Multiply chorded cycles

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Abstract

A classical result of Hajnal and Szemerédi, when translated to a complementary form, states that with sufficient minimum degree, a graph will contain disjoint large cliques. We conjecture a generalization of this result from cliques to cycles with many chords and prove this conjecture in several cases.

1 Introduction

A major branch of extremal graph theory involves finding disjoint subgraphs. In particular, much work has been focused on finding disjoint cycles in graphs. Trivially, one may observe that if $\delta(G) \geq 2$, there is a cycle in the graph G. In order to find more than one cycle using a minimum degree condition, one must appeal to the following classical result of Corrádi and Hajnal.

Theorem 1 (Corrádi and Hajnal, 1963 [3]) *If $\delta(G) \geq 2t$ and $|V(G)| \geq 3t$, then G contains t vertex disjoint cycles.*

In the case where $|V(G)| = 3t$, this guarantees a triangle factor: the vertices of G can be covered with disjoint triangles. Since triangles are cliques, a natural question is whether similar minimum degree conditions guarantee the existence of a K_{k+1} factor for $k \geq 3$. The celebrated theorem of Hajnal and Szemerédi answers this:

Theorem 2 (Hajnal and Szemerédi, 1970 [4]) *Given integers $s, k \geq 1$ and a graph G of order $s(k+1)$ with $\delta(G) \geq sk$, there exist s vertex disjoint copies of K_{k+1} in G.*

Both the Hajnal-Szemerédi theorem and the Corrádi-Hajnal theorem guaranteeing K_{k+1} factors have seen several generalizations and strengthenings, notably the Seymour conjecture (proved for sufficiently large graphs by Komlós, Sárkőzy, and Szemerédi using the regularity lemma in [5]) which states that if $\delta(G) \geq \frac{k}{k+1}n$ then G contains the kth power of a Hamiltonian cycle.

The Corrádi-Hajnal theorem, however, has one feature that the Hajnal-Szemerédi theorem does not. It guarantees structure in the graph whenever $\delta(G) \geq 2t$ even if $|V(G)| \gg 3t$. It does not guarantee the existence of triangles (indeed, if $|V(G)| > 4t$, then there need not be any triangles in the graph at all), but does none the less guarantee the existence of disjoint *cycles*. Cycles, here, can be seen as loose triangles and it is this loose structure that is guaranteed even when triangles are not. The Hajnal-Szemerédi theorem says nothing about sparse graphs, however, and the fundamental problem seems to be the notion of a loose clique. In this paper, we propose such a notion. A clique can also be viewed as a cycle with many chords. In particular, K_{k+1} is an $f(k)$-chorded cycles, where $f(k) = \frac{(k+1)(k-2)}{2}$. (Note that we take this to be $f(k)$, as opposed to $f(k+1)$ as K_{k+1} is k-regular). If the cycle is longer, still with $f(k)$ chords, this can be thought of as a "loose" clique in the sense that all the chords are present but the subgraph is not as confined as the clique.

It is trivial that if $\delta(G) \geq 2$ then G contains a cycle (and hence loose triangle), but it is less clear that if $\delta(G) \geq k$ that G contains a $f(k)$-chorded cycle (and hence a loose K_{k+1}). Addressing such a question, Ali and Staton have the following result.

Theorem 3 (Ali and Staton, 1999 [1]) *If $\delta(G) \geq k$, then G contains a cycle with at least $\left\lceil \frac{k(k-2)}{2} \right\rceil$ chords.*

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This is slightly weaker than what we would like, but we show (Theorem 5) that indeed an $f(k)$-chorded cycle is guaranteed if $\delta(G) \geq k$. In light of these results, the following conjecture provides a natural extension and amalgamation.

Conjecture 1 *Given integers $s, k \geq 1$ and a graph G of order $n \geq s(k+1)$ with $\delta(G) \geq sk$, there exist s vertex disjoint cycles in G, each with at least $f(k)$ chords.*

Note that Theorem 2 would be the special case of Conjecture 1 when $n = s(k+1)$. Our main result shows that Conjecture 1 holds when s, k and n are sufficiently large.

Theorem 4 *There exist s_0 and k_0 so that if $s \geq s_0$ and $k \geq k_0$, then there exists an $n_0 = n_0(s,k)$ so that if G has minimum degree at least sk and $|G| > n_0$, then G contains s vertex disjoint $f(k)$-chorded cycles.*

It is worth noting that Conjecture 1 can be both strengthened of weakened in several ways. As (largely) a weakening, one might ask only for subdivisions or minors. It is actually worth noting that the relationship between $f(k)$-chorded cycles and subdivisions is not quite clear - asking for chorded cycles requires that the paths between cycle vertices are of length 1, but more freedom is allowed in their placement. An interesting strengthening would be to require that the proper number of chords be *crossing*. The complete graph K_{k+1} has $\binom{k+1}{4}$ pairs of crossing chords, and so it is natural to add such a requirement to Conjecture 1. In this direction, all we know is that it is easy to show that if $\delta(G) \geq 3$, then G contains a cycle with a pair of crossing chords. An analogue for $\delta(G) \geq k$ (in analogy to Theorem 5) is still unknown.

In this work, we present several partial results toward a proof of Conjecture 1. In Section 2, we present some cases where the conjecture holds including the case when $s = 1$ and when the graph is triangle-free. In Section 3, we provide a result for finding one multiply chorded cycle using an average degree condition in place of the minimum degree. Section 4 contains a lemma involving 3-regular graphs and one concerning cycles with many chords which will be used in Section 5. The average degree result from Section 3 is also used as a lemma in Section 5, where we prove Theorem 4.

All notation is either defined when used or can be found in [2].

2 Preliminary Cases

Our first result of this section shows that Conjecture 1 holds easily for the case when $s = 1$.

Theorem 5 *If $\delta(G) \geq k$ for some positive integer k, then G contains a cycle C with at least $f(k) = \frac{(k-2)(k+1)}{2}$ chords.*

Proof: Consider a maximum length path and, over all such paths, let P have the property that the furthest neighbor of an end vertex v_1 is as far along P from that end as possible. Label the vertices of P as $v_1 v_2 \dots v_\ell \dots$ where v_ℓ is the furthest neighbor from v_1. Given a predecessor v_i of a neighbor of v_1, say $v_{\ell-1}$ for example, we call $P' = v_i v_{i-1} \dots v_1 v_{i+1} P$ a *restart* of P. Call a vertex v_j of P a *start vertex* if v_j is a predecessor of a neighbor of a start vertex after any number of restarts. Certainly all predecessors of neighbors of v_1 are start vertices.

By the assumption of P having the property that v_ℓ is furthest from v_1, there is no edge from any start vertex to a vertex v_m where $m > \ell$. Thus, if we let a be the number of start vertices, the cycle $v_1 v_2 \dots v_\ell v_1$ has at least $\frac{a(k-2)}{2}$ chords. Certainly $a \geq k$ but if $a \geq k+1$, this is already the desired cycle so we may assume $a = k$. It is easy to verify that having exactly k start vertices implies that every start vertex is adjacent to v_ℓ, meaning that we actually get at least

$$\frac{2(k-2) + (k-2)(k-3)}{2} + k - 2 = \frac{(k-2)(k+1)}{2}$$

chords as desired. □

Our next result provides a solution to Conjecture 1 in the class of triangle-free graphs.

Theorem 6 *If G is triangle-free and $\delta(G) \geq sk$ then there exist s disjoint cycles in G each with at least $\frac{(k+1)(k-2)}{2}$ chords.*

Proof: This proof is by induction on s. The case where $s = 1$ follows from Theorem 5, so suppose $s \geq 2$. By induction, there exists a set of $s-1$ disjoint cycles, each with at least $\frac{(k+1)(k-2)}{2}$ chords which avoids any set of at most k vertices. Let $\mathscr{C}$ be such a set of cycles with the additional assumptions that:

1. the sum of the lengths of cycles in $\mathscr{C}$ is as small as possible,
2. subject to 1, the number of chords in each cycle is as large as possible.

Let H be the set of vertices in $G \setminus V(\mathscr{C})$ and let $v \in H$ be a vertex of minimum degree within $G[H]$. If $d_H(v) \geq k$, then by Theorem 5, there exists another cycle as desired within H so we may assume $d_H(v) \leq k-1$. This means that $d_{\mathscr{C}}(v) \geq (s-1)k+1$ so v must have at least $k+1$ edges to a single cycle $C \in \mathscr{C}$. Let $x_1, \ldots, x_{k+1}$ be $k+1$ such neighbors in order on C.

We now show that we may replace a segment of C with v, thereby making the cycle shorter while preserving the desired number of chords. Suppose C has $g(k) \geq \frac{(k+1)(k-2)}{2}$ chords. Consider the set of all pairs of consecutive segments between neighbors of v on C. One such pair must have at most the average number of end-vertices of chords which means that, without loss of generality, the segment of C strictly between x_i and x_{i+2} (denoted by $C(x_i)$) contains at most $\frac{2g(k)}{k+1}$ end-vertices of chords. Note that we may replace the segment $C(x_i)$ for v, thereby removing at most $\frac{2g(k)}{k+1}$ chords and adding at least $k-2$ and making the cycle shorter. This is a contradiction if

$$g(k) - \frac{2g(k)}{k+1} + k - 2 \geq \frac{(k+1)(k-2)}{2}. \tag{1}$$

Thus, we observe that

$$\begin{aligned}
g(k) &\geq \frac{(k+1)(k-2)}{2} \\
&= \frac{(k+1)(k-2)(k-1)}{2(k-1)} \\
&= \frac{(k+1)(k^2-3k+2)}{2(k-1)} \\
&= \frac{(k+1)(k^2-k-2-2k+4)}{2(k-1)} \\
&= \frac{(k+1)[(k+1)(k-2)-2k+4]}{2(k-1)} \\
&= \frac{\frac{(k+1)(k-2)-2k+4}{2}}{\frac{k-1}{k+1}} \\
&= \frac{\frac{(k+1)(k-2)}{2} - k + 2}{1 - \frac{2}{k+1}}.
\end{aligned}$$

This implies $(1 - \frac{2}{k+1})g(k) \geq \frac{(k+1)(k-2)}{2} - k + 2$, thereby confirming (1) and completing the proof. $\square$

3 Average Degree

Theorem 7 *Let α denote the positive root of*

$$g(x) = 2x(x-2) - (d+1)(d-2).$$

Let $k = \left\lceil \sqrt{\frac{d(d-1)}{2}} \right\rceil$ denote the largest integer strictly less than α. Then:

a. If G has average degree at least $2k$, then G contains a $\frac{(d+1)(d-2)}{2}$-chorded cycle.

b. There exist graphs with average degree $2k - o(1)$ with no $\frac{(d+1)(d-2)}{2}$ chorded cycles.

Proof: The sharpness example for (b) is the complete bipartite graph $K_{n,k}$ with $k \leq n$, which has nk edges, and thus average degree

$$\frac{2nk}{n+k} = 2k - o(1).$$

Since any cycle in $K_{n,k}$ can contain at most k vertices from each part, there is no cycle with more than $k(k-2)$ chords. This is less than $\frac{(d+1)(d-2)}{2}$ since $k < \alpha$.

We now prove (a) by induction on n. If $n = k+1$, then $G = K_{k+1}$ and the result follows.

Let P be a longest path in G where $P = v_1, v_2, \ldots, v_\ell$ with v_1 adjacent to $v_{i_1}, \ldots, v_{i_j}$. Of all such longest paths, choose the one such that i_j is as large as possible. Consider the cycle $C = \{v_1, v_2, \ldots, v_{i_j}\}$. If C contains at least $\frac{(d-2)(d+1)}{2}$ chords, we are done. Thus, we may assume that C has fewer that $\frac{(d-2)(d+1)}{2}$ chords.

Let S be the set of all vertices v in C such that there exists a re-ordering of P starting with v which hits all vertices in C before any vertices in $P \setminus C$. The key property of S is the following.

Claim 1 *Suppose $v \in S$, then all neighbors of S are in C.*

Proof of Claim 1: All neighbors of v must be on P by the maximality of P. If one neighbor is not in C, then the rearrangement of P starting with v will contradict the maximality of i_j in the definition of P. □

Clearly $v_1 \in S$, but we have the following.

Claim 2 *If $u \in S$, let $u = u_1, u_2, \ldots, u_\ell$ denote the rearrangement of P starting with u so that the vertices of C are $\{u_1, \ldots, u_{i_k}\}$. Suppose u_s is a neighbor of u, then $u_{s-1} \in S$.*

Proof of Claim 2: Note that by Claim 1, $s \leq i_k$. Then the path

$$u_{s-1}, u_{s-2}, \ldots, u_1, u_s, u_{s+1}, \ldots, u_\ell$$

gives the desired rearrangement. □

An immediate consequence is that $|S| \geq \max_{v \in S} \deg(v)$.

Claim 3 *The degree of any vertex in G is at least $k+1$.*

Proof of Claim 3: Suppose $\deg(v) \leq k$. Then

$$2|E(G \setminus v)| \geq 2kn - 2k = 2k(n-1).$$

Thus, the average degree of $G \setminus v$ is at least $2k$ so the result holds by our inductive hypothesis. □

For $k = 1, 2, 3$ or 4, we have $d = 2, 3, 4$ or 5 respectively so the minimum degree is at least d. Then the desired result follows by Theorem 5. Thus, we may assume $d \geq 6$ and $k \geq 5$.

Since the minimum degree is at least $k+1$, we get $|S| \geq k+1$ and each vertex has degree at least $k+1$. In fact, more is true.

Lemma 1 $|S| \geq k+3$.

Since it is rather long, we postpone the proof of Lemma 1 until after completing the proof of Theorem 7.

Let $\mathcal{C}$ denote the set of chords of C with at least one end in S. Then $|\mathcal{C}| < \frac{(d-2)(d+1)}{2}$ by assumption. Consider removing S from the graph. Since all edges from vertices in S are to C, the only edges removed are the chords in $\mathcal{C}$ and the cycle edges incident to the vertices in S, of which there are at most $2|S|$. Therefore the number of edges in $G \setminus S$ is at least

$$\begin{aligned}
2|E(G \setminus S)| &\geq 2kn - 2|\mathcal{C}| - 4|S| \\
&\geq 2kn - (d+1)(d-2) + 2 - 4|S| \\
&= 2k\left(n - \frac{(d+1)(d-2)}{2k} + \frac{1}{k} - \frac{2|S|}{k}\right) \\
&= 2k\left(n - \frac{\alpha(\alpha-2)-1}{k} - \frac{2|S|}{k}\right).
\end{aligned}$$

The average degree is at least $2k$ so long as

$$\begin{aligned} |S|\left(1-\frac{2}{k}\right) &\geq \frac{\alpha(\alpha-2)-1}{k} \\ |S|(k-2) &\geq \alpha(\alpha-2)-1. \end{aligned} \tag{2}$$

Since $|S| \geq k+3$, we have

$$|S|(k-2) \geq (k+3)(k-2) \geq (\alpha+2)(\alpha-3) = \alpha^2-\alpha-6 \tag{3}$$

Thus (2) automatically holds as long as $\alpha \geq 5$. Since $\alpha = 1+\sqrt{\frac{d(d-1)}{2}}$, the result holds for $d \geq 7$. We can easily check the last few cases (which still hold, due to the inequality in (3) being strict in these cases). If $d = 6$, then $k = 4$, and $(k+3)(k-2) = 14 = \alpha(\alpha-2)-1$. This completes the proof, modulo the proof of Lemma 1. □

We now prove Lemma 1.

Proof of Lemma 1: Recall that we are under the induction assumption from Theorem 7. This means $\delta(G) \geq k+1$ and $|S| \geq k+1$. Let $C = \{v_1, \ldots, v_{i_j}\}$ where $v_{i_1}, \ldots, v_{i_j}$ denote the neighbors of v_1. Note that S contains v_{i_s-1} for all $s \in \{1, \ldots, j\}$.

Suppose first that $|S| = k+1$. Then $S = \{v_{i_s-1}\}$ for all $s \in \{1, \ldots, j = k+1\}$, that is, S is exactly the set of vertices preceding the neighbors of v_1. Furthermore, all vertices in S have degree exactly $k+1$.

Claim 4 *$G[S]$ is acyclic.*

Proof of Claim 4: If there is a cycle C' consisting of vertices in S, then there are at most $|C'|(k+1)-|C'| = |C'|k$ distinct edges incident to C' and hence, the removal of vertices in C' results in a smaller graph with average degree at least $2k$. We may then apply induction on n, completing the proof of the claim. □

Suppose $v_a \sim v_b$ (meaning v_a and v_b share an edge) with $a > 1, b > a+1$ and $v_a, v_b \in S$. Then, since $v_1 \sim v_{a+1}$ and $v_1 \sim v_{b+1}$, we know v_{a+1} and v_{b-1} are in S. This implies that $v_1 \sim v_b$ and $\{v_1, v_{a+1}, v_a, v_b\}$ is a 4-cycle in S, contradicting Claim 4. If $v_1 \sim v_b$ with $v_b \in S$ and $b > 2$, then $v_1 \sim v_{b+1}$, $v_{b-1} \in S$ and $v_2 \in S$ under the rearrangement $v_2, \ldots, v_b, v_1, v_{b+1}, \ldots, v_\ell$. The fact that $v_{b-1} \in S$ implies $v_2 \sim v_b$ but then there is a 3-cycle in S on v_1, v_2, v_b, again contradicting Claim 4. Finally note that if $v_2 \in S$, then $v_b \sim v_1$ for all vertices $v_b \in S$ since v_1 is the successor of v_2 in the rearrangement $v_b, v_{b-1}, \ldots, v_2, v_1, v_{b+1}, \ldots, v_\ell$.

Therefore, for any $v_a, v_b \in S$ we have that $v_a \not\sim v_b$. This implies that there are at least

$$(k+1)(k+1)-2(k+1) = (k-1)(k+1) \geq (\alpha-2)(\alpha) = \frac{(d+1)(d-2)}{2}$$

chords in C, as desired. Thus, we may assume $|S| \geq k+2$.

Suppose $|S| = k+2$. Note that we may assume v_1 has maximum degree among the vertices v_{a-1} where $v_a \sim v_1$. Hence, either $\deg(v_1) = k+1$ and there is one vertex of S that is not a predecessor of a neighbor of v_1, or $\deg(v_1) = k+2$ and all vertices in S are predecessors of neighbors of v_1.

If S is incident to fewer than $k(k+2)$ distinct edges then we are done as removing S will leave a graph with average degree at least $2k$. Thus, S is incident to at least $k(k+2)+1 = (k+1)^2$ distinct edges. On the other hand, if there are at least $(k+1)(k-1)$ distinct edges incident to S but not on C, we are done as $(k+1)(k-1) > \frac{(d+1)(d-2)}{2}$. Since there are at most $2(k+2)$ edges incident to S used in C, this implies that

$$(k+1)^2 \leq \text{number of distinct edges incident to } S \leq (k+1)^2+2, \tag{4}$$

or else we are done. Indeed, if there are fewer than $2(k+1)$ distinct edges on C incident to S, we are already done as (4) would yield a contradiction. Thus, we get the following fact.

Fact 1 *There is at most one edge of C between two vertices of S.*

On the other hand, since there are at least $(k+1)(k+2)$ edges incident to S (with repeats), there must be at least $k-1$ edges repeated, namely edges joining vertices in S.

Suppose $v_a \sim v_b$ with $v_a, v_b \in S$, $b > a+1$ and $v_{a+1}, v_{b+1} \sim v_1$. Then, as before, $v_{a+1}, v_{b-1} \in S$. However, this implies that the edges $v_a v_{a+1}$ and $v_{b-1} v_b$ are both edges of C between vertices of S, contradicting Fact 1.

Suppose $v_a, v_{a+1} \in S$ with $v_{a+1}, v_{a+2} \sim v_1$ and $a > 1$. Then we have $v_2 \in S$ by considering the path $v_2, v_3, \ldots, v_{a+1}, v_1, v_{a+2}, v_{a+3} \ldots, v_\ell$. This means $v_1 \sim v_2$ and $v_a \sim v_{a+1}$ are two edges of C between vertices of S, contradicting Fact 1.

If $d(v_1) = k+2$, then all vertices in S are predecessors of neighbors of v_1. Then since there must be at least $k-1$ edges repeated but at most one of them can be from a pair of vertices in S at distance 1 on the cycle, there must be an edge joining some pair $v_a, v_b \in S$ with $b > a+1$. This case was considered above.

If $d(v_1) = k+1$, then there is a unique vertex v_p which is not a predecessor of a neighbor of v_1. If there are no edges between v_p and any other vertex in S, the repeated edges must be between vertices who are predecessors of v_1 and we can easily find the desired cycle as in the previous case.

Finally, we may assume at least $k-2 \geq 2$ repeated edges are of the form $v_a \sim v_p$ for v_a a predecessor of a neighbor of v_1. If there exists a $v_a \sim v_p$ with $a < p$ and a $v_b \sim v_p$ with $b > p$ then $v_{p-1}, v_p, v_{p+1} \in S$ and hence, there are two edges of C between vertices of S. Thus, we may assume that $v_a \sim v_p$ and $v_b \sim v_p$ with $b > a > p$ or $b < a < p$. We then obtain the desired result as follows. Consider the rearrangement of P starting with v_p. The predecessors of v_a and v_b in this ordering are in S. One of these may be v_p, but one is v_q with $q \neq p$. This gives an adjacency of the form $v_q \sim v_a$ or $v_q \sim v_b$ for some $a \neq q$ or $b \neq q$, both of which are predecessors of neighbors of v_1 (since v_p was the *only* vertex in S which is not a predecessor of a neighbor of v_1). This is a situation we have already ruled out and so the proof is complete. □

Note that this result immediately implies this simplified corollary which we will use in proofs later.

Corollary 8 *If the average degree of G is at least $\sqrt{2}k+2$ then G contains a cycle with at least $\frac{(k+1)(k-2)}{2}$ chords.*

4 Helpful Lemmas

Our next lemma is very useful for bounding the number of edges inside a cycle.

Lemma 2 *Suppose C is a cycle on n vertices with $3c+3$ chords. Then there is a subgraph C' which is a cycle on fewer than n vertices which contains at least c chords.*

Proof: Suppose C is as stated and let $e = uv$ be a shortest chord of C. The removal of $\{u, v\}$ from e leaves 2 paths P_1 and P_2. Suppose $|P_1| \leq |P_2|$. Since e was chosen to be the shortest chord, we get that the cycle $P_1 \cup e$ contains no chords.

First note that there must be many edges from P_1 to P_2 since otherwise $P_2 \cup e$ provides the desired cycle C'. In particular, $P_2 \cup e$ must have at most $\frac{3c-1}{3}$ chords, so there must be at least $\frac{6c+10}{3} \geq 3$ edges from P_1 to P_2.

Let w be the vertex of P_1 which is closest to u (along C) such that w has an edge to a vertex $x \in P_2$ where x is not adjacent to either u or v along C. Note that such a vertex w exists since there are at least 3 edges between P_1 and P_2. Furthermore, let K be the set of edges from $P_1[w, v)$ to P_2 and note that $|K| > \frac{6c+7}{3}$.

Finally, we claim that either wP_1vP_2xw or wP_1vuP_1xw is the desired cycle C'. Since these cycles both contain $P_1[w, v)$ and together they cover P_2, one of these cycles contains about half the edges of K as chords. More specifically, one of these cycles contains at least $\frac{|K|-1}{2}$ chords. Furthermore, both of these cycles are strictly shorter than C. Thus, there exists the desired cycle C' with at least $\frac{|K|-1}{2} > c$ chords. □

In the proof of our next result, we will apply the following classical result of Erdős and Szekeres.

Theorem 9 (Erdős and Szekeres) *Any sequence of n integers contains either an increasing subsequence of length $\sqrt{n}$ or a decreasing subsequence of length $\sqrt{n}$.*

Although the following lemma is restricted to 3-regular graphs, the result can be applied in many situations as long as certain conditions hold which provide for a 3-regular subgraph.

Lemma 3 *Let G be a 3-regular graph of order n containing an induced cycle C where $G \setminus C$ forms an independent set. Then for any t, if $n \geq \frac{8t^3(t+2)^2}{3}$, there exists a cycle C' in G with at least t chords.*

Proof: Certainly since $|C| \geq 3$ and $G \setminus C$ contains no edges, there must be a cycle in G containing at least 1 chord. Thus, let $t \geq 2$. For each vertex $u \in C$, there is a unique vertex $v \in G \setminus C$ with $uv \in E(G)$. The vertex v also has two other neighbors $w, x \in C$ so we will call these two vertices *acquaintances* of u. We now prove a useful claim.

Claim 5 *There exists a segment $A \subseteq C$ of order $t^2(t+2)^2$ in which every vertex has both acquaintances in $C \setminus A$.*

Proof: Suppose not. Then every segment of order $t^2(t+2)^2$ contains a vertex with an acquaintance in the same segment because otherwise a simple counting argument yields a contradiction. Since $n \geq \frac{8t^3(t+2)^2}{3}$, we see that $|C| = 3n/4 \geq 2t^3(t+2)$. Thus, there exist $2t$ disjoint segments of C, each of order $t^2(t+2)^2$ and each containing a vertex with an acquaintance of that vertex. Choose one such segment A and let $u \in A$ be a vertex with an acquaintance $v \in A$ which is closest (along A). Let u' be the neighbor of u in $G \setminus C$. There is also another acquaintance w of u which may be in another segment, say B. We define a new cycle C_A by removing the segment strictly between u and v from C and using the path $uu'v$ to patch the cycle. This cycle contains at least one chord, namely, the edge $u'w$. In $C_A \setminus (A \cup B)$, there is yet another segment A' containing a vertex and one of its acquaintances. Using the same argument, we may repeat this whole process t times to produce the desired cycle C'. $\square_{Claim\ 5}$

Let A be a segment of C as guaranteed by Claim 5. Orient C in the clockwise direction and label the vertices according to their distance from A in this direction and let $\lambda(v)$ be the label of v. For each vertex $u \in A$, let v_u be the first, in this orientation, acquaintance of u. Let $\gamma(v_u)$ be the label of the other acquaintance of u. Let $w_1, w_2, \ldots, w_{t^4}$ be vertices of C such that $w_i = v_u$ for some $u \in A$ in order such that $\lambda(w_i) < \lambda(w_{i+1})$ for all i. By Theorem 9, there is either an increasing subsequence or a decreasing subsequence of the sequence $\gamma(w_i)$ which has length at least $\sqrt{t^2(t+2)^2} = t(t+2)$.

Let $v_1, v_2, \ldots, v_{t^2}$ be the subsequence of $w_1, w_2, \ldots, w_{t^4}$ corresponding to the subsequence guaranteed above and let x_i be the neighbor of v_i in $G \setminus C$ for all i. Furthermore, let y_i denote the acquaintance of v_i which is not in A and let u_i be the acquaintance of v_i which is in A for all i. Let $\overrightarrow{C}$ and $\overleftarrow{C}$ denote the cycle C in the clockwise or counterclockwise direction respectively.

First, suppose the subsequence implied by Theorem 9 is decreasing. Now either the cycle defined by $C_1' = v_1x_1y_1\overleftarrow{C}y_2x_2v_2\overrightarrow{C}v_3x_3y_3\cdots x_{t^2}u_{t^2}\overrightarrow{C}v_1$ or $C_2' = y_1x_1v_1\overrightarrow{C}v_2x_2y_2\overleftarrow{C}y_3x_3v_3\cdots x_{t^2}u_{t^2}\overleftarrow{C}y_1$ is the desired cycle depending on how many vertices u_i are clockwise versus counterclockwise from u_{t^2} on C. One of these two cycles will contain at least $\frac{t(t+2)}{2} \geq t$ chords since $t \geq 2$.

Next, assume the subsequence implied by Theorem 9 is increasing. First, suppose that, for some i, there are at least $t+1$ vertices v_j in between v_i and y_i on C. Then one of

$$C_1' = v_ix_iy_i\overrightarrow{C}y_{i+1}x_{i+1}v_{i+1}\overrightarrow{C}v_{i+2}x_{i+2}y_{i+2}\cdots y_{i+t}\overrightarrow{C}v_i$$

or

$$C_1' = v_ix_iy_i\overrightarrow{C}y_{i+1}x_{i+1}v_{i+1}\overrightarrow{C}v_{i+2}x_{i+2}y_{i+2}\cdots y_{i+t+1}\overrightarrow{C}v_i$$

is a cycle and furthermore, the desired cycle.

Finally, suppose that there are at most t vertices v_j in between v_i and y_i on C for all i. Let v_{i_1} be the first vertex of our subsequence which is beyond y_1 on C. Similarly let v_{i_j} be the first vertex of the subsequence which is beyond $y_{i_{j-1}}$ for all $1 \leq j \leq t$. Since there are at least $t(t+2)$ such vertices and we lost at most t in between along with the ends v_{i_j} of these segments, such vertices must all exist for all $1 \leq j \leq t$. Then the cycle $C' = v_1x_1y_1\overrightarrow{C}v_{i_1}x_{i_1}y_{i_1}\cdots y_{i_t}\overrightarrow{C}v_1$ is the desired cycle, completing the proof. $\square$

Using this lemma, we get a very useful corollary.

Corollary 10 *Let $G = C \cup H$ where C is a cycle and each vertex of H has at least 3 neighbors on C with all of these neighborhoods pairwise disjoint. If $|H| \geq 2(f(k))^5$, then there exists an $f(k)$-chorded cycle C' in G with $|C'| < |C|$.*

Proof: With $|H| \geq 2(f(k))^5$, there must be at least $6(f(k))^5$ vertices in the cycle. Consider a set D of $(f(k))^5$ vertices of H and contract all segments of C between neighbors of vertices in D to form a new cycle C_D. Note that $|D \cup C_D| = 4(f(k))^5 < |C|$. By Lemma 3, there exists an $f(k)$-chorded cycle C' in $D \cup C_D$. By construction,

$|C'| < |C|$ as desired. □

The next two results concern the number of edges between cycles. Lemma 4 will be used in the proof of Lemma 5.

Lemma 4 *Suppose P and Q are disjoint paths with at least $6f(k)+2$ edges between them. Then there exists an $f(k)$ chorded cycle between them.*

Proof: Order the vertices of $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$. Let p denote the first vertex on P with an edge to Q and let q denote a neighbor of p on Q. Then there exist at least $3f(k)+1$ edges from $p, \ldots p_n$ to either $q_1, \ldots, q$ or $q, \ldots, q_m$. Without loss of generality assume there are at least $3f(k)+1$ edges from $p, \ldots, p_n$ to $q_1, \ldots, q$.

Let q' denote the first vertex of Q with an edge to P and p' denote one of it's neighbors. Also let p'' denote the last vertex of P with an edge to $q_1, \ldots, q$ and q'' denote a neighbor of p'' in $q_1, \ldots, q$. Then we claim that one of the three cycles $p, \ldots, p'', q'', \ldots, q, p$ or $p, \ldots, p', q', \ldots, q, p$ or $p' \ldots p'', q'', \ldots, q', p'$ must contain at least $f(k)$ chords. Otherwise there would be at most $f(k)-1$ chords in each cycle which, along with the three edges themselves, makes a total of at most $3f(k)$ edges from $p, \ldots, p''$ to $q', \ldots, q$, a contradiction completing the proof. □

Lemma 5 *Suppose C_1 and C_2 are two $f(k)$-chorded cycles in a minimal cycle set with $e(C_1, C_2) \geq 28f(k)+20$. Then there exists a single vertex in one of these cycles (say C_1) with at least $e(C_1, C_2) - (12f(k)+3)$ edges to C_2.*

Proof: Without loss of generality, assume the vertex v of largest degree M to the opposite cycle lies in C_1. For a contradiction, suppose $M < e(C_1, C_2) - (12f(k)+3)$.

Claim 6 *We may assume $M < 2f(k)+6$.*

Proof: Without loss of generality, suppose $2f(k)+6 \leq M < e(C_1, C_2) - (12f(k)+3)$. Then there are at least $12f(k)+4$ edges between C_1 and C_2 that are not incident to v. Let $x_1, x_2 \in C_2 \cap N(v)$ with at least $f(k)+2$ neighbors of v between x_1 and x_2 in each direction on C_2. This means that at least one of $x_1^+ C_2 x_2$ or $x_2^+ C_2 x_1$ (suppose $x_1^+ C_2 x_2$) must be incident to at least $6f(k)+2$ edges from $C_1 \setminus \{v\}$. Applying Lemma 4 on $C_1 \setminus \{v\}$ and $x_1^+ C_2 x_2$, we see there is an $f(k)$-chorded cycle as a subgraph. Then the chorded cycle formed by the segment of C_2 between edges from v in the interval $x_2^+ C_2 x_1^-$ must have at least $f(k)$ chords. Since these two $f(k)$-chorded cycles both avoid x_1, this pair must be smaller than $|C_1 \cup C_2|$, contradicting the minimality of the cycle set. □

Label the vertices of C_1 in order with $x_1, x_2, \ldots, x_{m_1}$ and the vertices of C_2 in order with $y_1, y_2, \ldots, y_{m_2}$. Let x_i be the first vertex of C_1 such that $x_2 C_1 x_i$ has at least $2(6f(k)+2)$ edges to C_2. Since $M < 2f(k)+6$, we see that $x_2 C_1 x_i$ has at most $14f(k)+10$ edges to C_2. With at least $28f(k)+20$ edges between C_1 and C_2 and at most $2f(k)+6$ of these edges incident to x_1, this means there are at least $12f(k)+4$ edges from $x_{i+1} C_1 x_{m_1}$ to C_2. Similarly let y_j be the first vertex of C_2 such that $y_1 C_2 y_j$ has at least $12f(k)+4$ edges to $C_1 \setminus x_1$ and note that $y_{j+1} C_2 y_{m_2}$ also has at least $12f(k)+4$ edges to C_1.

Under these restrictions, there must either be at least $6f(k)+2$ edges between $x_2 C_1 x_i$ and $y_1 C_2 y_j$ and between $x_{i+1} C_1 x_{m_1}$ and $y_{j+1} C_2 y_{m_2}$ or at least $6f(k)+2$ edges between $x_2 C_1 x_i$ and $y_{j+1} C_2 y_{m_2}$ and between $x_{i+1} C_1 x_{m_1}$ and $y_1 C_2 y_j$. In either case, we may apply Lemma 4 between both pairs to find two $f(k)$-chorded cycles. Since these cycles avoid x_1, this pair must be smaller than $|C_1 \cup C_2|$, a contradiction. □

From this result, we get the following as a corollary.

Corollary 11 *Suppose $C_1, C_2, \ldots, C_t$ are a minimal collection of cycles each with at least $f(k)$-chords. Then there are at most $\binom{t}{2}(28f(k)+20) + (t-1)\sum |C_i|$ edges in total between the cycles.*

Proof: Let $C_1, C_2, \ldots, C_t$ be a minimal collection of $f(k)$-chorded cycles. By Lemma 5, if a pair of these cycles has at least $28f(k)+20$ edges between them, then there must exist a single vertex with degree at least $16f(k)+17$ from one of these cycles to the other. Thus, it suffices to show that there are at most $t-1$ such vertices of high degree so suppose there are at least t such vertices.

Define an auxiliary graph G' on vertices $v_1, \ldots, v_t$ where the vertex v_i represents the cycle C_i. For each vertex of high degree in C_i, choose a single cycle C_j to which this vertex has at least $16f(k)+17$ edges and add the edge

v_iv_j. Since there are at least t such vertices of high degree, there must exist a cycle in G'. By construction, the cycles C_i involved in this cycle of G' would easily allow us to construct a replacement set of cycles C_i' using fewer vertices, contradicting the minimality of the set $C_1, \ldots, C_t$. □

Our next lemma concerns the number of edges into the cycles from a vertex outside the collection of cycles.

Lemma 6 *Let $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ be a minimal collection of $f(k)$-chorded cycles and let $v \in G \setminus V(\mathscr{C})$. Then*

$$d_{\mathscr{C}}(v) \leq \min\{t(2\sqrt{2f(k)}+3), t(f(k)+3)\}.$$

Proof: If $d_{C_i}(v) \geq f(k)+4$ for some cycle C_i, we can easily create a cycle smaller than C_i using only v and its edges to C avoiding at least two vertices of C. This means that $d_{\mathscr{C}}(v) \leq t(f(k)+3)$.

Thus, suppose $d_{\mathscr{C}}(v) > t(2\sqrt{2f(k)}+3)$. Then there exists a cycle $C_i \in \mathscr{C}$ with $d_{C_i}(v) \geq 2\sqrt{2f(k)}+4$. Label the neighbors of v on C_i with $x_1, x_2, \ldots, x_\ell$ in order around C_i. Since there are at least $f(k)$ chords of C_i, there must be at least

$$2f(k) - 3\frac{2f(k)}{2\sqrt{2f(k)}} + 4 > 2f(k) - 2\sqrt{2f(k)}$$

endvertices of chords in $x_{j+3}C_ix_j$ for some j. Replacing the segment $x_jC_ix_{j+3}$ with the segment x_jvx_{j+3} makes a smaller cycle still with at least $f(k)$ chords, contradicting the minimality of the system $\mathscr{C}$. □

5 Proof of Theorem 4

Let $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ be a minimal set of t disjoint $f(k)$-chorded cycles in G where t is as large as possible. If $t \geq s$, this is the desired set of cycles so suppose $t < s$. Let $H = G \setminus V(\mathscr{C})$. We first show that $|H|$ is large.

Claim 7

$$|H| \geq \frac{n}{\sqrt{f(k)}}.$$

Proof: By Lemma 2, there are at most $3f(k)+2$ chords in each cycle so a total of $t(3f(k)+2) + \sum|C_i|$ edges within cycles. By Corollary 11, there are at most $\binom{t}{2}(28f(k)+20) + (t-1)\sum|C_i|$ edges between the cycles. Adding these two together, there are at most

$$t(3f(k)+2) + \sum|C_i| + \binom{t}{2}(28f(k)+20) + (t-1)\sum|C_i|$$

edges within $\mathscr{C}$. With minimum degree sk, there must be at least

$$\begin{aligned}
sk\sum|C_i| - 2&\left[t(3f(k)+2) + \sum|C_i| + \binom{t}{2}(28f(k)+20) + (t-1)\sum|C_i|\right] \\
&= (sk-2t)\sum|C_i| - 2\left[t(3f(k)+2) + \binom{t}{2}(28f(k)+20)\right] \\
&\geq s(k-2)\sum|C_i| - 30s^2f(k)
\end{aligned}$$

edges leaving $\mathscr{C}$ going to H. Since $k \geq 6$, Lemma 6 implies that each vertex of H can accept at most $t(2\sqrt{2f(k)}+3)$ of these edges from $\mathscr{C}$. This means that

$$\begin{aligned}
|H| &\geq \frac{s(k-2)\sum|C_i| - 30s^2f(k)}{s(2\sqrt{2f(k)}+3)} \\
&\geq \frac{4(n-|H|) - 30sf(k)}{(2\sqrt{2f(k)}+3)}.
\end{aligned}$$

Solving for $|H|$, we get

$$|H| \geq \frac{4n - 30sf(k)}{2\sqrt{2f(k)}+7} \geq \frac{3n}{2\sqrt{2f(k)}} > \frac{n}{\sqrt{f(k)}}$$

for $n \geq 26sf(k)$. $\square_{Claim\ 7}$

By Theorem 7, since H contains no $f(k)$-chorded cycle, we know the average degree within H is at most $\sqrt{2}k$ so by Markov's Inequality and Claim 7, there must be at least

$$(1 - \frac{1}{\sqrt{2}})|H| > \frac{\sqrt{2}-1}{\sqrt{2f(k)}}n$$

vertices in H with degree at least $(s-2)k$ into $\mathscr{C}$.

Let d be the smallest integer such that $K_{d,d}$ contains an $f(k)$-chorded cycle. Thus, $d < \frac{k}{\sqrt{2}} + 2$.

Claim 8 *If there are at least $(d-1)(10(f(k))^5)^d$ vertices in H, each with at least $d+1$ edges to a cycle C, then $|C| \leq 2d$.*

Proof: Let $H' \subseteq H$ be the vertices of H each with at least $d+1$ edges to C. Let A be a maximum set of vertices in H' each with 3 distinct neighbors on C (all the corresponding sets of 3 neighbors are pairwise disjoint). By Corollary 10, we see that $|A| < 2(f(k))^5$. Now let B be a maximum set of vertices in $H' \setminus A$ with 2 new neighbors each (pairwise disjoint sets and also disjoint from the previous sets of neighbors). By an argument similar to that applied in Lemma 3 and Corollary 10, we can also see that $|B| < 2(f(k))^5$.

With $|A \cup B| \leq 4(f(k))^5$, we see that the union of the distinct neighborhoods of these vertices in C, call this union C', has order at most $|C'| \leq 10(f(k))^5$. This means that all vertices of H' must have at least d edges to C'. Since $|H'| \geq (d-1)(10(f(k))^5)^d$, the pigeon hole principle implies that there is a set of d vertices in H' which share the same d neighbors in C', inducing a $K_{d,d}$. Thus, if $|C| > 2d$, it can be replaced by a spanning cycle of this $K_{d,d}$ contradicting the minimality of the cycle system. $\square_{Claim\ 8}$

With at least $\frac{\sqrt{2}-1}{\sqrt{2f(k)}}n$ vertices in H each sending at least $(s-2)k$ edges to $\mathscr{C}$ and at most $s-1$ cycles in $\mathscr{C}$, there must be at least one cycle, say C_1, with at least $(d-1)(10(f(k))^5)^d$ vertices in H, each having at least $\frac{(s-2)k}{s-1} \geq d+1$ edges to C_1 as long as $n \geq sk^{8k}$. By Claim 8, $|C_1| \leq 2d$.

Now there are still at least $\frac{\sqrt{2}-1}{\sqrt{2f(k)}}n$ vertices in H each sending at least $(s-2)k - |C_1|$ edges to $\mathscr{C} \setminus C_1$ and at most $s-2$ cycles remaining in $\mathscr{C} \setminus C_1$. Thus, there must be another cycle, say C_2, with many vertices of H each having at least

$$\frac{(s-2)k - |C_1|}{s-2} \geq \frac{(s-2)k - 2d}{s-2} \geq d+1$$

edges to C_2 as long as $n \geq s^2k^{8k}$. This implies $|C_2| \leq 2d$ as well.

This process can be repeated to make ℓ cycles small as long as

$$\frac{(s-2)k - \ell 2d}{s-(\ell+1)} \geq d+1. \tag{5}$$

With $s \geq s_0$ and $k \geq k_0$, this holds for $\ell \leq 3$ so we may assume C_1, C_2 and C_3 each have at most $2d$ vertices. By assumption, there are many vertices in H each with at least

$$\frac{(s-2)k}{s-1} + \frac{(s-2)k-2d}{s-2} + \frac{(s-2)k-4d}{s-3} \geq 4d \tag{6}$$

edges to $C_1 \cup C_2 \cup C_3$ since $s \geq s_0$ and $k \geq k_0$. By the pigeon hole principle, there must exist a set of $4d$ vertices in H which are all adjacent to the same set of at least $4d$ vertices in $C_1 \cup C_2 \cup C_3$ as long as $n \geq s^3 n^{8k}$. This allows us to build 4 copies of $K_{d,d}$ containing 4 disjoint $f(k)$-chorded cycles in place of C_1, C_2 and C_3, contradicting the maximality of t and completing the proof. $\square$

A choice of $s_0 = 29$ and $k_0 = 1000$ allows such a conclusion for (5) and (6). In fact, one could actually choose $s_0 = 17$ and $k_0 = 2600$ to satisfy (5) with $\ell = 4$. This adds an extra term to (6) allowing it to work with this choice of $s_0 = 17$ and $k_0 = 2600$. In general, we would need $n \geq s^\ell k^{8k}$.

References

[1] A. A. Ali and W. Staton. The extremal question for cycles with chords. *Ars Combin.*, 51:193–197, 1999.

[2] G. Chartrand and L. Lesniak. *Graphs & Digraphs.* Chapman & Hall/CRC, Boca Raton, FL, fourth edition, 2005.

[3] K. Corradi and A. Hajnal. On the maximal number of independent circuits in a graph. *Acta Math. Acad. Sci. Hungar.*, 14:423–439, 1963.

[4] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623. North-Holland, Amsterdam, 1970.

[5] J. Komlós, G. N. Sárközy, and E. Szemerédi. Proof of the Seymour conjecture for large graphs. *Ann. Comb.*, 2(1):43–60, 1998.