On The Structure of Barrier Graphs

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Abstract

Wireless sensor networks are commonly used to monitor various environmental conditions. Possible geometries for the region covered by a sensor include disks, wedges, and rays, among others, depending on the function of the sensor. In this paper we consider a network consisting of ray sensors deployed to detect intruders traversing a path, not necessarily straight, from a source α to a destination β. The coverage of the network with respect to α and β can be modeled by a tripartite graph, the *barrier graph* of the network. While all barrier graphs are tripartite, the converse is not true (for instance, C_5 is not a barrier graph).

The main result of this paper is a rigidity theorem on the structure of barrier graphs that results from constraints imposed by the geometry of the network. This allows us to show that almost all bipartite graphs are not barrier graphs, despite the fact that various classes of bipartite graphs, including trees, cycles of even length, and $K_{m,n}$ are barrier graphs. Furthermore, vertex cover of a barrier graph corresponds to a set of sensors whose removal allows a clear path from α to β. While all bipartite graphs with small vertex covers are barrier graphs (a fact we prove for sizes less than 4), the rigidity property also implies that graphs with vertex covers bigger than a certain constant are not barrier graphs.

1. Introduction

A wireless sensor network consists of a collection of spatially distributed sensors deployed to monitor various physical conditions (temperature, air pressure, movement, etc.) in their immediate surroundings. Applications are numerous and include, for example, region surveillance for intrusion detection, and communication control for the operation of cell phones.

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A proper understanding of the behavioral characteristics of these networks starts with an appropriate mathematical model. Various models of sensors have been considered: some use discs in the plane [2], with obvious analogy to cell towers, others use wedges to model floodlight-like behavior [1] or rays [9] to abstract the behavior of a laser beam design to perform intrusion detection.

Thus, each model is chosen to represent the network's coverage: the total area covered by discs, or the length of time certain places are illuminated by rotating floodlights, or locations that cannot be crossed without the culprit being detected. In the latter, a network is intended to present barriers between that intruder's base and the intended destination. When the barriers are modeled as rays the resulting network is called a *ray-barrier* network.

A key observation of Kirkpatrick, Yang, and Zilles [9] (KYZ) was that a collection of rays forms a barrier if and only if some pair of rays forms a barrier. This yields a graph, the *barrier graph*, with the rays as vertices and an edge for any pair that, on its own, forms a barrier. As discussed in [9], these are tripartite graphs, an example of which is seen in Figure 1. Note that the straight segment $\overline{\alpha\beta}$ in Figure 1a is used to compute the graph (Figure 1b) but in no way implies that a path from α to β needs to be straight. A barrier forces *all* paths from α to β to cross at least one ray.

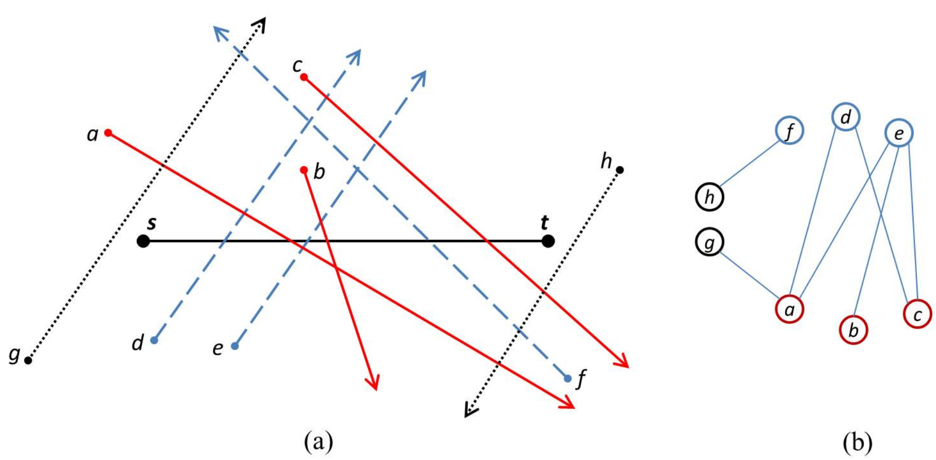


Figure 1: (a) A collection of ray sensors , and (b) their associated barrier graph.

In this paper we study the constraints that the geometry of such a network imposes on the combinatorial structure of the corresponding barrier graphs. The ways a collection of rays forms a barrier is constrained much more than intuition might suggest.

Our main result is that barrier graphs exhibit a rigid adjacency structure. This allows us to show that, for instance, almost all tripartite graphs – and even *almost all bipartite graphs*, are not barrier graphs (cf. Theorem 9).

This is surprising, in part, because many common classes of bipartite graphs – trees, complete bipartite graphs, and cycles of even length – all have straightforward realizations as ray-barrier networks.

Recently, Kirkpatrick and Bereg [2] introduced the notion of a network's *resilience*, which is the minimum number of sensors whose removal permits a path (not necessarily straight) between the chosen points. Since barriers are formed by pairs of vertices, a ray-barrier network's resilience is the size of a minimum vertex cover of the barrier graph.

In general graphs the minimum vertex cover problem is *NP-hard* [8], but as KYZ have shown, the ray barrier resilience of an arrangement of rays can be found efficiently. This suggests quite strongly that the underlying geometric structure greatly limits the graphs which can arise as a barrier graph.

Starting with a graph G, we can ask whether G is a barrier graph, i.e., whether there is some arrangement B of rays, together with points α and β, that makes G the corresponding barrier graph. If the answer is yes, we say that the tuple $\langle B, \alpha, \beta \rangle$ is a realization of G.

It is ultimately the question of which graphs can be realized as barrier graphs which we study in this paper. In addition to our result that almost all bipartite graphs aren't barrier graphs, we show that barrier graphs are perfect graphs, a class of graphs on which some algorithmically difficult problems are efficiently solvable. This immediately leads to a polynomial time algorithm for computing the resilience of a barrier graph that is different from the one proposed in KYZ.

The rest of this paper is organized as follows. In Section 2, we prove that while barrier graphs are always tripartite, not all tripartite graphs are barrier graphs, and that barrier graphs are always perfect graphs. In Section 3 we prove some supporting lemmas and then the main Rigidity Theorem, which strongly restricts the neighborhoods that can appear in a barrier graph. In section 4 we discuss some related problems and conclude the paper.

Our graph theoretical notation, throughout the paper, is standard. However, we frequently use $\mathcal{N}(v)$ to denote the neighborhood of a vertex v and, for a subset $X \subseteq V(G)$, we use $\mathcal{N}_X(v)$ to denote the neighbors of v within X.

2. The Tripartite Structure of Barrier Graphs

One of the key observations of KYZ is that not only does a collection of ray sensors yield a barrier graph, but this graph must be tripartite. In order to show this, they constructed a 3-coloring of a barrier graph as follows (see Figure 1 for an example). Consider any arrangement of rays and a pair of points α, β – the start and target points – and rotate everything so that the segment $\overline{\alpha\beta}$ is horizontal. Rays will be colored red if they intersect $\overline{\alpha\beta}$ from above, blue if they intersect $\overline{\alpha\beta}$ from below, and black if they don't intersect $\overline{\alpha\beta}$.

KYZ showed that with this coloring, barriers can only form from intersecting pairs of differently colored rays. Furthermore, if red and blue rays intersect they always form a barrier, but if one of the rays is black, then a barrier is only formed when the intersection is on the same side of the supporting line of $\overline{\alpha\beta}$ as the other ray's anchor point. Since edges in the barrier graph are exactly these barrier pairs, the colors give three sets of vertices connected by edges only to elements of a differently colored set. We will refer to this coloring through the rest of the paper and denote the supporting line of $\overline{\alpha\beta}$ by $\ell(\overline{\alpha\beta})$.

Viewing barrier graphs in this way provides a glimpse of how the graph structure depends on the rays' geometry, and allows us to find properties that prevent a graph from being a barrier graph, starting with the following proposition.

Proposition 1. *Let G be a barrier graph, and fix a realization with a particular set of rays and the path endpoints α and β. Then there is no induced length 2 path $r - k - b$ where r is red, k is black, and b is blue.*

Proof. Fix a realization of G, so that the coloring of the vertices of the barrier graph is fixed, and suppose $r - k - b$ is an induced path of length 2 in G, with r red, k black, and b blue.

Because r is red, it must be anchored above the bolded segment $\overline{\alpha\beta}$, and must also intersect $\overline{\alpha\beta}$. Therefore, its ray $\vec{r}$ lies somewhere in the wedge bounded by the dashed lines in Figure 2. Then, since k forms a barrier with r, it cannot be placed in the shaded region labeled $\star$ for, otherwise, it would

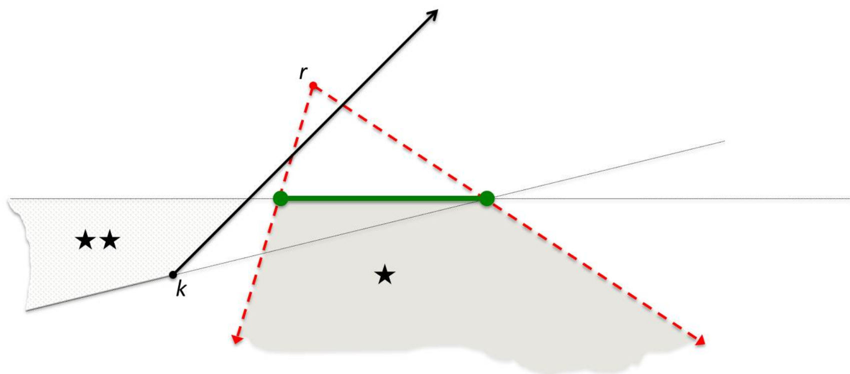


Figure 2: A barrier graph may not contain an induced $r - k - b$ path. The bold segment is the segment $\overline{\alpha\beta}$. The region labeled $\star$ cannot contain the anchor of k. The anchor of b must be located in the region labeled $\star\star$ as, otherwise, it will either not be blue or it will intersect k above $\ell(\overline{\alpha\beta})$. This forces b to also intersect r.

either intersect $\overline{\alpha\beta}$ (thus being blue instead of black) or intersect $\vec{r}$ below $\overline{\alpha\beta}$ (thus not forming a barrier with r).

k can either start above or below $\ell(\overline{\alpha\beta})$, as long as $\vec{k}$ intersects $\vec{r}$ above $\ell(\overline{\alpha\beta})$ and crosses $\ell(\overline{\alpha\beta})$ so that a proper intersection with $\vec{b}$ is possible. We proceed with the case that has the anchor of k below $\ell(\overline{\alpha\beta})$ and its supporting line to the left of $\overline{\alpha\beta}$ (see Figure 2). The argument for each of the other possibilities is similar.

Since $\vec{b}$ must intersect $\vec{k}$ below $\ell(\overline{\alpha\beta})$, it must be anchored in the region labeled $\star\star$, since were it anchored elsewhere, it would either not intersect $\overline{\alpha\beta}$ (and thus not be blue) or not intersect k below $\ell(\overline{\alpha\beta})$. But any blue ray anchored in $\star\star$ would intersect $\vec{r}$, which means that the vertices r, k, and b would induce a triangle instead of a path. So there could not have been such an induced subgraph. □

Proposition 2. *If G is a barrier graph, then any induced subgraph is a barrier graph.*

Proof. This follows immediately from the fact that any pair of rays forms a barrier independently of all the other rays in the collection. □

Proposition 3. *No graph containing a chordless cycle of odd order ≥ 5 is a barrier graph.*

Proof. Let $G = (V, E)$ be an odd cycle graph with $|V| \geq 5$ and with no chords, and suppose G is a barrier graph. Fix any realization of G, and let R, B, and K be the sets of red, blue, and black vertices respectively. R, B, and K are all nonempty since otherwise G is bipartite, which odd cycles are not.

For subsets $C, D \subseteq V$, define $C_D := \{c \in C \mid \mathcal{N}(c) \subseteq D\}$, i.e. those elements of C whose neighborhood consists only of elements of D.

Suppose every element of K has two neighbors of the same color.

Then K is the disjoint union $K_R \dot\cup K_B$. If we define $R' = R \cup K_B$ and $B' = B \cup K_R$, then R' and B' partition V into disjoint sets.

There are no edges between two elements of R' or between two elements of B', so we have found a bipartition of G, a contradiction.

But then there is an element $k \in K$ with differently colored neighbors $r \in R$ and $b \in B$. So, $r - k - b$ is an induced subgraph (since G is longer than a triangle and is chordless), which contradicts, by Proposition 1, that G was a barrier graph.

Thus, odd cycles of order ≥ 5 are not barrier graphs and, by Proposition 2, no graph with such an induced subgraph can be a barrier graph. $\square$

While Proposition 3 prevents certain tripartite graphs from being barrier graphs, it also leads to an interesting connection to perfect graphs. Recall that a graph is perfect if the chromatic number and clique number of every induced subgraph are equal.

The celebrated strong perfect graph theorem [4], of Chudnovsky, Robertson, Seymour and Thomas, gives an alternate structural characterization of perfect graphs by showing that they are the same as the so-called Berge graphs. A Berge graph is a graph which has no odd hole (induced cycle) or antihole (complement of an induced cycle) with length longer than 3 as an induced subgraph.

From Proposition 3 we have that barrier graphs have no odd hole of size ≥ 5; since C_5 is self-complementary, this also rules out 5-antiholes. Furthermore, given that antiholes of size $2k + 1$ have chromatic number $k + 1$, odd antiholes larger than 5 cannot occur due to the fact that barrier graphs are tripartite. This yields the following corollary.

Corollary 4. *Every barrier graph is a perfect graph.*

Perfect graphs have various properties that distinguish them from tripartite graphs. First, unlike tripartite graphs, we already have algorithms to recognize perfect graphs efficiently[3]. Furthermore, many algorithmically hard problems, such as graph coloring, maximum clique, and maximum independent set have polynomial time algorithms for the class of perfect graphs. We can take advantage of these algorithms in various ways. For instance, since S is an independent set for $G(V, E)$ if and only if $V - S$ is a vertex cover, it follows that a polynomial time algorithm for computing a maximum independent set of a perfect graph can be used, verbatim, to compute the resilience of a barrier graph, also in polynomial time.

This approach provides us with an compelling alternative to the algorithm in KYZ that is also efficient, but does not require explicit use of the geometric information. One could receive a barrier graph and compute its resilience in polynomial time without knowledge of the location and orientation of the sensors involved. This is impossible with the KYZ algorithm, which uses the underlying geometric information very strongly.

With the knowledge that barrier graphs are tripartite and perfect, a natural question is whether the converse is true: are all perfect tripartite graphs barrier graphs?

A natural class of perfect tripartite graphs is the class of bipartite graphs, which have both clique number and chromatic number equal to two. In the next section we will investigate the rigid structure of neighborhoods in barrier graphs, and will show that, in fact, almost all bipartite graphs are *not* barrier graphs.

3. The rigidity of barrier graphs

For the next few results, which connect the geometry of a barrier graph's realization to the kinds of neighborhoods that can appear in barrier graphs, we will use a standard geometric transformation $\mathcal{D}$, which maps lines in the plane to points in the plane and vice-versa. The image of an object r under $\mathcal{D}$ is referred to as the *dual* of r.

For the rest of this paper we assume that rays, line segments, and lines are in general position (none are vertical, no three coincide in a point, and no two intersect in more than one point).

Proposition 5. *Let $\mathcal{D}$ be the transformation that maps the point $p : (a, b)$ to the line $D(p) : y = ax - b$ and the line $\ell : y = ax + b$ to the point $D(\ell) : (a, -b)$. Then $\mathcal{D}$ has the following properties:*

(*i*) *[5] p is above (resp. below) $\ell \iff \mathcal{D}(p)$ is below (above) $\mathcal{D}(\ell)$.*

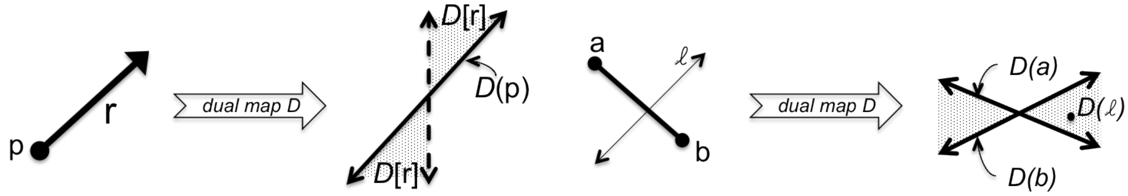


Figure 3: Dual transformation of a ray (top) and of a segment (bottom).

(ii) *[5] $\mathcal{D}$ preserves incidences between points and lines.*

(iii) *If the dual of a ray (resp. segment) r is defined as the union of the duals of the points comprising the ray (resp. segment), then the dual image $\mathcal{D}[r]$ is a "bow-tie" region, as shown in Figure 3.*

To see (*iii*), note that if a and b are the endpoints of a non-vertical segment then every x-value between the x-values of a and b is seen along the segment $\overline{ab}$, and thus every slope between the slope of $\mathcal{D}(a)$ and the slope of $\mathcal{D}(b)$ is the slope of some line in the dual image of $\overline{ab}$. Since each point on $\overline{ab}$ is incident with the supporting line of $\overline{ab}$, their dual lines all intersect the dual point of this supporting line, which is the center of the bow-tie.

The situation for rays is similar, except there is only one endpoint, the anchor a; if the ray points right then every slope $\geq$ the slope of $\mathcal{D}(a)$ is achieved and if it points left then every slope $\leq$ the slope of $\mathcal{D}(a)$ is achieved, and so the second boundary of the bow-tie is a vertical line and is not included in the bow-tie.

Lemma 6. *Let X be a set consisting of r rays and s line segments. Then there exists a set X' of lines that divide the dual plane into regions so that any two points which lie in the same region correspond to two lines in the primal plane that intersect the same elements of X in the same left-right order. Furthermore, X' can be chosen so that*

$$|X'| \leq \binom{r+s}{2} + 2(r+s).$$

Proof. Let X' consist of the following types of lines:

(I) For each ray in X, the dual of the anchor and a vertical line through the dual of the supporting line, and for each line segment in X, the dual of the endpoints (these are the "bow-tie" bounding lines).

(II) For each pair of elements in X the dual of the intersection of their supporting lines (if they intersect).

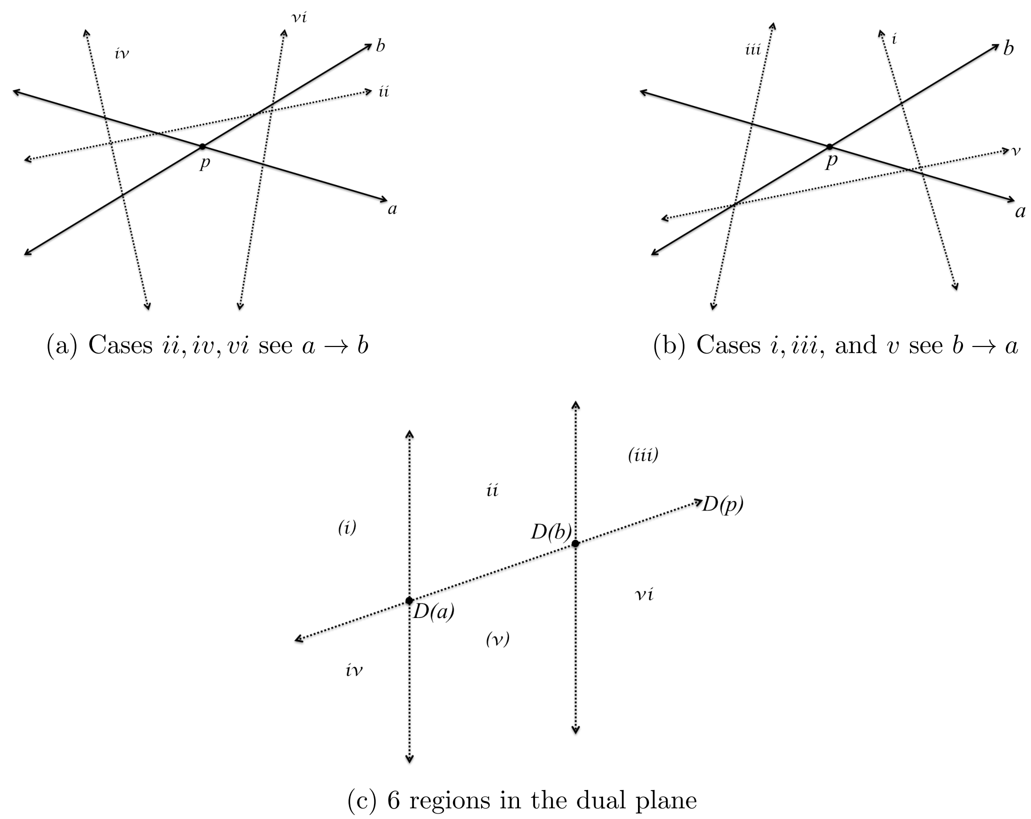


Figure 4: Lines of (a) intersect a, then b from left to right, and lines of (b) intersect b, then a from left to right. All lines which intersect a and b are one of these 6 types, and their duals are in the corresponding regions of (c), determined by Type II lines.

Clearly there are at most $\binom{s+r}{2}$ intersection points (and hence $\binom{s+r}{2}$ lines of Type II), and most $2(r+s)$ lines of Type I.

Consider, now, the partition $\mathcal{P}$ defined by X. The *regions* of this partition are simply the k-faces of the partition, i.e., the points of intersection of these lines, the open line segments connecting two points, and the open regions in the plane created by removal of these points and segments. We claim this partition has the desired property.

Consider an arbitrary pair of points x, y in the same region of $\mathcal{P}$. First note that the position of x (or y) with respect to Type I lines determines which line segments and rays of X the dual of x (or y) intersects. This is precisely determined by the bowtie regions of Proposition 5. That is, within a region all points are duals of lines which intersect the same elements of X. Now, fix a pair of elements $a, b \in X$ which are intersected by both x and y. We show they are intersected in the same order. It suffices to assume that a and b are both lines (possibly replacing a ray or segment with their supporting line.)

It is easy to see that the order in which a line intersects two others is determined by the relative order of the slopes of the lines, and whether the first line passes above or below the intersection point of the two others. This is illustrated in Figure 4, and is determined the position of the line's dual with respect to Type I & II lines. In particular, Type II lines carry the data on whether the first line passes above or below, while position with respect to the Type I lines (beyond simply carrying information about whether or not the line intersects the involved ray or segments) carries the information about the slope.

Thus, by our construction of $\mathcal{P}$, being in the same region of $\mathcal{P}$ implies that x and y have the same relation with a and b, and hence intersect a and b in the same left-right order. Note that a point which lies within one of the line segments might not intersect one or the other (or both) of a and b, or might intersect both at the same time. None the less, the information required to determine this is captured by the region.

Since a and b are arbitrary, this means that x and y intersect all elements of X in the same order as desired. ☐

Lemma 7. *Let X be a set of n rays or line segments in the plane. Let S_X denote the set of all subsets $A \subseteq X$ so that there exists a ray intersecting exactly the elements of A, and no others. Then*

$$|S_X| \leq \frac{1}{4} n \left(n^3 + 14n^2 + 15n + 14\right)\right) < \frac{(n+4)^4}{4}.$$

Proof. Let us suppose X has r rays and s line segments, where $r + s = n$, and consider the partition of the dual plane offered by Lemma 6. Consider a line in the primal plane. It intersects at most n elements of X in some order, and hence there are at most $2n$ subsets that a ray can intersect.

Recall that given two lines, if the duals of those lines fall into the same region of the dual plane in our partition, they intersect the same elements of X in the same order. Thus the subsets of X potentially stabbed by segments of a line are completely determined by the region of the dual plane.

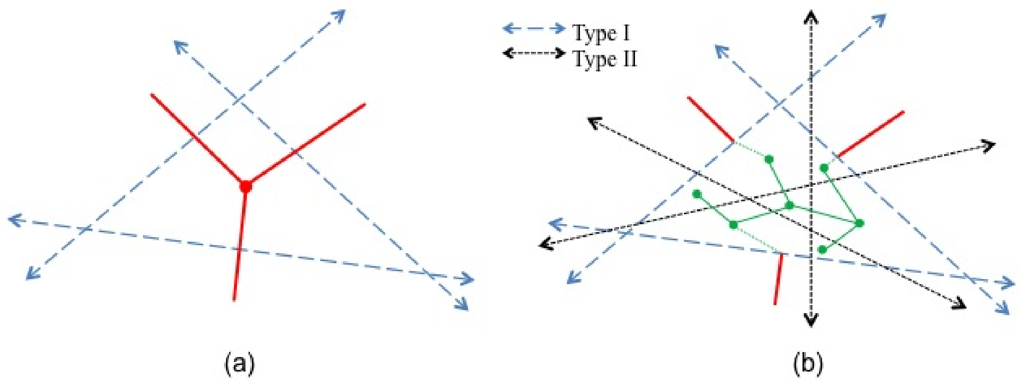


Figure 5: (a) The primary spanning tree of 2-faces for Type I lines, and (b) the secondary tree obtained by refining it for a particular 2-face region.

We will again use the Type I and Type II descriptions of the lines of the partition in Lemma 6.

Note that the collections of subsets corresponding to adjacent 2-faces separated by a Type I line differ by at most n subsets, since two points on different sides of a Type I line in the dual plane correspond to one line intersecting that element of X, and one line not intersecting it. On the other hand, since the rays are in general position, the collection of subsets for adjacent 2-faces separated by either a Type II line differ by at most 2 subsets since passing through such a line either induces a transposition of the order of two order-adjacent intersected elements of X or has no effect at all.

We now proceed to bound the total number of subsets encountered. We will first (carefully) fix any spanning tree of the 2-faces in the dual plane. We call this tree the *primary* spanning tree (shown in red in Figure 5 (a)). We then fix a root, and note that the total number of subsets encountered by rays within the 2-faces is at most $2n$ for the root, plus n times the number of Type I lines crossed by the spanning tree, plus 2 times the remaining number of line crossings.

In order to do this as efficiently as possible, we do the following: First, consider the regions of the dual plane determined by the (at most) $2n$ different Type I (bow-tie) lines. There are at most $\binom{2n+1}{2} + 1$ such regions, so the primary spanning tree has at most $\binom{2n+1}{2}$ edges.

Then we consider the regions defined by all the lines. This refines the

Type I partition of the plane: Within each of the Type I regions, we find an arbitrary *secondary* spanning tree of the regions in the refinement (shown in green in Figure 5 (b)).

For each (red) edge in the primary tree, we connect the two corresponding secondary (green) trees with a single edge, where the choice of vertex within each secondary tree is arbitrary. This spanning tree crosses Type I lines at most $\binom{2n+1}{2}$ times, and in general has one less than the number of 2-faces as edges.

Now let us determine the number of 2-faces. By Lemma 6, there are at most $\binom{n}{2} + 2n = \frac{n(n+3)}{2}$ lines. Since the number of 2-faces in an arrangement of k lines is at most $k(k+1)/2+1$, there are at most

$$\frac{n^2(n+3)^2}{8} + \frac{n(n+3)}{4} + 1.$$

2-face regions in the dual plane, and hence the spanning tree contains at most $\frac{n^2(n+3)^2}{8} + \frac{n(n+3)}{4}$ total edges. Thus the total number of subsets encountered within the 2-faces is at most

$$\begin{aligned} 2n + 2 \cdot \left(\frac{n^2(n+3)^2}{8} + \frac{n(n+3)}{4} \right) + n \cdot \binom{2n+1}{2} \\ = \frac{1}{4} n \left(n^3 + 14n^2 + 15n + 14 \right) \right) < \frac{(n+4)^4}{4}. \qquad (1) \end{aligned}$$

Finally, we must determine the contribution from lower dimensional regions in the dual plane. A point in the dual plane which occurs in a Type II line corresponds in the primal plane to a line through the intersection point. It is clear these account for fewer subsets of S_X as if a segment contains one of the intersecting lines it contains both. But also, these subsets are easily seen to be realizable by lines occurring in the neighboring 2-faces of the dual plane. Thus, for the purposes of differentiating between corresponding collections of subsets of X, we can ignore Type II lines.

Similarly, we can ignore Type I lines: points in the dual plane occurring in these lines correspond to lines through the 'endpoints' (where a point at infinity counts as an endpoint for rays) and the subsets realized by segments of these lines are captured in the neighboring 2-faces.

Thus the bound (1) suffices to complete the proof of the lemma. □

With these results in hand, it is finally possible to state and prove our main theorem. This next result strongly limits adjacencies within a barrier graph to any fixed subset. In a general graph, when a subset of vertices of size t is fixed, other vertices may potentially have any of the 2^t different adjacencies within the subset. The next result, however, implies a strong rigidity to the neighborhood structure within a barrier graph. It states that, with respect to any subset $X \subseteq V(G)$, vertices outside of X can only be polynomially many neighborhoods within X.

This is a strong rigidity theorem in the sense that it implies there are few distinct neighborhoods of vertices outside of X into X and, instead, there must be many vertices whose neighbors in X are the same. Quite remarkably, this is true even for fairly large sets X (for an n vertex graph G; there must be many neighborhood clones even into sets of size nearly $n^{1/4}$). The proof, however, follows quite easily from Lemma 7.

Theorem 8 (Barrier Rigidity)**.** *Suppose G is a barrier graph, and $X \subseteq V(G)$ with $|X| = t$. Let*

$$S_X = \{\mathcal{N}_X(y) : y \in V(G) \setminus X\}.$$

Then

$$|S_X| \leq \frac{1}{2} t \left(t^3 + 14t^2 + 15t + 14\right)) < \frac{(t+4)^4}{2}.$$

Proof. Let $f(t) = \frac{1}{4} t\,(t^3 + 14t^2 + 15t + 14))$.

Fix a realization of G, and vertex $y \in V(G) \setminus X$. The vertex y corresponds to a ray in the realization. Note that $\mathcal{N}_X(y)$ is determined by the intersection of that ray, along with rays and line segments determined by X.

In particular, if y is red or blue (say, red) in our coloring, then the neighborhood of y is determined by the intersections of the y-ray with the blue (resp. red) rays determined by X and the line segments or rays consisting of the black rays restricted to above the line $\alpha\beta$. Suppose there are r red, b blue and k black colored vertices in X. By Lemma 7, there are at most $f(b + k)$ subsets of X that can arise as such intersections.

Similarly, if y is colored black, the neighborhood of y is determined by the intersection of the y-ray with the line segments given by red rays above the $\alpha\beta$-line and blue rays below the $\alpha\beta$ line, giving a bound of $f(r + b)$.

In all, we arrive at a bound on the number of occurring subsets of

$$f(r + b) + f(r + k) + f(k + b) = f(t - k) + f(t - b) + f(t - r)$$

where we are subject to the constraint that $r + b + k = t$. Since f is a polynomial with positive coefficients, convexity implies that this quantity is maximized when $r = t$, $b = 0$, and $k = 0$, yielding a bound of $2f(t)$ as claimed. □

Theorem 8 implies strong conditions on the neighbors of vertices within a barrier graph. This is sufficient to show that barrier graphs are rare amongst bipartite graphs; and hence tripartite graphs as well. Indeed, we show an even stronger theorem: even when one of the sides of a random bipartite graph is small (all the way down to constant size), then barrier graphs get rare as the size of the graph tends to infinity.

Theorem 9. *Suppose G is chosen uniformly at random from all bipartite graphs with bipartition (X, Y) satisfying $16 \le |X| < |Y|$, and $|X| + |Y| = n$. Then the probability that G is a barrier graph is at most $2^{16}e^{-n2^{-17}} = o(1)$.*

Proof. For clarity, we set $t = 16$ for the purposes of the proof. Fix (arbitrarily) t vertices in X, call this set X'. We prove that if n is sufficiently large that, with probability $1 - 2^t e^{-n2^{-(t+1)}}$,

$$|\{\mathcal{N}_{X'}(y) : y \in Y\}| = 2^t.$$

By our choice of t, $2^t > \frac{1}{2}t\,(t^3 + 14t^2 + 15t + 14))$ and by the rigidity theorem this will imply that G is not a barrier graph.

Let Z denote the number of subsets of X' which do not appear as a neighborhood of a vertex in Y.

Then, by Markov's inequality and linearity of expectation

$$\mathbb{P}(Z \ge 1) \le \mathbb{E}[Z] = 2^t(1 - \frac{1}{2^t})^{|Y|} \le 2^t(1 - \frac{1}{2^t})^{n/2} \le 2^t e^{-n2^{-(t+1)}},$$

completing the proof. □

Remark. This proof reveals the reason for maintaining the somewhat less attractive bound in Theorem 8 – the weaker, but cleaner, bound while sufficient for most purposes would necessitate replacing the '16' by a '17' in the statement of Theorem 9.

An almost immediate corollary of this is the following:

Corollary 10. *Almost every bipartite graph is not a barrier graph.*

The only *slight* annoyance in the proof is that uniformly random bipartite graphs on n vertices are somewhat obnoxious to deal with – even the enumeration of bipartite graphs is somewhat difficult, see [7].

Proof. Let G be chosen uniformly at random from the set of (unlabeled) bipartite graphs on n vertices. We say that G admits an s-decomposition if there exists $X, Y \subseteq V(G)$ with $|X| = s$ and such that (X, Y) is a bipartition of G.

Let $\mathcal{A}_s$ denote the event that G admits an s-decomposition and $\mathcal{B}$ denote the event that G is a barrier graph. The content of Theorem 9 is that

$$\mathbb{P}(\mathcal{B}|\mathcal{A}_s) \leq 2^t e^{-n2^{-(t+1)}}.$$

if $s \geq t$, where $t = 16$ is as in Theorem 9.

On the other hand $G \in \mathcal{A}_s$ for some $0 \leq s \leq \lfloor n/2 \rfloor$; although note that the events $\mathcal{A}_s$ do not *partition* the space. Thus

$$\begin{aligned}
\mathbb{P}(\mathcal{B}) &\leq \sum_{s=1}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{B} \cap \mathcal{A}_s) \\
&= \sum_{s=1}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{B}|\mathcal{A}_s)\mathbb{P}(\mathcal{A}_s) \\
&\leq \sum_{s=1}^{t-1} \mathbb{P}(\mathcal{A}_s) + 2^t e^{-n2^{-(t+1)}} \sum_{s=t}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{A}_s). \qquad (2)
\end{aligned}$$

Finally, we estimate $\mathbb{P}(\mathcal{A}_s)$ rather crudely. There are $2^{s(n-s)}$ graphs which admit an s-decomposition and at least $2^{\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil}$ bipartite graphs, so

$$\mathbb{P}(A_s) \leq 2^{s(n-s) - \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil},$$

This is increasing for $s \leq \lfloor n/2 \rfloor$ and is always at most one. While the $\mathcal{A}_s$ do not partition the space, and hence we cannot simply bound $\sum_s \mathbb{P}(\mathcal{A}_s) = 1$, these bounds suffice. By estimating the $\mathbb{P}(\mathcal{A}_s)$ in each of the sums by the largest $\mathbb{P}(\mathcal{A}_s)$ appearing, we obtain from (2)

$$\begin{aligned}
\mathbb{P}(\mathcal{B}) &\leq \sum_{s=1}^{t-1} \mathbb{P}(\mathcal{A}_s) + 2^t e^{-n2^{-(t+1)}} \sum_{s=t}^{\lfloor n/2 \rfloor} \mathbb{P}(\mathcal{A}_s) \\
&\leq (t-1)2^{t(n-t) - \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil} + 2^t e^{-n2^{-(t+1)}} \cdot \frac{n}{2} \\
&= o(1),
\end{aligned}$$

completing the proof. $\square$

4. Discussion and Future Work

This work has focused on the structure of barrier graphs: we have investigated how the geometry of rays influences the structure of a barrier graph.

While we have focused on finding restrictions for showing that certain graphs are *not* realizable as barrier graphs and along the way ruled out almost all bipartite graphs, the question of constructing graphs which *are* barrier graphs is also interesting. Paths, even cycles, trees, complete bipartite and complete tripartite graphs, all have straightforward realizations.

Along these lines, an interesting problem suggested by our work is the following. Suppose we consider only graphs of a fixed minimum vertex cover size r (which, for the purposes of this discussion, we call resilience even if the graph is not realizable). For which r, if any, are *all* graphs of resilience r realizable? In the general case this proves to be a fairly uninteresting question as the 5 cycle, for instance, has resilience 3 and is not realizable. However the question seems much more interesting (and much less trivial) when restricted to the class of bipartite graphs.

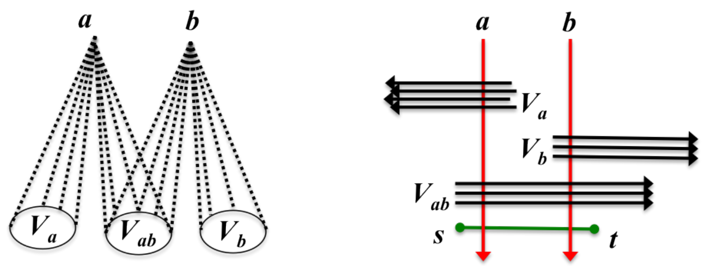
In particular, it is not difficult to prove the following:

Theorem 11. *All bipartite graphs of resilience 2 or 3 are realizable as barrier graphs*

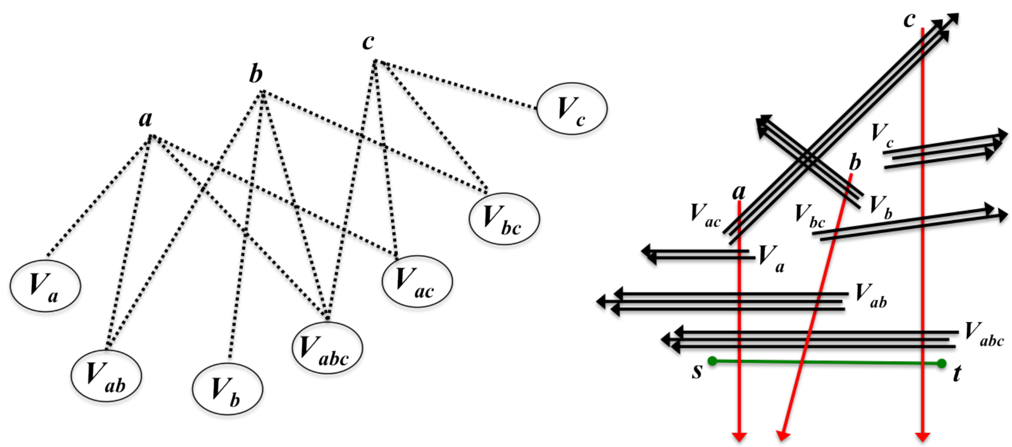
Proof. See Figure 6 for explicit constructions of some of most of the cases that may arise. The notation used in the figure is as follows: in a resilience 2 graph, if $\{a, b\}$ is a vertex cover of the graph, we write V_a for the vertices incident on a but not in the cover, V_{ab} for the vertices incident on both a and b but not in the cover, and so on – and similarly for other resiliences.

If the resilience is 2, there are two cases: the vertex cover $\{a, b\}$ is an independent set (Figure 6a); or $(a, b) \in E(g)$, in which case the bipartite graph is a tree.

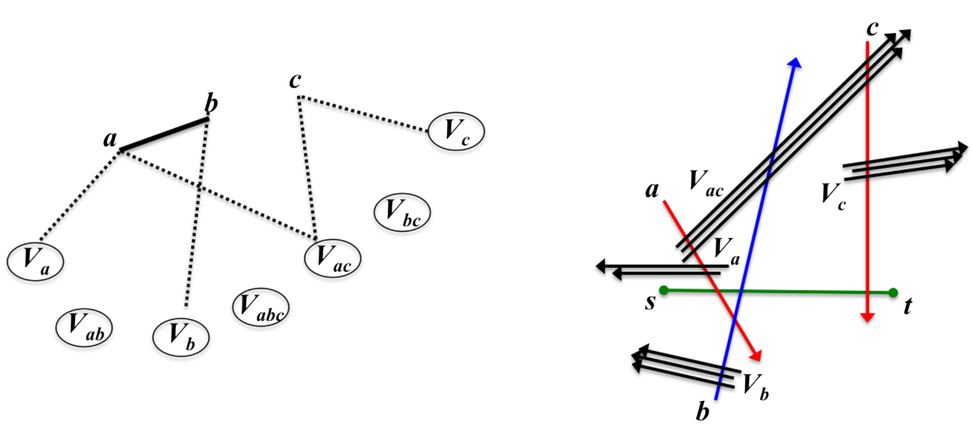
If the resilience is 3, the cases reduce to three: the vertex cover $\{a, b, c\}$ is independent (Figure 6b); or the graph induced by $\{a, b, c\}$ has one edge (Figure 6c); or the graph induced by $\{a, b, c\}$ is a path. The construction where the graph induced by $\{a, b, c\}$ is a path is identical to the one in Figure 6c only the ray for b must be modified so that it intersects with ray c. Note that in the case where the graph induced by $\{a, b, c\}$ has one edge, it is not necessary to construct both V_{ab} and V_{bc}: the reason for this is that vertices with both neighborhoods cannot be present in the same barrier



(a) General resilience 2 bipartite graph with independent vertex cover and its realization



(b) General resilience 3 bipartite graph with independent vertex cover and its realization



(c) General resilience 3 bipartite graph with one edge between elements of the vertex cover, and its realization

Figure 6: Generalized bipartite graphs with fixed resilience and their realizations

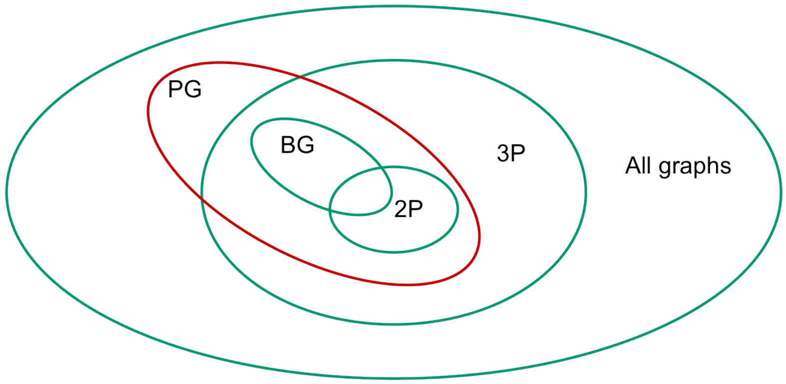


Figure 7: Barrier graphs in the context of well-known graph classes. PG = Perfect Graphs, BG = Barrier Graphs, 2P = Bipartite Graphs, 3P = Tripartite Graphs

graph, as otherwise the graph would not be bipartite (and a 5-cycle would be induced). □

On the other hand, Theorem 9 shows that not all bipartite graphs of large enough fixed resilience are realizable, so there is some largest resilience r such that all bipartite graphs of resilience r are realizable. Hence we propose the following question:

Question. *Find the minimum $r_\star$ such that there exists a bipartite graph G of resilience $r_\star$, so that G is not realizable as a barrier graph.*

We have shown that $r_\star$ exists, and that $4 \leq r_\star \leq 16$, where the upper bound on $r_\star$ follows from Theorem 9, and ultimately from Theorem 8. Determining $r_\star$, or even tightening the bounds, would be an interesting step in our understanding of bipartite barrier graphs. Some optimization of the bounds in Lemma 7 and Theorem 8 can reduce the upper bound on $r_\star$ slightly, but a substantial tightening seems to require a new idea.

Theorem 8 is a geometric result that strongly restricts the neighborhood structure of any barrier graph. This is progress toward an ultimate goal of a complete classification of barrier graphs. In Figure 7 is a visualization of the state of the classification as of this paper, in terms of known graph classes. Barrier graphs are always tripartite (as shown in [9]) and always perfect (as we showed), but not all bipartite, tripartite or perfect (e.g. K_4) graphs are barrier graphs.

One of the most interesting algorithmic questions remaining is recognition of barrier graphs. In general, even the recognition of tripartite graphs the problem is *NP-hard*. On the other hand, both recognition of perfect graphs [3, 4], and determining the chromatic number of perfect graphs is polynomial [6]. Thus it is possible to recognize whether a graph is tripartite and perfect in polynomial time. As we have shown, however, the converse is far from true. Thus while these results suggest that efficient recognition of barrier graphs might be possible, more work is needed to find a way to determine whether a graph is a barrier graph.

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