambda_1}^{\eta_0}(\kappa_0) \\
 \Downarrow & & \Downarrow \\
 \mathcal{G}_{\kappa-1}^\eta & \Rightarrow & \mathcal{G}_{\kappa_1-1}^{\eta_0}
 \end{array}$$

4.7.5. Implications between distributive laws and games.

$$\begin{array}{ccc}
\exists \text{ w.s. P2 in } \mathcal{G}_1^\eta(\kappa) & \implies & \exists \text{ w.s. P2 in } \mathcal{G}_{\kappa-1}^\eta \\
\Downarrow \nVdash & & \nVdash \\
(\kappa^{<\eta}, \kappa) - \text{d.l.} & \implies \nexists \text{ w.s. P1 in } \mathcal{G}_1^\eta(\kappa) \implies & (\eta, \kappa) - \text{d.l.} \\
\Uparrow & & \\
(\eta, \kappa^{<\eta}) - \text{d.l.} & &
\end{array}$$

$$\begin{array}{ccc}
\exists \text{ w.s. P2 in } \mathcal{G}_{\text{fin}}^\eta(\kappa) & \implies & \exists \text{ w.s. P2 in } \mathcal{G}_{\kappa-1}^\eta \\
\Downarrow \nVdash & & \nVdash \\
\text{weak } (\kappa^{<\eta}, \kappa) - \text{d.l.} & \implies \nexists \text{ w.s. P1 in } \mathcal{G}_{\text{fin}}^\eta(\kappa) \implies & \text{weak } (\eta, \kappa) - \text{d.l.}
\end{array}$$

$$\begin{array}{ccc}
\exists \text{ w.s. P2 in } \mathcal{G}_{<\lambda}^\eta(\kappa) & \implies & \exists \text{ w.s. P2 in } \mathcal{G}_{\kappa-1}^\eta \\
\Downarrow \nVdash & & \nVdash \\
((\kappa^{<\lambda})^{<\eta}, <\lambda, \kappa) - \text{d.l.} & \implies \nexists \text{ w.s. P1 in } \mathcal{G}_{<\lambda}^\eta(\kappa) \implies & (\eta, <\lambda, \kappa) - \text{d.l.}
\end{array}$$

$$\begin{array}{ccc}
\exists \text{ w.s. P2 in } \mathcal{G}_{\kappa-1}^\eta & & \\
\Downarrow \nVdash & & \\
\text{hyper-weak } (\kappa^{<\eta}, \kappa) - \text{d.l.} \implies \nexists \text{ w.s. P1 in } \mathcal{G}_{\kappa-1}^\eta \implies & & \text{hyper-weak } (\eta, \kappa) - \text{d.l.}
\end{array}$$

4.8. OPEN PROBLEMS

Examples 4.6.1 and 4.6.2 reveal a curious fact: For η regular and $\kappa \leq \eta$, (η, ∞) -distributivity does not imply P2 has a winning strategy in the game $\mathcal{G}_{\kappa-1}^\eta$, (as is seen in the diagram 4.7.5), even though it is much easier for P2 to win $\mathcal{G}_{\kappa-1}^\eta$ than \mathcal{G}_1^η , which is the game most closely related to (η, ∞) -distributivity. This result and the partial characterizations from Sections 4.2-4.6 lead to the following problems.

- (1) Show within ZFC that the existence of a winning strategy for P2 in $\mathcal{G}_{\kappa-1}^\eta$ is stronger than the hyper-weak (η, κ) -d.l.; or else find a model of ZFC in which they are equivalent.
- (2) Show (1) for the other three games and their related distributive laws.
- (3) Find a characterization of the $(\eta, < \lambda, \kappa)$ -d.l. for all cardinals η and $\lambda \leq \kappa$ in terms of $\mathcal{G}_{<\lambda}^\eta(\kappa)$ (possibly using additional properties).
- (4) Do (3) for the hyper-weak (η, κ) -d.l. for all cardinals η, κ in terms of $\mathcal{G}_{\kappa-1}^\eta$.
- (5) Find a complete, c.c.c. Boolean algebra in which P2 has a winning strategy in $\mathcal{G}_{\omega-1}^\omega$ but the weak (ω, ω) -d.l. fails.
- (6) Construct a Suslin algebra in which P2 has a winning strategy in $\mathcal{G}_{\kappa-1}^\eta$.
- (7) Construct a consistent counterexample to von Neumann's problem in which P2 has a winning strategy in $\mathcal{G}_{\text{fin}}^\omega(\omega)$.

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