Banach Spaces from Barriers in High Dimensional Ellentuck Spaces

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Two new hierarchies of Banach spaces $T_k(d,\theta)$ and $T(\mathcal{A}_d^k,\theta)$, k any positive integer, are constructed using barriers in high dimensional Ellentuck spaces [10] following the classical framework under which a Tsirelson type norm is defined from a barrier in the Ellentuck space [3]. The following structural properties of these spaces are proved. Each of these spaces contains arbitrarily large copies of ℓ_∞^n, with the bound constant for all n. For each fixed pair d and θ, the spaces $T_k(d,\theta)$ and $T(\mathcal{A}_d^k,\theta)$, $k \geq 1$, are ℓ_p-saturated, forming natural extensions of the ℓ_p space, where p satisfies $d\theta = d^{1/p}$. Moreover, the $T_k(d,\theta)$ spaces form a strict hierarchy over the ℓ_p space: For any $j < k$, the space $T_j(d,\theta)$ embeds isometrically into $T_k(d,\theta)$ as a subspace which is non-isomorphic to $T_k(d,\theta)$. The analogous result for the spaces $T(\mathcal{A}_d^k,\theta)$ is open.

46B03, 46B07, 03E05; 03E02, 05D10

1 Introduction

Banach space theory is rich with applications of fronts and barriers within the framework of the Ellentuck space (see for example [9], [23], and Part B of [3]). Infinite-dimensional Ramsey theory is a branch of Ramsey Theory initiated by Nash-Williams in the course of developing his theory of better-quasi-ordered sets in the early 1960's. This theory introduced the notions of fronts and barriers that turned out to be important in the context of Tsirelson type norms. During the 1970's, Nash-Williams' theory was reformulated and strengthened by the work of Silver [26], Galvin and Prikry [17], Louveau [21], and Mathias [22], culminating in Ellentuck's work in [15] introducing topological Ramsey theory on what is now called the Ellentuck space.

Building on seminal work of Carlson and Simpson in [7], Todorcevic distilled key properties of the Ellentuck space into four axioms which guarantee that a topological space satisfies infinite-dimensional Ramsey theory analogously to [15] (see Chapter

5 of [27]). These abstractions of the Ellentuck space are called topological Ramsey spaces. In particular, Todorcevic has shown that the theory of fronts and barriers in the Ellentuck space extends to the context of general topological Ramsey spaces. This general theory has already found applications finding exact initial segments of the Tukey structure of ultrafilters in [24], [13], [14], [12], and [10]. In this context, the second author constructed a new hierarchy of topological Ramsey spaces in [10] and [11] which form dense subsets of the Boolean algebras $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^\alpha$, for α any countable ordinal. Those constructions were motivated by the following.

The Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$, the Ellentuck space, and Ramsey ultrafilters are closely connected. A Ramsey ultrafilter is the strongest type of ultrafilter, satisfying the following partition relation: For each partition of the pairs of natural numbers into two pieces, there is a member of the ultrafilter such that all pairsets coming from that member are in the same piece of the partition. Ramsey ultrafilters can be constructed via standard methods using $\mathcal{P}(\omega)/\mathrm{Fin}$ and the Continuum Hypothesis or other set-theoretic techniques such as forcing. The Boolean algebra $\mathcal{P}(\omega^2)/\mathrm{Fin}^2$ is the next step in complexity above $\mathcal{P}(\omega)/\mathrm{Fin}$. This Boolean algebra can be used to generate an ultrafilter $\mathcal{U}_2$ on base set $\omega \times \omega$ satisfying a weaker partition relation: For each partition of the pairsets on ω into five or more pieces, there is a member of $\mathcal{U}_2$ such that the pairsets on that member are all contained in four pieces of the partition. Moreover, the projection of $\mathcal{U}_2$ to the first copy of ω recovers a Ramsey ultrafilter. In [6], many aspects of the ultrafilter $\mathcal{U}_2$ were investigated, but the exact structure of the Tukey, equivalently cofinal, types below $\mathcal{U}_2$ remained open. The topological Ramsey space $\mathcal{E}_2$ and more generally the spaces $\mathcal{E}_k$, $k \geq 2$, were constructed to produce dense subsets of $\mathcal{P}(\omega^k)/\mathrm{Fin}^k$ which form topological Ramsey spaces, thus setting the stage for finding the exact structure of the cofinal types of all ultrafilters Tukey reducible to the ultrafilter generated by $\mathcal{P}(\omega^k)/\mathrm{Fin}^k$ in [10].

Once constructed, it became clear that these new spaces are the natural generalizations of the Ellentuck space to higher dimensions, and hence are called high-dimensional Ellentuck spaces. This, in conjunction with the multitude of results on Banach spaces constructed using barriers on the original Ellentuck space, led the second author to infer that the general theory of barriers on these high-dimensional Ellentuck spaces would be a natural starting point for answering the following question.

Question 1.1 What Banach spaces can be constructed by extending Tsirelson's construction method by using barriers in general topological Ramsey spaces?

The work in this paper is a first step toward the broader goal of finding structural properties of Tsirelson analogues using infinite dimensional barriers on high and infinite dimensional Ellentuck spaces. The second author's motivation was the hope that this approach can shed new light on distortion problems. The constructions presented in this paper have as their starting point Tsirelson's groundbreaking example of a reflexive Banach space T with an unconditional basis not containing c_0 or ℓ_p with $1 \le p < \infty$ [28]. The idea of Tsirelson's construction became apparent after Figiel and Johnson [16] showed that the norm of the dual of the Tsirelson space satisfies the following equation:

$$\left\|\sum\nolimits_n a_n e_n\right\| = \max\left\{\sup\nolimits_n |a_n|, \frac{1}{2}\sup\sum_{i=1}^{m}\left\|E_i\left(\sum\nolimits_n a_n e_n\right)\right\|\right\}, \tag{1}$$

where the sup is taken over all sequences $(E_i)_{i=1}^m$ of successive finite subsets of integers with the property that $m \le \min(E_1)$ and $E_i\left(\sum_n a_n e_n\right) = \sum_{n\in E_i} a_n e_n$.

The first systematic abstract study of Tsirelson's construction was achieved by Argyros and Deliyanni [1]. Their construction starts with a real number $0 < \theta < 1$ and an arbitrary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ that is the downwards closure of a barrier in Ellentuck space. Then, one defines the *Tsirelson type space* $T(\mathcal{F},\theta)$ as the completion of $c_{00}(\mathbb{N})$ with the implicitly given norm (1) replacing $1/2$ by θ and using sequences $(E_i)_{i=1}^m$ of finite subsets of positive integers which are $\mathcal{F}$-admissible, i.e., there is some $\{k_1, k_2, \dots, k_m\} \in \mathcal{F}$ such that $k_1 \le \min(E_1) \le \max(E_1) < k_2 \le \cdots < k_m \le \min(E_m) \le \max(E_m)$.

It became customary in the literature to identify Tsirelson's original space with its dual. So, in the above notation, Tsirelson's original space is denoted by $T(\mathcal{S}, 1/2)$, where $\mathcal{S} = \{F \subset \mathbb{N} : |F| \le \min(F)\}$ is the Schreier family. In addition to $\mathcal{S}$, the *low complexity hierarchy* $\{\mathcal{A}_d\}_{d=1}^{\infty}$ with

$$\mathcal{A}_d := \{F \subset \mathbb{N} : |F| \le d\}$$

is noteworthy in the realm of Tsirelson type spaces. In fact, Bellenot proved in [4] the following remarkable theorem:

Theorem 1.2 (Bellenot [4]) *If $d\theta > 1$, then for every $x \in T(\mathcal{A}_d,\theta)$,*

$$\frac{1}{2d}\|x\|_p \le \|x\|_{T(\mathcal{A}_d,\theta)} \le \|x\|_p,$$

where $d\theta = d^{1/p}$ and $\|\cdot\|_p$ denotes the ℓ_p-norm.

We refer the reader to [8, 20] for a systematic study of Tsirelson's space and Tsirelson type spaces.

There are three non-trivial directions in which this construction may be extended using high-dimensional Ellentuck spaces, two of which are considered in this paper. We construct two new hierarchies of Banach spaces extending the low complexity hierarchy on the Ellentuck space to the low complexity hierarchy on the finite dimensional Ellentuck spaces. This produces various structured extensions of the ℓ_p spaces.

In Section 2 we review the construction of the finite-dimensional Ellentuck spaces in [10], introducing new notation and representations more suitable to the context of this paper. Our new Banach spaces $T_k(d,\theta)$ and $T(\mathcal{A}_d^k,\theta)$ are constructed in Section 3, using finite rank barriers on the k-dimensional Ellentuck spaces. These spaces may be thought of as structured generalizations of ℓ_p, where $d\theta = d^{1/p}$, as they extend the construction of the Tsirelson type space $T_1(d,\theta)$, which by Bellenot's Theorem 1.2 is exactly ℓ_p.

We prove the following structural results about the spaces $T_k(d,\theta)$. They contain arbitrarily large copies of ℓ_∞^n, where the bound is fixed for all n (Section 4), and that there are many natural block subspaces isomorphic to ℓ_p (Section 5). Moreover, they are ℓ_p-saturated (Section 6). The spaces $T_k(d,\theta)$ are not isomorphic to each other (Section 7), but for each $j < k$, there are subspaces of $T_k(d,\theta)$ which are isometric to $T_j(d,\theta)$ (Section 8). Thus, for fixed d,θ, the spaces $T_k(d,\theta)$, $k \geq 1$, form a natural hierarchy in complexity over ℓ_p, where $d\theta = d^{1/p}$.

The second class of spaces we consider involves the most stringent definition of norm. These spaces, $T(\mathcal{A}_d^k,\theta)$, are constructed in 3 using admissible sets which are required to be separated by sets which are finite approximations to members of the $\mathcal{E}_k$. The norms on these spaces are thus bounded by the norms on the $T_k(d,\theta)$ spaces. In Section 9, the spaces $T(\mathcal{A}_d^k,\theta)$ are shown to have the same properties as the as shown for $T_k(d,\theta)$, the only exception being that we do not know whether $T(\mathcal{A}_d^j,\theta)$ embeds as an isometric subspace of $T(\mathcal{A}_d^k,\theta)$ for $j < k$. The paper concludes with open problems for further research into the properties of these spaces.

Acknowledgement. Dobrinen was partially supported by National Science Foundation Grants DMS-1301665 and DMS-1600781

2 High Dimensional Ellentuck Spaces

In [10], the second author constructed a new hierarchy $(\mathcal{E}_k)_{2\leq k<\omega}$ of topological Ramsey spaces which generalize the Ellentuck space in a natural manner. In this section, we reproduce the construction, though with slightly different notation more suited to the context of Banach spaces.

Recall that the Ellentuck space [15] is the triple $([\omega]^\omega, \subseteq, r)$, where the finitization map $r : \omega \times [\omega]^\omega \to [\omega]^{<\omega}$ is defined as follows: For each $X \in [\omega]^\omega$ and $n < \omega$, $r(n, X)$ is the set of the least n elements of X. Usually $r(n, X)$ is denoted by $r_n(X)$. We shall let $\mathcal{E}_1$ denote the Ellentuck space.

We now begin the process of defining the high dimensional Ellentuck spaces $\mathcal{E}_k$, $k \geq 2$. The presentation here is slightly different than, but equivalent to, the one in [10]. We have chosen to do so in order to simplify the construction of the Banach spaces. In logic, the set of natural numbers $\{0, 1, 2, \dots\}$ is denoted by the symbol ω. In keeping with the logic influence in [10], we shall use this notation. We start by defining a well-ordering on the collection of all non-decreasing sequences of members of ω which forms the backbone for the structure of the members in the spaces.

Definition 2.1 For $k \geq 2$, denote by $\omega^{\not\downarrow \leq k}$ the collection of all non-decreasing sequences of members of ω of length less than or equal to k.

Definition 2.2 (The well-order $<_{\text{lex}}$) Let $(s_1, \dots, s_i)$ and $(t_1, \dots, t_j)$, with $i, j \geq 1$, be in $\omega^{\not\downarrow \leq k}$. We say that $(s_1, \dots, s_i)$ is lexicographically below $(t_1, \dots, t_j)$, written $(s_1, \dots, s_i) <_{\text{lex}} (t_1, \dots, t_j)$, if and only if there is a non-negative integer m with the following properties:

(i) $m \leq i$ and $m \leq j$;

(ii) for every positive integer $n \leq m$, $s_n = t_n$; and

(iii) either $s_{m+1} < t_{m+1}$, or $m = i$ and $m < j$.

This is just a generalization of the way the alphabetical order of words is based on the alphabetical order of their component letters.

Example 2.3 Consider the sequences $(1, 2), (2)$, and $(2, 2)$ in $\omega^{\not\downarrow \leq 2}$. Following the preceding definition we have $(1, 2) <_{\text{lex}} (2) <_{\text{lex}} (2, 2)$. We conclude that $(1, 2) <_{\text{lex}} (2)$ by setting $m = 0$ in Definition 2.2; similarly, $(2) <_{\text{lex}} (2, 2)$ follows by setting $m = 1$, as any proper initial segment of a sequence is lexicographically below that sequence.

Definition 2.4 (The well-ordered set $(\omega^{\not\downarrow \leq k}, \prec)$) Set the empty sequence $()$ to be the $\prec$-minimum element of $\omega^{\not\downarrow \leq k}$; so, for all nonempty sequences s in $\omega^{\not\downarrow \leq k}$, we have $() \prec s$. In general, given $(s_1, \dots, s_i)$ and $(t_1, \dots, t_j)$ in $\omega^{\not\downarrow \leq k}$ with $i, j \geq 1$, define $(s_1, \dots, s_i) \prec (t_1, \dots, t_j)$ if and only if either

(1) $s_i < t_j$, or

(2) $s_i = t_j$ and $(s_1, \ldots, s_i) <_{\text{lex}} (t_1, \ldots, t_j)$.

Notation. Since $\prec$ well-orders $\omega^{\not\downarrow \leq k}$ in order-type ω, we fix the notation of letting $\vec{s}_m$ denote the m-th member of $(\omega^{\not\downarrow \leq k}, \prec)$. Let $\omega^{\not\downarrow k}$ denote the collection of all non-decreasing sequences of length k of members of ω. Note that $\prec$ also well-orders $\omega^{\not\downarrow k}$ in order type ω. Fix the notation of letting $\vec{u}_n$ denote the n-th member of $(\omega^{\not\downarrow k}, \prec)$. For $s, t \in \omega^{\not\downarrow \leq k}$, we say that s is an *initial segment* of t and write $s \sqsubset t$ if $s = (s_1, \ldots, s_i)$, $t = (t_1, \ldots, t_j)$, $i < j$, and for all $m \leq i$, $s_m = t_m$. Recall the concatenation operation: Given sequences $s = (s_1, \ldots, s_i)$ and $t = (t_1, \ldots, t_j)$, $s^\frown t$ denotes the *concatenation* of s and t, which is the sequence $(s_1, \ldots, s_i, t_1, \ldots, t_j)$ of length $i + j$. As is standard, for a natural number n, $s^\frown n$ denotes the sequence $(s_1, \ldots, s_i, n)$.

Definition 2.5 (The spaces $(\mathcal{E}_k, \leq, r)$, $k \geq 2$, Dobrinen [10]) An $\mathcal{E}_k$*-tree* is a function $\widehat{X}$ from $\omega^{\not\downarrow \leq k}$ into $\omega^{\not\downarrow \leq k}$ that preserves the well-order $\prec$ and initial segments $\sqsubset$. For $\widehat{X}$ an $\mathcal{E}_k$-tree, let X denote the restriction of $\widehat{X}$ to $\omega^{\not\downarrow k}$. The space $\mathcal{E}_k$ is defined to be the collection of all X such that $\widehat{X}$ is an $\mathcal{E}_k$-tree. We identify X with its range and usually will write $X = \{v_1, v_2, \ldots\}$, where $v_1 = X(\vec{u}_1) \prec v_2 = X(\vec{u}_2) \prec \cdots$. The partial ordering on $\mathcal{E}_k$ is defined to be simply inclusion; that is, given $X, Y \in \mathcal{E}_k$, $X \leq Y$ if and only if (the range of) X is a subset of (the range of) Y. For each $n < \omega$, the n-th restriction function r_n on $\mathcal{E}_k$ is defined by $r_n(X) = \{v_1, v_2, \ldots, v_n\}$ that is, the $\prec$-least n members of X. When necessary for clarity, we write $r_n^k(X)$ to highlight that X is a member of $\mathcal{E}_k$. We set

$$\mathcal{AR}_n^k := \{r_n(X) : X \in \mathcal{E}_k\} \qquad \text{and} \qquad \mathcal{AR}^k := \{r_n(X) : n < \omega, X \in \mathcal{E}_k\}$$

to denote the set of all n-th approximations to members of $\mathcal{E}_k$, and the set of all finite approximations to members of $\mathcal{E}_k$, respectively.

Remark Let $s \in \omega^{\not\downarrow \leq k}$ and denote its length by $|s|$. Since $\widehat{X}$ preserves initial segments, it follows that $|\widehat{X}(s)| = |s|$. Thus, $\mathcal{E}_k$ is the space of all functions X from $\omega^{\not\downarrow k}$ into $\omega^{\not\downarrow k}$ which induce an $\mathcal{E}_k$-tree. Notice that the identity function is identified with $\omega^{\not\downarrow k}$ and therefore $\omega^{\not\downarrow k}$ is regarded as a member of $\mathcal{E}_k$. It is of course the maximal member of $\mathcal{E}_k$: Every $X \in \mathcal{E}_k$ satisfies $X \leq \omega^{\not\downarrow k}$. Notice also that $\omega^{\not\downarrow \leq k}$ is an $\mathcal{E}_k$-tree, and this tree is induced by $\omega^{\not\downarrow k}$. The set $\omega^{\not\downarrow k}$ is the prototype for all members of $\mathcal{E}_k$ in the sense that every member X of $\mathcal{E}_k$ will be a subset of $\omega^{\not\downarrow k}$ which has the same structure as $\omega^{\not\downarrow k}$, according to the interaction between the two orders $\prec$ and $\sqsubset$. Definition 2.5 essentially is generalizing the key points about the structure, according to $\prec$ and $\sqsubset$, of the identity function on $\omega^{\not\downarrow \leq k}$.

The family of all non-empty finite subsets of $\omega^{\not\downarrow k}$ will be denoted by $\mathrm{FIN}(\omega^{\not\downarrow k})$. Clearly, $\mathcal{AR}^k \subset \mathrm{FIN}(\omega^{\not\downarrow k})$. If $E \in \mathrm{FIN}(\omega^{\not\downarrow k})$, then we denote the minimal and maximal elements of E with respect to $\prec$ by $\min_\prec(E)$ and $\max_\prec(E)$, respectively.

Example 2.6 (The space $\mathcal{E}_2$) The members of $\mathcal{E}_2$ look like ω many copies of the Ellentuck space. The well-order $(\omega^{\not\downarrow \leq 2}, \prec)$ begins as follows:

$$() \prec (0) \prec (0,0) \prec (0,1) \prec (1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2) \prec (2,2) \prec \cdots$$

The tree structure of $\omega^{\not\downarrow \leq 2}$, under lexicographic order, looks like ω copies of ω, and has order type the countable ordinal $\omega \cdot \omega$. Here, we picture the finite tree $\{\vec{s}_m : 1 \leq m \leq 21\}$, which indicates how the rest of the tree $\omega^{\not\downarrow \leq 2}$ is formed. This is exactly the tree formed by taking all initial segments of the set $\{\vec{u}_n : 1 \leq n \leq 15\}$.

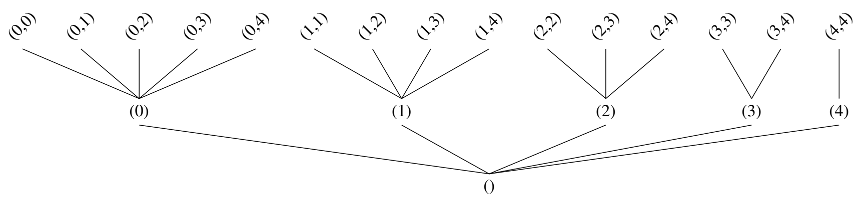

Figure 1: Initial structure of $\omega^{\not\downarrow \leq 2}$.

Next we present the specifics of the structure of the space $\mathcal{E}_3$.

Example 2.7 (The space $\mathcal{E}_3$) The well-order $(\omega^{\not\downarrow \leq 3}, \prec)$ begins as follows:

$$\begin{aligned} &() \prec (0) \prec (0,0) \prec (0,0,0) \prec (0,0,1) \prec (0,1) \prec (0,1,1) \prec (1) \\ &\prec (1,1) \prec (1,1,1) \prec (0,0,2) \prec (0,1,2) \prec (0,2) \prec (0,2,2) \\ &\prec (1,1,2) \prec (1,2) \prec (1,2,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (0,0,3) \prec \cdots \end{aligned}$$

The set $\omega^{\not\downarrow \leq 3}$ is a tree of height three with each non-maximal node branching into ω many nodes. The maximal nodes in the following figure are technically the set $\{\vec{u}_n : 1 \leq n \leq 20\}$, which indicates the structure of $\omega^{\not\downarrow \leq 3}$.

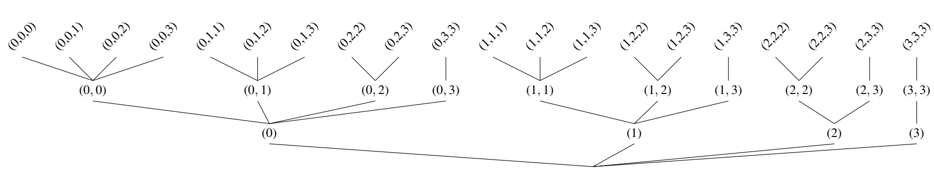

Figure 2: Initial structure of $\omega^{\not\downarrow \leq 3}$.

2.1 Upper Triangular Representation

We now present an alternative and very useful way to visualize elements of $\mathcal{E}_2$. This turned out to be fundamental to developing more intuition and to understanding the Banach spaces that we define in the following section. We refer to it as the *upper triangular representation of* $\omega^{\not\downarrow 2}$:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	...
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	...
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	...
													(13,13)	(13,14)	(13,15)	
														(14,14)	(14,15)	
															(15,15)	
																$\ddots$

Figure 3: Upper triangular representation of $\omega^{\not\downarrow 2}$.

The well-order $(\omega^{\not\downarrow 2}, \prec)$ begins as follows: $(0,0) \prec (0,1) \prec (1,1) \prec (0,2) \prec (1,2) \prec (2,2) \prec \cdots$. In comparison with the tree representation shown in Figure 1, the upper triangular representation makes it simpler to visualize this well-order: Starting at $(0,0)$ we move from top to bottom throughout each column, and then to the right to the next column.

The following figure shows the initial part of an $\mathcal{E}_2$-tree $\widehat{X}$. The highlighted pieces represent the restriction of $\widehat{X}$ to $\omega^{\not\downarrow 2}$.

$$
\begin{array}{ccccccccccccccccccccccc}
() & \prec & (0) & \prec & (0,0) & \prec & (0,1) & \prec & (1) & \prec & (1,1) & \prec & (0,2) & \prec & (1,2) & \prec & (2) & \prec & (2,2) & \prec & (0,3) & \prec & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
() & \prec & (2) & \prec & (2,4) & \prec & (2,6) & \prec & (6) & \prec & (6,6) & \prec & (2,8) & \prec & (6,8) & \prec & (9) & \prec & (9,9) & \prec & (2,10) & \prec & \cdots
\end{array}
$$

Figure 4: Initial part of an $\mathcal{E}_2$-tree.

Under the identification discussed after Definition 2.5, we have that

$$X = \{(2,4), (2,6), (6,6), (2,8), (6,8), (9,9), (2,10), \ldots\}$$

is an element of $\mathcal{E}_2$. Using the upper triangular representation of $\omega^{\not\uparrow 2}$ we can visualize $r_{10}(X)$:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	...
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	**(2,4)**	(2,5)	**(2,6)**	(2,7)	**(2,8)**	(2,9)	**(2,10)**	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	**(6,8)**	(6,9)	(6,10)	**(6,11)**	(6,12)	(6,13)	(6,14)	(6,15)	...
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	**(9,12)**	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	...
													(13,13)	(13,14)	(13,15)	
														(14,14)	(14,15)	
															(15,15)	
																⋱

Figure 5: $r_{10}(X)$ for a typical $X \in \mathcal{E}_2$.

2.2 Special Maximal Elements of $\mathcal{E}_k$

There are special elements in $\mathcal{E}_k$ that are useful for describing the structure of some subspaces of the Banach spaces that we define in the following section. Given $v \in \omega^{\not\uparrow k}$ we want to construct a special $X_v^{\max} \in \mathcal{E}_k$ that has v as its first element and that is maximal in the sense that every other $Y \in \mathcal{E}_2$ with v as its $\prec$-minimum member is a subset of X.

For example, if $v = (0,4) \in \omega^{\not\uparrow 2}$, then a finite approximation of $X_v^{\max} \in \mathcal{E}_2$ looks like this:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	...
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	...
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	
												(12,12)	(12,13)	(12,14)	(12,15)	...
													(13,13)	(13,14)	(13,15)	
														(14,14)	(14,15)	
															(15,15)	
																⋱

Figure 6: A finite approximation of $X^{\max}_{(0,4)} \in \mathcal{E}_2$.

Now let us illustrate this with $k = 3$ and $v = (0, 2, 7)$. Since we want v as the first element of $X^{\max}_v$, we identify it with $(0, 0, 0)$ and then we choose the next elements as small as possible following Definition 2.5:

$$
\begin{aligned}
&() \prec (0) \prec (0,2) \prec (0,2,7) \prec (0,2,8) \prec (0,8) \prec (0,8,8) \prec (8)\\
&\quad \prec (8,8) \prec (8,8,8) \prec (0,2,9) \prec (0,8,9) \prec (0,9) \prec (0,9,9)\\
&\quad \prec (8,8,9) \prec (8,9) \prec (8,9,9) \prec (9) \prec (9,9) \prec (9,9,9) \prec (0,2,10) \prec \cdots
\end{aligned}
$$

Therefore, under the identification discussed after Definition 2.5, we have

$$X^{\max}_v = \{(0,2,7), (0,2,8), (0,8,8), (8,8,8), (0,2,9), \ldots\}.$$

In general, for any $k \geq 2$, $X^{\max}_v$ is constructed as follows.

Definition 2.8 Let $k \geq 2$ be given and suppose $v = (n_1, n_2, \ldots, n_k)$. First we define the $\mathcal{E}_k$-tree $\widehat{X}_v$ that will determine $X^{\max}_v$. $\widehat{X}_v$ must be a function from $\omega^{\not\nearrow \leq k}$ to $\omega^{\not\nearrow \leq k}$ satisfying Definition 2.5 and such that $\widehat{X}_v(0, 0, \ldots, 0) = v$. So, for $m, j \in \mathbb{N}^+, j \leq k$, define the following auxiliary functions: $f_j(0) := n_j$ and $f_j(m) := n_k + m$. Then, for $t = (m_1, m_2, \ldots, m_l) \in \omega^{\not\nearrow \leq k}$ set $\widehat{X}_v(t) := (f_1(m_1), f_2(m_2), \ldots, f_l(m_l))$. Finally, define $X^{\max}_v$ to be the restriction of $\widehat{X}_v$ to $\omega^{\not\nearrow k}$.

It is routine to check that:

Lemma 2.9 *Let $v = (n_1, \ldots, n_k)$. Then $\widehat{X}_v$ is an $\mathcal{E}_k$-tree, and $w = (m_1, \ldots, m_k) \in \omega^{\not\nearrow k}$ belongs to $X^{\max}_v$ if and only if either $m_1 > n_k$ or else there is $1 \leq i \leq k$ such that $(m_1, m_2, \ldots, m_i) = (n_1, n_2, \ldots, n_i)$, and if $i < k$ then $m_{i+1} > n_k$.*

We use this Lemma to prove that $X_v^{\max}$ contains all elements of $\mathcal{AR}^k$ that have v as initial value.

Proposition 2.10 *Let $E \in \mathcal{AR}^k$ and $v = \min_{\prec}(E)$. Then $E \subset X_v^{\max}$.*

Proof Let $E \in \mathcal{AR}^k$ with $v = \min_{\prec}(E)$. Then there exists an $\mathcal{E}_k$-tree $\widehat{X}$ such that $\widehat{X}(0, \dots, 0) = (n_1, \dots, n_k) = v$ and $E = r_j(X)$ for some j. If E has more than one member, then its second element is $\widehat{X}(0, \dots, 0, 1) = (n_1, \dots, n_{k-1}, n'_k)$, for some $n'_k > n_k$. By Lemma 2.9, $\widehat{X}(0, \dots, 0, 1) \in X_v^{\max}$.

Suppose that $w = (p_1, \dots, p_k)$ is any member of E besides v. We will show that $w \in X_v^{\max}$. Let $(m_1, \dots, m_k) \in \omega^{\not\downarrow k}$ be the sequence such that $w = \widehat{X}(m_1, \dots, m_k)$. Suppose first that $m_1 > 0$. Then

$$(0, \dots, 0, 1) \prec (m_1) \prec (m_1, m_2, \dots, m_k).$$

Applying $\widehat{X}$ and recalling that $\widehat{X}$ preserves $\prec$ and $\sqsubset$, we conclude that

$$(n_1, \dots, n_{k-1}, n'_k) \prec (p_1) \prec (p_1, p_2, \dots, p_k).$$

Comparing the first two elements we see that either $p_1 > n'_k$ or else $p_1 = n'_k$ and $p_1 > n_1$. In both cases, $p_1 \geq n'_k > n_k$, which, by Lemma 2.9, gives that $(p_1, p_2, \dots, p_k) = w \in X_v^{\max}$.

Suppose now that $m_1 = \cdots = m_i = 0$ and $m_{i+1} > 0$, where $i + 1 \leq k$. Then

$$(0, \dots, 0, 1) \prec (m_1, \dots, m_i, m_{i+1}) = (0, \dots, 0, m_{i+1}) \prec (0, \dots, 0, m_{i+1}, \dots, m_k).$$

Applying $\widehat{X}$ we obtain

$$(n_1, \dots, n_{k-1}, n'_k) \prec (n_1, \dots, n_i, p_{i+1}) \prec (n_1, \dots, n_i, p_{i+1}, \dots, p_k).$$

Arguing as before, we conclude that $p_{i+1} \geq n'_k > n_k$, and then Lemma 2.9 yields that $w \in X_v^{\max}$. $\square$

Corollary 2.11 *If $w \in X_v^{\max}$, then $X_w^{\max} \subseteq X_v^{\max}$. In particular, if $E \in \mathcal{AR}^k$ and $\min_{\prec}(E) \in X_v^{\max}$, then $E \subset X_v^{\max}$.*

Notice that the only elements in $\widehat{X}_v$ that are $\prec$-smaller than v are the initial segments of v.

Corollary 2.12 *If $s \prec v$ and $s \in \operatorname{ran}(\widehat{X}_v)$ then $s \sqsubset v$.*

3 The Banach Spaces $T_k(d,\theta)$ and $T(\mathcal{A}_d^k,\theta)$

3.1 Preliminary Definitions

Set $\mathbb{N} := \{0,1,\dots\}$ and $\mathbb{N}^+ := \mathbb{N}\setminus\{0\}$. For the rest of this paper, fix $d,k \in \mathbb{N}^+$, $k \geq 2$ and $\theta \in \mathbb{R}$ with $0 < \theta < 1$. Given $E,F \in \mathrm{FIN}(\omega^{\not\downarrow k})$, we write $E < F$ (resp. $E \leq F$) to denote that $\max_{\prec}(E) \prec \min_{\prec}(F)$ (resp. $\max_{\prec}(E) \preceq \min_{\prec}(F)$), and in this case we say that E and F are successive. Similarly, for $v \in \omega^{\not\downarrow k}$, we write $v < E$ (resp. $v \leq E$) whenever $\{v\} < E$ (resp. $\{v\} \leq E$).

By $c_{00}(\omega^{\not\downarrow k})$ we denote the vector space of all functions $x : \omega^{\not\downarrow k} \to \mathbb{R}$ such that the set $\mathrm{supp}\,(x) := \{v \in \omega^{\not\downarrow k} : x(v) \neq 0\}$ is finite. Usually we write x_v instead of $x(v)$. We can extend the orders defined above to vectors $x,y \in c_{00}(\omega^{\not\downarrow k})$: $x < y$ (resp. $x \leq y$) iff $\mathrm{supp}\,(x) < \mathrm{supp}\,(y)$ (resp. $\mathrm{supp}\,(x) \leq \mathrm{supp}\,(y)$).

From the notation in Section 2 we have that $\omega^{\not\downarrow k} = \{\vec{u}_1,\vec{u}_2,\dots\}$. Denote by $(e_{\vec{u}_n})_{n=1}^{\infty}$ the canonical basis of $c_{00}(\omega^{\not\downarrow k})$. To simplify notation, we will usually write e_n instead of $e_{\vec{u}_n}$. So, if $x \in c_{00}(\omega^{\not\downarrow k})$, then $x = \sum_{n=1}^{\infty} x_{\vec{u}_n} e_{\vec{u}_n} = \sum_{n=1}^{m} x_{\vec{u}_n} e_{\vec{u}_n}$ for some $m \in \mathbb{N}^+$. Using the above convention, we will write $x = \sum_{n=1}^{\infty} x_n e_n = \sum_{n=1}^{m} x_n e_n$. If $E \in \mathcal{AR}^k$, we put $Ex := \sum_{v\in E} x_v e_v$.

3.2 Construction of $T_k(d,\theta)$ and $T(\mathcal{A}_d^k,\theta)$

The Banach spaces that we introduce in this section have their roots (as all subsequent constructions [4], [25], [1], [19], [2]) in Tsirelson's fundamental discovery of a reflexive Banach space T with an unconditional basis not containing c_0 or ℓ_p with $1 \leq p < \infty$ [28].

Recall that in the construction of the Banach spaces $T(\mathcal{S},\theta)$ and $T(\mathcal{A}_d,\theta)$, the norm is determined using sequences of the form

$$v_1 \leq E_1 < v_2 \leq E_2 < \cdots < v_m \leq E_m,$$

where $m \leq d$ and each set E_i is some finite set of natural numbers. How one extends this method for constructing norms to high dimensional Ellentuck spaces depends on how one conceives of the sets $\{v_1,\dots,v_m\}$ and E_i, $1 \leq i \leq m$. The set $\{v_1,\dots,v_m\}$ may be thought of either as simply a set of cardinality m, or as an initial segment of a member of the barrier of rank d on the Ellentuck space. Likewise, one may think of the sets E_i as simply finite sets, or as finite initial segments of members of the Ellentuck space. Thus, there are four natural ways to generalize the norm construction to the k-dimensional Ellentuck space, $\mathcal{E}_k$, or for topological Ramsey spaces in general. Fixing d and letting $m \le d$, we may (a) let $\{v_1, \dots, v_m\}$ range over all m-sized subsets of $\omega^{\not\downarrow k}$, or we may (b) restrict to those $\{v_1, \dots, v_m\}$ which are in $\mathcal{AR}^k_m$. Likewise we may (i) simply let the sets E_i range over all finite subsets of $\omega^{\not\downarrow k}$, or (ii) we may restrict the E_i to only range over members of $\mathcal{AR}^k$. Combination (a) and (i) yields the space $T(\mathcal{A}_d, \theta)$, so nothing new is gained by that approach. Combining (a) and (ii), we define the new space $T_k(d, \theta)$. The space $T(\mathcal{A}^k_d, \theta)$ is constructed using (b) and (ii). This is the most restrictive of the possible constructions. The space constructed using (b) and (i) was considered by the third author in his disseration [18].

Recalling the fact that $(\omega^{\not\downarrow k}, \prec)$ is a linear order with order type exactly that of the natural numbers, the classical notion of barrier on the natural numbers transfers to $(\omega^{\not\downarrow k}, \prec)$. We say that a subset $\mathcal{B}$ of $\mathrm{FIN}(\omega^{\not\downarrow k})$ is a *barrier on* $\omega^{\not\downarrow k}$ if (a) for each infinite subset $Y \subseteq \omega^{\not\downarrow k}$, there is some $E \in \mathcal{B}$ which is an initial segment (in the $\prec$ ordering) of Y, and (b) for each pair $E \neq F$ in $\mathcal{B}$, $E \not\subseteq F$. Given a barrier $\mathcal{B}$ on $(\omega^{\not\downarrow k}, \prec)$, the set

$$C_1(\mathcal{B}) = \{F \subseteq \mathrm{FIN}(\omega^{\not\downarrow k}) : (\exists E \in \mathcal{B})\, F \subseteq E\}$$

is a *compact family on* $\omega^{\not\downarrow k}$. Notice that $[\omega^{\not\downarrow k}]^d$ is the barrier of rank d on $(\omega^{\not\downarrow k}, \prec)$, and that $C_1([\omega^{\not\downarrow k}]^d) = [\omega^{\not\downarrow k}]^{\le d}$.

The notion of barrier for abstract topological Ramsey spaces appears in Chapter 5 of [27]. For high dimensional Ellentuck spaces, the notion of barrier reduces to the following.

Definition 3.1 A family $\mathcal{B} \subset \mathcal{AR}^k$ is a *barrier* on $\mathcal{E}_k$ if

(a) For every $Y \le X$ there exists n such that $r_n(Y) \in \mathcal{B}$; and

(b) $E \not\subseteq F$, for all $E \neq F$ in $\mathcal{B}$.

The barrier on $\mathcal{E}_k$ of rank d is exactly $\mathcal{AR}^k_d$, the set of all d-th approximations to members of $\mathcal{E}_k$. Barriers of infinite rank are defined recursively in a manner similarly to barriers of infinite rank on the natural numbers. However, as the precise definition takes several paragraphs and as infinite rank barriers are not used in this paper, we omit their definition here.

Notation: Given a barrier $\mathcal{B}$ on $\mathcal{E}_k$, let

$$C(\mathcal{B}) = \{F \in \mathcal{AR}^k : (\exists E \in \mathcal{B})\, F \subseteq E\}.$$

This is the analogue for $\mathcal{E}_k$ of a compact family determined by a barrier on the natural numbers, and we call $C(\mathcal{B})$ a *compact family on* $\mathcal{E}_k$. Notice that $C(\mathcal{AR}^k_d) = \bigcup_{m \le d} \mathcal{AR}^k_m$, which we denote as $\mathcal{A}^k_d$.

Definition 3.2 Let $\mathcal{F}$ be a compact family on $\omega^{\not k}$ or on $\mathcal{E}_k$. We say that $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ is *$\mathcal{F}$-admissible* if and only if there exists $\{v_1, v_2, \ldots, v_m\} \in \mathcal{F}$ such that $v_1 \le E_1 < v_2 \le E_2 < \cdots < v_m \le E_m$.

For $x = \sum_{n=1}^{\infty} x_n e_n \in c_{00}(\omega^{\not k})$ and $j \in \mathbb{N}$, we define a non-decreasing sequence of norms on $c_{00}(\omega^{\not k})$ as follows:

- $|x|_0^{\mathcal{F}} := \max\limits_{n \in \mathbb{N}^+} |x_n|$,

- $|x|_{j+1}^{\mathcal{F}} := \max \left\{ |x|_j^{\mathcal{F}}, \theta \max \left\{ \sum\limits_{i=1}^{m} |E_i x|_j^{\mathcal{F}} : 1 \le m \le d, (E_i)_{i=1}^m \ \mathcal{F}\text{-admissible} \right\} \right\}$.

For fixed $x \in c_{00}(\omega^{\not k})$, the sequence $(|x|_j^{\mathcal{F}})_{j\in\mathbb{N}}$ is bounded above by the $\ell_1(\omega^{\not k})$-norm of x. Therefore, we can set

$$\|x\|^{\mathcal{F}} := \sup_{j \in \mathbb{N}} |x|_j^{\mathcal{F}} .$$

We write $\|\cdot\|_{T_k(d,\theta)}$ to denote the norm obtained using $\mathcal{F} = C_1([\omega^{\not k}]^d)$, the compact family determined by the rank d barrier on $(\omega^{\not k}, \prec)$, and write $\|\cdot\|_{T(\mathcal{A}_d^k,\theta)}$ to denote the norm obtained using $\mathcal{F} = C(\mathcal{AR}_d^k)$. Clearly $\|\cdot\|_{T_k(d,\theta)}$ and $\|\cdot\|_{T(\mathcal{A}_d^k,\theta)}$ are norms on $c_{00}(\omega^{\not k})$.

Definition 3.3 The completion of $c_{00}(\omega^{\not k})$ with respect to the norm $\|\cdot\|_{T_k(d,\theta)}$ is denoted by $(T_k(d,\theta), \|\cdot\|)$. Likewise, the completion of $c_{00}(\omega^{\not k})$ with respect to the norm $\|\cdot\|_{T(\mathcal{A}_d^k,\theta)}$ is denoted by $(T(\mathcal{A}_d^k,\theta), \|\cdot\|)$.

Notice that when $k = 1$, $T_1(d,\theta)$ is the space Bellenot considered [4].

For $v \in \omega^{\not k}$ and $x \in T_k(d,\theta)$ we also write $v < x$ whenever $v < \mathrm{supp}\,(x)$. From the preceding definition we have the following:

Proposition 3.4 *If $\mathcal{F}$ is a compact family on $\omega^{\not k}$ or on $\mathcal{E}_k$, $x \in c_{00}(\omega^{\not k})$ and $|\mathrm{supp}\,(x)| = n$, then $|x|_n^{\mathcal{F}} = |x|_{n+1}^{\mathcal{F}} = \cdots$.*

Therefore, we conclude that for every $x \in c_{00}(\omega^{\not k})$ we have

$$\|x\|^{\mathcal{F}} = \max_{j \in \mathbb{N}} |x|_j^{\mathcal{F}} .$$

Also, note that the formulas defining $|x|_j^{\mathcal{F}}$ do not depend on the signs of x, so the unit basis of $c_{00}(\omega^{\not\downarrow k})$ is a 1-unconditional basic sequence, and by definition the unit basis generates $T_k(d,\theta)$ and $T(\mathcal{A}_d^k,\theta)$. Therefore we have the following:

Proposition 3.5 *$(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis of $T_k(d,\theta)$ and of $T(\mathcal{A}_d^k,\theta)$.*

Proposition 3.6 *For $x=\sum_{n=1}^{\infty} x_n e_n \in T_k(d,\theta)$ it follows that*

$$\|x\|_{T_k(d,\theta)} = \max\left\{\|x\|_\infty, \theta \sup\left\{\sum_{i=1}^{m}\|E_i x\|_{T_k(d,\theta)} : 1\le m\le d, (E_i)_{i=1}^m \text{ } d\text{-admissible}\right\}\right\},$$

where $\|x\|_\infty := \sup_{n\in\mathbb{N}^+}|x_n|$. Likewise for $T(\mathcal{A}_d^k,\theta)$.

4 Subspaces of $T_k(d,\theta)$ isomorphic to ℓ_∞^N

The Banach space $T_k(d,\theta)$ "lives" in $\omega^{\not\downarrow k}$, at the top of $\omega^{\not\downarrow\le k}$. We will see that it's structure is determined by subspaces indexed by elements in the lower branches. Let $s\in\omega^{\not\downarrow\le k}$. The tree generated by s and the Banach space associated to it are given by

$$\tau^k[s] := \left\{v\in\omega^{\not\downarrow k} : s\sqsubseteq v\right\} \qquad \text{and} \qquad T^k[s] := \overline{\operatorname{span}}\{e_v : v\in\tau^k[s]\},$$

respectively. In this section, let $N\in\mathbb{N}^+$ and $s_1,\dots,s_N\in\omega^{\not\downarrow\le k}$ be such that $|s_1|=\cdots=|s_N|<k$ and $s_1\prec\cdots\prec s_N$. The following is a very useful result.

Corollary 4.1 *If $v\in\omega^{\not\downarrow k}$ satisfies $s_N\prec v$, then there is at most one $i\le N$ such that $X_v^{\max}\cap\tau^k[s_i]\ne\emptyset$.*

Proof Suppose that $v=(n_1,n_2,\dots,n_k)$ and let $w=(m_1,m_2,\dots,m_k)\in X_v^{\max}$. It follows from Lemma 2.9 that either $m_1>n_k$ or that there is $0<i<k$ such that $(m_1,\dots,m_i)=(n_1,\dots,n_i)$ and $m_{i+1}>n_k$.

If $m_1>n_k$, w does not belong to any $\tau^k[s_i]$ because n_k is greater than any coordinate of the s_i's and as a result, none of the s_i's can be an initial segment of w. Heence it follows from the other option of w that the the only way an s_i is an initial segment of w is if s_i is also an initial segment of v. Since all the s_i's have the same length, at most one of them is an initial segment of v and the result follows. $\square$

It is useful to have an analogous result to the preceding corollary but related to approximations $E \in \mathcal{AR}^k$ instead of special maximal elements of $\mathcal{E}_k$:

Lemma 4.2 *Suppose $E \in \mathcal{AR}^k$ and set $v := \min_{\prec}(E)$. If $s \prec v$ and $s \sqsubset v$, then $E \cap \tau^k[s] \neq \emptyset$.*

We will study the Banach space structure of the subspaces of $T_k(d,\theta)$ of the form $Z := T^k[s_1] \oplus T^k[s_2] \oplus \cdots \oplus T^k[s_N]$. Since $(e_{\vec{u}_n})_{n=1}^{\infty}$ is 1-unconditional, we can decompose Z as $F \oplus C$, where

$$F = \overline{\operatorname{span}}\{e_v \in Z : v \in \omega^{\not\uparrow k}, v \preceq s_N\} \quad \text{and} \\ C = \overline{\operatorname{span}}\{e_v \in Z : v \in \omega^{\not\uparrow k}, s_N \prec v\}. \tag{2}$$

By setting $k = 2, N = 4, s_1 = (4), s_2 = (6), s_3 = (8)$, and $s_4 = (10)$, the following figure shows the elements of $\omega^{\not\uparrow 2}$ used to generate the subspaces F (dashed outline) and C (thicker outline) in which we decompose the subspace $T^2[(4)] \oplus T^2[(6)] \oplus T^2[(8)] \oplus T^2[(10)]$:

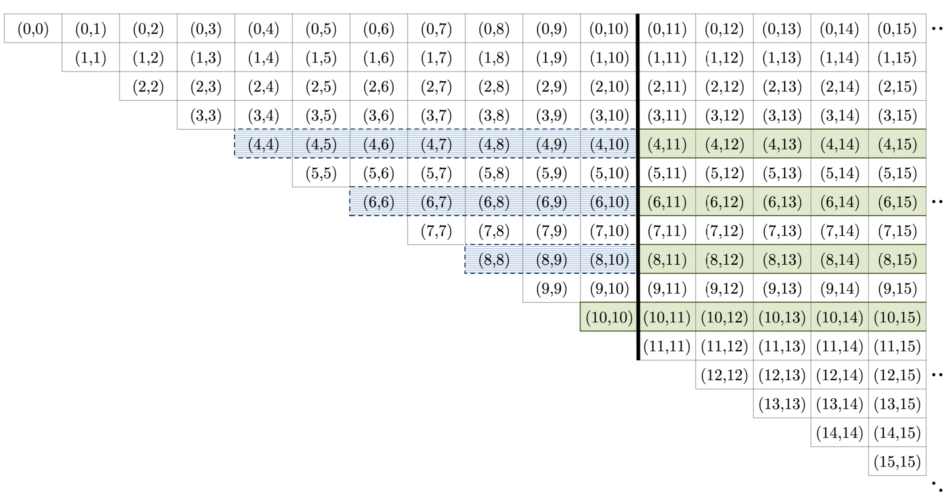

Figure 7: Elements of $\omega^{\not\uparrow 2}$ used to generate $T^2[(4)] \oplus T^2[(6)] \oplus T^2[(8)] \oplus T^2[(10)]$.

Applying Corollary 4.1 and Lemma 4.2 we have:

Lemma 4.3 *Let $E \in \mathcal{AR}^k$ be such that $s_N \prec \min_{\prec}(E)$. If $E[T_k(d,\theta)] := \overline{\operatorname{span}}\{e_w : w \in E\}$, then either $E[T_k(d,\theta)] \cap C = \emptyset$, or there is exactly one $i \leq N$ such that $E[T_k(d,\theta)] \cap C \subset T^k[s_i]$.*

Proof Suppose that $E[T_k(d,\theta)] \cap C \neq \emptyset$ and set $v := \min_{\prec}(E)$. Then, by Corollary 4.1, there is exactly one $i \in \{1,\dots,N\}$ such that $X_v^{\max} \cap \tau^k[s_i] \neq \emptyset$; consequently, $s_i \sqsubset v$ and $E \cap \tau^k[s_j] = \emptyset$ for any $j \in \{1,\dots,N\}, j \neq i$. By hypothesis, $s_i \preceq s_N \prec v$. Applying Lemma 4.2 we conclude that $E \cap \tau^k[s_i] \neq \emptyset$. Hence, $E[T_k(d,\theta)] \cap C \subset T^k[s_i]$. $\square$

This lemma helps us establish the presence of arbitrarily large copies of ℓ_∞^N inside $T_k(d,\theta)$:

Theorem 4.4 *Suppose that $s_1 \prec s_2 \prec \cdots \prec s_N$ belong to $\omega^{\not\uparrow<k}$ and that $|s_1| = \cdots = |s_N| < k$. Let $v \in \omega^{\not\uparrow k}$ with $s_N \prec v$ and suppose that $x \in \sum_{i=1}^N \oplus T^k[s_i]$ satifies $v < x$. If we decompose x as $x_1 + \cdots + x_N$ with $x_i \in T^k[s_i]$, then*

$$\max_{1\le i\le N} \|x_i\| \le \|x\| \le \frac{\theta(d-1)}{1-\theta} \max_{1\le i\le N} \|x_i\| .$$

In particular, if $\|x_1\| = \cdots = \|x_N\| = 1$, span$\{x_1,\dots,x_N\}$ is isomorphic to ℓ_∞^N in a canonical way and the isomorphism constant is independent of N and of the x_i's.

Proof Since the basis of $T_k(d,\theta)$ is unconditional, $\max_{1\le i\le N}\|x_i\| \le \|x\|$. We will check the upper bound. Let $m \in \{1,\dots,d\}$ and $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ be an admissible sequence such that $\|x\| = \theta \sum_{i=1}^m \|E_i x\|$.

Without loss of generality we assume that $E_1 x \neq 0$, so that $s_N \prec \min_{\prec}(E_2)$. By Lemma 4.3, when $j \ge 2$, we have $E_j x = E_j x_i$ for some $i \in \{1,\dots,N\}$. Then it follows that $\|E_j x\| \le \max_{1\le i\le N}\|x_i\|$. Consequently,

$$\|x\| \le \theta \|E_1 x\| + \theta(d-1) \max_{1\le i\le N} \|x_i\| .$$

Repeat the argument for $E_1 x$. Find $m' \in \{1,\dots,d\}$ and an admissible sequence $(F_i)_{i=1}^{m'} \subset \mathcal{AR}^k$ such that $\|E_1 x\| = \theta \sum_{i=1}^{m'} \|F_i(E_1 x)\|$. We can assume that $F_1(E_1 x) \neq 0$, and applying Lemma 4.3 once again we conclude that for $j \ge 2$, $\|F_j(E_1 x)\| \le \max_{1\le i\le N}\|x_i\|$. Then,

$$\|x\| \le \theta \left(\theta \|F_1(E_1 x)\| + \theta(d-1) \max_{1\le i\le N} \|x_i\| \right) + \theta(d-1) \max_{1\le i\le N} \|x_i\| .$$

Iterating this process we conclude that

$$\|x\| \le \sum_{n=1}^{\infty} \theta^n (d-1) \max_{1\le i\le N} \|x_i\| \le \frac{\theta(d-1)}{1-\theta} \max_{1\le i\le N} \|x_i\| .$$

$\square$

Example 4.5 To illustrate the previous Theorem, suppose that $k = 2$, that $s_1 < s_2 < \cdots < s_N$ and that $|s_i| = 1$ for every $i \leq N$. If we use the upper triangular representation of $\omega^{\not 2}$, the s_i's represent rows. Theorem 4.4 says that if the x_i's are supported in the ith row, but starting after the s_Nth column, then $\mathrm{span}\{x_1, \ldots, x_N\}$ is isomorphic to ℓ_∞^N, with isomorphism constant independent of N. The image below illustrates $s_1 = 0, s_1 = 2, s_2 = 3$ and $s_4 = 6$ and the shaded horizontal lines represent the support of the x_i's.

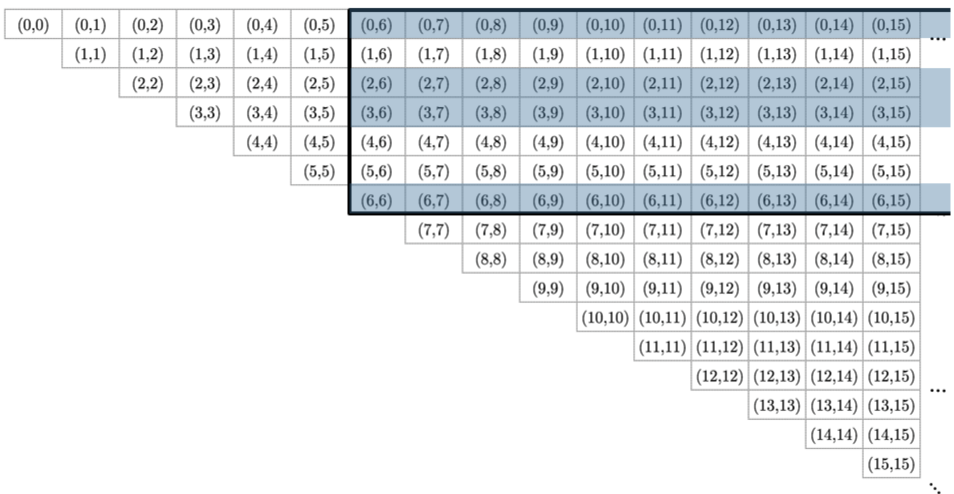

Figure 8: Elements of $\omega^{\not 2}$ that generate isomorphic copies of ℓ_∞^N.

In particular, each "column" of the upper triangular representation of $\omega^{\not 2}$ satisfies this condition. That is, $\mathrm{span}\{e_{(i,n)} : i \leq n\} \overset{C(\theta,d)}{\approx} \ell_\infty^{n+1}$, where $C(d,\theta)$ is the constant from Theorem 4.4.

5 Block Subspaces of $T_k(d,\theta)$ isomorphic to ℓ_p

For the rest of this paper suppose that $d\theta > 1$ and let $p \in (1,\infty)$ be determined by the equation $d\theta = d^{1/p}$. Bellenot proved that $T_1(d,\theta)$ is isomorphic to ℓ_p (see Theorem 1.2). The same result was then proved by Argyros and Deliyanni in [1] with different arguments which can be extended to more general cases like ours. In this section we show that we can find many copies of ℓ_p spaces inside $T_k(d,\theta)$ for $k \geq 2$:

Theorem 5.1 *Suppose that $(x_i)_{i=1}^\infty$ is a normalized block sequence in $T_k(d,\theta)$ and that we can find a sequence $(v_i)_{i=1}^\infty \subset \omega^{\not k}$ such that:*

(1) $v_1 \leq x_1 < v_2 \leq x_2 < v_3 \leq x_3 < v_4 \leq x_4 < \cdots$
(2) $\mathrm{supp}\,(x_i) \subset X_{v_i}^{\max}$ *and* $v_{i+1} \in X_{v_i}^{\max}$ *for every* $i \geq 1$.

Then, (x_i) *is equivalent to the basis of* ℓ_p.

Notice that Corollary 2.11 implies that for every $j \geq i$, $v_j \in X_{v_i}^{\max}$ and $\mathrm{supp}\left(x_j\right) \subset X_{v_i}^{\max}$. Theorem 5.1 allows us to identify natural subspaces of $T_k(d,\theta)$ isomorphic to ℓ_p. For example, it implies that the top trees of $T_k(d,\theta)$ are isomorphic to ℓ_p. In section 8 we will see that the top trees are isometrically isomorphic to $T_1(d,\theta)$.

Corollary 5.2 *If* $s \in \omega^{\not\uparrow \leq k}$ *and* $|s| = k-1$, *then* $T^k[s]$ *is isomorphic to* ℓ_p.

Proof Suppose that $s = (s_1, \ldots, c_{k-1})$. Define $v_1 = (s_1, \ldots, c_{k-1}, c_{k-1})$, $v_2 = (s_1, \ldots, c_{k-1}, c_{k-1}+1)$, $v_3 = (s_1, \ldots, c_{k-1}, c_{k-1}+2), \cdots$ and let $x_i = e_{v_i}$. If follows from Lemma 2.9 that the (v_i)'s and (x_i)'s satisfy the hypothesis of Theorem 5.1 and the result follows. $\square$

The "diagonal" subspaces of $T_k(d,\theta)$ are also isomorphic to ℓ_p spaces. For $s = (s_1, \ldots, c_l) \in \omega^{\not\uparrow \leq k}$ with $l < k$, define

$$\begin{aligned}
v_1 &= (s_1, \ldots, c_l, c_l, \ldots, c_l) \\
v_2 &= (s_1, \ldots, c_l, c_l+1, \ldots, c_l+1) \\
v_3 &= (s_1, \ldots, c_l, c_l+2, \ldots, c_l+2) \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

and $x_i = e_{v_i}$ for $i \geq 1$. Then we use Lemma 2.9 to verify that the (v_i)'s and (x_i)'s satisfy the hypothesis of Theorem 5.1 and we conclude

Corollary 5.3 $D[s] = \overline{\mathit{span}}\{e_{v_i} : i \geq 1\}$ *is isomorphic to* ℓ_p, *for every* $s \in \omega^{\not\uparrow < k}$

We prove Theorem 5.1 in two steps. First we prove the lower ℓ_p-estimate using Bellenot's space $T_1(d,\theta)$. Then we prove the upper ℓ_p-estimate in a more general case. Denote by (t_i) the canonical basis of $T_1(d,\theta)$. In order to avoid confusion, we will write $\|\cdot\|_1$ to denote the norm on $T_1(d,\theta)$.

Proposition 5.4 *Under the same hypotheses of Theorem 5.1, for a finitely supported* $z = \sum_i a_i x_i \in T_k(d,\theta)$, *we have:*

$$\frac{1}{2d}\left(\sum_i |a_i|^p\right)^{1/p} \leq \left\|\sum_i a_i t_i\right\|_1 \leq \left\|\sum_i a_i x_i\right\|_{T_k(d,\theta)}.$$

Proof Let $x = \sum_i a_i t_i \in T_1(d,\theta)$. Following Bellenot [4], either $\|x\|_1 = \max_i |a_i|$, or there exist $m \in \{1,\dots,d\}$ and $E_1 < E_2 < \cdots < E_m$ such that $\|x\|_1 = \sum_{j=1}^m \theta \|E_j x\|_1$. For each $j \in \{1,\dots,m\}$, either $\|E_j x\|_1 = \max_i\{|a_i| : i \in E_j\}$, or there exist $m' \in \{1,\dots,d\}$ and $E_{j1} < E_{j2} < \cdots < E_{jm'}$ subsets of E_j such that $\|E_j x\|_1 = \sum_{l=1}^{m'} \theta \|E_{jl} x\|_1$. Since the sequence (a_i) has only finitely many non-zero terms, this process ends and x is normed by a tree.

We will prove the result by induction on the height of the tree. If $\|x\|_1 = \max_i |a_i|$, the result follows. Since the basis of $T_k(d,\theta)$ is unconditional and the (x_i)'s are a normalized block basis of $T_k(d,\theta)$, we have that $\max_i |a_i| \le \|\sum_i a_i x_i\|$.

Suppose that the result is proved for elements of $T_1(d,\theta)$ that are normed by trees of height less than or equal to h and that x is normed by a tree of height $h+1$. Then, there exist $m \in \{1,\dots,d\}$ and $E_1 < E_2 < \cdots < E_m$ such that $\|x\|_1 = \sum_{j=1}^m \theta \|E_j x\|_1$ and each $E_j x$ is normed by a tree of height less than or equal to h.

We will find a corresponding admissible sequence in $\mathcal{AR}^k$. For each $j \in \{1,\dots,m\}$, let $n_j = \min(E_j)$ and define

$$F_j = \{w \in X^{\max}_{v_{n_j}} : w \prec v_{n_{j+1}}\} \in \mathcal{AR}^k.$$

Then $F_1 < F_2, \cdots < F_m$, and we conclude that (F_j) is *d-admissible*. Moreover we easily check from the hypothesis of Theorem 5.1 and by Corollary 2.11 that if $i \in E_j$, then $\operatorname{supp}(x_i) \subset F_j$. Hence, using the induction hypothesis and the unconditionality of the basis of $T_k(d,\theta)$ we conclude that

$$\|E_j x\|_1 = \left\|\sum\nolimits_{l\in E_j} a_l t_l\right\|_1 \le \left\|\sum\nolimits_{l\in E_j} a_l x_l\right\| \le \|F_j z\|.$$

Therefore,

$$\left\|\sum\nolimits_i a_i t_i\right\|_1 = \sum_{j=1}^m \theta \|E_j x\|_1 \le \sum_{j=1}^m \theta \|F_j z\| = \left\|\sum\nolimits_i a_i x_i\right\|.$$

The result follows now applying Theorem 1.2. □

The proof of the upper bound inequality of Theorem 5.1 is harder and we need some preliminary results.

5.1 Alternative Norm

To establish a upper ℓ_p-estimate we will adapt an alternative and useful description of the norm on $T_1(d,\theta)$ introduced by Argyros and Deliyanni [1] to our spaces. In that regard, the following definition plays a key role.

Definition 5.5 Let $m \in \{1, \dots, d\}$. A sequence $(F_i)_{i=1}^m \subset \mathrm{FIN}(\omega^{\not k})$ is called *almost admissible* if there exists a d-admissible sequence $(E_n)_{n=1}^m$ in $\mathcal{AR}^k$ such that $F_i \subseteq E_i$, for $i \le m$.

A standard alternative description of the norm of the space $T_k(d,\theta)$, closer to the spirit of Tsirelson space, is as follows. Let $K_0 := \{\pm e_i^* : i \in \mathbb{N}^+\}$, and for $n \in \mathbb{N}$,

$$K_{n+1} := K_n \bigcup \{\theta(f_1 + \cdots + f_m) : m \le d, (f_i)_{i=1}^m \subset K_n\},$$

where $(\mathrm{supp}\,(f_i))_{i=1}^m$ is almost admissible. Then, set $K := \bigcup_{n\in\mathbb{N}} K_n$. Now, for each $n \in \mathbb{N}$ and fixed $x \in c_{00}(\omega^{\not k})$, define the following non-decreasing sequence of norms:

$$|x|_n^* := \max\{f(x) : f \in K_n\}.$$

Lemma 5.6 *For every $n \in \mathbb{N}$ and $x \in c_{00}(\omega^{\not k})$ we have $|x|_n = |x|_n^*$.*

Proof Clearly, $|x|_0 = |x|_0^*$ for every $x \in c_{00}(\omega^{\not k})$. So, let $n \in \mathbb{N}^+$. Suppose $|y|_j = |y|_j^*$ for every $j \in \mathbb{N}, j < n$ and every $y \in c_{00}(\omega^{\not k})$.

If $|x|_n = |x|_{n-1}$, then $|x|_n = |x|_{n-1}^* \le |x|_n^*$. Suppose $|x|_n \ne |x|_{n-1}$. Let $m \in \{1, \dots, d\}$ and $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ be an admissible sequence such that $|x|_n = \theta\sum_{i=1}^m |E_i x|_{n-1}$. Then, $|x|_n = \theta\sum_{i=1}^m |E_i x|_{n-1}^* = \theta\sum_{i=1}^m f_i(E_i x)$ for some $(f_i)_{i=1}^m \subset K_{n-1}$. Define, for each $i \in \{1, \dots, m\}$, a new functional f_i' satisfying $f_i'(y) = f_i(E_i y)$ for every $y \in c_{00}(\omega^{\not k})$. This implies that $\mathrm{supp}\,(f_i') = \mathrm{supp}\,(f_i') \cap E_i$. Then, $(f_i')_{i=1}^m \subset K_{n-1}$ with $(\mathrm{supp}\,(f_i'))_{i=1}^m$ almost admissible and $f_i'(E_i x) = f_i(E_i x)$. So,

$$\theta\sum_{i=1}^m f_i(E_i x) = \theta\sum_{i=1}^m f_i'(E_i x) \le |E_i x|_n^* \le |x|_n^*;$$

therefore, $|x|_n \le |x|_n^*$.

Now, let $f = \theta(f_1 + \cdots + f_m)$ for some $m \in \{1, \dots, d\}$ and $(f_i)_{i=1}^m \subset K_{n-1}$ with $(\mathrm{supp}\,(f_i))_{i=1}^m$ almost admissible. Then,

$$f(x) = \theta\sum_{i=1}^m f_i(x) \le \theta\sum_{i=1}^m |\mathrm{supp}\,(f_i)\,x|_{n-1}^* = \theta\sum_{i=1}^m |\mathrm{supp}\,(f_i)\,x|_{n-1}.$$

Since $(\mathrm{supp}\,(f_i))_{i=1}^m$ is almost admissible, there exists an admissible sequence $(E_i)_{i=1}^d \subset \mathcal{AR}^k$ such that $\mathrm{supp}\,(f_i) \subseteq E_{n_i}$, where $n_1, \dots, n_m \in \{1, \dots, m\}$ and $n_1 < \cdots < n_m$.

So,

$$\theta \sum_{i=1}^{m} |\mathrm{supp}\,(f_i)\, x|_{n-1} \leq \theta \sum_{i=1}^{m} |E_{n_i} x|_{n-1} \leq |x|_n\,;$$

hence, by definition of $|\cdot|_n^*$, we conclude that $|x|_n^* \leq |x|_n$. □

Consequently, an alternative description of the norm on $T_k(d,\theta)$ is:

Proposition 5.7 *For every $x \in T_k(d,\theta)$,*

$$\|x\| = \sup\{f(x) : f \in K\}\,.$$

5.2 Upper Bound for Theorem 5.1

For $m \in \{1,\dots,d\}$ we say that $f_1,\dots,f_m \in K$ are successive if $\mathrm{supp}\,(f_1) < \mathrm{supp}\,(f_2) < \cdots < \mathrm{supp}\,(f_m)$.

If $f \in K$, then there exists $n \in \mathbb{N}$ such that $f \in K_n$. The "complexity" of f increases as n increases. That is to say, for example, that the complexity of $f \in K_1$ is less than that of $g \in K_{10}$. This is captured in the following definition.

Definition 5.8 Let $n \in \mathbb{N}^+$ and $\phi \in K_n \setminus K_{n-1}$. An *analysis of* ϕ is a sequence $(K_l(\phi))_{l=0}^n$ of subsets of K such that:

(1) $K_l(\phi)$ consists of successive elements of K_l and $\bigcup_{f\in K_l(\phi)} \mathrm{supp}\,(f) = \mathrm{supp}\,(\phi)$.

(2) If $f \in K_{l+1}(\phi)$, then either $f \in K_l(\phi)$ or there exist $m \in \{1,\dots,d\}$ and successive $f_1,\dots,f_m \in K_l(\phi)$ with $(\mathrm{supp}\,(f_i))_{i=1}^m$ almost admissible and $f = \theta(f_1+\cdots+f_m)$.

(3) $K_n(\phi) = \{\phi\}$.

Note that, by definition of the sets K_n, each $\phi \in K$ has an analysis. Moreover, if $f_1 \in K_l(\phi)$ and $f_2 \in K_{l+1}(\phi)$, then either $\mathrm{supp}\,(f_1) \subseteq \mathrm{supp}\,(f_2)$ or $\mathrm{supp}\,(f_1) \cap \mathrm{supp}\,(f_2) = \emptyset$.

Let $\phi \in K_n \setminus K_{n-1}$ and let $(K_l(\phi))_{l=0}^n$ be a fixed analysis of ϕ. Suppose $(x_j)_{j=1}^N$ is a finite block sequence on $T_k(d,\theta)$.

Following [1], for each $j \in \{1,\dots,N\}$, set $l_j \in \{0,\dots,n-1\}$ as the smallest integer with the property that there exists at most one $g \in K_{l_j+1}(\phi)$ with $\mathrm{supp}\,(x_j) \cap \mathrm{supp}\,(g) \neq \emptyset$.

Then, define the *initial part* and *final part* of x_j with respect to $(K_l(\phi))_{l=0}^n$, and denote them respectively by x'_j and x''_j, as follows. Let

$$\left\{f \in K_{l_j}(\phi) : \operatorname{supp}(f) \cap \operatorname{supp}\left(x_j\right) \neq \emptyset\right\} = \{f_1, \dots, f_m\},$$

where $f_1, \dots, f_m$ are successive. Set

$$x'_j = (\operatorname{supp}(f_1))x_j \qquad \text{and} \qquad x''_j = (\cup_{i=2}^m \operatorname{supp}(f_i))x_j.$$

The following is a useful property of the sequence $(x'_j)_{j=1}^N$ (see [5]). The analogous property is true for $(x''_j)_{j=1}^N$.

Proposition 5.9 *For $l \in \{1, \dots, n\}$ and $j \in \{1, \dots, N\}$, set*

$$A_l(x'_j) := \left\{f \in K_l(\phi) : \operatorname{supp}(f) \cap \operatorname{supp}\left(x'_j\right) \neq \emptyset\right\}.$$

Then, there exists at most one $f \in A_l(x'_j)$ such that $\operatorname{supp}(f) \cap \operatorname{supp}\left(x'_i\right) \neq \emptyset$ *for some $i \neq j$.*

Proof Let $A_l(x'_j) = \{f_1, \dots, f_m\}$, where $m \geq 2$ and $f_1, \dots, f_m$ are successive. Obviously, only $\operatorname{supp}(f_1)$ and $\operatorname{supp}(f_m)$ could intersect $\operatorname{supp}\left(x'_i\right)$ for some $i \neq j$. We will prove that it is not possible for f_m.

Suppose, towards a contradiction, that $\operatorname{supp}(f_m) \cap \operatorname{supp}\left(x'_i\right) \neq \emptyset$ for some $i > j$. Given that $m \geq 2$, we must have $l \leq l_j$. Consequently, there exists $g \in K_{l_j}(\phi)$ such that $\operatorname{supp}(f_m) \subseteq \operatorname{supp}(g)$. Since $\operatorname{supp}(g) \cap \operatorname{supp}\left(x_j\right) \neq \emptyset$ and $\operatorname{supp}(g) \cap \operatorname{supp}(x_i) \neq \emptyset$ for some $i > j$, the definition of x''_j implies that $\operatorname{supp}(g) \cap \operatorname{supp}\left(x_j\right) \subseteq \operatorname{supp}\left(x''_j\right)$. Therefore, $\operatorname{supp}(f_m) \cap \operatorname{supp}\left(x'_j\right) = \emptyset$, a contradiction. □

Following [3] and [5] we now provide an upper ℓ_p-estimate that implies the upper ℓ_p-estimate of Theorem 5.1:

Proposition 5.10 *Let $(x_j)_{j=1}^N$ be a finite normalized block basis on $T_k(d, \theta)$. Denote by $(t_n)_{n=1}^\infty$ the canonical basis of $T_1(d, \theta)$. Then, for any $(a_j)_{j=1}^N \subset \mathbb{R}$, we have:*

$$\left\|\sum_{j=1}^N a_j x_j\right\| \leq \frac{2}{\theta}\left(\sum_{j=1}^N |a_j|^p\right)^{1/p}$$

Proof In order to avoid confusion, we will write $\|\cdot\|_1$ to denote the norm on $T_1(d,\theta)$. By Proposition 5.7 and Theorem 1.2 it suffices to show that for every $\phi \in K$,

$$\left|\phi\left(\sum_{j=1}^{N} a_j x_j\right)\right| \leq \frac{2}{\theta}\left\|\sum_{j=1}^{N} a_j t_j\right\|_1 .$$

By unconditionality we can assume that $x_1,\ldots,x_N$ and ϕ are positive. Suppose $\phi \in K_n \setminus K_{n-1}$ for some $n \in \mathbb{N}^+$, and let $(K_l(\phi))_{l=0}^n$ be an analysis of ϕ (see Definition 5.8). Next, split each x_j into its initial and final part, x'_j and x''_j, with respect to $(K_l(\phi))_{l=0}^n$.

We will show by induction on $l \in \{0,1,\ldots,n\}$ that for all $J \subseteq \{1,\ldots,N\}$ and all $f \in K_l(\phi)$ we have

$$\left|f\left(\sum_{j\in J} a_j x'_j\right)\right| \leq \frac{1}{\theta}\left\|\sum_{j\in J} a_j t_j\right\|_1 \quad \text{and} \quad \left|f\left(\sum_{j\in J} a_j x''_j\right)\right| \leq \frac{1}{\theta}\left\|\sum_{j\in J} a_j t_j\right\|_1 .$$

We prove the first inequality given that the other one is analogous. Let $J \subseteq \{1,\ldots,N\}$ and set $y = \sum_{j\in J} a_j x'_j$.

If $f \in K_0(\phi)$, then $f = e_i^*$ for some $i \in \mathbb{N}^+$. We want to prove that

$$|e_i^*(y)| \leq \frac{1}{\theta}\left\|\sum\nolimits_{j\in J} a_j t_j\right\|_1 .$$

Suppose that $e_i^*(y) \neq 0$. So, there exists exactly one $j_i \in J$ such that $e_i^*(x'_{j_i}) \neq 0$. Applying Proposition 5.7 we have

$$|e_i^*(y)| = \left|e_i^*(a_{j_i} x'_{j_i})\right| \leq \left\|a_{j_i} x'_{j_i}\right\| \leq |a_{j_i}|\,\|x_{j_i}\| \leq \left\|\sum\nolimits_{j\in J} a_j t_j\right\|_1$$

since the basis of $T_k(d,\theta)$ is unconditional, $\|x_{j_i}\| = 1$, and by definition

$$\max_{j\in J} |a_j| \leq \left\|\sum\nolimits_{j\in J} a_j t_j\right\|_1 .$$

Now suppose that the desired inequality holds for any $g \in K_l(\phi)$. We will prove it for $K_{l+1}(\phi)$. Let $f \in K_{l+1}(\phi)$ be such that $f = \theta(f_1 + \cdots + f_m)$, where $f_1,\ldots,f_m$ are successive elements in $K_l(\phi)$ with $(\operatorname{supp}(f_i))_{i=1}^m$ almost admissible. Then, $1 \leq m \leq d$. Without loss of generality assume that $f_i(y) \neq 0$ for each $i \in \{1,\ldots,m\}$. Define the following sets:

$$I' := \left\{ i \leq m : \exists j \in J \text{ with } f_i(x'_j) \neq 0 \text{ and } \operatorname{supp}(f) \cap \operatorname{supp}\left(x'_j\right) \subseteq \operatorname{supp}(f_i) \right\}$$

and

$$J' := \left\{ j \in J : \exists i \in \{1, \ldots, m-1\} \text{ such that } f_i(x'_j) \neq 0 \text{ and } f_{i+1}(x'_j) \neq 0 \right\}.$$

We claim that $|I'| + |J'| \leq m$. Indeed, if $j \in J'$, there exists $i \in \{1, \ldots, m-1\}$ such that $f_i(x'_j) \neq 0$ and $f_{i+1}(x'_j) \neq 0$. From the proof of Proposition 5.9 it follows that $f_{i+1}(x'_h) = 0$ for every $h \neq j$, which implies that $i+1 \notin I'$. Hence, we can define an injective map from J' to $\{1, \ldots, m\} \setminus I'$ and we conclude that $|I'| + |J'| \leq m$.

Finally, for each $i \in I'$, set $D_i := \left\{ j \in J : \operatorname{supp}(f) \cap \operatorname{supp}\left(x'_j\right) \subseteq \operatorname{supp}(f_i) \right\}$. Notice that for all $i \in I'$ we have $D_i \cap J' = \emptyset$. Then,

$$f(y) = \theta \left[\sum\nolimits_{i \in I'} f_i \left(\sum\nolimits_{j \in D_i} a_j x'_j \right) + \sum\nolimits_{j \in J'} f(a_j x'_j) \right],$$

and consequently

$$|f(y)| \leq \theta \left[\sum\nolimits_{i \in I'} \left| f_i \left(\sum\nolimits_{j \in D_i} a_j x'_j \right) \right| + \sum\nolimits_{j \in J'} \left| f(a_j x'_j) \right| \right].$$

However, by the induction hypothesis,

$$\left| f_i \left(\sum\nolimits_{j \in D_i} a_j x'_j \right) \right| \leq \frac{1}{\theta} \left\| \sum\nolimits_{j \in D_i} a_j t_j \right\|_1.$$

Moreover, for each $j \in J'$, we have $\left| f(a_j x'_j) \right| \leq \left\| a_j x'_j \right\| \leq \|a_j x_j\| = |a_j| = \|a_j t_j\|_1$. Hence,

$$\begin{aligned} |f(y)| &\leq \theta \left[\frac{1}{\theta} \sum\nolimits_{i \in I'} \left\| \sum\nolimits_{j \in D_i} a_j t_j \right\|_1 + \frac{1}{\theta} \sum\nolimits_{j \in J'} \|a_j t_j\|_1 \right] \\ &= \theta \left[\frac{1}{\theta} \sum\nolimits_{i \in I'} \left\| D_i \left(\sum\nolimits_{j \in J} a_j t_j \right) \right\|_1 + \frac{1}{\theta} \sum\nolimits_{j \in J'} \|a_j t_j\|_1 \right]. \end{aligned}$$

Given that for every $i \in I', D_i \cap J' = \emptyset$ and $|I'| + |J'| \leq m \leq d$, the family $\{D_i\}_{i \in I'} \cup \{\{j\}\}_{j \in J'}$ is d-admissible in $\mathcal{AR}^1$. So, by the definition of $\|\cdot\|_1$, we have

$$\begin{aligned} |f(y)| &\leq \theta \left[\frac{1}{\theta} \sum\nolimits_{i \in I'} \left\| D_i \left(\sum\nolimits_{j \in J} a_j t_j \right) \right\|_1 + \frac{1}{\theta} \sum\nolimits_{j \in J'} \|a_j t_j\|_1 \right] \\ &\leq \frac{1}{\theta} \left\| \sum\nolimits_{j \in J} a_j t_j \right\|_1. \end{aligned}$$

□

6 $T_k(d,\theta)$ is ℓ_p-saturated

In this section we prove that every infinite dimensional subspace of $T_k(d,\theta)$ has a subspace isomorphic to ℓ_p.

Recall that the subspaces $T^k[s]$ for $s \in \omega^{\not\downarrow\leq k}$ with $|s| < k$ decompose naturally into countable sums. Namely, if $s = (a_1, a_2, \ldots, a_l) \in \omega^{\not\downarrow\leq k}$ and $l < k$, then $\tau^k[s] = \bigcup_{j=a_l}^{\infty} \tau^k[s \frown j]$, and therefore $T^k[s] = \sum_{j=a_l}^{\infty} \oplus T^k[s \frown j]$.

The next lemma tells us that we can find elements $v \in \tau^k[s]$ such that $X_v^{\max}$ contains arbitrary tails of the decomposition of $\tau^k[s]$. Its proof follows from the definition of the $\mathcal{E}_k$-tree $\widehat{X}_v$ that determines $X_v^{\max}$ (see paragraph preceding Lemma 2.9).

Lemma 6.1 *Let $s = (a_1, a_2, \ldots, a_l) \in \omega^{\not\downarrow\leq k}$ with $l < k$. If $m \in \mathbb{N}$ with $m > a_l$ and $v = s \frown (m, m, \ldots, m) \in \omega^{\not\downarrow k}$, then $X_v^{max} \cap \tau^k[s] = \bigcup_{j=m}^{\infty} \tau^k[s \frown j]$.*

We now present the main result of this section:

Theorem 6.2 *Suppose that Z is an infinite dimensional subspace of $T_k(d,\theta)$. Then, there exists $Y \subseteq Z$ isomorphic to ℓ_p.*

Proof Let Z be an infinite dimensional subspace of $T_k(d,\theta)$. After a standard perturbation argument, we can assume that Z has a normalized block basic sequence (x_n).

We will show that a subsequence of (x_n) is isomorphic to ℓ_p. From Proposition 5.10 we have that

$$\left\|\sum_n a_n x_n\right\| \leq \frac{2}{\theta}\left(\sum_n |a_n|^p\right)^{1/p}.$$

To obtain the lower bound we will find a subsequence and a projection Q onto a subspace of the form $T^k[s]$ such that $\left(Q\left(x_{n_j}\right)\right)$ has a lower ℓ_p-estimate.

To this end, assume that $Z \subset T^k[s]$ for some $s \in \omega^{\not\downarrow\leq k}$ with $|s| < k$. Decompose $T^k[s] = \sum_{j=1}^{\infty} \oplus T^k[s_j]$, where for each $j \in \mathbb{N}^+$, $s \sqsubset s_j$, $|s_j| = |s| + 1$, and $s_j \prec s_{j+1}$. For each $j \in \mathbb{N}^+$ let $Q_j : T^k[s] \to T^k[s_j]$ be the projection onto $T^k[s_j]$. Then we have the two cases:

Case 1: $\forall j \in \mathbb{N}^+, Q_j x_n \to 0$.

Case 2: $\exists j_0 \in \mathbb{N}^+$ such that $Q_{j_0} x_n \not\to 0$.

Let us look at Case 1 first. Let v_1 be the first element of $\tau^k[s]$. Since there exists p_1 such that $\operatorname{supp}(x_1) \subset \bigcup_{j=1}^{p_1} \tau^k[s_j]$, applying Lemma 6.1 we can find $q_1 > p_1$ and $v_2 \in \tau^k[s]$ such that $v_1 \leq x_1 < v_2$ and $X_{v_2}^{\max} \cap \tau^k[s] = \bigcup_{j=q_1}^{\infty} \tau^k[s_j]$. Since $Q_j x_n \to 0$ for $1 \leq j \leq q_1$ we can find $n_2 > 1$ and $y_2 \in T^k[s]$ such that $y_2 \approx x_{n_2}$ and $Q_j y_2 = 0$ for $1 \leq j \leq q_1$. Then we have

$$v_1 \leq x_1 < v_2 < y_2 \text{ and } \operatorname{supp}(y_2) \subset X_{v_2}^{\max}.$$

We now repeat the argument. Since there exists p_2 such that $\operatorname{supp}(y_2) \subset \bigcup_{j=1}^{p_2} \tau^k[s_j]$, applying Lemma 6.1 we can find $q_2 > p_2$ and $v_3 \in \tau^k[s]$ such that $v_2 < y_2 < v_3$ and $X_{v_3}^{\max} \cap \tau^k[s] = \bigcup_{j=q_2}^{\infty} \tau^k[s_j]$. Since $Q_j x_n \to 0$ for $1 \leq j \leq q_2$, we can find $n_3 > n_2$ and $y_3 \in T^k[s]$ such that $y_3 \approx x_{n_3}$ and $Q_j y_3 = 0$ for $1 \leq j \leq q_2$. Then we have

$$v_1 \leq x_1 < v_2 < y_2 < v_3 < y_3 \text{ and } \operatorname{supp}(y_2) \subset X_{v_2}^{\max}, \operatorname{supp}(y_3) \subset X_{v_3}^{\max}.$$

Proceeding this way we find a subsequence (x_{n_i}) and a sequence (y_i) such that y_i is close enough to x_{n_i}. Consequently, $\overline{\operatorname{span}}\{y_i\} \approx \overline{\operatorname{span}}\{x_{n_i}\}$ and

$$v_1 \leq x_1 < v_2 < y_2 < v_3 < y_3 < \cdots \text{ and } \operatorname{supp}(y_i) \subset X_{v_i}^{\max} \text{ for } i > 1.$$

By Proposition 5.4, there exist $C_1, C_2 \in \mathbb{R}$ such that

$$\left\| \sum\nolimits_i a_i x_{n_i} \right\| \geq C_1 \left\| \sum\nolimits_i a_i y_{n_i} \right\| \geq C_2 \left(\sum\nolimits_i |a_i|^p \right)^{1/p}.$$

Let us look at Case 2 now. Find a subsequence (n_i) and $\delta > 0$ such that $\delta \leq \|Q_{j_0} x_{n_i}\| \leq 1$.

Let $W = \overline{\operatorname{span}}\{Q_{j_0} x_{n_i}\}$. We now apply the argument in Case 1 to the sequence $Q_{j_0} x_{n_1} < Q_{j_0} x_{n_2} < Q_{j_0} x_{n_3} < \cdots$. That is, first decompose $T^k[s_{j_0}] = \sum_{j=1}^{\infty} \oplus T^k[t_j]$, where for every $j \in \mathbb{N}^+$, $s_{j_0} \sqsubset t_j$, $|t_j| = |s_{j_0}| + 1$, $t_j \prec t_{j+1}$. Then, look at the two cases for the sequence $\left(Q_{j_0} x_{n_i}\right)$. If Case 1 is true, $\left(Q_{j_0} x_{n_i}\right)$ has a subsequence with a lower ℓ_p-estimate, and therefore $\left(x_{n_i}\right)$ has a subsequence with a lower ℓ_p-estimate; and if Case 2 is true, we can repeat the argument for some t_j that has length strictly larger than the length of s_{j_0}. If Case 1 continues to be false, after a finite number of iterations of the same argument, the length of t_j will be equal to $k-1$, and therefore, applying Corollary 5.2, $T^k[t_j]$ would be isomorphic to ℓ_p. The result follows. □

7 The spaces $T_k(d,\theta)$ are not isomorphic to each other

In this section we prove one of the main results of the paper.

Theorem 7.1 *If $k_1 \neq k_2$, then $T_{k_1}(d,\theta)$ is not isomorphic to $T_{k_2}(d,\theta)$.*

The proof goes by induction and it shows that when $k_1 > k_2$, $T_{k_1}(d,\theta)$ does not embed in $T_{k_2}(d,\theta)$. The idea is that if we had an isomorphic embedding, we would map an ℓ_∞^N-sequence into an ℓ_p^N-sequence for arbitrarily large N's. The induction step requires a stronger and more technical statement that appears in Proposition 7.4 below.

The proof uses the notation of the trees $\tau^k[s]$ and their Banach spaces $T^k[s]$ (see Section 4). We start with some lemmas. The first one is an easy consequence of the fact that the basis of $T_k(d,\theta)$ is 1-unconditional.

Lemma 7.2 *If $s \in \omega^{\not\downarrow \leq k}$, and $|s| < k$, there exist $s_1 \prec s_2 \prec s_3 \prec \cdots$ such that $|s_i| = |s| + 1$ and $\tau^k[s] = \bigcup_{i=1}^\infty \tau^k[s_i]$. Consequently, we decompose $T^k[s] = \sum_{i=1}^\infty \oplus T^k[s_i]$ and for $m \in \mathbb{N}^+$, there is a canonical projection $P_m : T^k[s] \to \sum_{i=1}^m \oplus T^k[s_i]$.*

Proof If $s = (a_1, \ldots, a_l)$, then $s_1 = (a_1, \ldots, a_l, a_l), s_2 = (a_1, \ldots, a_l, a_l + 1), s_3 = (a_1, \ldots, a_l, a_l + 2), \cdots$. $\square$

Lemma 7.3 *Let $s \in \omega^{\not\downarrow \leq k}$ with $|s| < k$. Let $v = \min \tau^k[s]$. Then $\tau^k[s] \subset X_v^{\max}$.*

Proof If $s = (a_1, \ldots, a_l)$, then we have that $v = (a_1, \ldots, a_l, a_l, \ldots, a_l)$ and the result follows from Lemma 2.9. $\square$

We are ready to state and prove the main proposition.

Proposition 7.4 *Let $s \in \omega^{\not\downarrow \leq k_1}$ with $|s| < k_1$ and decompose $T^{k_1}[s] = \sum_{i=1}^\infty \oplus T^{k_1}[s_i]$ according to Lemma 7.2. Let $M \in \mathbb{N}^+$ and $t_1, \ldots, t_M \in \omega^{\not\downarrow \leq k_2}$ such that $|t_1| = \cdots = |t_M| < k_2$.*

If $k_1 - |s| > k_2 - |t_1|$, then for every $n \in \mathbb{N}^+$, $\sum_{i=n}^\infty \oplus T^{k_1}[s_i]$ does not embed into $T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$.

Proof We proceed by induction. For the base case we assume that $k_2 - |t_1| = 1 < k_1 - |s|$. By Corollary 5.2, $T^{k_2}[t_i]$ is isomorphic to ℓ_p, and consequently so is $T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$. On the other hand, Theorem 4.4 guarantees that $T^{k_1}[s]$ has arbitrarily large copies of ℓ_∞^N.

Suppose now that the result is true for $m \in \mathbb{N}^+$ and let $k_2 - |t_1| = m + 1 < k_1 - |s|$.

We will show a simpler case first, when $M = 1$. Suppose, towards a contradiction, that there exists $n \in \mathbb{N}^+$ and an isomorphism

$$\Phi : \sum_{i=n}^{\infty} \oplus T^{k_1}[s_i] \to T^{k_2}[t_1].$$

Decompose $T^{k_2}[t_1] = \sum_{j=1}^{\infty} \oplus T^{k_2}[r_j]$ according to Lemma 7.2. Find N large enough and $v \in \omega^{\not\, k_1}$ such that $s_n \prec s_{n+1} \prec \cdots \prec s_{n+N-1} \prec v$. We will find normalized $x_1 \in T^{k_1}[s_n], x_2 \in T^{k_1}[s_{n+1}], \ldots, x_N \in T^{k_1}[s_{n+N-1}]$ such that $v < x_i$ for $i \leq N$ and we will use Theorem 4.4 to conclude that $\operatorname{span}\{x_1, \ldots, x\} \approx \ell_\infty^N$. Recall that the isomorphism constant is independent of N and of the x_i's.

Let v_1 be the first element of $\tau^{k_2}[t_1]$, and let $x_1 \in T^{k_1}[s_n]$ be such that $\|x_1\| = 1$ and $v < x_1$. Find a finitely supported $y_1 \in T^{k_2}[t_1]$ such that $y_1 \approx \Phi(x_1)$. Applying Lemma 6.1 we can find $v_2 \in \tau^{k_2}[t_1]$ such that $v_1 \leq y_1 < v_2$ and $X_{v_2}^{\max} \cap \tau^{k_2}[t_1] = \bigcup_{j=m_1+1}^{\infty} \tau^{k_2}[r_j]$ for some $m_1 \in \mathbb{N}$.

Since $k_1 - |s_{n+1}| > k_2 - |r_1| = m$ we can apply the induction hypothesis. In particular, the map

$$P_{m_1} \Phi_{|T^{k_1}[s_{n+1}]} : T^{k_1}[s_{n+1}] \to T^{k_2}[r_1] \oplus \cdots \oplus T^{k_2}[r_{m_1}].$$

is not an isomorphism. As a result, there exists $x_2 \in T^{k_1}[s_{n+1}]$ such that $\|x_2\| = 1$ and $P_{m_1}\Phi(x_2) \approx 0$. To add the property $v < x_2$, we decompose $T^{k_1}[s_{n+1}] = \sum_{i=1}^{\infty} \oplus T^{k_1}[u_i]$ as in Lemma 7.2 and apply the induction hypothesis to $\sum_{i=p}^{\infty} \oplus T^{k_1}[u_i]$ for p large enough.

Now that we have a normalized $x_2 \in T^{k_1}[s_{n+1}]$ that satisfies $v < x_2$ and $P_m\Phi(x_2) \approx 0$, we find a finitely supported $y_2 \in T^{k_2}[t_1]$ such that $y_2 \approx \Phi(x_2)$ and $P_{m_1} y_2 = 0$. Notice that $v_1 \leq y_1 < v_2 < y_2$ and that Lemma 7.3 gives that $\operatorname{supp}(y_2) \subset X_{v_2}^{\max}$.

We now repeat the argument. Use Lemma 6.1 to find $v_3 \in \tau^{k_2}[t_1]$ such that $y_2 < v_3$ and $X_{v_3}^{\max} \cap \tau^{k_2}[t_1] = \bigcup_{j=m_2+1}^{\infty} \tau^{k_2}[r_j]$. Then we find a normalized $x_3 \in T^{k_1}[s_{n+2}]$ such that $v < x_3$ and $P_{m_2}\Phi(x_3)$ is essentially zero. Finally, we find a finitely supported $y_3 \in T^{k_2}[t_1]$ such that $y_3 \approx \Phi(x_3)$ and $P_{m_2} y_3 = 0$.

Proceeding this way, for every $i \leq N$, we find normalized $x_i \in T^{k_1}[s_{n+i-1}]$ with $v < x_i$, $v_i \in \omega^{\not\, k_2}$, and $y_i \in T^{k_2}[t_1]$ such that $y_i \approx \Phi(x_i)$ and

$$v_1 \leq y_1 < v_2 < y_2 < \cdots < v_N < y_N \text{ and } v_{i+1}, \operatorname{supp}(y_i) \subset X_{v_i}^{\max}.$$

By Theorem 5.1, $(y_i)_{i=1}^N$ is isomorphic to the canonical basis of ℓ_p^N. Hence, Φ maps ℓ_∞^N isomorphically into ℓ_p^N. Since N is arbitrary, this contradicts that Φ is continuous (see equation 3 below) and we conclude the case $M = 1$.

Let $M > 1$ and suppose, towards a contradiction, that there exists $n \in \mathbb{N}^+$ and an isomorphism

$$\Phi : \sum_{i=n}^{\infty} \oplus T^{k_1}[s_i] \to T^{k_2}[t_1] \oplus T^{k_2}[t_2] \oplus \cdots \oplus T^{k_2}[t_M].$$

For each $j \in \mathbb{N}^+$ let $Q_j : \sum_{i=1}^M T^{k_2}[t_i] \to T^{k_2}[t_j]$ be the canonical projection. Decompose $T^{k_2}[t_j] = \sum_{i=1}^\infty T^{k_2}[r_i^j]$ as in Lemma 7.2 and for each $m \in \mathbb{N}^+$, let $P_m^j : T^{k_2}[t_j] \to \sum_{i=1}^m T^{k_2}[r_i^j]$ be the canonical projection onto the first m blocks.

The proof is similar to the case $M = 1$. Find N large enough and $v \in \omega^{\not\downarrow k_1}$ such that $s_n \prec s_{n+1} \prec \cdots \prec s_{n+N-1} \prec v$. Find $x_1 \in T^{k_1}[s_n]$ such that $\|x_1\| = 1$ and $v < x_1$ and find a finitely supported $y_1 \in T^{k_2}[t_1] \oplus \cdots \oplus T^{k_2}[t_M]$ such that $y_1 \approx \Phi(x_1)$.

For each $j \leq M$, let $v_1^j = \min_\prec(\tau^{k_2}[t_j])$. Use Lemma 6.1 to find $v_2^j \in \tau^{k_2}[t_j]$ such that $Q_j(y_1) < v_2^j$ and $X_{v_2}^{\max} \cap \tau^{k_2}[t_j] = \bigcup_{i=m_1^j+1}^\infty \tau^{k_2}[r_i^j]$ for some $m_1^j \in \mathbb{N}$. Let $P_1 = \sum_{j=1}^M P_{m_1^j}^j$ be the projection onto the first blocks of each of the $T^{k_2}[t_j]$'s.

Since $k_1 - |s_{n+1}| > k_2 - |r_1| = m$ we can apply the induction hypothesis. In particular, the map $P_1\Phi_{|T^{k_1}[s_{n+1}]}$ is not an isomorphism and we can find $x_2 \in T^{k_2}[s_{n+1}]$ such that $\|x_2\| = 1$ and $P_1\Phi(x_2) \approx 0$. Arguing as in the case $M = 1$, we can also assume that $v < x_2$. We then find a finitely supported $y_2 \in \sum_{j=1}^M \oplus T^{k_2}[t_j]$ such that $y_2 \approx \Phi(x_2)$ and $P_1 y_2 = 0$.

Proceeding this way, for every $i \leq N$, we find normalized $x_i \in T^{k_1}[s_{n+i-1}]$ with $v < x_i$ and $y_i \in \sum_{j=1}^M \oplus T^{k_2}[t_j]$ such that $y_i \approx \Phi(x_i)$. Moreover, for every $j \leq M$, we can find $v_i^j \in \omega^{\not\downarrow k_2}$ such that

$$v_1^j \leq Q_j(y_1) < v_2^j < Q_j(y_2) < \cdots < v_N^j < Q_j(y_N) \text{ and } v_{i+1}^j, \operatorname{supp}\left(Q_j(y_i)\right) \subset X_{v_i^j}^{\max}.$$

By Theorem 5.1, there exists $C_1 > 0$ independent of N such that for every $j \leq M$,

$$\frac{1}{C_1}\left(\sum_{i=1}^N \|Q_j(y_i)\|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{i=1}^N Q_j(y_i)\right\| \leq C_1\left(\sum_{i=1}^N \|Q_j(y_i)\|^p\right)^{\frac{1}{p}}.$$

Using the triangle inequality for $y_i = \sum_{j=1}^{M} Q_j(y_i)$, Holder's inequality $\left(\sum_{i=1}^{N} |a_i| \leq N^{1/q}\left(\sum_{i=1}^{N} |a_i|^p\right)^{1/p}, \frac{1}{p}+\frac{1}{q}=1\right)$, Theorem 5.1, and the fact that the projections Q_j are contractive, we get

$$\begin{aligned}
\sum_{i=1}^{N} \|y_i\| &\leq \sum_{i=1}^{N}\sum_{j=1}^{M} \|Q_j(y_i)\| \leq N^{1/q} \sum_{j=1}^{M} \left(\sum_{i=1}^{N} \|Q_j(y_i)\|^p\right)^{1/p} \\
&\leq C_1 N^{1/q} \sum_{j=1}^{M} \left\| \sum_{i=1}^{N} Q_j(y_i) \right\| \leq C_1 N^{1/q} \sum_{j=1}^{M} \left\| \sum_{i=1}^{N} y_i \right\| \\
&= C_1 N^{1/q} M \left\| \sum_{i=1}^{N} y_i \right\| \approx C_1 N^{1/q} M \left\| \Phi\left(\sum_{i=1}^{N} x_i\right) \right\| \\
&\leq C_1 N^{1/q} M \|\Phi\| \left\| \sum_{i=1}^{N} x_i \right\|. \qquad (3)
\end{aligned}$$

Since N is arbitrary, $\sum_{i=1}^{N} \|y_i\|$ is of order N, and $\|\sum_{i=1}^{N} x_i\|$ stays bounded, we see that Φ cannot be bounded, contradicting our assumption. □

8 $T_k(d,\theta)$ embeds isometrically into $T_{k+1}(d,\theta)$

For fixed d and θ, the spaces $T_k(d,\theta)$, $k \geq 1$, form a natural hierarchy in complexity over ℓ_p. In this section we prove that when $j < k$, the Banach space $T_j(d,\theta)$ embeds isomorphically into $T_k(d,\theta)$. The basis for these results is the special feature that the j-dimensional Ellentuck space $\mathcal{E}_j$ embeds into the k-dimensional Ellentuck space $\mathcal{E}_k$ in many different ways. First, $(\omega^{\not\downarrow \leq j}, \prec)$ embeds into $(\omega^{\not\downarrow \leq k}, \prec)$ as a trace above any given fixed stem of length $k-j$ in $\omega^{\not\downarrow \leq k}$. Second, $(\omega^{\not\downarrow \leq j}, \prec)$ also embeds into $(\omega^{\not\downarrow \leq k}, \prec)$ as the projection of each member in $\omega^{\not\downarrow \leq k}$ to its first j coordinates. There are many other ways to embed $(\omega^{\not\downarrow \leq j}, \prec)$ into $(\omega^{\not\downarrow \leq k}, \prec)$, and each of these embeddings will induce an embedding of $T_j(d,\theta)$ into $T_k(d,\theta)$, as these embeddings preserve both the tree structure and the $\prec$ order. This is implicit in the constructions of the spaces $\mathcal{E}_k$ in [10] and explicit in the recursive construction of the finite and infinite dimensional Ellentuck spaces in [11].

The following notation will be useful. Let $\Phi : \omega^{\not\downarrow k} \to \omega^{\not\downarrow k+1}$ be defined by $\Phi(v) = (0)^\frown v$, where $^\frown$ is the concatenation operation. One can easily check that Φ preserves

$\prec$ and $\sqsubset$. We can naturally extend the definition of Φ to the finitely supported vectors of $T_k(d,\theta)$ by $\Phi\left(\sum_i a_i e_{v_i}\right) = \sum_i a_i e_{\Phi(v_i)}$.

Lemma 8.1 *Let $E \in \mathcal{AR}^k$ and suppose that $v = \min_\prec E$. Then there exists $F \in \mathcal{AR}^{k+1}$ such that $\phi(E) \subset F$, $\min_\prec(\Phi(E)) = \min_\prec(F)$ and $\max_\prec(\Phi(E)) = \max_\prec(F)$.*

Proof Let $v = \min_\prec(E)$. Proposition 2.10 says that every $w \in E$ belongs to $X_v^{\max}$. Since Φ adds a 0 at the beginning of each sequence, the characterization of Lemma 2.9 implies that for every $w \in E$, $\Phi(w)$ belongs to $X^{\max}_{\Phi(v)}$. Then the initial segment of $X^{\max}_{\Phi(v)}$ defined by $F = \{s \in X^{\max}_{\Phi(v)} : s \preceq \max_\prec \Phi(E)\}$ satisfies all the conditions of the Lemma. □

Corollary 8.2 *Let $x \in T_k(d,\theta)$ be finitely supported. Then $\|\Phi(x)\|_{T_{k+1}(d,\theta)} \geq \|x\|_{T_k(d,\theta)}$.*

Proof We use induction over the length of the support of x. If $|\mathrm{supp}\,(x)| = 1$ the two norms are equal. Then we assume that the result is true for all vectors of $T_k(d,\theta)$ that have fewer than n elements in their support and we take $x \in T_k(d,\theta)$ with $|\mathrm{supp}\,(x)| = n$.

If $\|x\|_{T_k(d,\theta)} = \|x\|_{c_0}$, the result is obviously true. If $\|x\|_{T_k(d,\theta)} > \|x\|_{c_0}$, there exist $E_1, \dots, E_d \in \mathcal{AR}^k$ such that $E_1 \preceq E_2 \preceq \cdots \preceq E_d$ and $\|x\| = \theta \sum_{i=1}^d \|E_i x\|_{T_k(d,\theta)}$. Notice that this implies that $|\mathrm{supp}\,(E_i x)| < n$ for every $i \leq d$.

By Lemma 8.1, there are $F_1, \dots, F_d \in \mathcal{AR}^{k+1}$ such that $F_1 \preceq F_2 \preceq \cdots \preceq F_d$ and $\Phi(E_i) \subset F_i$ for $i \leq d$. Since

$$\|E_i x\|_{T_k(d,\theta)} \leq \|\Phi(E_i x)\|_{T_{k+1}(d,\theta)} = \|\Phi(E_i)\Phi(x)\|_{T_{k+1}(d,\theta)} \leq \|F_i \Phi(x)\|_{T_{k+1}(d,\theta)}$$

we conclude that

$$\|x\|_{T_k(d,\theta)} = \theta \sum_{i=1}^d \|E_i x\|_{T_k(d,\theta)} \leq \theta \sum_{i=1}^d \|F_i \Phi(x)\|_{T_{k+1}(d,\theta)} \leq \|\Phi(x)\|_{T_{k+1}(d,\theta)}.$$

□

To prove the reverse inequality, the following notation will be useful. For $n \in \omega$, let $tr_n : \omega^{\not\downarrow \leq k+1} \to \omega^{\not\downarrow \leq k}$ be defined by $tr_n(n_1, n_2, \dots, n_i) = (n_2, \dots, n_i)$ if $n_1 = n$ and $tr_n(n_1, n_2, \dots, n_i) = \emptyset$ if $n_1 \neq n$. If $E \subset \omega^{\not\downarrow k+1}$, $tr_n(E) = \{tr_n(v) : v \in E, tr_n(v) \neq \emptyset\}$. Notice that $tr_n(E) = \emptyset$ if for every $v \in E$, $tr_n(v) = \emptyset$.

Lemma 8.3 *Let $x = \sum_{i=1}^{N} a_i e_{v_i} \in T_{k+1}(d,\theta)$. Given $F_1 \preceq \cdots \preceq F_m$, $m \leq d$ an admissible sequence for $T_{k+1}(d,\theta)$, there is an admissible sequence $E_1 \preceq \cdots \preceq E_m$ for $T_k(d,\theta)$ such that each $E_i = \mathrm{tr}_0(F_i)$ and $\Phi(E_i x) = F_i y$.*

Proof We will prove that if $F \in \mathcal{AR}^{k+1}$ and $F\Phi(x) \neq 0$, then $E = tr_0(F) \in \mathcal{AR}^k$ and $\Phi(Ex) = F\Phi(x)$. The rest of the lemma follows easily from this.

Let $F \in \mathcal{AR}^{k+1}$ with $F\Phi(x) \neq 0$. Then there exists an $\mathcal{E}_{k+1}$-tree $\widehat{X}$ such that F is an initial segment of X. Since $F\Phi(x) \neq 0$, the first element of F is of the form $(0, n_2, \ldots, n_{k+1})$ for some $n_2 \leq \cdots \leq n_{k+1}$. Define $\widehat{Y} : \omega^{\not\downarrow \leq k} \to \omega^{\not\downarrow \leq k}$ by $\widehat{Y}(v) = tr_0\left(\widehat{X}\big((0)^\frown v\big)\right)$. Since $\widehat{X}$ preserves $\prec$ and $\sqsubset$, it follows that $\widehat{Y}$ preserves those orders as well and hence $\widehat{Y}$ is an $\mathcal{E}_k$-tree. Since $\widehat{Y}$ preserves $\prec$, it follows that $E = tr_0(F)$ is an initial segment of Y (i.e., $E \in \mathcal{AR}^k$). Since $(0)^\frown v \in F$ iff $v \in E$, we have

$$F\Phi(x) = \sum_{(0)^\frown v_i \in F} a_i e_{\Phi(v_i)} = \sum_{v_i \in E} a_i e_{\Phi(v_i)} = \Phi\left(\sum_{v_i \in E} a_i e_i\right) = \Phi(Ex).$$

To complete the proof of the Lemma, suppose that $F_1 \preceq \cdots \preceq F_m$, $m \leq d$ is an admissible sequence with $E_i = tr_0(F_i) \neq \emptyset$ for $i \leq m$. Let $i < j$, $v \in E_i$ and $w \in E_j$. Then $(0)^\frown v \in F_i$ and $(0)^\frown w \in F_j$, which implies that $(0)^\frown v \prec (0)^\frown w$. And from here we conclude that $v \prec w$. Since the elements are arbitrary we have that $E_1 \preceq \cdots \preceq E_m$. □

Corollary 8.4 *Let $x \in T_k(d,\theta)$ be finitely supported. Then $\|\Phi(x)\|_{T_{k+1}(d,\theta)} \leq \|x\|_{T_k(d,\theta)}$.*

Proof We use induction over the length of the support of x. If $|\mathrm{supp}\,(x)| = 1$ the two norms are equal. Then we assume that the result is true for all vectors of $T_k(d,\theta)$ that have fewer than n elements in their support and we take $x \in T_k(d,\theta)$ with $|\mathrm{supp}\,(x)| = n$.

If $\|\Phi(x)\|_{T_{k+1}(d,\theta)} = \|\Phi(x)\|_{c_0}$, the result is obviously true. If $\|\Phi(x)\|_{T_{k+1}(d,\theta)} > \|\Phi(x)\|_{c_0}$, there exist $F_1, \ldots, F_m \in \mathcal{AR}^{k+1}$, $m \leq d$, such that $F_1 \preceq \cdots \preceq F_m$ and $\|\Phi(x)\|_{T_{k+1}(d,\theta)} = \theta \sum_{i=1}^{m} \|F_i\Phi(x)\|_{T_{k+1}(d,\theta)}$. Notice that this implies that for $i \leq m$, $|\mathrm{supp}\,(F_i\Phi(x))| < n$. Moreover, we can assume that $F_i\Phi(x) \neq 0$.

By Lemma 8.3, there are $E_1, \ldots, E_m \in \mathcal{AR}^k$ such that $E_1 \preceq \cdots \preceq E_m$ and $F_i\Phi(E_i) = \Phi(E_i x)$. By the induction hypothesis, $\|\Phi(E_i x)\|_{T_{k+1}(d,\theta)} \leq \|E_i x\|_{T_k(d,\theta)}$. Then

$$\|\Phi(x)\|_{T_{k+1}(d,\theta)} = \theta \sum_{i=1}^{m} \|F_i \Phi(x)\|_{T_{k+1}(d,\theta)} \leq \theta \sum_{i=1}^{m} \|E_i x\|_{T_k(d,\theta)} \leq \|x\|_{T_k(d,\theta)}.$$

□

Combining the previous corollaries, we obtain the following result:

Theorem 8.5 $\Phi : T_k(d,\theta) \to T_{k+1}(d,\theta)$ *is an isometric isomorphism.*

Iterating this theorem, and using the notation of the beginning of Section 4 we describe isometrically all subspaces of $T_k(d,\theta)$ of the form $T_k[s]$ for $s \in \omega^{\not\downarrow \leq k}$ and $|s| < k$.

Corollary 8.6 *Suppose that* $s \in \omega^{\not\downarrow \leq k}$ *with* $|s| < k$. *Then* $T_k[s] \subset T_k(d,\theta)$ *is isometrically isomorphic to* $T_{k-|s|}(d,\theta)$.

Remark Another way of embedding $T_k(d,\theta)$ into $T_{k+1}(d,\theta)$ is by sending each member $s = (s_1, \dots, s_k) \in \omega^{\not\downarrow k}$ to $\Psi(s) = (s_1, \dots, s_k, s_k) \in \omega^{\not\downarrow k+1}$. One can check that Ψ maps $T_k(d,\theta)$ isometrically into $T_{k+1}(d,\theta)$. In fact, for each $k_1 < k_2$, there are infinitely many different ways of embedding $\mathcal{E}_{k_1}$ into $\mathcal{E}_{k_2}$, and each one of these

9 The Banach space $T(\mathcal{A}_d^k, \theta)$

In this section, we investigate the most complex of the natural constructions of Banach spaces using the Tsirelson methods on the high dimensional Ellentuck spaces. These spaces, $T(\mathcal{A}_d^k, \theta)$, were constructed in Section 3 using admissible sets with the order of the E_i's determined by elements of $\mathcal{A}_d^k = \bigcup_{m \leq d} \mathcal{AR}_m^k$ as follows: Let $m \in \{1, 2, \dots, d\}$. We say that $(E_i)_{i=1}^m \subset \mathcal{AR}^k$ is $\mathcal{A}_d^k$*-admissible* if and only if there exists $\{v_1, v_2, \dots, v_m\} \in \mathcal{A}_d^k$ such that $v_1 \leq E_1 < v_2 \leq E_2 < \cdots < v_m \leq E_m$. (Recall Definition 3.2.)

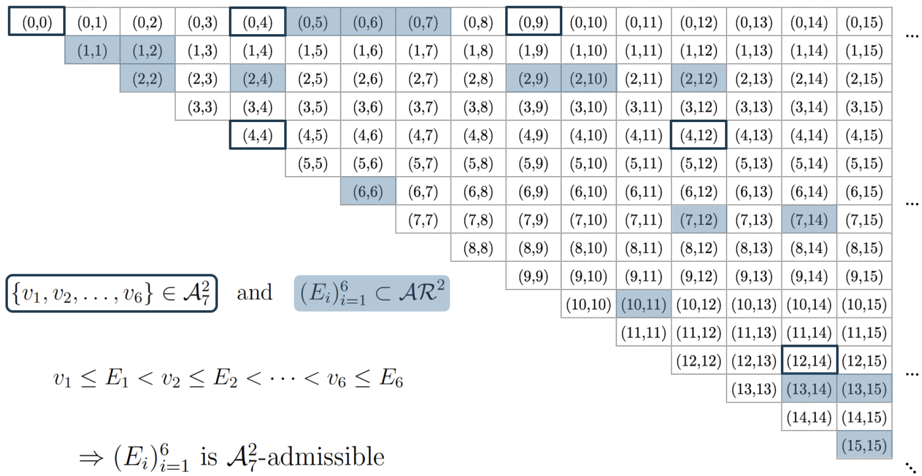

Figure 9: An $\mathcal{A}_7^2$-admissible sequence.

Recall Definition 3.3 of the norm $\|\cdot\|_{T(\mathcal{A}_d^k,\theta)}$ and Propositions 3.4 - 3.6.

The Banach space $T(\mathcal{A}_d^k,\theta)$ shares many properties with the Banach space $T_k(d,\theta)$, their proofs being very similar. In this section we indicate what parts of the proofs from previous sections for $T_k(d,\theta)$ apply also to $T(\mathcal{A}_d^k,\theta)$, and we provide the proofs that are different. In particular we have the following: (1) For $k \ge 2$, $T(\mathcal{A}_d^k,\theta)$ has arbitrarily large copies of ℓ_∞^n with uniform isomorphism constant. (2) $T(\mathcal{A}_d^k,\theta)$ is ℓ_p-saturated. (3) If $j \ne k$, then $T(\mathcal{A}_d^j,\theta)$ and $T(\mathcal{A}_d^k,\theta)$ are not isomorphic. We do not know if $T(\mathcal{A}_d^j,\theta)$ embeds into $T(\mathcal{A}_d^k,\theta)$ for $j < k$.

We start with the results of Section 4. For $s \in \omega^{\not\downarrow <k}$, define $T[\mathcal{A}_d^k, s] = \overline{\text{span}}\{e_v : v \in \tau^k[s]\}$. Since Corollary 4.1, Lemma 4.2, and Lemma 4.3 depend only on properties of $\mathcal{AR}^k$, and since these are the results used in the proof of Theorem 4.4, we obtain the following result:

Theorem 9.1 *Suppose that $s_1 \prec s_2 \prec \cdots \prec s_N$ belong to $\omega^{\not\downarrow <k}$ and that $|s_1| = \cdots = |s_N| < k$. Let $v \in \omega^{\not\downarrow k}$ with $s_N \prec v$ and suppose that $x \in \sum_{i=1}^N \oplus T[\mathcal{A}_d^k, s_i]$ satifies $v < x$. If we decompose x as $x_1 + \cdots + x_N$ with $x_i \in T[\mathcal{A}_d^k, s_i]$, then*

$$\max_{1\le i\le N} \|x_i\| \le \|x\| \le \frac{\theta(d-1)}{1-\theta} \max_{1\le i\le N} \|x_i\| .$$

In particular, if $\|x_1\| = \cdots = \|x_N\| = 1$, span$\{x_1,\dots,x_N\}$ is isomorphic to ℓ_∞^N in a canonical way and the isomorphism constant is independent of N and of the x_i's.

We now move to the results of Section 5. The main result is the same but the proof for the lower ℓ_p-estimate is different.

Theorem 9.2 *Suppose that $(x_i)_{i=1}^{\infty}$ is a normalized block sequence in $T(\mathcal{A}_d^k, \theta)$ and that we can find a sequence $(v_i)_{i=1}^{\infty} \subset \omega^{\not\downarrow k}$ such that:*

(1) $v_1 \le x_1 < v_2 \le x_2 < v_3 \le x_3 < v_4 \le x_4 < \cdots$

(2) $\operatorname{supp}(x_i) \subset X_{v_i}^{\max}$ *and* $v_{i+1} \in X_{v_i}^{\max}$ *for every* $i \ge 1$.

Then, (x_i) is equivalent to the basis of ℓ_p.

We start with the upper ℓ_p-estimate. The construction of the Alternative Norm (see Subsection 5.1) is almost identical. The main difference is the definition of almost admissible sequences.

Definition 9.3 Let $m \in \{1, \ldots, d\}$. A sequence $(F_i)_{i=1}^{m} \subset \mathrm{FIN}(\omega^{\not\downarrow k})$ is called $\mathcal{A}_d^k$-*almost admissible* if there exists an $\mathcal{A}_d^k$-admissible sequence $(E_n)_{n=1}^{d}$ such that $F_i \subseteq E_{n_i}$, where $n_1, \ldots, n_m \in \{1, \ldots, d\}$ are such that $n_1 < n_2 < \cdots < n_m$.

This results in an alternative/dual description of the norm of $T(\mathcal{A}_d^k, \theta)$. The sets K_n are smaller than the corresponding sets for $T_k(d, \theta)$, but the proofs are identical, resulting in the upper ℓ_p-estimate identical to Proposition 5.10.

The lower ℓ_p-estimate is harder, because we have fewer admissible sequences than in $T_k(d, \theta)$. We need to have enough "room" to find $\mathcal{A}_d^k$-admissible sequences, since we need to place the sets between an element of $\mathcal{A}_d^k$. To do so, we will prove a lower ℓ_p-estimate for the sequences (x_{2n}) and (x_{2n-1}). Since the closed span of (x_{2n}) and (x_{2n+1}) are complemented in the closed span of (x_n), the general result follows. We will obtain the estimate for (x_{2n}). The other case is similar.

We start with the following lemma.

Lemma 9.4 *Suppose that $(q_i)_{i=1}^{\infty} \subset \mathbb{N}$ is such that $q_1 < q_2 < \cdots$. Then, there exists $X = \{w_1, w_2, \ldots\} \in \mathcal{E}_k$ such that for every $i \in \mathbb{N}^+$, $\max(w_i) = q_i$ and w_i is a sequence all of whose terms are in the set $\{q_1, q_2, \ldots\}$.*

Proof Following Definition 2.5 we will construct inductively an $\mathcal{E}_k$-tree $\widehat{X}$ that determines X. Recall that for fixed $k \ge 2$, the first k members of $(\omega^{\not\downarrow \le k}, \prec)$ are $\vec{s}_0 = ()$, $\vec{s}_1 = (0)$, and in general, for $1 \le k' \le k$, $\vec{s}_{k'}$ is the sequence of 0's of length k'. Begin by setting $\widehat{X}(\vec{s}_0) := ()$, and for each $1 \le k' \le k$, set $\widehat{X}(\vec{s}_{k'})$ to be the sequence of length

k' with all entries being q_1. Note that $\vec{u}_0$ is the sequence of 0's of length k, and that $\max(\widehat{X}(\vec{u}_0)) = q_0$.

Suppose we have defined $\widehat{X}(\vec{s}_{m'})$ for all $m' \leq m$ so that whenever $\vec{s}_{m'} = \vec{u}_j$ for some j, then $\max(\widehat{X}(\vec{u}_j)) = q_j$. Define $\widehat{X}(\vec{s}_{m+1})$ based on the following cases:

Case 1: $|\vec{s}_{m+1}| = |\vec{s}_m| + 1$. Letting $(n_1, \ldots, n_{|\vec{s}_m|})$ denote $\widehat{X}(\vec{s}_m)$, set $\widehat{X}(\vec{s}_{m+1}) := (n_1, \ldots, n_{|\vec{s}_m|}, n_{|\vec{s}_m|})$.

Case 2: $|\vec{s}_m| = k$. Let j be the index such that $\vec{s}_m = \vec{u}_j$. By the induction hypothesis, $\max(\widehat{X}(\vec{s}_m)) = q_j$. Take m' to be the index such that $\vec{s}_{m'} \sqsubset \vec{s}_{m+1}$ with $|\vec{s}_{m'}| = |\vec{s}_{m+1}| - 1$. Then $\widehat{X}$ is already defined on $\vec{s}_{m'}$, so define $\widehat{X}(\vec{s}_{m+1}) := \widehat{X}(\vec{s}_{m'})^\frown q_{j+1}$. □

Lemma 9.5 *Assume that all the hypotheses of Theorem 9.2 are satisfied. If $m \in \{1, \ldots, d\}$ and $E_1 < E_2 < \cdots < E_m$ are finite subsets of $\mathbb{N}^+$, then there exists an $\mathcal{A}_d^k$-admissible sequence $(F_i)_{i=1}^m \subset \mathcal{AR}^k$ such that*

$$E_i \subset \left\{ j \in \mathbb{N}^+ : \operatorname{supp}\left(x_{2j}\right) \subseteq F_i \right\}.$$

Proof For $i \in \{1, \ldots, m\}$, set $n_i := \min(E_i)$. It is helpful to keep the following picture in mind throughout this proof:

$$\cdots \leq x_{2(n_i - 1)} < v_{2n_i - 1} \leq x_{2n_i - 1} < v_{2n_i} \leq x_{2n_i} < \cdots \leq x_{2(n_{i+1} - 1)} < \cdots.$$

Set $q_i := \max(v_{2n_i - 1})$. By hypothesis (1) of Theorem 9.2, it follows that $q_1 < q_2 < \cdots < q_m$. Applying Lemma 9.4, we can find $\{w_1, w_2, \ldots, w_m\}$ in $\mathcal{AR}_d^k$ such that $\max(w_i) = q_i$ and all the terms of w_i are in $\{q_1, q_2, \ldots, q_m\}$. Consequently,

$$\cdots \leq x_{2(n_i - 1)} < w_i \prec v_{2n_i} \leq x_{2n_i} < \cdots. \tag{4}$$

By hypothesis, $X_{v_{2n_i}}^{\max}$ contains the support of x_j for any $j \geq 2n_i$. Hence we define:

$$F_i = \left\{ w \in X_{v_{2n_i}}^{\max} : w \prec v_{2n_{i+1}} \right\} \tag{5}$$

as the initial segment of $X_{v_{2n_i}}^{\max}$ up to (but not including) $v_{2n_{i+1}}$. By construction, $F_i \in \mathcal{AR}^k$, $\min_{\prec}(F_i) = v_{2n_i}$, and F_i contains the supports of

$$x_{2n_i}, x_{2(n_i+1)}, x_{2(n_i+2)}, \ldots, x_{2(n_{i+1}-1)}.$$

Thus, from equation (4) and (5), we have:

$$w_1 \prec v_{2n_1} \leq F_1 < w_2 \prec v_{2n_2} \leq F_2 < \cdots \leq F_{m-1} < w_m \prec v_{2n_m} \leq F_m,$$

and we conclude that $F_1, F_2, \ldots, F_m \in \mathcal{AR}^k$ is the desired $\mathcal{A}^k_d$-admissible sequence. $\square$

With this, the proof of Proposition 5.4 applies and we obtain the lower ℓ_p-estimate for the normalized block sequence (x_{2n}). A similar argument gives a lower ℓ_p-estimate for the normalized block sequence (x_{2n-1}) and we conclude the sketch of the proof of Theorem 9.2.

Since Theorems 9.1 and 9.2 hold for $T(\mathcal{A}^k_d, \theta)$, we have all the elements to show that the different $T(\mathcal{A}^k_d, \theta)$'s are not isomorphic to each other. The proof of Theorem 7.1 applies and we obtain the following:

Theorem 9.6 *If $k_1 \neq k_2$, then $T(\mathcal{A}^{k_1}_d, \theta)$ is not isomorphic to $T(\mathcal{A}^{k_2}_d, \theta)$.*

10 Further Directions

In this paper, we considered two different methods for constructing norms on high dimensional Ellentuck spaces. One required both the admissible sets and their end-points to be finite approximations to members of $\mathcal{E}_k$, and the other only required the admissible sets to be finite approximations. Theorems 7.1 and 8.5 show that this latter norm construction produces a hierarchy of Banach spaces $T_k(d, \theta)$ which embed isometrically as subspaces into Banach spaces constructed from higher order Ellentuck spaces.

Question 10.1 For fixed d and θ and $k_1 < k_2$, does $T(\mathcal{A}^{k_1}_d, \theta)$ embed as an isometric subspace of $T(\mathcal{A}^{k_2}_d, \theta)$?

Preliminary analysis shows that if we let d' be sufficiently greater than d, we can show that the norm on $T(\mathcal{A}^{k_1}_d, \theta)$ is bounded by the norm on the trace subspace above (0) in $T(\mathcal{A}^{k_2}_{d'}, \theta)$, where d' is computed from d in a straightforward manner using methods from [10]. However, we have not checked whether or not this produces and isometric subspace.

Question 10.2 For fixed d, θ, k how different are $T_k(d, \theta)$ and $T(\mathcal{A}^k_d, \theta)$? Are they isomorphic to each other? Does one of them embed into the other?

We finish with some questions about the behaviors of norms over certain sequences of these spaces.

Question 10.3 What is the behavior of the norms on $T_k(\mathcal{B}, \theta)$, constructed using barriers $\mathcal{B}$ on $(\omega^{\not\downarrow k}, \prec)$ of infinite rank?

Finally, we ask about Banach spaces constructed on infinite dimensional Ellentuck spaces from [11].

Question 10.4 What new properties of the sequence of Banach spaces emerge as we construct $T_\alpha(d, \theta)$, where α is any countable ordinal and $\mathcal{E}_\alpha$ is the α-dimensional Ellentuck space?

We also ask this question for the spaces built using $\mathcal{F} = C(\mathcal{B})$, where $\mathcal{B}$ is a barrier on $\mathcal{E}_\alpha$.

As the spaces $T_k(d, \theta)$ and $T(\mathcal{A}_d^k, \theta)$ were shown to extend the ℓ_p space into natural hierarchies, it will be interesting to see what properties emerge in these classes of new Banach spaces.

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