On Independent Doubly Chorded Cycles

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Abstract

In a graph G, we say a cycle $C: v_1, v_2, \ldots, v_k, v_1$ is *chorded* if its vertices induce an additional edge (chord) which is not an edge of the cycle. The cycle C is *doubly chorded* if there are at least two such chords. In this paper we show a sharp degree sum condition that implies the existence of k vertex disjoint doubly chorded cycles in a graph.

1 Introduction

We consider only simple graphs. Let G be a graph and P_t be a path on t vertices. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. Let A and B be subgraphs of a graph G, then $e(A, B)$ denotes the number of edges that have one end in A and the other end in B, so $e(G)$ denotes the number of edges in G. Given a vertex set W, we say that a cycle C is a *proper* cycle if C does not span W. Let $N(u)$ denote the set of neighbors of the vertex u, that is, the vertices adjacent to u in the graph. For a noncomplete graph G, we define

$$\sigma_2(G) = \min \{\deg(u) + \deg(v) \mid u \text{ and } v \text{ are nonadjacent}\},$$

with the convention that for the complete graph $\sigma_2(G) = \infty$. We say an edge that joins two vertices of a cycle C is a *chord* of C if the edge is not itself an edge of the cycle. We then say that C is a *chorded cycle*. A *k-degenerate* graph is one in which every induced subgraph contains a vertex of degree at most k. We denote by K_4^- the graph obtained from K_4 by removing one edge. For terms not defined here see [2].

The study of cycles and systems of vertex disjoint cycles in graphs is well established. Recently, there have been numerous papers considering cycles with additional properties such as containing a specific set of vertices, or containing a specific set of vertices in a specific order (see the survey [6]). Another natural additional property for cycles is that of containing at least one chord or at least some number $t \geq 1$ of chords.

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In 1961 Pósa [11] suggested the problem of finding, in a graph G, degree conditions that imply the existence of a cycle with at least one chord. J. Czipzer proved (see Lovász [10], problem 10.2) that any graph G with minimum degree $\delta(G) \geq 3$ contains a chorded cycle, that is, a cycle with an additional edge. Corrádi and Hajnal [4] proved that any graph G with $|V(G)| \geq 3r$ and $\delta(G) \geq 2r$ contains r vertex disjoint cycles. Finkel [9] showed that if G is a graph with $|V(G)| \geq 4r$ and $\delta(G) \geq 3r$, then G contains r vertex disjoint chorded cycles.

Bialostocki, Finkel and Gyárfás [1] made the following conjecture and verified it for the cases $r = 0$, $s = 2$ and for $s = 1$. Let r, s be nonnegative integers and let G be a graph with $|V(G)| \geq 3r + 4s$ and $\delta(G) \geq 2r + 3s$. Then G contains a collection of $r + s$ vertex disjoint cycles with s of these cycles chorded. This conjecture was settled completely in [3] where a slightly stronger σ_2 condition was used.

Theorem 1. [3] *Let r and s be integers with $r + s \geq 1$, and let G be a graph of order at least $3r + 4s$. If $\sigma_2(G) \geq 4r + 6s - 1$, then G contains a collection of $r + s$ vertex disjoint cycles such that s of them are chorded.*

More recently it was shown in [7] that a graph on at least $4k$ vertices such that $|N(u) \cup N(v)| \geq 4k + 1$ for any pair of non-adjacent vertices u and v, contains k vertex disjoint chorded cycles.

Note that it is only slightly more difficult to show that $\delta(G) \geq 3$ implies a cycle with at least two chords exists in G. Thus, in some sense, it is more natural to consider conditions implying the existence of such cycles (which we call *doubly chorded* cycles or DCC's for short). Since a spanning cycle of K_4 has two chords, we are in some sense, seeking versions of K_4 where the spanning cycle has been loosened. This point of view was taken in [8].

In [13] the following was shown.

Theorem 2. *If G is a graph of order $n \geq 4k$ and minimum degree at least $\lfloor 7k/2 \rfloor$, then the graph G contains k vertex disjoint doubly chorded cycles.*

The goal of this paper is to show the following stronger result.

Theorem 3. *If G is a graph on $n \geq 6k$ vertices with $\sigma_2(G) \geq 6k - 1$, then G contains k vertex disjoint doubly chorded cycles.*

Theorem 3 is sharp in the sense that the degree sum cannot be lowered. The condition $n \geq 6k$ is needed for the proof. The complete bipartite graph $K_{3k-1,n-3k+1}$ has degree sum $6k - 2$ and fails to contain k vertex disjoint doubly chorded cycles, since any such cycle in this graph must contain at least three vertices from each partite set.

2 Proof of Theorem 3

In this section we prove the main result, Theorem 3. Along the way we state several lemmas that are needed. We postpone the proofs of some of these lemmas until the next section.

Proof. Suppose $k = 1$. If $\delta(G) \geq 3$, the result follows for $k = 1$ from Theorem 2. If $\delta(G) = 1$, say the vertex x is adjacent only to y. Now all other vertices have degree at least 4 in order to satisfy the degree condition. Deleting x and y leaves a subgraph with minimum degree at least 3, and again we apply Theorem 2. If $\delta(G) = 2$, say x is adjacent to y and z. Now all other vertices have degree at least 3 by our degree condition. If y and z each have degree at least 3, then take a second copy of G, say G' and join x to its corresponding vertex in G' by an edge. The new graph has minimum degree 3 and so contains a doubly chorded cycle. Clearly, this cycle sits in one copy of G. Next suppose that one of y or z also has degree 2 (as both cannot), say y. Contract the edge xy and call the resulting vertex w and the resulting graph G''. Take two copies of G'' and join the corresponding copies of w by an edge. This new graph has minimum degree 3 (unless x, y and z form a triangle) and as before, contains a doubly chorded cycle. Note that upon expanding the vertex w back to an edge, it is easy to see we do not hurt our doubly chorded cycle. If x, y and z form a triangle, then remove x and y, leaving z of degree at least one. Now take 3 copies of $G - \{x, y\}$ and join each of the copies of z, forming a triangle. This new graph has minimum degree at least three (since $\deg(z) \geq 3$) and clearly the doubly chorded cycle must reside in one copy of $G - \{x, y\}$. So we may assume $k \geq 2$.

Our proof proceeds by contradiction. Let G be an edge maximal counterexample to the result. Thus, the addition of any edge to G will produce the desired system of k doubly chorded cycles. Hence, G must contain $k-1$ vertex disjoint doubly chorded cycles. Over all such possible collections of doubly chorded cycles, we choose one, say, $\mathcal{C}$: $C_1, \ldots, C_{k-1}$, subject to the constraint that $|\bigcup_{i=1}^{k-1} V(C_i)|$ is minimum. We assume that $|C_1| \geq \ldots \geq |C_{k-1}|$. Let $H = G \setminus \left(\bigcup_{i=1}^{k-1} C_i\right)$.

The key to this proof is the following lemma, whose proof is rather involved and will be deferred until the next section.

Lemma 1. *Suppose C and C' are two vertex disjoint doubly chorded cycles with $\ell = |C| \geq |C'|$ and $\ell \geq 7$. If $e(C, C') \geq 3\ell + 1$, then the graph induced by $C \cup C'$ contains two vertex disjoint doubly chorded cycles whose union is smaller than $|C| + |C'|$.*

The following will also be useful.

Lemma 2. *Let C be a cycle with at least 5 chords. Then C contains a proper doubly chorded cycle.*

Proof. Suppose C is a cycle with at least 5 chords and suppose it has a clockwise ordering of vertices. Suppose e is a minimum length chord of C, and order the vertices of $C = \{v_1, v_2, \ldots, v_{|C|}\}$ so that e connects v_1 and v_i so that i is as small as possible. If there are no chords incident to v_k for $1 < k < i$, then clearly $v_1, v_i, v_{i+1}, \ldots, v_1$ is a shorter doubly chorded cycle, so choose $1 < k < i$ minimum so that v_k is incident to a chord, e'. Since e is a shortest chord, the chord e' is of the form $v_k v_j$ where $j > i$.

Consider the three segments $S_1 = [v_k, v_i]$, $S_2 = [v_i, v_j]$ and $S_3 = [v_j, v_1]$. First suppose that $k > 2$. We have three proper cycles which are constructed from these segments: S_1, S_2, e'; S_2, S_3, e and S_3, e, S_1^-, e'. Since there are no edges incident to the segment (v_1, v_k), all of the at least three remaining chords are contained within or between these segments. Further note that none of the above three cycles contain two or more chords or, since they are proper, we are done.

We have three chords remaining. If one chord lies within a segment, then no other chord can enter that segment or a DCC is formed. This implies that the other two chords lie between segments or one lies in each of the other two segments. But in each case, a shorter DCC exists, since seach segment lies in a cycle with one of the other two segments. Thus, there is exactly one chord joining each pair of segments. In each case, however, we can still find a shorter doubly chorded cycle, as we now exhibit. Let $x = x_1x_2$ denote the chord from S_1 to S_2 (so that $x_1 \in [v_k, v_i]$ and $x_2 \in [v_i, v_j]$). Likewise, let $y = y_1y_2$ denote the chord from S_1 to S_3 and $z = z_1z_2$ denote the chord from S_2 to S_3.

We now look at possibilities for how x, y and z intersect:

Case 1. Suppose $x_2 \leq z_1$.
Consider the cycle: x_1, x, x_2, C, x_1. This contains the chords z and e', and is proper since v_i is not included.

Case 2. Suppose $z_2 \leq y_2$.
The cycle z_1, z, z_2, C, z_1 contains the chords y and e, and is proper as v_j is not included.

Case 3. Suppose $y_1 \leq x_1$.
The cycle x_1, x, x_2, C, x_1 contains the chords y and e', and is proper as v_i is not included.

Case 4. Suppose $x_1 < y_1$, $z_1 < x_2$, and $y_2 < z_2$.
Then the cycle $y_2, C^-, z_1, z_2, C, y_1, y_2$ has e' and x as chords and avoids v_i unless $y_1 = v_i$ or $z_1 = v_i$. If $y_1 = v_i$ then the cycle $v_1, C, x_1, x_2, C^-, v_i, y_2, C, v_1$ has e and z as chords and avoids v_j, so is proper. Note this still works if $z_1 = y_1 = v_i$. Finally, if $z_1 = v_i$ and $y_1 < v_i$, then the cycle $v_1, C, y_1, y_2, C^-, v_i, v_1$ has e' and x as chords and avoids z_2, so it is proper. This comletes the $k > 2$ case.

Next suppose $k = 2$ (and $j = i + 1$). Now the cycle S_3, e, S_1^-, e' is not proper, so the above argument does not hold. However, we can consider e' as the shortest chord and repeat the above argument. This works unless the minimum chord is of the form $e'' = v_3v_{i+2}$. Then, we can consider e'' as the shortest chord and try to repeat the original argument. Again this works unless the minimum chord is of the form $e''' = v_4v_{i+3}$. But now, the cycle $v_1, v_2, v_3, v_4, v_{i+3}, v_{i+2}, v_{i+1}, v_i, v_1$ is proper (unless the cycle has order exactly 8) and has chords e' and e''. In this case, the fifth chord lies between $[v_1, v_4]$ and $[v_5, v_8]$ and no matter how it is placed, a proper doubly chorded cycle is easy to find. This completes the proof of the Lemma. □

One additional lemma will be useful, as when it applies it will allow us to transfer our σ_2 condition into a more applicable degree condition.

Lemma 3. *Suppose that C is a doubly chorded cycle with $|C| \geq 7$ and containing no proper doubly chorded cycle. Then the complement $\overline{C}$ of C can be covered by a collection of connected vertex-disjoint regular subgraphs (not necessarily induced and not necessarily of the same degree of regularity for different subgraphs) of order at least two.*

Proof. Note that C has at most four chords by Lemma 2. In C, the degree of any vertex is at most four, since any vertex with at least three chords out of it produces a proper doubly chorded cycle. Therefore, $\delta(\overline{C})$ is at least $|V(C)| - 5$. Hence, if $|V(C)| = n \geq 10$, then by Dirac's Theorem, $\overline{C}$ is hamiltonian and therefore covered by a 2-regular graph.

For the remaining cases ($n = 7, 8, 9$) we use the above observations to note that we may assume that $\overline{C}$ is not hamiltonian (or we are done), and has minimum degree $\delta(\overline{C}) \geq n - 5$ and maximum degree $\Delta(\overline{C}) \leq n - 3$.

Suppose then that $n = 9$. Then $\delta(\overline{C}) \geq 4$ and $\Delta(\overline{C}) \leq 6$. Then the graph $\overline{C}$ is traceable by Ore's Theorem [12]. Let $P: \ v_1, v_2, \ldots, v_9$ be such a path. We note that if v_1 is adjacent to v_i, then v_9 is not adjacent to v_{i-1}, or else $\overline{C}$ is hamiltonian. Further, if v_1 is adjacent to any of v_3, v_5 or v_7, then an odd cycle and a matching (or single edge) cover $V(\overline{C})$ and we are again done. Thus, we may assume these adjacencies do not occur. Similarly, we may assume v_9 is not adjacent to v_3, v_5 or v_7.

As v_1 has three adjacencies on P besides v_2, these adjacencies must be to v_4, v_6 and v_8. Similarly, v_9 must be adjacent to v_2, v_4 and v_6. Now v_7 has degree at least 4 and is not adjacent to either v_1 or v_9. If v_7 is adjacent to v_5, then v_1, v_2, v_3, v_4, v_1 and $v_5, v_7, v_8, v_9, v_6, v_5$ are two cycles covering the graph. A similar argument applies if v_7 is adjacent to v_3. Thus, v_7 must be adjacent to v_2 and v_4. A similar argument shows that v_5 must be adjacent to v_2 and v_8. Finally, a similar argument shows v_3 is adjacent to v_6 and v_8, or we have the components we seek. But the graph we now have is $K_{4,5}$ which cannot be $\overline{C}$ as its complement is disconnected, not a cycle. Hence, a contradiction is reached completing the case when $n = 9$.

Next suppose $n = 8$ with $\delta(\overline{C}) \geq 3$ and $\Delta(\overline{C}) \leq 5$. If $\overline{C}$ is not connected, then the graph is $K_4 \cup K_4$ and these two components suffice. So assume $\overline{C}$ is connected. If $\overline{C}$ is traceable, then an argument similar to the $n = 9$ case shows the regular components exist. Thus, we may assume that $\overline{C}$ is not traceable.

Let P be a longest path in $\overline{C}$. Suppose $|P| = 7$, say $P : v_1, v_2, \ldots, v_7$. Then v_8 is not on this path but has at least three adjacencies on this path, and clearly, these must be v_2, v_4 and v_6. Now v_1 must have two more adjacencies on P and these are not v_3, v_5 or v_7 or a longer path would exist. Thus, v_1 is adjacent to v_4 and v_6. By symmetry, v_7 is adjacent to v_2 and v_4.

Now consider v_3 and v_5, each having an additional adjacency on P. If they are themselves adjacent, then the cycle $v_1, v_2, v_8, v_6, v_7, v_4, v_1$ and the edge v_3v_5 form the two regular components. Otherwise, we already know v_1v_3, v_3v_7, v_1v_5 and v_7v_5 do not exist, thus, v_3v_6 and v_2v_5 are edges of G. But the graph thus formed cannot be $\overline{C}$ as it implies C is not connected. Thus, the longest path cannot contain seven vertices. If the longest path P has six vertices, then the two vertices off this path cannot be adjacent or $\overline{C}$ would contain a perfect matching. But then, each of the vertices off the path would be adjacent to three of v_2, v_3, v_4, v_5 and thus have consecutive adjacencies on P. But then P is not the longest path. Similar arguments show $|P|$ cannot be five or less. Thus, this case is completed.

If $n = 7$ and $\delta(\overline{C}) \geq 2$ and $\Delta(\overline{C}) \leq 4$ we note that the graph must contain at least 10 edges. This follows since if $\overline{C}$ had at most 9 edges, then C would have at least 12 edges and hence be a cycle with at least 5 chords. But then a proper doubly chorded cycle would exist by Lemma 2. By inspection of the list of graphs of order 7 and size 10 or more contained in [14], either the graph admits the components we seek or cannot be the complement of C for one of the several reasons for contradictions given in earlier cases of this proof. Thus, we conclude that such a cover always exists. $\square$

Lemma 4. *There is no vertex $x \in H$ and cycle $C \in \mathcal{C}$ so that* $\deg_C(x) \geq 5$.

Proof. Suppose that $deg_C(x) \geq 5$. Then if $|V(C)| \geq 6$, it is easy to find a shorter doubly chorded cycle, contradicting our choice of $\mathcal{C}$. If $|V(C)| = 5$, then a K_4 would be formed using x, again contradicting our choice of $\mathcal{C}$.

□

Lemma 5. *Suppose some vertex $x \in H$ has $\deg_C(x) = 4$ for some $C \in \mathcal{C}$. Then $|C| \leq 5$.*

Proof. Suppose $|V(C)| \geq 9$. Then it is easy to find a shorter doubly chorded cycle using x and omitting a segment of C with at least two vertices.

Next suppose $|V(C)| = 8$, say $V(C) = \{v_1, v_2, \ldots, v_8\}$. Then, without loss of generality, $N_C(x) = \{v_1, v_3, v_5, v_7\}$ or a shorter doubly chorded cycle can again be found.. Now note that if any chord of C is contained within $v_1, v_2, \ldots, v_5$, then a shorter doubly chorded cycle exists using these vertices and x. Similarly, if a chord exists within the vertices $v_3, v_4, \ldots, v_7$ or within v_5, v_6, v_7, v_8, v_1 or v_7, v_8, v_1, v_2, v_3 we can again find a shorter doubly chorded cycle. Thus, the two chords of C must be v_2v_6 and v_4v_8. But now, $x, v_3, v_4, v_5, v_6, v_2, v_1, x$ is a 7-cycle with chords xv_5 and v_2v_3, contradicting our choice of $\mathcal{C}$.

Next assume $|V(C)| = 7$. Let $V(C) = \{v_1, v_2, \ldots, v_7\}$ and, without loss of generality, assume $N_C(x) = \{v_1, v_3, v_5, v_7\}$. As before, a chord in any of the segments $v_1 - v_5$, $v_3 - v_7$, $v_5 - v_1$, or $v_7 - v_3$ produces a shorter doubly chorded cycle. But now, the only possible chord which does not produce such a cycle is v_2v_6. Hence, this case also cannot happen.

Now suppose $|V(C)| = 6$, say $V(C) = \{v_1, v_2, \ldots, v_6\}$. Note that no vertex off C can be adjacent to four consecutive vertices of C, or a shorter doubly chorded cycle would exist. First assume $N_C(x) = \{v_1, v_3, v_5, v_6\}$. Note that no chord can be contained within the vertices $v_3 - v_6$ or within the vertices $v_6 - v_3$ or a shorter doubly chorded cycle is immediate. Thus, the only possible chords are v_1v_5, v_1v_4, v_2v_4 and v_2v_5. If v_1v_5 is a chord, then a K_4 exists, again a contradiction. If v_1v_4 is a chord, then x, v_1, v_4, v_5, v_6, x is a 5-cycle with chords v_1v_6 and xv_5, a contradiction. Thus, there are no chords from v_1. If the chords are v_2v_4 and v_2v_5, then x, v_5, v_4, v_2, v_3, x is a 5-cycle with chords v_3v_4 and v_2v_5, a contradiction.

Next assume that $N_C(x) = \{v_1, v_2, v_4, v_5\}$. Note that this is the only other case we must consider. Now there can be no chords within the vertices $v_1 - v_4$, or within $v_2 - v_5$ or $v_4 - v_1$ or $v_5 - v_2$ or a shorter doubly chorded cycle exists. Thus, the only possible chord is v_3v_6, a contradiction completing this case. □

Lemma 6. *If $\mathcal{C}$ contains a cycle of length at least 7, then $|H| \geq 9$.*

Proof. Recall, $H = G - (C_1 \cup C_2 \cup \ldots \cup C_{k-1})$ is the remainder after a minimal set of $k-1$ vertex disjoint cycles, $\mathcal{C}$ is removed from G. Order the cycles $C_i \in \mathcal{C}$ so that $|C_1| \geq |C_2| \geq \ldots |C_{k-1}|$. Suppose $|C_1| \geq 7$ and $|H| \leq 8$. We show that a shorter cycle system exists, proving the claim.

By Lemma 3, C_1 can be covered by a collection R_i, each a d_i-regular subgraph of the complement of C_1 and for each such subgraph

$$\sum_{e=xy \in R_i} (\deg\ x + \deg\ y) = d_i \sum \ \deg\ x \geq \sigma_2(G) d_i |R_i|/2.$$

Thus, $\sum_{x \in V(C_1)} \deg x \geq \sigma_2(G) \sum |R_i|/2 = \sigma_2(G)|C_1|/2$.

By Lemma 2 and minimality, $e(C_1) \leq |V(C_1)| + 4$, so

$$e(C_1, G - C_1) \geq |V(C_1)|(\sigma_2(G)/2 - 1) - 4 \geq 3|C_1|(k-2) + 4.5|C_1| - 4.$$

By Lemma 5, if any vertex of H has 4 or more adjacencies on C_1, then a doubly chorded cycle smaller than C_1 exists, a contradiction. Now

$$e(C_1, G - (C_1 \cup H)) \geq 3|C_1|(k-2) + 4.5|C_1| - 4 - 3|H|.$$

Combining with our assumptions that $|C_1| \geq 7$ and $|H| \leq 8$ implies that $e(C_1, G - (C_1 \cup H)) > 3|C_1|(k-2)$.

Therefore, C_1 sends more than $3|C_1| + 1$ edges to another cycle in $\mathcal{C}$. Now, by Lemma 1, we obtain two smaller cycles, replacing two cycles of $\mathcal{C}$, contradicting our choice of $\mathcal{C}$.

□

Corollary 1. *Without loss of generality, $|H| \geq 6$. Furthermore, if $\mathcal{C}$ contains at least one 5-cycle then $|H| \geq 7$, and if $\mathcal{C}$ contains a 4-cycle or two 5-cycles then $|H| \geq 8$.*

Proof. This follows as $|H| = |V(G)| - \sum_{C_i \in \mathcal{C}} |C_i|$ and $|V(G)| \geq 6k$ by assumption. Under the assumption that $|H| < 9$, then the maximum cycle length is 6. □

Beyond this basic control over $|H|$, we require some additional lemmas describing when and how we may perform exchanges that preserve $|H|$, allowing us to assert further control over the properties of H. A simple fact, following immediately from the preceeding lemmas is the following.

Lemma 7. *Suppose $x, y \in V(H)$ are such that $\deg_C(x) + \deg_C(y) \geq 7$ for some cycle $C \in \mathcal{C}$. Then, without loss of generality $\deg_C(x) = 4$ and $\deg_C(y) \geq 3$.*

A more complicated version is the following. While the ultimate statement is quite technical, it is set up in a way to conveniently use later. On an initial reading one might ignore the conditions $2(a), 2(b)$ and $2(c)$ which the lemma asserts, instead focusing on the first condition. A number of immediate consequences are described below, and are also easier on the reader.

Lemma 8. *Suppose $C \in \mathcal{C}$ is such that there is some vertex $x \in H$ with four neighbors in C. Suppose $y \in H$ is incident to $N \subseteq C$ with $|N| = 3$. Let $X_N = \{z : (C - z) \cup \{y\}\}$ is a DCC.*

1. $(N_c(x) \setminus N) \subseteq X_N$
2. *Also, with a single exception, the following occurs: $|N_C(x) \cap X_N| \geq 2$ and at least one of*
   (a) $|N_C(x) \cap X_N| > 2$
   (b) $|N_C(x) \cap X_N| = 2$ *and $N \setminus (N_C(x) \cap X_N)$ spans an edge of C (this edge may be a chord)*
   (c) $C \setminus (N \cup (C \setminus (N_C(x) \cap X_N))) = \{r\}$, *and there exists an $s \in (C \setminus (N_C(x) \cap X_N))$ with $r \sim s$ and if $x \neq y$, $\{x, y\} \cup (C \setminus \{r, s\})$ induces a DCC.*

Furthermore, in the exceptional case it is still the case that $X_N = C \setminus N$.

Remark: One may take $x = y$ in the application of this lemma – in which case N can be any three vertex of $N_C(x)$.

Proof. Suppose $|V(C)| = 5$, say $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality assume $N_C(x) = \{v_1, v_2, v_3, v_4\}$. Then v_1v_3 and v_2v_4 cannot be chords of C or a K_4 would exist, contradicting our choice of cycles. Thus the two (or more) chords of C must come from v_1v_4, v_2v_5, and v_3v_5. To complete the proof of the lemma, we illustrate for all possible N, and $z \in (C \setminus N)$ the DCC formed on $(C - z) \cup \{y\}$. Note that we do not include N which induces a triangle in C, as then there would be a K_4 which contradicts minimality of $\mathcal{C}$. For each N, we exhibit sufficient vertices in X_N to verify that one of $2(a), 2(b)$, or $2(c)$ holds. We label which of these it holds, and in case (b) we list the edge, in case (c) we list v_1, v_2.

First we consider the case where the chords are v_2v_5 and v_3v_5, in Table 1.

N	$z \in X_N$	Cycle on $(C - z) \cup \{y\}$	chords	satisfies
v_1, v_2, v_3	v_4	y, v_1, v_2, v_5, v_3, y	yv_2, v_2v_3	$2(b)$
	v_1	y, v_2, v_5, v_4, v_3, y	v_2v_3, v_3v_5	Edge: v_2v_3
v_1, v_2, v_4	v_4	y, v_1, v_5, v_3, v_2, y	v_2v_1, v_2v_5	$2(a)$
	v_3	y, v_4, v_5, v_1, v_2, y	yv_1, v_2v_5	
	v_1	y, v_2, v_3, v_5, v_4, y	v_2v_5, v_3v_4	
v_1, v_3, v_4	v_1	y, v_3, v_2, v_5, v_4, y	v_3v_5, v_3v_4	$2(b)$
	v_2	y, v_1, v_5, v_4, v_3, y	yv_4, v_3v_5	Edge: v_3v_4
v_1, v_3, v_5	v_4	y, v_1, v_2, v_3, v_5, y	yv_3, v_2v_5	$2(b)$
	v_2	y, v_1, v_5, v_4, v_3, y	yv_5, v_3v_5	Edge: v_1v_5
v_1, v_4, v_5	v_2	y, v_1, v_5, v_3, v_4, y	yv_5, v_4v_5	$2(a)$
	v_3	y, v_1, v_2, v_5, v_4, y	yv_5, v_1v_5	
	v_4	y, v_1, v_2, v_3, v_5, y	v_1v_5, v_2v_5	
	v_1	y, v_4, v_3, v_2, v_5, y	v_3v_5, v_4v_5	
v_2, v_3, v_4	v_1	y, v_2, v_5, v_4, v_3, y	v_2v_3, v_3v_5	$2(b)$
	v_4	y, v_2, v_1, v_5, v_3, y	v_2v_3, v_2v_5	Edge: v_2v_3

Table 1: Cycles for chords v_2v_5 and v_3v_5.

Next we consider the case where the chords are v_1v_4 and v_3v_5 in Table 2.

Note that the singular case referenced in the statement is the case where $N = v_1, v_3, v_4$.

The case with chords v_1v_4 and v_2v_5 is completely symmetric.

If $|C| = 4$, the statement is completely clear as $\{y\} \cup N$ induces a DCC, and hence the singular $z \in (C \setminus N)$ is the desired N. □

Two immediate consequences are exchange Lemmas which state instances in which a vertex can be swapped from H into a cycle.

N	z	Cycle on $(C-z)\cup\{y\}$	chords	satisfies
v_1, v_2, v_3	v_2	y, v_1, v_5, v_4, v_3, y	v_1v_4, v_3v_5	$2(c)$
	v_4	y, v_1, v_5, v_3, v_2, y	v_1v_2, yv_3	$r = v_4,\ s = v_5$
	v_5	y, v_1, v_4, v_3, v_2, y	yv_3, v_2v_1	
v_1, v_2, v_4	v_2	y, v_1, v_5, v_3, v_4, y	v_1v_4, v_4v_5	$2(b)$
	v_3	y, v_2, v_1, v_5, v_4, y	yv_1, v_1v_4	Edge: v_1v_4
	v_5	y, v_1, v_2, v_3, v_4, y	yv_2, v_1v_4	
v_1, v_2, v_5	v_4	y, v_1, v_2, v_3, v_5, y	yv_2, v_1v_5	$2(a)$
	v_3	y, v_2, v_1, v_4, v_5, y	yv_1, v_1v_5	
	v_2	y, v_1, v_4, v_3, v_5, y	v_1v_5, v_4v_5	
v_1, v_3, v_4	v_2	y, v_1, v_5, v_4, v_3, y	v_1v_4, yv_4	Exception
	v_5	y, v_1, v_2, v_3, v_4, y	yv_3, v_1v_4	
v_1, v_3, v_5	v_2	y, v_1, v_5, v_4, v_3, y	v_1v_4, v_3v_5	$2(b)$
	v_4	y, v_1, v_2, v_3, v_5, y	v_1v_5, yv_3	Edge: v_1v_5
v_2, v_3, v_4	v_1	y, v_2, v_3, v_5, v_4, y	yv_3, v_3v_4	$2(b)$
	v_2	y, v_3, v_5, v_1, v_4, y	v_3v_4, v_4v_5	Edge: v_3v_4
	v_5	y, v_2, v_1, v_4, v_3, y	yv_4, v_2v_3	
v_2, v_3, v_5	v_1	y, v_2, v_3, v_4, v_5, y	yv_3, v_3v_5	$2(a)$
	v_2	y, v_3, v_4, v_1, v_5, y	v_3v_5, v_4v_5	
	v_4	y, v_2, v_1, v_5, v_3, y	yv_5, v_2v_3	
v_2, v_4, v_5	v_1	y, v_2, v_3, v_4, v_5, y	yv_4, v_3v_5	$2(b)$
	v_3	y, v_2, v_1, v_5, v_4, y	yv_5, v_1v_4	Edge: v_4v_5

Table 2: Cycles for chords v_1v_4 and v_3v_5.

Lemma 9 (Single Bypass Lemma)**.** *Suppose $x \in V(H)$ and $C \in \mathcal{C}$ satisfies $\deg_C(x) \geq 4$. Then for any vertex $z \in C$, $(C-z)\cup\{x\}$ is a doubly chorded cycle.*

Proof. This follows immediately from Lemma 8. For $z \in N_C(x)$, applying the lemma with $N = (N_C(x) \setminus \{x\}$ yields the conclusion by the first bulleted conclusion. In the case where $|C| = 5$, and $z \notin N_C(x)$ a DCC is clear where both chords are incident to x. $\square$

Lemma 10. *Suppose $x, y \in V(H)$ are nonadjacent vertices with $deg_C(x) = 4$ and $deg_C(y) = 3$ for some $C \in \mathcal{C}$. Then there exists vertices $z_x, z_y \in V(C)$ such that z_x is adjacent to x and z_y is adjacent to y and both $(C-z_x)\cup\{y\}$ and $(C-z_y)\cup\{x\}$ induce doubly chorded cycles.*

Proof. This follows almost immediately from Lemma 8. Note that for $N = N_C(y)$, that some vertex in $N_C(x)$ is always one of the admissible z – this is z_x. It is easily seen that any vertex in $N_C(y)$ can serve as z_y by Lemma 9. $\square$

In some instances in the main part of the proof, more complicated exchanges are necessary as well. Our main tool is the following lemma which gives conditions on when two vertices in H may be exchanged for two vertices in a cycle.

Lemma 11 (Double Bypass Lemma)**.** *Suppose C is a doubly chorded 5 cycle. Further suppose that there are vertices $x, y, z, w \notin C$ with $\deg_C(x) = 4$ $\deg_C(y) = \deg_C(z) = 2$ and $\deg_C(w) = 3$. Suppose that $N_C(y) \cap N_C(w) = \emptyset$ and $N_C(z) \subseteq N_C(w)$. Further suppose that neither $N_C(x)$ or $N_C(w)$ span a triangle in C, and neither $N_C(z)$ or $N_C(y) \cap N_C(x)$ span an edge of C. Then there exist two vertices $u, v \in C$ so that $u \sim y$, $v \sim z$, $u \sim v$ and so that $(C \setminus \{u, v\}) \cup \{x, w\}$ is a DCC.*

Proof. Let $V(C) = \{c_1, c_2, c_3, c_4, c_5\}$ and $N_C(x) = \{c_1, c_2, c_3, c_4\}$. Then, without loss of generality, the chords are either (c_1c_4, c_3c_5) or (c_2c_5, c_3c_5). Note that the facts that $N_C(z) \subseteq N_C(w)$ and $N_C(z)$ cannot span an edge of C mean that if $N_C(w)$ consists of three consecutive vertices, then $N_C(z)$ contains the first vertex and the third vertex in $N_C(w)$, and if $N_C(w)$ consists of two consecutive vertices plus a third, nonconsecutive vertex, then the third vertex must be in $N_C(z)$. We present Table 3 displaying u and v for various $N_C(w)$. Note that $N_C(w)$ cannot span a triangle. Also, as many $N_C(w)$ are the same by symmetry, we only include one representative from each symmetry class. □

$N_C(w)$	(u, v)	Cycle on $(C \setminus \{u, v\}) \cup \{x, w\}$	Chords
c_1, c_2, c_3	(c_5, c_1)	x, c_2, w, c_3, c_4, x	xc_3, c_2c_3
c_1, c_2, c_4	(c_5, c_4)	x, c_1, w, c_2, c_3, x	xc_2, c_1c_2
c_1, c_2, c_5	(c_4, c_5)	x, c_1, w, c_2, c_3, x	xc_2, c_1c_2
c_1, c_3, c_5	(c_2, c_3) if chords c_1c_4 and c_3c_5	x, c_1, w, c_5, c_4, x	c_1c_4, c_1c_5
	(c_4, c_3) if chords c_2c_5 and c_3c_5	x, c_1, w, c_5, c_2, x	c_1c_2, c_1c_5
c_1, c_4, c_5	(c_2, c_1)	x, c_3, c_5, w, c_4, x	c_3c_4, c_4c_5
c_2, c_3, c_5	(c_4, c_5)	x, c_1, c_2, w, c_3, x	xc_2, c_2c_3

Table 3: Two vertices (u, v) and cycles for chords (c_1c_4, c_3c_5) or (c_2c_5, c_3c_5).

We finally reach the true crux of the proof: In this lemma we build on the exchange lemmas established above, resulting in a strong conclusion about the structure that we can assume on H.

Lemma 12. *Without loss of generality H contains a Hamiltonian path, on $v_1, \ldots, v_{|H|}$. Furthermore, this can be chosen so that either $v_1 \sim v_{|H|}$ or $v_2 \sim v_{|H|}$.*

Proof.
Claim 0: We may assume, without loss of generality, that H is connected.

Proof of Claim 0: If not, then as long as H is disconnected, there exists nonadjacent vertices x and y in different components and with at least 7 edges to one cycle of $\mathcal{C}$ by Lemma 7, and by repeated application of the Exchange Lemma (Lemma 10), H can be made connected. □

Suppose, of all connected H with minimal cycle system size, we choose the one with the longest path. Let P be a longest path in H. If such exists, we choose P so that the first vertex is incident to one of the last two vertices – in which case we will be are done.

Claim 1: We may assume, without loss of generality, that the first and last vertex of P are of degree at most 2.

Proof of Claim 1*:* Let $P = v_1, v_2, \ldots, v_t$. There are possibly many spanning paths on this vertex set; we choose P so the degree of v_1 and v_t are at most 2 if possible. Suppose it is not possible that the degree of v_1 is 2. Then every vertex which can start the path (we call these *start vertices*) has degree at least 3. Of all such paths, choose v_1 so that its neighbors are v_i and v_j with j as large as possible. Then v_{i-1} is a potential start to the path. By our assumption $\deg(v_{i-1}) \geq 3$ and the maximality of P, it must send a chord into the path and by our assumption that v_j is as large as possible, it must send the chord into the cycle $v_1, \ldots, v_j$, which is now easily seen to be a DCC with the other chord being v_1v_i. Thus if H does not contain a DCC, we may assume that $\deg(v_1) \leq 2$.

Arguing similarly, taking P to be the arrangement of $v_1, \ldots, v_t$ with v_1 to be the fixed vertex of degree 2, and looking at potential end vertices of the path we observe we may assume that v_t has degree at most 2 as well. □

Claim 2: We may assume that if $\deg(v_1) = 2$ and $\deg(v_2) = 4$, and v_q is the highest indexed neighbor of v_2, then there is a vertex v_r of degree 2 with $1 < r < q$.

Proof of Claim 2*:* We build on the proof of Claim 1, to observe that there are at least two start vertices of degree 2 unless the first four vertices induce a K_4^- where v_1 and v_4 are the non adjacent vertices of the K_4^-. Let $v'_1, v'_2, \ldots, v'_t$ be the path P on vertex set $v_1, \ldots, v_t$ so that $v_t = v'_t$, but v'_1's neighbor is v'_i for i as large as possible.

If $\deg_P(v'_1) = 3$, then v'_1 has neighbors v'_i and v'_j where $j < i$. Then both v'_{i-1} and v'_{j-1} are start vertices. If $j \neq i-1$, either of these vertices having degree 3 would create at DCC, so we have two start vertices of degree 2 as claimed. Hence $j = i-1$. But then v'_2 is a start vertex, as one can take $v'_2, v'_3 \ldots, v'_j, v'_1, v'_i, v'_{i+1}, \ldots$. Here, both v'_2 and v'_{i-1} must have degree 2 or we have a DCC. Thus we are done unless $2 = i-1$, that is unless $i = 3$. In this case v'_1, v'_2, v'_3, v'_4 are a K_4^-. Since $v_1, \ldots, v_t$ is obtained by taking the same path with v'_2 as v_1, we have the purported structure in this case.

If $\deg_P(v'_1) = 2$, then v'_1 has neighbor v'_i. If $\deg_P(v'_{i-1}) = 2$, we are done. Otherwise, v'_{i-1} has another neighbor on P, v'_j. Note that $j < i$ by the maximality of i. Then v'_{j+1} is also a start vertex as witnessed by the path $v'_{j+1}, v'_{j+2}, \ldots, v'_{i-1}, v'_j, v'_{j-1}, \ldots, v'_1, v'_i, v'_{i+1}, \ldots$. Hence v'_{j+1} cannot have any neighbors other than v'_j and v'_{j+2} in $v'_1, \ldots, v'_i$ without creating a DCC and none in $v'_{i+1}, v'_{i+2}, \ldots$ by the maximality of i. Thus v'_{j+1} has degree 2 and v'_1 and v'_{j+1} are the two degree 2 vertices.

If the initial four vertices give the purported K_4^- in such a way, note that v_2 cannot have any additional neighbors without creating a DCC, so if $\deg(v_2) = 4$ we are not in this case.

Note that $\{v_1, \ldots, v_i\} = \{v'_1, \ldots, v'_i\}$ and thus the two exhibited vertices of degree 2 are v_1 and v_s for some $s < i$. If v_t, the highest indexed neighbor of v_2 has $t \leq i$, then $v'_1, \ldots, v'_i$ contains both of the chords from v_2 and is hence a DCC. Thus $s < i < t$ and we have the conclusion of the claim. □

Now we turn to proving that the path is Hamiltonian. Our first step is to rule out the existence of small degree vertices off of the path.

Claim 3: $\deg_H(v) \geq 4$ for all $v \in (H \setminus P)$.

Proof of Claim 3: We may assume that the vertices in P do not span a cycle, or we are done by maximality of P. Thus v_1 is not incident to v_t. Suppose $v \notin P$. Note $v \not\sim v_1$ and $v \not\sim v_t$. If $\deg_H(v) \leq 3$, then consider the three pairs $\{v_1, v_t\}$, $\{v_1, v\}$, $\{v_t, v\}$. All three of these pairs are of pairwise non-adjacent vertices. Applying our σ_2 condition to v_1 and v yields

$$\deg(v_1) + \deg(v) \geq (6k - 1) = 6(k - 1) + 5. \tag{1}$$

If $\deg_C(v_1) + \deg_C(v) \geq 7$ for any $C \in \mathcal{C}$ the exchange lemma implies that the path can be lengthened; exchanging v for a vertex incident to v_1. That fact, along with (1) imply that v_1 and v together have *exactly* 6 incidences into every $C \in \mathcal{C}$. The same holds for v_t and v.

Now applying the σ_2 condition to v_1 and v_t, and noting that $\deg_H(v_1)+\deg_H(v_t) \leq 4$, we observe that there is some cycle C with $\deg_C(v_1) + \deg_C(v_t) \geq 7$. The fact that v_1 along with v and also v_2 along with v have the same number of edges into every cycle imply that $\deg_C(v_1) = \deg_C(v_t) = 4$ and $\deg_C(v) = 2$, and $|C| \leq 5$. Note that v_t and v must have a common neighbor in C, and the Single Bypass Lemma implies that we may exchange v_1 for some $z_{v_1} \in C$ with $z_{v_1} \sim v_t$ and $z_{v_1} \sim v$. This contradicts the maximality of the path. □

Claim 4: Every component of $H \setminus P$ has cardinality at least 4.

Proof of Claim 4: By Claim 3, $\deg_H(v) \geq 4$ for every vertex not on P. Suppose X is a connected component of $H \setminus P$. If $|X| = 1$, then the single vertex has four neighbors on the path yielding a DCC. If $|X| = 2$ then the pair of vertices in X each have 3 neighbors on the path and this is settled by the $(3, 3) \hookrightarrow P$ case of Lemma 14. The $|X| = 3$ case is similarly settled by the $(2, 2, 2) \hookrightarrow P$ case of Lemma 14. Thus every component outside of $H \setminus P$, if any, is of cardinality at least 4. □

Claim 5: We may assume that one of the edges $v_1 \sim v_t$, $v_2 \sim v_t$ or $v_1 \sim v_{t-1}$ is present.

Note that Claim 5 completes the Lemma, as it along with the maximality of P rule out any components in $H \setminus P$.

We say an that P is *set up* if it either v_1, v_2, v_3, v_4 or $v_{t-3}, v_{t-2}, v_{t-1}, v_t$ induce a K_4^- so that v_1 (or v_t) is a vertex of degree 2. We assume that the P meets all the assumed qualifications:

(†) P is of maximum length, has end vertices of degree 2, and all components of $H \setminus P$ have cardinality at least 4 and that subject to these, if possible, P is set up.

Proof of Claim 5: Suppose none of the purported edges are present.

Note that $\deg_H(v_1), \deg_H(v_t) \leq 2$ by assumption and $\deg_H(v_2), \deg_H(v_{t-1}) \leq 4$ as otherwise there would be a doubly chorded cycle. Since $v_1 \not\sim v_{t-1}$ and $v_2 \not\sim v_t$, we may apply our σ_2 condition to see that

$$\deg(v_1) + \deg(v_2) + \deg(v_{t-1}) + \deg(v_t) \geq 2(6k - 1) = 12(k - 1) + 10. \tag{2}$$

Case 1: $\deg_H(v_1) + \deg_H(v_2) + \deg_H(v_{t-1}) + \deg_H(v_t) < 10$.

In this case, combining with (2) and averaging over cycles implies that there are at least 13 edges between v_1, v_2, v_{t-1} and v_t and some $C \in \mathcal{C}$. Note this means that $\deg_C(v_i) \geq 4$ for some v_i and hence $|C| \leq 5$. Without loss of generality, we may assume $\deg_C(v_1)+\deg_C(v_2) \geq \deg_C(v_{t-1})+\deg_C(v_t)$. This means that $\deg_C(v_1)+\deg_C(v_2) \geq 7$ and hence either $\deg_C(v_1) = 4$ or $\deg_C(v_2) = 4$, while the other vertex must have degree at least 3.

Suppose first that $\deg_C(v_1) = 4$. If $\deg_C(v_2) + \deg_C(v_t) > 5$ or $\deg_C(v_2) + \deg_C(v_{t-1}) > 5$ we are done. Indeed, this implies that for some $i \in \{t-1, t\}$, v_2 and v_i have a common neighbor $z \in C$. By the Single Bypass Lemma, we can exchange v_1 for that z, obtaining the desired path structure. One of these always occurs however, as one of $\deg_C(v_t)$ or $\deg_C(v_{t-1})$ is at least

$$\frac{13 - 4 - \deg_C(v_2)}{2} = \frac{9 - \deg_C(v_2)}{2}.$$

Now assume $\deg_C(v_1) = 3$ but $\deg_C(v_2) = 4$. If $|C| = 4$, and one of v_t or v_{t-1} is incident to the vertex in C which is not incident to v_1 may finish by exchanging v_1 for that vertex (which is also incident to v_2.) Thus, designating the vertices on C to be c_1, c_2, c_3, c_4 we may assume that the neighbors of v_1, v_{t-1} and v_t are all $\{c_1, c_2, c_3\}$. If either end is set up, then we are done. Indeed, suppose v_1, v_2, v_3, v_4 is set up: Depending on the configuration of the K_4^-, either $v_1, c_1, v_2, v_3, v_4, v_1$ is a cycle with chords v_1v_2 and v_2v_4 or $v_1, c_1, v_2, v_4, v_3, v_1$ is a cycle with chords v_1v_2 and v_2v_3. Similar cycles exist if the other end is set up. Otherwise, we may exchange v_{t-1} and v_t for c_1 and c_4, with c_4, c_1, v_1, v_2 as the new initial vertices of the path – which now has a set-up end.

In order to see this yields a set-up path matching the conditions (†), we must verify that it can be chosen with end degree 2 while maintaining the K_4^- at the end. Note that it is clear c_4 has degree 2 already, as if it had another adjacency in H, there would clearly be a doubly chorded cycle or longer path. We only need to verify that we can find an ordering that *ends* with a vertex of degree 2 but that still has the initial K_4^-. We run the argument of Claim 1, taking the ordering of the path starting with c_4, c_1, v_1, v_2 and so that the vertex ending the path has its neighbor as close to the start as possible. If the argument does not run smoothly preserving the K_4^-, there would have to be a potential start vertex for the end of the path that sent a chord into c_4, c_1, or v_1. The vertex c_1 here is the only actual option as if v_1 had another neighbor on P it would form a DCC. But if c_1 were incident to a start vertex we would have a structure that satisfies the conclusion of the claim.

The case $|C| = 5$ remains. Without loss of generality, v_2 is incident to c_1, c_2, c_3, c_4 and v_1 is incident to c_5, c_1 and one of c_3 or c_4 – if v_1 is incident to c_4 then the chords of C are c_5c_2 and c_5c_3. In any case, note that there is a DCC including v_1 and $C \setminus c_i$ for any c_i not incident to v_1. That implies that the neighbors of v_t and v_{t-1} must match those of v_1 as otherwise an exchange can be done – v_1 for a common neighbor of both v_2 and one of v_t or v_{t-1}. But then a K_4 is formed from v_t, v_{t-1} along with c_1 and c_5, contradicting the minimality of $\mathcal{C}$.

Case 2: $\deg_H(v_1) + \deg_H(v_2) + \deg_H(v_{t-1}) + \deg_H(v_t) \in \{10, 11\}$.

In this case, either v_1, v_2, v_{t-1}, and v_t cumulatively send 13 edges to *some* cycle $C \in \mathcal{C}$ or they cumulatively send 12 edges into *every* cycle $C \in \mathcal{C}$ except for perhaps one where they send 11 edges. The prior case, where they send at least 13 into one cycle is handled in Case 1. Thus we focus on the second possibility. We note that $v_1 \not\sim v_t$ and applying the σ_2 condition there we find some cycle C so that v_1 and v_t cumulatively send at least 7 edges. Hence we may assume $\deg_C(v_1) = 4$ and

$|C| \leq 5$. The vertices together send at least 11 edges to C, and this is what we handle in this case.

First assume that $\deg_C(v_t) = 4$ as well. Then without loss of generality $\deg_C(v_2) \geq 2$. Thus v_2 and v_t have a common neighbor in C, and exchanging v_1 for the common neighbor finishes this case. Thus we may assume that $\deg_C(v_t) = 3$, whence $\deg_C(v_2) + \deg_C(v_{t-1}) \geq 4$. If $\deg_C(v_2) > 2$ then we may exchange v_1 for a common neighbor of v_2 and v_t. If $|C| = 4$, then this covers the case where $\deg_C(v_2) = 2$ as well.

If $|C| = 4$ and $\deg_C(v_2) = 1$, then $\deg_C(v_{t-1}) \geq 3$. We fail an immediate exchange only if the neighbors of v_{t-1} in C are the same as those of v_t and different than the neighbor of v_2. In this case, either one end is set up or we can set up one of the ends as in Case 2.

If $|C| = 5$ and $\deg_C(v_2) = 1$ then again $\deg_C(v_{t-1}) \geq 3$. Let $N = N_C(v_t)$ and we apply of Lemma 8 with this choice, where $x = v_1$ and $y = v_t$ in this application. Let X_N be as in the statement of Lemma 8.

If v_{t-1} and v_1 have a common neighbor, say z, in X_N, then by replacing C with the DCC $(C - z) \cup \{v_t\}$ gives a Hamiltonian H along with a cycle system of the same size. This is exactly what happens when $2(a)$ or $2(b)$ occur. If N satisfies $2(a)$, the facts that $|X_N \cap N_C(v_1)| \geq 3$, $\deg_C(v_{t-1}) \geq 3$, and $|C| = 5$ immediately yields a common neighbor of v_1 and v_{t-1} in X_N.

If N satisfies $2(b)$ or $2(c)$, it is theoretically possible that v_1 and v_{t-1} have no common neighbor. In this case since $|X_N \cap N_C(x)| \geq 2$ this is enough to determine $N_C(v_{t-1})$. It must be the case that $N_C(v_{t-1}) = N \setminus (N_C(x) \cap X_N)$. In case $2(b)$, this spans an edge of C (this edge may possibly be a chord of C). This means that v_t and v_{t-1} span a K_4, which contradicts the minimality of $\mathcal{C}$.

If N satisfies $2(c)$, note that if v_2 is incident to any neighbor of either v_t or v_{t-1} on C then we may exchange v_1 for that vertex and obtain an H as the lemma hypothesizes. Since $N_C(v_{t-1}) = N \setminus (N_C(x) \cap X_N)$, this means that the neighbor of v_2 is the vertex 'r' of condition $2(c)$. In this case, the conclusion of the Lemma is exactly that we may exchange v_1 and v_2 for the $\{r, s\}$ in C and we obtain a Hamiltonian H.

There is a single remaining case, the exceptional case, when $N_C(v_t) = \{c_1, c_3, c_4\}$. As already discussed, v_{t-1} cannot be incident to any vertex in $X_N \cap N_C(v_1) = \{c_2\}$ and v_2 cannot be incident to any vertex in $N_C(v_{t-1}) \cup N_C(v_t)$. Note that it also cannot be the case that both $\{c_3, c_4\}$ or $\{c_1, c_4\}$ are in $N_C(v_{t-1})$ as otherwise v_t and v_{t-1} will span a K_4. This restricts the case to $N_C(v_{t-1}) = \{c_1, c_3, c_5\}$ and $N_C(v_2) = \{c_2\}$.

Here, if either end of P is set up, we have exhibited two DCCs (similar to the previous case, depending on the end set up and the configuration of the K_4^-). If neither end is setup, replacing $c_1c_2c_3c_4c_5$ with the DCC $v_tc_3c_4c_5v_{t-1}v_t$ leads to a set up path starting with $c_1, c_2, v_1, v_2, \ldots$. This completes the case where $\deg_C(v_2) = 1$, and $\deg_C(v_{t-1}) \geq 3$.

Now suppose $\deg_C(v_2) = 0$, then $\deg_C(v_{t-1}) = 4$. In this case we exchange v_t for a common neighbor of v_1 and v_{t-1} – that this is possible follows from the degrees and Lemma 8.

The final possibility is if $|C| = 5$ and $\deg_C(v_2) = 2$. Then $\deg_C(v_{t-1}) \geq 2$. If v_2 shares a neighbor with v_{t-1} or v_t we are done by the Single Bypass Lemma. Thus v_{t-1}'s neighbors are two of the neighbors of v_{t-1}.

In this case we use the Double Bypass Lemma to exchange v_1 and v_t for two vertices in C – one of which is incident to v_2 and one of which is incident to v_{t-1}. This yields a cycle of the same length as P and completes Case 2.

Case 3: $\deg_H(v_1) + \deg_H(v_2) + \deg_H(v_{t-1}) + \deg_H(v_t) = 12$.

This is the pessimal case: in this case, we must have $\deg_H(v_2) = \deg_H(v_{t-1}) = 4$. Note that since all components outside of H have cardinality at least 3, v_2 and v_{t-1} cannot be incident to them so all neighbors of v_2 and v_{t-1} in H are on P. Let v_s be the highest indexed neighbor for v_2 and v_t be the lowest indexed neighbor of v_{t-1}. It is easy now to observe that if $t < s$, there is a DCC, so $s \leq t$. (This DCC depends somewhat on how the neighbors of v_2 and v_t interlace, but all are formed according to the following rule: starting at v_2 follow P until the highest indexed neighbor of v_{t-1} with index strictly less than s. Follow this edge to v_{t-1}, then proceed back to the smallest indexed neighbor of v_2 which isn't already in use then follow the edge back to v_2.)

By Claim 2, applied to both ends of the path, there are two additional vertices of degree 2, say v_q and v_r with $q < s$ and $r > t$. Note that $\{v_1, v_q\}$, $\{v_r, v_t\}$ and $\{v_2, v_{t-1}\}$ are all disjoint pairs of vertices and applying our σ_2 condition we see that these vertices have total degree at least

$$3 \cdot (6k - 1) \geq 18(k - 1) + 15.$$

Furthermore, their degrees in H are 16. Therefore, either these 6 vertices send a total of 19 edges to some $C \in \mathcal{C}$ or they send at least 17 edges to every cycle in $C \in \mathcal{C}$.

Suppose the latter holds. Applying the σ_2 condition to just v_1 and v_t there is some $C \in \mathcal{C}$ where $\deg_C(v_1) + \deg_C(v_2) \geq 7$. To this cycle,

$$\deg_C(v_1) + \deg_C(v_2) + \deg_C(v_{t-1}) + \deg_C(v_t) < 11$$

or we are in the situation dealt with in Case 2. Thus $\deg_C(v_q) + \deg_C(v_r) \geq 7$. Without loss of generality $\deg_C(v_1) = 4$. Then $\deg_C(v_q) + \deg_C(v_t) \geq 6$, so v_q and v_t have a shared neighbor. Furthermore taking the portion of the path between v_q and v_t along with the shared neighbor is a DCC, with two chords incident to v_{t-1} lie along this path. v_1 incident to the rest of C is also a DCC by the Single Bypass Lemma.

Otherwise, suppose the former holds and consider the cycle C so that these 6 vertices send at least 19 edges to C. If

$$\deg_C(v_1) + \deg_C(v_2) + \deg_C(v_{t-1}) + \deg_C(v_t) \geq 13,$$

the Case 1 argument applies so $\deg_C(v_q) + \deg_C(v_r) \geq 7$.

Note that if some $v \in \{v_1, v_2, v_{t-1}, v_t\}$ has $\deg_C(v) = 4$, then a very similar argument to the case we just dealt with – if $v \in \{v_1, v_2\}$ one of v_{t-1} or v_t will have a common neighbor with v_q.

On the other hand, $\deg_C(v_q) + \deg_C(v_r) \leq 8$ so at least one of $\{v_1, v_2\}$ or $\{v_{t-1}, v_t\}$ – say, $\{v_1, v_2\}$ – have total degree 6 into the cycle. If this is the case, v_q and one of v_{t-1} or v_t have a common neighbor, x. As before x along with P between v_1 and v_{t-1} (or v_t) create a DCC. Furthermore, $\{v_1, v_2\} \cup (C \setminus x)$ yields a DCC as the minimum degree of v_1 and v_2 into $(C \setminus x)$ is 2 and $(2, 2) \hookrightarrow \{P_3, P_4\}$ yields a DCC by Lemma 14. This is all possibilities, completing Case 3. $\square$

This completes the proof of the lemma. $\square$

The final lemma will complete the proof of Theorem 3 by using the structure of H that we have established to k disjoint doubly chorded cycles and hence contradicting the fact that we assumed we had a counterexample.

Lemma 13. *There exists a cycle $C \in \mathcal{C}$ so that $H \cup C$ contains two vertex disjoint DCCs.*

Proof. By Lemma 12, H can be assumed to be either a Hamiltonian cycle or a Hamiltonian path with $v_2 \sim v_t$.

In the first case, there can be at most one chord, so H contains at most $|H| + 1$ edges. In the second case, there can be at most one chord in the cycle, and v_1 can be incident to at most 3 vertices on the cycle. But if v_1 has 3 neighbors in the cycle and there also exists a chord in the cycle, a DCC is easily found – so there are always at most $|H| + 2$ edges within H. Note that our observations also imply $|H| \geq 7$. Indeed, it is easy to see that there are two non-adjacent vertices of degree 2 in H. By our degree condition and averaging, we see that there is a vertex of degree 4 into some cycle – this shows that $\mathcal{C}$ contains a 5-cycle and hence $|H| \geq 7$ by the Corollary to Lemma 6.

Now we consider the total degrees of vertices in H. First note that if two vertices in H have degree less than $3k$, they must be adjacent by the σ_2 condition. Since H does not contain a K_4, there are at most 3 such vertices.

Suppose first there are 3 such vertices, and their degrees are $3k - t_1$, $3k - t_2$ and $3k - t_3$, with $t_1 \geq t_2 \geq t_3$. All other vertices have degree at least $3k + (t_3 - 1)$ by the fact that H contains no K_4. All but at most two have degree at least $3k + (t_1 - 1)$ as H has maximum (interior) degree 4. Furthermore, all but at most one has degree at least $3k + (t_2 - 1)$. We can see this as follows: if two vertices had degree less than $3k + (t_2 - 1)$, then H would contain a $K_{3,2}$ with an edge connecting the vertices in the partite set of size 2. This cannot happen: H is connected, and if it contains two vertices of interior degree 4 then all other vertices of H have interior degree 2. But since $|H| > 5$ and H is connected, the $K_{3,2}$ we have cannot occur as part of H – no other edge can be connected to it without violating one of our conditions.

Thus the total degree in H is at least

$$\begin{aligned}(3k - t_1) + (3k - t_2) + (3k - t_3) + [3k + (t_2 - 1)] + [3k + (t_3 - 1)] + (|H| - 5)(3k + (t_1 - 1))\\ = 3k|H| - 2 + (|H| - 5)(t_1 - 1) - t_1\\ \geq 3k|H| - 3 = 3(k-1)|H| + 3|H| - 3.\end{aligned}$$

The same count works if there are fewer such vertices (essentially, this follows by setting $t_3 = 0$ or both $t_2 = 0$ and $t_3 = 0$). Indeed, in these cases a slightly stronger count applies.

Now, if $|H| = 7$, the degree sum within the cycle is at most $2(|H| + 2) = 2|H| + 4$, and hence there are at least

$$3(k-1)|H| + 3|H| - 3 - [2(|H| + 2)] = 3(k-1)|H| + |H| - 7 = 3(k-1)|H|$$

edges to the cycles, and hence either 22 edges to some cycle or exactly 21 edges to *every* cycle.

Suppose there is a Hamiltonian path in H starting at a vertex sending at least 4 edges to some cycle C, and there are at least 21 edges between C and H. Then $|C| \leq 5$ and there is some vertex in C with at least $\lceil \frac{(21-4)}{|C|} \rceil \geq 4$ other vertices in H. This gives two doubly chorded cycles – one involving the vertex in the cycle and the remainder of H and the other involving the start vertex and the remainder of the cycle.

Note that v_1 starts a Hamiltonian path, as does any successor or predecessor of neighbor of v_1. Thus v_t and v_3 also start a Hamiltonian path. If they are connected, note that v_{t-1} and v_4 also start a Hamiltonian path. This, plus the fact that there's at most one chord in the cycle, is enough to conclude that there are two nonadjacent vertices starting a Hamiltonian path with degree at most 2 in H. By the σ_2 condition, these vertices send at least 7 edges to some cycle, and hence there *is* a vertex starting a Hamiltonian path with 4 edges to a single cycle. If H sent at least 21 edges to this path we are done, so we may assume it sends fewer. Thus there is some different cycle, where H has at least 22 adjacencies. But then there is some vertex in H sending at least 4 vertices to the cycle. This vertex cannot start a Hamiltonian path and – in particular – this cycle is a different cycle than the one just considered. Thus there are two vertices $h_1, h_2 \in H$ and two cycles $C_1, C_2 \in \mathcal{C}$ so that h_1 has four neighbors in C_1 and h_2 has four neighbors in C_2. This means that $|C_1| \leq 5$ and $|C_2| \leq 5$, so $|H| \geq 8$ by the Corollary to Lemma 6.

If $|H| \geq 8$, then the degree from H to the cycles is at least $3(k-1)|H| + 1$, and hence there are at least $3|H| + 1$ edges from H to some particular $C \in \mathcal{C}$. Thus there exists some vertex $v \in H$ with at least 4 neighbors in C, and so $|C| \geq 4$. Likewise there is some vertex $c \in C$ with at least $\lceil \frac{(3|H|+1)-4}{|C|} \rceil \geq 5$ neighbors in H other than v. If v starts a Hamiltonian path this is clearly enough by the above. Otherwise, v starts a path of length $|H| - 1$, not including v_1. But then this vertex in the $c \in C$ has 4 neighbors other than v on this path, and this gives two DCCs as desired. $\square$

This immediately implies Theorem 3. $\square$

3 Proof of Lemma 1

Our next goal is to finally prove Lemma 1, which is fairly involved.

Recall that we have two cycles C and C' with $\ell = |C| \geq |C'|$, $\ell \geq 7$ and $e(C, C') \geq 3|C| + 1$, and our objective is to find two doubly chorded cycles whose union is smaller than $|C| + |C'|$.

The key to the proof is simply observing that many collections of edges between two paths yield (in many cases proper) doubly chorded cycles. We will require a library of cases in order to complete the proof of Lemma 1. Since these cases are plentiful, it is helpful to introduce some notation.

We use the notation $(d_1, d_2, \ldots, d_i) \hookrightarrow P_k$ to indicate a path with consecutive vertices of degree $(d_1, d_2, \ldots, d_i)$ into a P_k. We use the symbol $\star$ to indicate a *non-empty* sequence of arbitrary degrees. We use the notation $P_k \hookrightarrow_d P_j$ to indicate an arbitrary d edges from a P_k to P_j. If no length is indicated then P represents a path of arbitrary length. In the following lemma, we indicate various degree sequences into short paths which always admit doubly chorded cycles and, in many cases proper doubly chorded cycles.

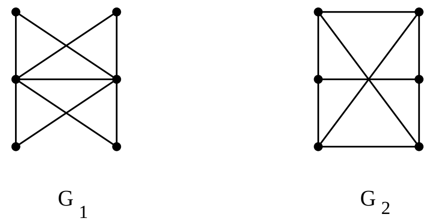

Figure 1: Two exceptions to the Lemma 14.

Lemma 14. *1. The following ensure a doubly chorded cycle:* $\{(2,2),(2,\star,2)\} \hookrightarrow P_3$*,* $(2,2) \hookrightarrow P_4$*,* $\{(3,2),(3,\star,2)\} \hookrightarrow P$*,* $P_2 \hookrightarrow_5 P$*,* $(3,1,1) \hookrightarrow P$*,* $\{(2,1,2),(2,1,\star,2)\} \hookrightarrow P$*,* $P_3 \hookrightarrow_6 P$*,* $(2,2,1,1) \hookrightarrow P$ *and* $\{P_4, P_5\} \hookrightarrow_6 P_5$*.*

2. $P_3 \hookrightarrow_5 P_3$ *yields a doubly chorded cycle except for the degree sequence* $(1,3,1)$ *and a proper doubly chorded cycle except in the case* $(2,1,2)$*, as pictured in Figure 1.*

3. *The following ensure a proper doubly chorded cycle:* $(3,\star,2) \hookrightarrow \{P_3, P_4\}$*,* $P_2 \hookrightarrow_5 P_4$*,* $P_2 \hookrightarrow_6 P$*,* $\{(3,1,1),(3,\star,1,1)\} \hookrightarrow P_3$*,* $(3,1,1) \hookrightarrow P_4$*,* $(2,1,2) \hookrightarrow P_4$*,* $(2,1,1,2) \hookrightarrow P_4$*,* $P_3 \hookrightarrow_6 P_5$*,* $\{P_3\} \hookrightarrow_6 \{P_3, P_4\}$*,* $\{P_3, P_4, P_5\} \hookrightarrow_7 P$*,* $P_6 \hookrightarrow_8 P$*,* $P_7 \hookrightarrow_9 P$*,* $P \hookrightarrow_{10} P'$*,* $(1,2,2,1) \hookrightarrow P_5$ *and* $(2,2,1,1) \hookrightarrow P_5$*.*

4. $P_3 \hookrightarrow_5 P_4$ *yields a proper doubly chorded cycle except for* $(1,3,1)$*.*

5. $P_4 \hookrightarrow_5 P_4$ *yields a doubly chorded cycle except for the degree sequences* $(1,3,1,0)$*,* $(1,3,0,1)$*,* $(1,2,1,1)$*, or an inside* $(2,0,2,1)$ *(see Figure 2).*

6. $P_4 \hookrightarrow_6 P_4$ *yields a doubly chorded cycle, which is necessarily proper in all cases except for the degree sequence* $(2,1,1,2)$*.*

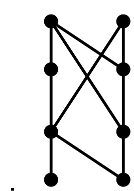

Figure 2: The inside degree configuration.

We defer the proof of Lemma 14, which involves fairly extensive case analysis, to the end of the section.

To restrict cases later, it is useful to bound the maximum degree between C and C'.

Lemma 15. *The maximum degree of any vertex in* C *or* C' *to the other is at most* 5*.*

Proof. Let $m \geq 6$ denote the maximum degree of a vertex in one of C, C' to the other, and let v have that degree. We don't know which of C, C' that v lies on, so we say that v lies on C^*, with

m edges to C^{**}. Then there are at least $3\ell + 1 - m \geq 2\ell + 1$ edges between C^{**} and $C^* \setminus v$. By averaging, some adjacent pair of vertices in C^{**} sends at least 5 edges to $C^* \setminus v$. If $m \geq 7$, we are done, as this pair creates a doubly chorded cycle with $C^* \setminus v$, by Lemma 14 part (1), and v must be adjacent to at least 5 other vertices of C^{**}, so it is easy to ensure a proper doubly chorded cycle. If $m = 6$ we are only in trouble if our adjacent pair of vertices and both of their neighbors on the cycle are adjacent to v. There are at most 3 pairs of this type, which is achieved only if 6 neighbors of v are consecutive. These pairs may have at most 5 edges to C^* as otherwise there is a proper doubly chorded cycle containing them and $C^* \setminus v$ (this is the third part of Lemma 14). Each of the other pairs have at most 4 edges. Thus we have an upper bound on the number of edges between C^{**} and $C^* \setminus v$ of $\frac{1}{2}(5 \times 3 + 4 \times (\ell - 3)) = 2\ell + \frac{3}{2}$. On the other hand, we have a lower bound of $3\ell + 1 - 6 = 3\ell - 5$. Since $\ell \geq 7$, and $15.5 < 16$, this is a contradiction proving the lemma; unless $|C^{**}| = 6$. In this case if there is an edge of C^{**} with at least 6 edges to $C^* \setminus v$, the same argument works. Thus, each edge of C^{**} must send at most 5 edges to $C^* \setminus v$. But, there are at least $3 \times 7 + 1 - 6 = 16$ edges to C^{**}, and so one of the edges must have at least 6 edges to $C^* \setminus v$. $\square$

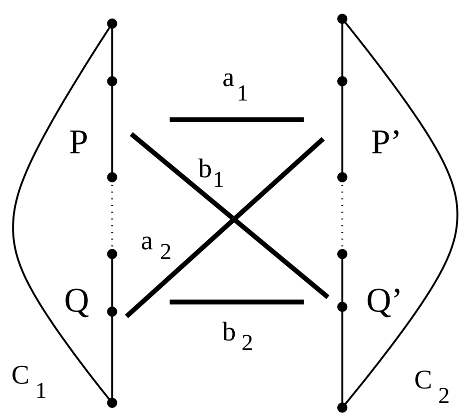

Figure 3: Path partitions of the two cycles with edges sent to each part.

We frequently use the following simple fact.

Lemma 16. *Let $C \in \mathcal{C}$ denote a cycle of length ℓ with maximum degree at most* 5 *and with* $\deg(C) = \sum_{v \in C} \deg(v)$. *Then C can be partitioned into two paths P and Q (with degree of any path defined in a manner similar to that of deg(C)) with $|P| = \lceil \frac{\ell}{2} \rceil$ and $|Q| = \lfloor \frac{\ell}{2} \rfloor$, and* $|\deg(P) - \deg(Q)| \leq 5$.

Proof. This is a standard intermediate value theorem proof. Let $m = \lceil \frac{\ell}{2} \rceil$, and order the vertices of C, say $c_1, c_2, \ldots, c_\ell$ (with indices mod ℓ). Let $P^i = \{c_i, \ldots, c_{i+m-1}\}$ and $Q^i = C \setminus P^i$. Note that $\deg(P^i) + \deg(Q^i) = \deg(C)$. Consider $x_i = \deg(P^i) - \deg(Q^i)$. In the ℓ even case, $x_1 = -x_{m+1}$. Since $|x_i - x_{i+1}| \leq 10$ by the maximum degree condition, some x_i has $|x_i| \leq 5$ and this is our desired partitioning. For ℓ odd, we note that $|x_1 + x_{m+1}| = 2deg(C) \leq 10$. In particular, if $|x_1| > 5$, then x_{m+1} has an opposite sign and again we find our desired partition. Otherwise, $|x_1| \leq 5$ and already P^1, Q^1 is our desired partition. $\square$

At this point we partition C into two paths with degrees (r, s) to C', respectively, and C'

into two paths with degrees (t, u) to the paths of C, respectively (see Figure 3). We choose the balanced partition guaranteed by Lemma 16, except in the cases when the cycle has length $\ell = 7, 8$. If $\ell = 7$ or $\ell = 8$, we choose the decomposition on the cycle that is most balanced in terms of $|\deg(P) - \deg(Q)|$ (which may not be the one guaranteed by the pairs) and we assume that P is as short as possible. We have the caveat that, if there exists a $P_2 \cup P_{\ell-2}$ decomposition so that $|\deg(P_2) - \deg(P_{\ell-2})| \leq 5$, we take this even if there is a more balanced partitioning. (Note that we only do this when $\ell = 7$, for $(9, 13)$ and $\ell = 8$ for $(10, 15)$.)

Claim 1. *Assume $\ell = 7$. In the case of $(9, 13)$ we may assume that $|P| = 2$. In the case of $(10, 12)$, if $|P| = 4$, then we may assume that the degree of one of the end points of P is at most 1.*

Proof. We begin by noting that if $(9, 13)$ is the most balanced setup and the order of the first path is greater than 2, then only degrees 4 and 5 (or 0) may be present (or a more balanced partition is easily obtained). Indeed, suppose there is a positive degree less than 4. Start at this vertex and add the degrees of adjacent vertices, clockwise around the cycle until the degree sum is larger than 9. Then the degree sum must be at least 13, or it would be a more balanced partition. If it is exactly 13, moving the vertex of degree less than 4 to the other side yields a more balanced partition. Otherwise, using Lemma 15, the sum is exactly 14 and this implies that the initial degree is 1 and the last degree is 5. That is, we have $(1, d_1, \ldots, d_i, 5)$ with $\sum_{t=1}^{i} d_t = 8$. We may further assume that $i = 2$: if $i > 2$, then repeating in a counter-clockwise direction around the cycle yields an identical situation where $i = 2$ – here we strongly use that $\ell = 7$. Now, $d_1 + d_2 = 8$. If $d_1 \in \{4, 5\}$, then there is the desired 9 segment of order two or a more balanced partition. Hence $d_1 = 3$, and $d_2 = 5$. Now consider the three vertices between the vertex of degree 1 and the vertex of degree 5 when transversing the cycle in the other direction. These vertices have degree sum 5, but also it is easy check that none of them may have non-zero degree without violating the balanced condition. Indeed, consider the vertex closest to the degree 1 vertex with non-zero degree. If this has degree in $1, 2$ or 3, there is a segment with degree sum $10, 11$, or 12 respectively starting with the vertex of degree $d_1 = 3$. If this has degree 4 or 5, then there is a segment of degree 10 or 11 starting with the vertex $d_2 = 5$.

Thus the remaining possibility is that all positive degrees are at least 4. Furthermore, each degree 5 vertex must be surrounded by degree four vertices, not a priori adjacent. However, this says that the degrees of the vertices in the cycle are $5, 4, 5, 4, 4$, possibly with some degree zero vertices in between. However, since the order of the cycle is at most 7, a degree 5 and a degree 4 vertex must be adjacent, given $|P| = 2$.

Next we consider the case $|P| = 4$. Clearly, then, $|Q| = 3$, because the maximum degree is 5. Furthermore, the endpoints of P have positive degree, as otherwise they could be added to Q. The only degree sequence options for Q are $(4, 4, 4)$, $(5, 3, 4)$ and $(5, 2, 5)$; these are the only (sorted) options because an adjacent $(5, 4)$ would give the preferred $(9, 13)$ partition and adjacent $(5, 5)$ would give a P of order 2. In the first case, it is clear that the only neighbors of the 4 are either another 4 or 1, otherwise a P of order 3 would exist, a $(5, 4)$ would exist or a more balanced $(11, 11)$ partition would exist. However, if a degree 4 vertex is adjacent, one may shift over and repeat the argument and as not every vertex can have degree 4, one eventually gets a 1 as desired. The other cases are similar. In the $(5, 3, 4)$ case it is clear that the only positive number allowable incident to the 5 is 1 as others yield a more preferred partition. In the $(5, 2, 5)$ case only 1 or 2 is permissible next to the 5's. If 2 is present in both fives, there are two remaining spots which must add degree 6. The options for these are $(5,1)$ (which would yield the P beginning with 1), $(4,2)$ or $(3,3)$ (both of which yield more preferred partitions.) $\square$

Claim 2. *Assume $\ell = 8$. In the case of a P, Q decomposition of type $(10,15)$ we may assume that $|P| = 2$.*

Proof. The proof is similar to the $(9,13)$ case, but easier. We claim that if $(10,15)$ is the most balanced decomposition then every vertex with positive degree must be of degree 5. Indeed, if there were a vertex of degree less than 5, the strategy above yields a more balanced decomposition. But then there are five vertices of degree 5 on a cycle of length 8 so two must be adjacent. This gives the desired $(10,15)$ decomposition. $\square$

Finally, we are ready to prove Lemma 1.

Proof of Lemma 1. Recall, we have two cycles with $\ell = |C| \geq |C'|$ and $\ell \geq 7$. We partition C and C' into paths with degrees (r,s) and (t,u), balanced as above. We next note that each of the pairs for the two cycles can further be partitioned into two parts based on the number of edges sent to each subpath of the other cycle. We consider these pairs as (a_1,b_1) and (a_2,b_2) as shown in Figure 3. Thus, for C we have $r = a_1 + b_1$ and $s = a_2 + b_2$. While for cycle C' we have $t = a_1 + a_2$ and $u = b_1 + b_2$.

At this point, there are many cases based on ℓ, and the partitions (a_1,b_1) and (a_2,b_2). Our primary tool to handled these cases is Lemma 14. Recall that in all cases, we wish to show that the graph induced by $C \cup C'$ contains a pair of doubly chorded cycles with at least one of them proper. We will proceed based on ℓ.

Case 1: Suppose $\ell = 7$.

Possible pairs are reflected in the interior parts of the tables. The tables show all the cases based upon the path partitions for C and C'. Note that the first column lists the (r,s) pair for C versus all possible (t,u) pairs for C'. The interior pairs are the (a_1,a_2) and (b_1,b_2) splits based upon the corresponding (r,s) and (t,u) values for each cycle.

The remainder of the proof is to verify that in each case, one of the splits (a_1,b_2) and (b_1,a_2) allows us to find two disjoint doubly chorded cycles that, in total, use fewer vertices than C and C' together.

In most cases, the results of Lemma 14 make this transparent. In the cases where $(9,13)$ is against other pairs the fact that $P_2 \hookrightarrow_6 P$ gives a proper doubly chorded cycle, and $\{P_3, P_4, P_5\} \hookrightarrow_7 P$ gives proper doubly chorded cycles makes finding a proper doubly chorded cycle easy. In the tables, we indicate these cases by underlining the choices that most easily produce the desired result, and bolding the segment which guarantees a proper doubly chorded cycle. We will not include all the details here.

There remains a few more difficult cases where we must argue a bit more.

In the $(10,12)$ vs. $(10,12)$ case where we have $(4,6)$ vs. $(6,6)$, note that we have two cases, $(4,\underline{6})$ vs. $(\underline{\mathbf{6}},6)$ and $(4,\underline{\mathbf{6}})$ vs. $(\underline{6},6)$ – which case occurs depends on which of the segments involved

Table 4: C pair $(9, 13)$ versus all (t, u) pairs for C'.

(9,13)			
9	$(\underline{4},5)$	$(\le 3, \ge \underline{6})$	
13	$(5,\underline{\mathbf{8}})$	$(\ge \underline{\mathbf{6}}, \le 7)$	
10	$(\ge \underline{5}, \le 5)$	(4,6)	$(\le 3, \ge \underline{\mathbf{7}})$
12	$(\le 4, \ge \underline{\mathbf{8}})$	(5,7)	$(\ge \underline{\mathbf{6}}, \le 6)$
11	$(\ge \underline{5}, \le 6)$	$(4, \underline{\mathbf{7}})$	$(\le 3, \ge \underline{\mathbf{8}})$
11	$(\le 4, \ge \underline{\mathbf{7}})$	$(\underline{5}, 6)$	$(\ge \underline{\mathbf{6}}, \le 5)$

Table 5: C pair $(10, 12)$ versus all (t, u) pairs for C'.

(10,12)				
11	$(\le 4, \ge \underline{\mathbf{7}})$	(5,6)	$(\ge \underline{6}, \le 5)$	
11	$(\ge \underline{6}, \le 5)$	(5,6)	$(\le 4, \ge \underline{\mathbf{7}})$	
10	$(\le 3, \ge \underline{\mathbf{7}})$	(4,6)	(5,5)	$(\ge \underline{6}, \le 4)$
12	$(\ge \underline{\mathbf{7}}, \le 5)$	(6,6)	(5,7)	$(\le 4, \ge \underline{\mathbf{8}})$

is shorter, but since the total length is 7, at least one of the segments of the first cycle must have length at most 3. The fact that $\{P_2, P_3\} \hookrightarrow_6 \{P_3, P_4, P_5\}$ yields a proper DCC is enough to show that the system is proper. Similar reasoning reveals that in the $(11, 11)$ vs $(11, 11)$ case where we have $(6, 5)$ vs $(5, 6)$ both '$6'$s guarantee a DCC, and since one of the segments must have length 3 one guarantees a proper DCC.

In the $(9, 13)$ vs. $(10, 12)$ case, where we have $(4, 6)$ versus $(5, 7)$, if the 10 segment has order 2, so that 4 edges guarantees a doubly chorded cycle, $(\underline{4}, 6)$ and $(5, \underline{\mathbf{7}})$ gives the desired cycles. Otherwise, we need that the 6 edges from the 10 segment to the 13 segment of the other cycle gives a proper doubly chorded cycle. If the 10 segment has order 3, then $P_3 \hookrightarrow_6 P_5$ guarantees a proper doubly chorded cycle by Lemma 14. In the case where the 10 segment has order 4, we have that one end has degree at most 1 into the P_5, by Claim 1. If it has degree zero, then we have $P_3 \hookrightarrow_6 P_5$, yielding the proper cycle. If it has degree 1, excluding that vertex, we have $P_3 \hookrightarrow_5 P_5$, and we are done unless there is no doubly chorded cycle (as we have already excluded a vertex). Note that $(3, 1, 1) \hookrightarrow P$ yields a doubly chorded cycle by Lemma 14, as does $(2, 1, 2) \hookrightarrow P$, $(3, 2) \hookrightarrow P$ and $(3, \star, 2) \hookrightarrow P$. If the maximum degree is 3, we are left with $(1, 3, 1) \hookrightarrow P$, which after adding back the vertex of degree one to the path, gives a $(3, 1, 1) \hookrightarrow P$ and hence the desired proper doubly chorded cycle. In the cases with maximum degree 2, except for $(2, 1, 2)$, the maximum degree two cases not covered, after adding back the degree 1 vertex, are $(2, 2, 1, 1) \hookrightarrow P_5$ and $(1, 2, 2, 1) \hookrightarrow P_5$, both of which are shown by Lemma 14 to give a proper doubly chorded cycle, finishing the case.

By far the most difficult is the $(10, 12)$ vs. $(11, 11)$ case where we have $(5, 6)$ vs. $(5, 6)$. We know by 16 that the 10 segment has order 2, 3 or 4. If the 10 segment has order 2, since then $P_2 \hookrightarrow_5 \{P_3, P_4\}$ yields a doubly chorded cycle, and $P_3 \hookrightarrow_6 P_5$ yields a proper doubly chorded cycle by Lemma 14, we find our desired pair of doubly chorded cycles.

If the 10 segment has order 3, by Lemma 14, $P_3 \hookrightarrow_5 \{P_3, P_4\}$ yields a doubly chorded cycle, except in the case where the P_3 has degree sequence $(1, 3, 1)$, and $P_4 \hookrightarrow_6 \{P_3, P_4\}$ yields a proper

Table 6: C pair $(11, 11)$ versus all (t, u) pairs for C'.

(11,11)		
11	$(\leq 4, \geq \underline{\mathbf{7}})$	(6, 5)
11	$(\geq \underline{\mathbf{7}}, \leq 4)$	(5, 6)

doubly chorded cycle except in the case $(2, 1, 1, 2) \hookrightarrow P_4$. Since the 10 segment of C cannot have degrees $(1, 3, 1)$ into both segments of C', we find our desired pair of doubly chorded cycles except in a singular case – that in which both P and Q are divided into segments of length 3 and 4. If the segments of length 3 are connected with the degree sequence $(1, 3, 1)$ as we can find the desired DCCs between the segments of length 3 and 4 on opposite sides.

The bad case here is where both length 3 segments are connected with degree sequence $(2, 1, 2)$ and both length 4 segments are connected with degree sequence $(2, 1, 1, 2)$. In this case the P_3 segment of the $(10, 12)$ path has degrees $(3, 4, 3)$. If either of the ends of the P_4 on the $(10, 12)$ cycle had degree 3 or 4 wed be done. This gives either a more balanced partition of the top or a P_3 with a $(4, 3, 3)$ degree configuration that doesnt support this bad case. If either of the ends of the P_4 on the $(10, 12)$ cycle had degree 4, we'd also be done. Since this degree 5 vertex sends 2 edges into the P_4 on the $(11, 11)$ side, it must send 3 into the P_3 on the $(11, 11)$ side. Combining it with it's neighbor on the P_3 gives $(3, 2) \hookrightarrow P_3$, giving a DCC and the remainder gives a $P_5 \hookrightarrow_7 P_4$ which guarantees a proper DCC. Thus both ends of the P_4 on the $(10, 12)$ have degree 2 (both into the P_4 on the other side, and in sum.) Then combining one of those vertices with the adjacent two vertices in the P_3 give a degree sequence of $(2, 1, 3) \hookrightarrow P_4$, giving a proper DCC. The middle two vertices of the P_4 have a total degree sum of $12 - 2 \cdot 2 = 8$ and only send two edges to the P_4 which gives a $P_2 \hookrightarrow_6 P_3$ and hence also a proper DCC. This completes the case of this $(10, 12)$ vs $(11, 11)$ split where the 10 segment has order 3.

The case where the 10 segment of C has order 4 also requires some additional argument. Let us denote by P the segment of order 4 in C and by Q_1 and Q_2 the two segments in our partition of C'. If we can find a DCC between P and either Q_1 or Q_2 then we are done, as the remaining $P_3 \hookrightarrow_6 \{Q_1, Q_2\}$ will provide the desired proper doubly chorded cycle. Unfortunately we are not guaranteed the existence of a DCC between P and either Q_1 or Q_2. Instead, we will argue as follows: in the event that no DCC lies between P and either Q_1 or Q_2, we will use Lemma 14 to assert structural information about the edge set. We will then use this information to find a new partition Q_1' and Q_2' of C' so that there exist DCCs both between P and Q_1' and between P and Q_2'. Then as the remaining P_3 of C must send at least 6 edges to one of Q_1' and Q_2', we will be done.

We proceed as follows. Consider the two partitions Q_1 and Q_2, both of which send 5 edges into P. If both $|Q_1| = |Q_2| = 3$ and neither yields a DCC with P, then by Lemma 14 part (4), the degree sequence of both is $(1, 3, 1)$ so the degree sequence of the cycle C' into P is $(1, 3, 1, 1, 3, 1)$. With such a choice, we repartition according to $(1, \mathbf{3}, \mathbf{1}, \mathbf{1}, 3, 1)$ (taking the bolded vertices to be Q_1' and the non-bolded to be Q_2'). Both Q_1' and Q_2' have degrees $(3, 1, 1)$ into P and thus yield DCCs.

Thus we assume $|Q_1| = 4$ and $|Q_2| = 3$. If neither yields a DCC with P, by Lemma 14 part (4), the degree sequence of Q_2 is $(1, 3, 1)$ and the degree sequence of Q_1 is one of $(1, 3, 1, 0)$, $(1, 3, 0, 1)$, or $(1, 2, 1, 1)$ or an 'inside' $(2, 0, 2, 1)$. (Technically, this is up to symmetry, but we orient the cycle so that this is the case and since $(1,3,1)$ is symmetric we can do this without loss of generality). Below we list all possible cases with our choice for Q_1' in bold and Q_2' in non-bold.:

(1, **3**, **1**, **0**, **1**, 3, 1)	(1, **3**, **0**, **1**, **1**, 3, 1)	(1,**2**,**1**,**1**,**1**,3,1)	(**2**, **0**, **2**, 1 ,1, 3, **1**)

This yields Q_1' and Q_2' both of which are guaranteed to yield DCCs with P, completing the argument. One final word of explanation is warranted: Quite crucially, we note that if the original degree sequence of Q_1 involved an 'inside' $(2,0,2,1)'$ the $(\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{1})$ of Q_1' is *not* 'inside' and this explains why, in this case, Q_1' yields a DCC with P while the original Q_1 did not.

In the $(10,12)$ vs. $(10,12)$ case where we have $(5,5)$ vs. $(5,7)$, we are done if the 5 edges between the two 10 parts create a doubly chorded cycle, as the 7 edges between the other parts make a proper doubly chorded cycle. If it does not, then the arguments are similar to $(5,6)$ vs. $(5,6)$.

This completes the proof when $\ell = 7$. Fortunately, for larger ℓ the number of edges increases so the proofs are mostly easier.

Case 2**:** Suppose $\ell = 8$.

Recall that we have a $(10,15)$ decomposition where the 10 edge path has order 2 by Claim 2. We must verify the cases, but they are easy using Lemma 14. Recall that $P_2 \hookrightarrow_5 P$ and $\{P_4, P_5\} \hookrightarrow_6 P_5$ yield doubly chorded cycles, and $P_2 \hookrightarrow_6 P$, $P_3 \hookrightarrow_6 P_5$, $\{P_3, P_4\} \hookrightarrow_6 \{P_3, P_4\}$ and $\{P_3, P_4, P_5\} \hookrightarrow_7 P$, yield proper doubly chorded cycles. These covers nearly all of the possible partitions; again we give tables showing how the partitions are covered.

Table 7: C pair $(10,15)$ versus all (t,u) pairs for C'.

(10,15)		
10	$(\leq 3, \geq \underline{\mathbf{7}})$	(4,6)
15	$(\geq \underline{\mathbf{7}}, \leq 8)$	(6,4)
11	$(\geq \underline{5}, \leq 6)$	$(\leq 4, \geq \underline{\mathbf{7}})$
14	$(\leq 5, \geq \underline{\mathbf{9}})$	$(\geq \underline{\mathbf{6}}, \leq 8)$
12	$(\geq \underline{5}, \leq 7)$	$(\leq 4, \geq \underline{\mathbf{8}})$
13	$(\leq 5, \geq \underline{\mathbf{8}})$	$(\geq \underline{\mathbf{6}}, \leq 7)$

Table 8: C pairs $(11,14)$ and $(12,13)$ versus all (t,u) pairs for C'.

(11,14)				(12, 13)		
11	$(\leq 4, \geq \underline{\mathbf{7}})$	(5,6)	$(\geq \underline{6}, \leq 5)$	12	$(\leq 5, \geq \underline{\mathbf{7}})$	$(\geq \underline{6}, \leq 6)$
14	$(\geq \underline{6}, \leq 8)$	(6,8)	$(\leq 5, \geq \underline{\mathbf{9}})$	13	$(\geq \underline{\mathbf{7}}, \leq 6)$	$(\leq 6, \geq \underline{\mathbf{7}})$
12	$(\leq 5, \geq \underline{\mathbf{7}})$	$(\geq \underline{6}, \leq 6)$				
13	$(\geq \underline{6}, \leq 7)$	$(\leq 5, \geq \underline{\mathbf{8}})$				

In the $(11,14)$ vs. $(11,14)$ case where we have $(5,6)$ vs. $(6,8)$, the argument is somewhat more complicated as well. If either of the segments supporting 6 is of length 3 (note both segments are of length at least 3) then we may use the fact that $P_3 \hookrightarrow_6 \{P_3, P_4, P_5\}$ yields a proper doubly chorded cycle and the fact that $\{P_4, P_5\} \hookrightarrow_6 \{P_3, P_4, P_5\}$ yields a DCC to get a proper system.

A more complicated scenario occurs when both segments of both cycles have length 4, as $P_4 \hookrightarrow_6 P_4$ guarantees only a DCC, but not a proper one in the case of $(2, 1, 1, 2) \hookrightarrow P_4$. Here, suppose the first cycle is $x_1, \ldots, x_8$ and the second cycle is $y_1, \ldots, y_8$.

We assume $x_1, \ldots, x_4$ has a total of 11 edges incident to it and $x_5, \ldots, x_8$ has 14. Likewise, assume $y_1, \ldots, y_4$ has 11 edges incident to it, and $y_5, \ldots, y_8$ has 14. The problem case is when there are 6 edges between $x_1, \ldots, x_4$ and $y_5, \ldots, y_8$ and likewise between $y_1, \ldots, y_4$ and $x_5, \ldots, x_8$. If neither of the DCC's from the $P_4 \hookrightarrow_6 P_4$ are proper, the degrees of $x_1, \ldots, x_4$ into $y_5, \ldots, y_8$ must be $(2, 1, 1, 2)$ and likewise the degrees of $x_5, \ldots, x_8$ must be $(2, 1, 1, 2)$.

Now, suppose that either x_1, or x_4 has degree at most 1 into $y_1, \ldots, y_4$. Without loss of generality, say x_4 has this property. Now consider the new partition (x_4, x_5, x_6, x_7) and (x_8, x_1, x_2, x_3).

(x_4, x_5, x_6, x_7) has a total of $2 + (8 - 3) = 7$ edges into y_5, y_6, y_7, y_8. This follows as there are a total of 2 edges from x_4 into this set (by our degree case) and a total of 8 edges from x_5, x_6, x_7, x_8 into this set. Since x_8 has degree 2 into y_1, y_2, y_3, y_4 it has degree at most 3 into x_5, x_6, x_7, x_8, giving the count.

On the other hand (x_8, x_1, x_2, x_3) has $5 - 1 + 2 = 6$edges into (y_1, y_2, y_3, y_4). This is as we already know that x_8 has degree 2 in this set. x_1, x_2, x_3, x_4 has degree 5 into this set, and we assume that x_4 has degree 1 into this set. Combining gives the desired bound. Thus we have $P_4 \hookrightarrow_6 P_4$ giving a DCC, and $P_4 \hookrightarrow_7 P_4$ giving a proper DCC.

Thus the degree of both x_4, and x_1 is at least 2 into y_1, y_2, y_3, y_4. Consider the same pairs: (x_4, x_5, x_6, x_7) still has 7 neighbors in y_5, y_6, x_7, y, y_8 and gives a proper DCC. x_8, x_1 into y_1, y_2, y_3, y_4 is $(2, 2) \hookrightarrow P_4$ which gives a DCC. This completes the case.

Case 3**:** Suppose $\ell = 9$.

Consider that our (r, s) decomposition is into paths of order 4 and 5 by Lemma 16, and we have at least $3\ell + 1 = 28$ edges. Thus, our balanced partite sets are $(12, 16), (13, 15)$ and $(14, 14)$. Recall that $\{P_4, P_5\} \hookrightarrow_6 P_5$ yield doubly chorded cycles, and $\{P_4, P_5\} \hookrightarrow_7 P$ yield proper doubly chorded cycles by Lemma 1. Thus, it is easy to verify that we always get a pair that guarantee our proper doubly chorded cycle.

Case 4**:** Suppose $\ell \geq 10$.

Again, for $\ell = 10$, we get parts of order 5, and the same verification as above works. After this point, the verification gets easier. As above, we consider the balanced partition of both cycles, as guaranteed by Lemma 16. This gives a (P, Q) decomposition of the C with $|P| = \lceil \frac{\ell}{2} \rceil$ and a $Q = \lfloor \frac{\ell}{2} \rfloor$, along with a (P', Q') decomposition of C'. The total degree of C into C' is at least $3\ell + 1$. Consider the one of P and Q with the lower degree; it sends at least half of it's edges to one of P' or Q'. Now consider the 'parallel pair'. By the minimality of P, this must have degree at least as high as the initially considered pair. For $\ell = 11$, these are both at least 8. Since $P_5 \hookrightarrow_7 P$ and $P_6 \hookrightarrow_8 P$ yield DCCs, this completes the proof. For $\ell = 12$, these are both at least 8 as well, and that $P_6 \hookrightarrow_8 P$ yields a DCC as well.

For $\ell = 13$, the minimum degree of the first pair is 9 as the minimum degree in the balanced partition is 18. If the degree is at least 10 the fact that $P \hookrightarrow_{10} P$ yields a proper DCC. However, if the degree in the first pair is 9, the degree from whichever of P or Q which has minimum degree must be 9 into *both* P' and Q'. This also implies that the other of P or Q must have degree at least 9 into both P' and Q'. By pairing up the larger cardinality of P and Q with the smaller cardinality of P' and Q' and vice versa, we may use the fact that $P_6 \hookrightarrow_8 P$ yields a proper DCC twice to find the desired DCCs. Finally, if $\ell > 14$, the proof is trivial: the minimum pair degree is at least 10 and the fact that $P \hookrightarrow_{10} P$ yields a proper DCC finishes the proof.

□

Finally, we turn to give the proof of Lemma 14.

Proof of Lemma 14. Throughout the proof, for a degree sequence $(d_1, \ldots, d_i) \hookrightarrow P_k$ we let v_j denote the jth vertex, that is, the vertex of degree d_j, and $x_1 < x_2 < \ldots < x_{d_1}$ denote the neighbors of v_1, and likewise $y_1 < y_2 < \ldots y_{d_2}$, and $z_1 < z_2 < \ldots < z_{d_3}$, $w_1 < w_2 < \ldots w_{d_4}$ denote the neighbors of the vertices with degrees d_2 and d_3, d_4 in increasing order with respect to some ordering of the P_k. For sequences with $\star$, the $v_j, x_\ell, y_{\ell'}$, etc. only refer to the neighbors of vertices with specified degrees. Note that v_j is denoted for positive degree d_j.

Claim (α)**:** $\{(2,2),(2,\star,2)\} \hookrightarrow P$ yields a DCC unless $x_1 < y_1 < y_2 < x_2$ (up to reordering v_1 and v_2).

Proof. Let Q be a path with degree sequence $(2,2)$ or $(2,\star,2)$ and vertices $v_1 < v_2$, each of degree 2. If $y_1 \le x_1 < y_2 \le x_2$ (say), then $v_1, x_2, P^-, y_1, v_2, Q^-, v_1$ is a cycle with chords v_2y_2 and v_1x_1. If $x_1 = y_1 < x_2 < y_2$, consider $v_1, x_1, P, y_2, v_2, Q^-, v_1$, which is a DCC even when $x_1 < x_2 \le y_1 < y_2$. □

This immediately yields that $(2,2) \hookrightarrow P_3$ and $(2,\star,2) \hookrightarrow P_3$ yield a DCC, and also that $(2,2) \hookrightarrow P_4$ does as well. If $(2,2) \hookrightarrow P_4$, then either there is a clear DCC (as in the proof above) or by Claim (α) because $P = P_4$ we need only consider the case $x_1 < y_1 < y_2 < x_2$, and $v_1, x_1, y_1, v_2, y_2, x_2, v_1$ is a DCC with chords y_1y_2 and v_1v_2.

$(2,0,2,1) \hookrightarrow P_4$ yields a DCC except for an inside $(2,0,2,1)$ by Claim (α).

Claim (β)**:** $\{(3,2),(3,\star,2)\} \hookrightarrow P$ yields a DCC.

Proof. Let Q be a path with degree sequence as above and vertices $v_1 < v_2$. After removing the edge v_1x_1, we have either a DCC or $x_2 < y_1 < y_2 < x_3$ or $y_1 < x_2 < x_3 < y_2$ by Claim (α). Then, in either case, adding x_1 yields a DCC. Indeed, in the first case $v_1, x_1, P, y_2, v_2, Q^-, v_1$ has chords v_1x_2 and v_2y_1 and in the second case it has chords v_1x_2 and v_1x_3. □

From here it is easy to see that $(3,\star,2) \hookrightarrow \{P_3, P_4\}$ yields a proper DCC. Indeed, for $(3,\star,2) \hookrightarrow P_3$, it is clear that the resulting cycle is proper if $x_1 = y_1 < x_2 = y_2 < x_3$, as x_3 is not required for a DCC. Thus, the remaining case is $x_1 = y_1 < x_2 < x_3 = y_2$. This has cycle $v_1, x_1, v_2, y_2, x_2, v_1$ which has chords x_1x_2 and v_1x_3. (Note that this is only proper because $\star$ is posited to be a *non-empty* sequence.) Taken into P_4, the same argument applies in the first case (replacing $x_2 = y_2$ with $x_2 \le y_2$.) In the second case, note that either $x_1x_2 \in E(P_4)$ or $x_2x_3 \in E(P_4)$, so the cycle above works. Next, assume $y_1 < x_1 < x_2 < x_3$. This gives an obvious cycle in which case y_1 or x_3 can be avoided. The remaining cases are of the form $x_1 < x_2 = y_1 < y_2 < x_3$ (or the mirror with $x_2 = y_2$). But these also have obvious cycles avoiding x_3.

Since it is easy to see that $(4) \hookrightarrow P$ yields a DCC, this and Claim (β) give that $P_2 \hookrightarrow_5 P$ yields a DCC and both $P_2 \hookrightarrow_5 P_4$ and $P_2 \hookrightarrow_6 P$ yield proper DCCs. Indeed, the only case of $P_2 \hookrightarrow_6 P$ that is not automatic is that $(3,3) \hookrightarrow P$. Removing the edges v_1x_1 and v_2y_1 yields either $x_2 < y_2 < y_3 < x_3$ or $y_2 < x_2 < x_3 < y_3$, by Claim (α). Adding back x_1 in the first case or y_1 in the second case yields a proper DCC avoiding x_3 or y_3 respectively.

Claim (γ)**:** $\{(3,1,1),(3,\star,1,1)\} \hookrightarrow P_3$, and $(3,1,1) \hookrightarrow P_4$ yield proper DCCs.

Proof. Let Q be a path with degree sequence as above and vertices $v_1 < v_2 < v_3$. First consider the case $\{(3,1,1),(3,\star,1,1)\} \hookrightarrow P_3$. Note that the P_3 is precisely $x_1 < x_2 < x_3$ and if $y_1 \in \{x_1, x_3\}$, then if $y_1 = x_1$ then $v_1, x_3, x_2, x_1, v_2, Q^-, v_1$ is a proper DCC avoiding v_3 and has chords v_1x_1 and v_1x_2; while if $y_1 = x_3$ then $v_1, Q^-, v_2, x_3, x_2, x_1, v_1$ is a proper DCC avoiding v_3 and with chords v_1x_2 and v_1x_3. Thus $y_1 = x_2$. With this, it is easy to check that regardless of whether z_1 is x_1, x_2 or x_3 then there is a proper DCC. In the case $(3,1,1) \hookrightarrow P_4$ the argument is similar. Let g denote the vertex not adjacent to v_1 on the P_4. Without loss of generality either $g < x_1 < x_2 < x_3$ or $x_1 < g < x_2 < x_3$. In the first case, note that (as before) $y_1 = x_2$ or we are done, and similarly, regardless of z_1, there is a proper DCC. In the second case, $y_1 \in \{g, x_2\}$ or we are done as before. However, regardless of the placement of z_1, similar proper DCCs exist. The more difficult case is $y_1 = g$ and $z_1 = x_3$. Then $v_1, x_2, g, v_2, v_3, x_3, v_1$ is a proper DCC avoiding x_1 and with chords v_1v_2 and x_2x_3. □

Claim (δ)**:** $(3,1,1) \hookrightarrow P$, $(3,1,\star,1) \hookrightarrow P_4$, and $(3,\star,1,1) \hookrightarrow P_4$ each yield a DCC.

Proof. If $y_1 \le x_1$ or $y_1 \ge x_3$, this is clear. If $y_1 = x_2$, it is as in Claim (γ). Thus we may assume that, without loss of generality $x_1 < y_1 < x_2$. Now consider z_1. Again, $x_1 < z_1 < x_3$ or we are done. If $x_2 \le z_1$, then a doubly chorded cycle $v_1, x_1, P, z_1, v_3, v_2, v_1$ is clear. Similarly if $z_1 \le y_1$, then $v_1, x_3, P^-, z_1, v_3, v_2, v_1$ is a DCC. The remaining case is $x_1 < y_1 < z_1 < x_2 < x_3$. But then $v_1, x_1, P, y_1, v_2, v_3, z_1, P, x_3, v_1$ has chords v_1x_2 and v_1v_2.

For the second part, let Q be a path with degree sequence $(3,1,\star,1)$ and vertices $v_1 < v_2 < v_3$. As in Claim (γ) we have either $g < x_1 < x_2 < x_3$ or $x_1 < g < x_2 < x_3$. In the first case, if y_1 or z_1 is one of g, x_1, x_3 there exist DCCs. Thus, $y_1 = z_1 = x_2$ and $v_1, Q, v_3, x_2, x_1, v_1$ is a cycle with chords v_1x_2 and v_2x_2. In the second case, we may assume $y_1, z_1 \in \{g, x_2\}$. Suppose $y_1 = g$. If $z_1 = g$, then $v_1, Q, v_3, g, x_2, x_3, v_1$ is a cycle with chords v_1x_2 and v_2g. If $z_1 = x_2$, then $v_1, Q, v_3, x_2, g, x_1, v_1$ is a cycle with chords v_1x_2 and v_2g. Suppose $y_1 = x_2$. Then there exist DCCs regardless of z_1 by the same argument as the case $y_1 = g$.

The proof of the third part is analogous to that of the second part. □

Claim (ϵ)**:** $(2,1,2), \hookrightarrow P_4$ yields a proper DCC.

Proof. Let g_1 and g_2 denote the vertices not adjacent to v_1 on P_4. Without loss of generality, then, the cases are $x_1 < x_2 < g_1 < g_2$, and $x_1 < g_1 < x_2 < g_2$, and $x_1 < g_1 < g_2 < x_2$, and $g_1 < x_1 < x_2 < g_2$. In the first two cases note that we are done unless $z_2 = g_2$, (as then we have $(2, \star, 2) \hookrightarrow P_3$ of Claim (α) and the cycle is now proper as g_2 is not included). Suppose $x_1 < x_2 < g_1 < z_2$. In the case that $z_1 \in \{x_2, g_1\}$, then it is easy to find proper DCCs regardless of the placement of y_1, avoiding either x_1 or z_2. The more difficult case is when $x_1 = z_1 < x_2 < g_1 < z_2$. If $y_1 \in \{x_1, x_2\}$ it is easy to build proper DCCs avoiding $\{g_1, z_2\}$. If $y_1 = z_2$, then $v_1, x_2, x_1, v_3, z_2, v_2, v_1$ avoids g_1 and has chords v_1x_1 and v_3v_2. If $y_1 = g_1$, then $v_1, x_1, v_3, v_2, g_1, x_2, v_1$ avoids z_2 and has chords x_1x_2 and v_1v_2. Next suppose $x_1 < g_1 < x_2 < z_2$. If $z_1 \in \{g_1, x_2\}$, we can find proper DCCs avoiding either x_1 or g_2, depending on y_1. Thus, we may assume $z_1 = x_1$. If $y_1 \in \{x_1, g_1, x_2\}$, it is easy to find a proper DCC avoiding g_2. In the case where $y_1 = g_2 = z_2$, then $v_1, x_1, v_3, v_2, z_2, x_2, v_1$ has chords v_3z_2 and v_1v_2 and avoids g_1. Next suppose $x_1 < g_1 < g_2 < x_2$. If $x_1 = z_1$ and $z_2 = g_2$, or $z_1 = g_1$ and $z_2 = x_2$, then this is similar to the cases already examined. If $x_1 = z_1$ and $x_2 = z_2$, then if $y_1 = x_1$ (or symmetrically $y_1 = x_2$) the cycle $v_1, x_1, v_2, v_3, z_2, v_1$ has chords v_1v_2 and v_3z_1 and avoids g_1, g_2. If $y_1 = g_1$ (or symmetrically $y_1 = g_2$), then $v_1, x_1, g_1, v_2, v_3, x_2, v_1$ has chords v_1v_2 and v_3x_1 and avoids g_2 and hence is a proper doubly chorded cycle. If $z_1 = g_1$ and $z_2 = g_2$ or $z_1 = x_1$ and $z_2 = g_1$ (or symmetrically $z_1 = g_2$ and $z_2 = x_2$), then regardless of the location of y_1, it is easy to build a proper DCC. The case when $g_1 < x_1 < x_2 < g_2$ is similar. □

Claim (ζ)**:** $\{(2, 1, 2), (2, 1, \star, 2)\} \hookrightarrow P$ yields a DCC.

Proof. Note that as we just require a DCC, and not a proper one, we may assume that we have $x_1 < z_1 < z_2 < x_2$ (or $z_1 < x_1 < x_2 < z_2$) or we are done by Claim (α). Regardless of y_1, we can find a doubly chorded cycle. □

By Claims (β), (γ) and (ζ) (resp. (ϵ)), and since $(2, 2) \hookrightarrow P_3$ (resp. P_4) yields a DCC by Claim (α) and $(3, \star, 2) \hookrightarrow P_3$ (resp. P_4) yields a proper DCC by Claim (β), the assertion of Lemma 14 (2) (resp. (4)) holds.

Claim (η)**:** $(2, 1, 1, 2) \hookrightarrow P_4$ yields a DCC.

Proof. This actually follows from Claim (ϵ)'s argument. □

Claim (θ)**:** $P_3 \hookrightarrow_6 P$ yields a DCC.

Proof. Since $(4) \hookrightarrow P$ yields a DCC, we have max degree 3. Since $\{(3, 2), (3, \star, 2)\} \hookrightarrow P$ yields a DCC by Claim (β), the only remaining case is $(2, 2, 2) \hookrightarrow P$, which follows from Claim (ζ). □

Claim (ι)**:** $P_3 \hookrightarrow_6 P_5$ yields a proper DCC.

Proof. If $(3, 2)$ or (4) arises in the degree sequence of the P_3 we are done as we do not use all of the P_3. Thus, our options are $(3, 1, 2) \hookrightarrow P_5$, or $(3, 0, 3) \hookrightarrow P_5$, or $(2, 2, 2) \hookrightarrow P_5$. Consider $(3, 1, 2) \hookrightarrow P_5$, which is quite similar to Claim (β). Let g_1, g_2 denote the vertices not adjacent to v_1. Then without loss of generality, the possibilities are $x_1 < x_2 < x_3 < g_1 < g_2$, $x_1 < x_2 < g_1 < x_3 < g_2$,

$x_1 < g_1 < x_2 < g_2 < x_3$ and $x_1 < x_2 < g_1 < g_2 < x_3$ and $x_1 < g_1 < x_2 < x_3 < g_2$ and $g_1 < x_1 < x_2 < x_3 < g_2$. In the first two cases it is clear that $z_2 = g_2$ (as otherwise we have $(3, \star, 2) \hookrightarrow P_4$ of Claim (β)). It is easy to find proper DCC's then, regardless of z_1. In the case where $x_1 < g_1 < x_2 < g_2 < x_3$, it is easy to find proper DCCs unless $z_1 = x_1$ and $z_2 = x_3$. In this case, we find our desired DCCs using v_2y_1 as the second chord, regardless of y_1. The cases for $x_1 < x_2 < g_1 < g_2 < x_3$, $x_1 < g_1 < x_2 < x_3 < g_2$ and $g_1 < x_1 < x_2 < x_3 < g_2$ are straightforward. In the case $(3, 0, 3) \hookrightarrow P_5$, the argument is similar. It remains to consider $(2, 2, 2) \hookrightarrow P_5$. The case $(2, 2) \hookrightarrow P_5$ already yields the desired proper DCC unless either $x_1 < y_1 < g < y_2 < x_2$, where g denotes the gap (or similarly $x_1 < g < y_1 < y_2 < x_2$), or $y_1 < x_1 < g < x_2 < y_2$ (or similarly $y_1 < g < x_1 < x_2 < y_2$ by Claim (α). In the first case $z_1 = x_1$ and $z_2 = x_2$ or we are done by Claim (α). Then $v_1, x_1, y_1, v_2, v_3, z_2, v_1$ is the proper DCC avoiding $\{g, y_2\}$ and with chords v_1v_2 and v_3x_1. In the second case the argument is similar. □

Claim (κ)**:** $(2, 2, 1, 1) \hookrightarrow P$ yields a DCC.

Proof. We may assume that $x_1 < y_1 < y_2 < x_2$ or $y_1 < x_1 < x_2 < y_2$ or the $(2, 2) \hookrightarrow P$ already gives the desired cycle by Claim (α). In the first case, note that if $z_1 \leq y_1$ or $y_2 \leq z_1$, DCCs are easy to find. A similar argument applies for w_1. Thus $y_1 < z_1 < y_2$ and $y_1 < w_1 < y_2$. If $z_1 \leq w_1$, then $v_1, x_1, P, w_1, v_4, v_3, v_2, v_1$ is a DCC with chords v_2y_1 and v_3z_1. Otherwise, if $z_1 > w_1$, we use $v_1, x_2, P^-, w_1, v_4, v_3, v_2, v_1$. Thus, we are in the case $y_1 < x_1 < x_2 < y_2$. Here we have $y_1 < z_1 < y_2$ and $y_1 < w_1 < y_2$. Note that, possibly by reversing P, we may assume that $z_1 \leq w_1$. We now simply list the cases and cycles and chords in the table that follows. □

By Claims (α), (β), (δ), (ζ) and (σ), the assertion of Lemma 14 part (5) holds.

Case	Cycles	Chords
$y_1 < z_1 < x_1 < x_2 < y_2$, $z_1 < w_1 < x_2$	$v_4, w_1, P, x_1, v_1, x_2, P, y_2, v_2, y_1, P, z_1, v_3, v_4$ (Note: w_1, P^-, x_1 if $x_1 < w_1$)	v_1v_2, v_2v_3
$y_1 < x_1 = z_1 < w_1 < x_2 < y_2$	$v_4, w_1, P, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$	v_1v_2, v_3x_1
$y_1 < z_1 = w_1 \leq x_1 < x_2 < y_2$	$v_4, w_1, P, x_1, v_1, x_2, P, y_2, v_2, v_3, v_4$	v_3z_1, v_1v_2
$y_1 < z_1 \leq x_1 < x_2 \leq w_1 < y_2$	$v_4, w_1, P^-, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$	v_3z_1, v_1v_2
$y_1 < x_1 < z_1 \leq x_2 < y_2$, $z_1 < w_1 < y_2$	$v_4, w_1, P, x_2, v_1, v_2, y_1, P, z_1, v_3, v_4$ (Note: $v_4, w_1, P^-, x_1, v_1, v_2, v_3, v_4$ if $x_2 < w_1$)	v_1x_1, v_2v_3 v_1x_2, v_3z_1
$y_1 < x_1 < z_1 = w_1 \leq x_2 < y_2$	$v_4, w_1, P, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$	v_1v_2, v_3z_1
$y_1 < x_1 < x_2 < z_1 \leq w_1 < y_2$	$v_4, w_1, P^-, x_2, v_1, x_1, P^-, y_1, v_2, v_3, v_4$	v_1v_2, v_3z_1

Claim (λ)**:** $\{P_3\} \hookrightarrow_6 \{P_3, P_4\}$ yields a proper DCC while $P_4 \hookrightarrow P_4$ yields a DCC which is proper except in the $(2, 1, 1, 2) \hookrightarrow P_4$ case.

Proof. Consider the case $P_k \hookrightarrow_6 P_j$ where $k \leq j$. Note that $(4) \hookrightarrow P_4$ yields a DCC so the max degree (of either) is 3. Since $(3, 2) \hookrightarrow P_3$ yields a DCC by Claim (β), and $(3, \star, 2) \hookrightarrow \{P_3, P_4\}$ and $(3, 2) \hookrightarrow P_4$ yield proper DCCs by Claim (β), if the maximum degree of P_k is 3, then all other vertices have degree 1 and we have only to consider $P_4 \hookrightarrow_6 P_4$. But $(3, 1, 1) \hookrightarrow P_4$ yields a proper DCC by Claim (γ). Thus, the maximum degree is 2. Note that $(2, 2) \hookrightarrow \{P_3, P_4\}$ yields a DCC by Claim (α), so if P_k has two adjacent vertices of degree 2 we are done. Thus, we are done unless

$k = j = 4$. Then our options are $(2,1,2,1)$ or $(2,1,1,2)$. Since we have already shown in Claims (ζ) and (η) that $(2,1,2) \hookrightarrow P_4$ yields a DCC and $(2,1,1,2) \hookrightarrow P_4$ yields a DCC we are done. □

Claim (μ)**:** $\{P_3, P_4, P_5\} \hookrightarrow_7 P$ yields a proper DCC.

Proof. Note that it suffices to show that this yields a DCC. If there is such a cycle, there is necessarily a proper one by Lemma 2. We have already shown that $P_3 \hookrightarrow_6 P$ yields a DCC by Claim (θ), so this case is trivial. In the cases of $P_4 \hookrightarrow_7 P$ or $P_5 \hookrightarrow_7 P$, note that $(4) \hookrightarrow P$ yields a DCC and $\{(3,2),(3,\star,2),(3,1,1)\} \hookrightarrow P$ all yield DCCs by Claims (β) and (δ), and hence the maximum degree is 2. But $\{(2,1,2),(2,1,\star,2),(2,2,1,1)\} \hookrightarrow P$ yield DCCs by Claims (ζ) and (κ) and one of these must occur as $P_5 \hookrightarrow_7 P$ must have at least 2 vertices of degree 2 (and if there are vertices of degree 0, then there are additional vertices of degree 2). □

Claim (ν)**:** $P_6 \hookrightarrow_8 P$ yields a proper DCC.

Proof. Again we have that $\{(4),(3,2),(3,\star,2),\ (3,1,1)\} \hookrightarrow P$ yield DCCs, so we are done unless the P_6 has maximum degree 2. Note that we then cannot avoid $\{(2,1,2),(2,1,\star,2),(2,2,1,1)\}$ and hence are guaranteed a DCC (and hence a proper one by Lemma 2). □

Claim (ξ)**:** $P_7 \hookrightarrow_9 P$ yields a proper DCC.

Proof. Analogous to Claim (ν). □

Claim (o)**:** $P \hookrightarrow_{10} P'$ yields a proper DCC.

Proof. As in the Claim (μ) it suffices to show the existence of one DCC. In this case, we order the vertices of P and P'. Then the chords give a permutation $\sigma \in S_{10}$ (the symmetric group) induced by the endpoints of the edges between the paths, breaking ties arbitrarily. The Erdős-Szekeres Theorem [5] guarantees an increasing or decreasing sequence of length 4 which easily gives the desired DCC. □

Claim (π)**:** $(1,2,2,1) \hookrightarrow P_5$ yields a proper DCC.

Proof. If the $(2,2) \hookrightarrow P_5$ does not already yield a DCC, then we have, two cases to consider. First, if $y_1 < z_1 < g < z_2 < y_2$, where g is a gap, it is trivial to find a proper DCC unless $w_1 = g$, so that we have $y_1 < z_1 < w_1 < z_2 < y_2$. But then $v_2, y_1, z_1, v_3, v_4, w_1, z_2, y_2, v_2$ avoids v_1 and has chords z_1w_1 and v_2v_3. Next, if $y_1 < z_1 < z_2 < g < y_2$, then it is easy to find a proper DCC by considering w_1. □

Claim (ρ)**:** $(2,2,1,1) \hookrightarrow P_5$ yields a proper DCC.

Proof. If $(2,2) \hookrightarrow P_5$ does not already yield a DCC, then we have $x_1 < y_1 < g < y_2 < x_2$, or $y_1 < x_1 < g < x_2 < y_2$ where g is a gap. The first case is analogous to Claim (π), and v_4 can be avoided. In the second case, if $z_1 \in \{y_1, g, y_2\}$, DCCs are easy to find; the most difficult being when $z_1 = g$ and a proper DCC is $v_1, x_1, y_1, v_2, v_3, z_1, x_2, v_1$ with chords v_1v_2 and x_1z_1. Thus we may assume that $z_1 = x_1$ (or symmetrically $z_1 = x_2$). Likewise it is easy to see that we either have that $w_1 = x_1$ or $w_1 = x_2$, or we may omit one of y_1, y_2 and find a proper DCC. The more difficult case is $w_1 = g$. Then $v_1, x_1, v_3, v_4, g, x_2, y_2, v_2, v_1$ is a cycle with chords v_1x_2 and v_2v_3 avoiding y_1. In either case it is quite easy to find a proper DCC avoiding both y_1 or y_2. □

Claim (σ)**:** $(2,1,1,1) \hookrightarrow \{P_3, P_4\}$ yields a DCC.

Proof. We begin by noting that $(2,1,1,1) \hookrightarrow P_3$ yields a DCC. Indeed, if there were no DCC, then the P_3 would have degree sequence $(1,3,1)$. We may assume, without loss of generality, that x_1 is the first vertex of the P_3. Then w_1,the neighbor of v_4, must be one of the bottom two vertices of the P_3. Hence, the cycle containing the paths along with v_1x_1 and v_4w_1 avoids only (possibly) the edge incident to the bottom vertex of the P_3 and is thus a DCC. Now we consider the case $(2,1,1,1) \hookrightarrow P_4$. We repeatedly use the fact that if we can find a cycle containing 7 of the 8 vertices, omitting only an end vertex of degree 1, then the cycle is doubly chorded. This fact follows by counting edges as such a cycle has 5 path edges and 4 cross edges, for a total of 9 edges induced on a 7 vertex cycle. We denote the ordered vertices on P_4 by u_1, u_2, u_3, u_4. First we consider the case where $x_1 = u_1$. The vertex u_4 has degree at least 1, as otherwise we are in the case $(2,1,1,1) \hookrightarrow P_3$. If $u_4 = z_1$ or $u_4 = w_1$, either the resulting graph is hamiltonian or contains a DCC on 7 vertices. Thus either $u_4 = x_2$ or $u_4 = y_1$ (or both). If $u_4 = x_2$, then by symmetry, $w_1 = u_3$ and we have a cycle on 7 vertices. Therefore u_4 must have degree 2 and hence $u_4 = y_1$. But then $v_1, u_4, v_2, v_3, v_4, u_3, u_2, u_1, v_1$ is hamiltonian. Thus, we may assume that $u_4 \neq x_2$ but instead that $u_4 = y_1$.

Since then the degree of u_4 is 1, we may assume that $w_1 \neq u_3$ (as otherwise there would be a cycle avoiding only u_4). Thus $w_1 = u_1$ or $w_1 = u_2$. First suppose $w_1 = u_1$. If $x_2 = u_2$, then $v_1, u_1, v_4, v_3, v_2, u_4, u_3, u_2, v_1$ is hamiltonian. Suppose $x_2 = u_3$. Likewise if $z_1 = u_2$, then $v_1, v_2, u_4, u_3, u_2, v_3, v_4, u_1, v_1$ is hamiltonian. Therefore, u_2 has degree 0 across and by edge counting the cycle $v_1, u_1, v_4, v_3, v_2, u_4, u_3, v_1$ which contains every vertex except u_2 and induces every cross edge, is a DCC. If $x_2 = u_4$, then $v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, v_1$ is hamniltonian. Next suppose $w_1 = u_2$. Consider the second neighbor of v_1. Suppose $x_2 = u_2$. Then we may assume $z_1 = u_3$, or we can find a DCC easily. Then $v_1, v_2, u_4, u_3, v_3, v_4, u_2, v_1$ is a cycle with chords v_2v_3 and u_2u_3. Suppose $x_2 = u_3$. Then $v_1, u_1, u_2, v_4, v_3, v_2, u_4, u_3, v_1$ is hamiltonian. Suppose $x_2 = u_4$. If $z_1 \in \{u_2, u_3, u_4\}$, then $v_1, v_2, v_3, v_4, u_2, u_3, u_4, v_1$ is a cycle with chords v_2u_4 and v_3z_1. If $z_1 = u_1$, then $v_1, v_2, u_4, u_3, u_2, v_4, v_3, u_1, v_1$ is hamiltonian. This completes the case where $x_1 = u_1$ (and by symmetry the case where $x_2 = u_4$).

The remaining case is when $x_1 = u_2$ and $x_2 = u_3$. Again, both u_1 and u_4 must have positive degree (as otherwise we reduce to the case where $(2,1,1,1) \hookrightarrow P_3$). If one of u_1, u_4 has degree 3, we reduce to the case where $(2,1,1,1) \hookrightarrow P_3$. If one of u_1, u_4 has degree 2, then we also have the degree sequence $(2,1,1,1)$ within the P_4, and (up to symmetry) this gives $y_1 = z_1 = u_1$ and $w_1 = u_4$. Then $v_1, u_2, u_1, v_2, v_3, v_4, u_4, u_3, v_1$ is hamiltonian. Thus, u_1 and u_4 both have degree 1. If $w_1 = u_4$, then this gives a cycle avoiding just u_1 (which has degree 1). By symmetry, the case

$w_1 = u_1$ is the same. Therefore, $y_1 = u_1$ and $z_1 = u_4$. Then $v_1, u_2, u_1, v_2, v_3, u_4, u_3, v_1$ is a cycle with chords v_1v_2 and u_2u_3. This completes the case. □

Claim (τ)**:** $\{P_4, P_5\} \hookrightarrow_6 P_5$ yields a DCC.

Proof. As in Claim (λ) cases, we may assume that the maximum degree in the $P_4 \hookrightarrow_6 P_5$ or $P_5 \hookrightarrow_6 P_5$ is 2. The cases not covered by a combination of Claims $(\zeta), (\pi), (\rho)$ are $(2,0,2,0,2) \hookrightarrow P_5$ and $(2,2,0,1,1) \hookrightarrow P_5$ (or its reverse sequence or some permutation of the middle three terms of these two sequences), and $\{(1,2,2,0,1),\ (1,2,0,2,1),\ (1,0,2,2,1)\} \hookrightarrow P_5$, and the case where the minimum degree in the P_5's is 1.

For $(2,0,2,0,2) \hookrightarrow P_5$, the fact that $(2,0,2) \hookrightarrow P_5$ yields no DCC implies that (with g the vertex nonadjacent to either vertex of degree two) we either have $x_1 < y_1 < g < y_2 < x_2$, $x_1 < y_1 < y_2 < g < x_2$ or $x_1 < y_1 < y_2 < x_2 < g$ or similar inequalities with the roles of x and y reversed. If the x_i are in the exterior vertices, then the y_i and z_i are forced to be interior. But the y_i being interior force the z_i to be exterior, since, for example in the $x_1 < y_1 < g < y_2 < x_2$ case, if $z_1 = y_1$ and $z_2 = g$, then $v_1, v_2, v_3, v_4, v_5, y_1, g, y_2, x_2, v_1$ is proper (as x_1 is omitted) and has chords v_3y_1 and v_3y_2. Similar cycles can be found in the other cases for the adjacencies of v_5. But the x_i and z_i both exterior implies a DCC exists as shown earlier. If x_i is interior, then y_i are exterior, but again there are DCCs once z_i is included in either the interior or exterior.

For $(2,2,0,1,1) \hookrightarrow P_5$, this almost follows the proof of Claim (ρ). As in Claim (ρ) if $(2,2) \hookrightarrow P_5$ does not already produce a DCC, then $x_1 < y_1 < g < y_2 < x_2$ or $y_1 < x_1 < g < x_2 < y_2$. In the first case, if $z_1 = x_2$, then $v_1, v_2, v_3, v_4, x_2, y_2, g, y_1, x_1, v_1$ has chords v_2y_1 and v_2y_2. If $z_1 = y_2$ then a similar cycle with the same chords is obvious. If $z_1 = g$, then $v_1, x_1, y_1, g, v_4, v_3, v_2, y_2, x_2, v_1$ has chords v_1v_2 and v_2y_1. If $z_1 = y_1$, then $v_1, x_1, y_1, v_4, v_3, v_2, y_2, x_2, v_1$ has chords v_1v_2 and v_2y_1. Finally, if $z_1 = x_1$, then $x_1, v_4, v_3, v_2, v_1, x_2, y_2, g, y_1, x_1$ has chords v_2y_1 and v_2y_2. In the second case, if $z_1 = y_2$, then $v_1, x_1, y_1, v_2, v_3, v_4, y_2, x_2, v_1$ has chords v_1v_2 and v_2y_2. If $z_1 = y_1$, then $v_1, v_2, v_3, v_4, y_1, x_1, g, x_2, v_1$ has chords v_1x_1 and v_2y_1. If $z_1 = g$, then $v_1, x_1, g, v_4, v_3, v_2, y_2, x_2, v_1$ has chords v_1v_2 and gx_2. If $z_1 = x_1$ or x_2, we can produce DCCs by considering w_1.

The cases of the reverse sequence or where the middle terms are permuted are also all similar, and amount to merely interchanging the degrees along the path. In each case a DCC is easily found. The cases $(1,2,2,0,1)$, $(1,2,0,2,1)$ and $(1,0,2,2,1)$ are all similar to Claim (π).

Finally, we consider the case where the minimum degree in the P_5's is 1. Note that there is only one vertex of degree 2 in both P_5's. The proofs of these cases are slightly different than those we have considered before. Let $p_1, \ldots, p_5$ denote the ordered vertices of the first P_5 and $q_1, \ldots, q_5$ denote the ordered vertices of the other path. We assume (possibly reordering the p_i, q_i or swapping the roles of the two P_5's) that p_1's neighbor is q_j for j as small as possible. Note that the two paths and the edges between them comprise 10 vertices and 14 edges. If a hamiltonian cycle exists, then there are 4 chords. If a cycle avoiding only one vertex exists, then if that vertex is an end vertex of one of the paths, (so that it has degree at most 3 in G) there are at least 2 chords for the cycle. This is also true if an internal vertex of degree at most 3 in G is avoided.

Case 1: Suppose $p_1q_1 \in E$.

Then consider neighbors of p_5. If p_5q_5 is an edge, the graph is hamiltonian, and there is a DCC.

If p_5q_4 is an edge, the cycle $C:\ p_1, q_1, q_2, q_3, q_4, p_5, p_4, p_3, p_2, p_1$ avoids q_5 and no other vertex and so is a DCC.

Subcase 1.1: Suppose $p_5q_3 \in E$.

Then consider the p_j (max j) adjacent to q_5. By symmetry, $j \in \{1, 2, 3\}$ or we would be in an earlier case. Now consider the cycles $C_1 : p_j, \ldots, p_5, q_3, q_4, q_5, p_j$ and $C_2 : p_1, p_2, \ldots, p_5, q_3, q_2, q_1, p_1$ and $C_3 : p_1, q_1, \ldots, q_5, p_j, \ldots, p_1$. These three cycles can be thought of as breaking the paths and edges into path segments $S_1 = [p_j, p_5]$, $S_2 = [q_3, q_5]$ and $S_3 = p_j, p_{j-1}, \ldots, p_1, q_1, q_2, q_3$. Note that p_j and q_3 each belong to two of these segments. Now there are three more edges between the paths. We would have a DCC unless these three edges each join a distinct pair of the segments. Note that under this restriction, p_j and q_3 cannot be incident to any of these three edges.

If this is the case, consider p_4. If $p_4q_4 \in E$, then $p_1, \ldots, p_j, q_5, q_4, p_4, p_5, q_3, q_2, q_1, p_1$ is hamiltonian if $j = 3$, or misses only p_3 if $j = 2$. If p_3 sends only one edge to the other path we are done. Otherwise, there are two edges from p_3. If p_3 sends an edge to S_2, C_1 is a DCC. Thus, $p_3q_1 \in E$ and $p_3q_2 \in E$. Then C_2 is a DCC. If $j = 1$, then each of p_2 and p_3 are incident to (at least) one of the remaining edges. If both edges go into S_3, then C_2 is a DCC. Thus, one of these edges goes into S_2 (one of q_4 or q_5), then C_1 is a DCC.

Next suppose $p_4q_3 \in E$. Now any edge p_rq_2, $r \in \{1, 2, 3, 4, 5\}$ implies that C_2 has chords q_2p_r and p_4q_3.

Now suppose that $p_4q_2 \in E$. Then $p_1, q_1, q_2, p_4, p_5, q_3, q_4, q_5, p_j, \ldots, p_1$ is hamiltonian if $j = 3$. This cycle avoids only p_3 if $j = 2$ and is a DCC if $deg(p_3) = 3$. Thus, suppose that p_3 is incident to two edges to the other path. One of these must go to q_4. Now $p_5, p_4, q_2, q_1, p_1, p_2, p_3, q_4, q_3, p_5$ avoids only q_5. If $j = 1$, then note that p_2 and p_3 must each send at least one edge to the other path. If either sends an edge to $\{q_1, q_2, q_3\}$, then C_2 is a DCC using that edge and p_4q_2. Thus, both edges must go to q_4 and now C_1 is a DCC. This completes the p_4q_2 subcase.

Next suppose that $p_4q_1 \in E$. Assume $j = 3$, i.e., that q_5p_3 is an edge. Then p_2 must send at least one edge to the other path. If this edge goes to any of $\{q_1, q_2, q_3\}$, then C_2 is clearly a DCC. Thus, p_2 has an adjacency in $\{q_4, q_5\}$. Then $p_1, q_1, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$ has both p_2q_4 (or p_2q_5) and p_3p_4 as chords. Thus we next assume $j = 2$. Then p_3 has at least one adjacency to the other path. If $p_3q_r \in E$, $r \in \{1, 2, 3\}$, then C_2 is a DCC with chords p_3q_r and p_4q_1. If $r = 4$, then $p_2, q_5, q_4, p_3, p_4, p_5, q_3, q_2, q_1, p_1, p_2$ is hamiltonian. Now assume $j = 1$, i.e., $q_5p_1 \in E$. If p_2 has an adjacency in $\{q_1, q_2, q_3\}$ then C_2 is a DCC. Thus, $p_2q_4 \in E$ and the cycle $p_1, q_1, q_2, q_3, p_5, \ldots, p_2, q_4, q_5, p_1$ is hamiltonian. By symmetry these are all the necessary cases when $p_1q_1 \in E$. This completes the cases when $p_4q_1 \in E$ and Subcase 1.1.

Subcase 1.2: Suppose $p_5q_2 \in E$ (so by symmetry, q_5 is adjacent to p_2 or p_1).

If $q_5p_1 \in E$, then $p_1, \ldots, p_5, q_2, \ldots q_5, p_1$ avoids only q_1, so it is a DCC. Now assume instead that $q_5p_2 \in E$. Note that each of p_3 and p_4 needs to send at least one edge to the other path. If two such edges are incident in $\{q_1, q_2\}$, then $p_1, \ldots, p_5, q_2, q_1, p_1$ is a DCC. If two such edges are incident in $\{q_2, q_3, q_4\}$ (note q_5 not possible in this case) then $p_2, \ldots, p_5, q_2, \ldots, q_5, p_2$ is a DCC. This implies that each of p_3 and p_4 sends exaclty one edge to the other path into $\{q_1\}$ or $\{q_3, q_4\}$. If $p_3q_1 \in E$, then a hamiltonian cycle is easy to find. Thus $p_4q_1 \in E$. But then a cycle avoiding only p_3 (which must have degree 3, for otherwise p_3q_3 and p_3q_4 are edges and a hamiltonian cycle is easily found) and we are again done.

Subacse 1.3: Suppose $p_5q_1 \in E$.

Then by symmetry, $q_5p_1 \in E$ and $p_1, \ldots, p_5, q_1, \ldots, q_5, p_1$ is hamiltonian. This completes Case 1.

Case 2: Suppose $p_1q_2 \in E$ (and by symmetry, p_1q_1 and p_5q_5 are not edges).

Thus, the neighbors of q_1 and q_5 lie in $\{p_2, p_3, p_4\}$.

Subcase 2.1: Suppose q_1p_2 and q_5p_4 are in E.

Suppose $p_5q_4 \in E$. Then $p_1, p_2, \ldots, p_5, q_4, q_3, q_2, p_1$ is a DCC unless at least one of the remaining two edges goes to q_1 or q_5. But, by our symmetry assumptions, that edge is incident to q_1. Then q_1 must be adjacent to p_3 or p_4. If $p_3q_1 \in E$, then $p_1, q_2, q_3, q_4, p_5, p_4, p_3, q_1, p_2, p_1$ avoids only q_5 and hence is a DCC. If $p_4q_1 \in E$, then $p_3q_3 \in E$ by the assumption on the degree sequence and hence, $p_2, p_3, p_4, q_5, q_4, q_3, q_2, q_1, p_2$ is a DCC.

Next suppose that $p_5q_3 \in E$. Assume that $p_4q_4 \in E$. Then for any edge p_3q_r, $r \in \{1, 2, 3, 4, 5\}$, $p_2, p_3, p_4, q_5, q_4, \ldots, q_r, p_2$ has chords p_3q_r and p_4q_4. If $p_3q_4 \in E$, then $p_5, p_4, q_5, q_4, p_3, p_2, q_1, q_2, q_3, p_5$ avoids only p_1. If instead, $p_2q_4 \in E$, then any edge p_3q_r, $r \in \{1, 2, 3, 4, 5\}$ gives $p_2, p_3, p_4, q_5, \ldots, q_1, p_2$ as a DCC. Now suppose that $p_1q_4 \in E$. Then $p_1, q_4, q_5, p_4, p_5, q_3, q_2, q_1, p_2, p_1$ avoids only p_3 which has degree 3, and thus is a DCC.

Next suppose that $p_5q_2 \in E$. Then each of q_3 and q_4 sends an edge to $\{p_1, p_2, p_3, p_4\}$, or we would be in an earlier case, and so $p_1, p_2, p_3, p_4, q_5, q_4, q_3, q_2, p_1$ is a DCC. This completes Subcase 2.1.

Subcase 2.2: Suppose q_1p_2 and q_5p_3 are in E.

Suppose $p_5q_3 \in E$. Now consider the adjacency of q_4 (which cannot include p_5). Suppose $q_4p_4 \in E$. Then $p_1, q_2, q_3, p_5, p_4, q_4, q_5, p_3, p_2, p_1$ avoids only q_1. Next suppose that $q_4p_3 \in E$. Now p_4 has an adjacency on the other path. Say $p_4q_3 \in E$. Then $p_3, p_4, p_5, q_3, q_4, q_5, p_3$ has p_3q_4 and p_4q_3 as chords. If $p_4q_2 \in E$, then $p_1, q_2, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$ has q_2q_3 and p_3q_4 as chords. If $p_4q_1 \in E$, then $p_1, q_2, q_1, p_4, p_5, q_3, q_4, q_5, p_3, p_2, p_1$ is hamiltonian. Next suppose that $q_4p_2 \in E$. Then $p_1, q_2, q_3, p_5, p_4, p_3, q_5, q_4, p_2, p_1$ avoids only q_1. Finally, suppose $q_4p_1 \in E$. Then $p_1, q_4, q_5, p_3, p_4, p_5, q_3, q_2, q_1, p_2, p_1$ is hamiltonian.

Finally, if $p_5q_2 \in E$, a similar argument applies. This completes Subcase 2.2.

Subcase 2.3: Suppose that $q_1p_2 \in E$ and $q_5p_2 \in E$.

By symmetry, p_5 is adjacent to q_2. Now p_3 and p_4 each have at least one edge to the other path, and these edges end in $\{q_3, q_4\}$. But then, $p_2, \ldots, p_5, q_2, \ldots, q_5, p_2$ is a DCC. This completes Subcase 2.3.

Similar arguments hold for p_1q_2 when either q_1p_3 or q_1p_4 are assumed along with a possible adjacency for q_5. By our symmetry assumptions, this completes Case 2.

Case 3: Suppose $p_1q_3 \in E$.

By symmetry and our earlier cases, p_5q_3, q_1p_3 and q_5p_3 are also edges. Now p_2 and p_4 must have their adjacencies in $\{q_2, q_4\}$. If p_2q_2 and p_4q_4 are edges, then $p_1, p_2, q_2, q_1, p_3, q_5, q_4, p_4, p_5, q_3, p_1$ is hamiltonian. While if p_2q_4 and p_4q_2 are the edges, then reverse one of the paths to obtain the previous cycle. This completes Case 3. $\square$

By our symmetry and path reversal assumptions, this completes the proof. $\square$

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