Graph Curvature and Local Discrepancy

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Abstract

In recent years, discrete notions of curvature have been defined and exploited to understand various geometric properties of graphs; especially regarding heat flow, and spectral properties. In this paper, we study various combinatorial properties implied by satisfying the Bakry-Émery curvature dimension inequality $CD(\infty, K)$. In particular we derive a local discrepancy inequality, similar in spirit to the expander mixing lemma from spectral graph theory, which certifies a type of 'local pseudo-randomness' of the edge set of the graph, for graphs satisfying a curvature lower bound. In addition, several other consequences are derived regarding graph connectivity and cycle statistics of the graph.

Keywords— Bakry-Émery Curvature, Discrete Curvature, Graphs

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1 Introduction

A fundamental issue with the computational study of large networks is that their sheer size makes many algorithmic approaches unrealistic. Instead, then, of computing various graph properties exactly, one often studies simpler properties which 'certify' that various graph properties hold – perhaps not giving the best possible value of a parameter, but enabling a quick guarantee.

One important route to such information is through spectral graph theory. The spectra of various matrices associated with graphs is known to capture various structural and geometric properties. For instance, the well-known Cheeger inequality relates isoperimetry in a graph (that is, the size of normalized cuts) with the first non-trivial eigenvalue of the Laplacian. Kirkhoff's matrix tree theorem [14] relates the spectra of the (combinatorial) Laplacian with number of spanning trees – which, again, can be thought of as a measure of how well connected a graph is. As another example, the discrepancy inequality commonly known as the expander mixing lemma uses the eigenvalues of the adjacency matrix (for regular graphs) or the normalized Laplacian (for general graphs) to certify the pseudo-random properties of the edge set of a graph. These facts underlie many commonly used graph partitioning, clustering, and drawing algorithms.

One particularly interesting aspect of the spectral graph theory of the Laplacian matrices is the strong analogies between Laplace operators on graphs and the Laplace-Beltrami operator on Riemannian manifolds – perhaps most notably through the Cheeger inequality, but also through inequalities relating eigenvalues and diameters (for instance). As a result of this, recent years have seen researchers try to adapt other concepts on manifold to the discrete setting. A particular point of interest has been on the development of notions of curvature for graphs. One markedly successful aspect of this work has been the adaptation of the so-called curvature dimension inequality $CD(N, K)$, introduced by Bakry and Émery in a different setting, along with some variants to graphs. Curvature lower bounds in this sense, which is more precisely defined in the next section, have been used to prove bounds on eigenvalues, and diameter, along with being used to study heat flow on graphs and, though this, to establish other geometric properties of networks.

Many of these works have been largely analytic in their character – using curvature to study networks via studying eigenfunctions and the mixing of random walks (through heat flow). In this work, we take a more directly combinatorial view of curvature and aim to use the definition of curvature to establish a number of purely combinatorial consequences of a curvature lower bound in a more direct way.

Our main result is that, just as eigenvalues of a graph certify the pseudo-randomness of the edge set, that a curvature lower bound at a point establishes a local pseudo-randomness – and that the more 'positively curved' a graph looks at a point, the more the edge sets in neighborhoods of that vertex behave (in a precise sense) pseudo-randomly. This result, and some variants, are then used to obtain a number of combinatorial properties of graphs. For instance, a bound on the connectivity of a connected graph is given in terms of a curvature lower bound – interestingly, curvature (being a local property) cannot detect whether a given graph is connected but it can detect that a connected graph is well connected. Some other properties are also studied – using the curvature to bound the number of some small subgraphs, in a sense showing that curvature improves a classical result of Corrádi and Hajnal, is also given.

The remainder of the paper is organized as follows. In Section 2 we introduce the curvature-dimension inequality introduced by Bakry and Émery [2], and give a reformulation of this inequality more useful for our purposes. In Section 3 we derive our main result, a discrepancy inequality depending on curvature. Several corollaries and variants are also presented to highlight the flexibility of results of this type. Finally, in Section 4 we present several combinatorial consequences of our result, dealing with the connectivity of a graph and its local neighborhoods, and small subgraph containment.

2 Preliminaries

In this paper, $G = (V, E)$ will denote a simple graph. We denote the degree of the vertex $x \in V(G)$ by $\mathrm{d}(x)$ and for a subset $X \subset V$ we denote by $\mathrm{d}_X(x)$ the number of neighbors of x in X. The minimum degree is denoted by δ. We write $x \sim y$ when $xy \in E(G)$. Given two vertices in the graph, $d(x, y)$ denotes the combinatorial distance between x and y, and we let $N_1(x) = \{y : d(x, y) = 1\}$ be the neighborhood of x and let $N_2(x) = \{z : d(x, z) = 2\}$ be the second neighborhood of x. When x has been fixed, we will often write these neighborhoods simply as N_1 and N_2.

We only consider graphs which are locally finite; that is, $\mathrm{d}(x) < \infty$ for all $x \in V(G)$ (which encompass all finite graphs). Given a measure $\mu : V \to \mathbb{R}$, the μ-Laplacian on G is the operator $\Delta : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ defined by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} (f(y) - f(x)).$$

In the case $\mu(x) = 1$ for all $x \in V(G)$, we have $\Delta = -L$, where L is the combinatorial graph Laplacian. Also of interest is when $\mu(x) = \mathrm{d}(x)$ for all $x \in V(G)$, in which case Δ is the normalized graph Laplacian with a change of sign. Both operators are used in various applications. For the types of combinatorial results that are our goal, we will use the case $\mu = 1$; the combinatorial (standard) graph Laplacian.

The gradient form $\Gamma = \Gamma^{\Delta}$ is defined by

$$\begin{aligned} 2\Gamma(f,g)(x) &= \frac{1}{2}\left(\Delta(f \cdot g) - f \cdot \Delta(g) - \Delta(f) \cdot g\right)(x) \\ &= \frac{1}{2\mu(x)} \sum_{y \sim x} (f(y) - f(x))(g(y) - g(x)) \end{aligned}$$

for all $f, g \in \mathbb{R}^{|V|}$. For brevity's sake we write $\Gamma(f) = \Gamma(f,f)$. The iterated gradient form $\Gamma_2 = \Gamma_2^{\Delta}$ is defined by

$$\Gamma_2(f,g) = \Delta\Gamma(f,g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g)$$

for all $f, g \in \mathbb{R}^{|V|}$. Again, we write $\Gamma_2(f) = \Gamma_2(f,f)$.

In their 1985 paper [2], Bakry and Émery demonstrated that in the manifold setting curvature lower bounds may be understood completely through the Laplace-Beltrami operator. They did so through a modification of Bochner's identity, a fundamental identity in Riemannian Geometry. Bakry and Émery suggested using this consequence of curvature in the manifold setting as a definition in the more general setting of the Markov Semigroup, which encompasses both diffusion and random walks on graphs. This approach has been met with much success in graph theory (see e.g. [3, 10, 12, 13, 19]), albeit with some modifications needed in some applications due to the graph Laplace operator not satisfying a chain rule. These successes are, in no small part, due to the close ties between the Laplace-Beltrami Operator and the Laplacian, as described above. Through use of the differential operators Γ and Γ_2, constructed from the Laplacian, we state the Bakry-Émery notion of discrete curvature below.

Definition 1 (Bakry-Émery Curvature)**.** Let G be a locally finite graph. For $K \in \mathbb{R}$ and $N \in (0, \infty]$, we say that a vertex $x \in V$ satisfies Bakry-Émery's *curvature-dimension inequality* $CD(N,K)$, if for any $f : V \to \mathbb{R}$, we have

$$\Gamma_2(f)(x) \geq \frac{1}{N}(\Delta f(x))^2 + K\Gamma(f)(x),$$

where N is a dimension parameter and K is regarded as a lower Ricci curvature bound at x. A graph G is said to satisfy $CD(N,K)$ if every vertex satisfies $CD(N,K)$. Since graphs do not have a well-defined dimension, a natural choice simplifying this inequality is to take $n = \infty$. In this paper the curvature of a vertex or graph G is defined as the maximum value K for which $CD(\infty, K)$ holds, for that vertex or globally.

Often graph curvature is studied on highly symmetric graphs. For vertex transitive graphs, such as Cayley graphs, the curvature of all vertices is the same. For less structured graphs, neighborhood structures differ among vertices and the curvatures of different vertices differs. In these cases, global graph curvature is a lower bound on curvature computed at the vertex level. In this paper we explore local combinatorial implications of such curvature lower bounds. As we shall see, this gives a way to certify that graphs satisfy certain interesting properties.

As defined above, curvature is formulated via the Laplacian (Δ) and the gradient (Γ), each of which only depend on the the structure of the first neighborhood of vertices. Since the iterated gradient (Γ_2) is a composition of these two operators, the curvature is determined solely by the structure of the balls of radius 2. Therefore, we can translate $CD(\infty, K)$ into more combinatorial terms, highlighting the contribution of each edge type (that is, the edges within the first neighborhood of x, the edges between the first and second neighborhoods of x, etc.) to the curvature dimension inequality.

The proposition below allows us to begin to rework the CD inequality.

Proposition 1. G satisfies $CD(\infty, K)$ at x if and only if for all $f : V(G) \to \mathbb{R}$,

$$\sum_{z \in N_2} \sum_{\substack{y \sim z \\ y \in N_1}} \left[\frac{1}{4}(f(z) - f(y))^2 - \frac{1}{2}(f(z) - f(y))(f(y) - f(x))\right] + \sum_{\substack{y, y' \in N_1 \\ \{y, y'\} \in E(G)}} (f(y) - f(y'))^2$$
$$\geq \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right)\Gamma(f)(x) - \frac{1}{2}\Delta(f(x))^2.$$

Proof. Expanding Γ_2 according to the combinatorial definitions of Δ and Γ yields

$$
\begin{aligned}
\Gamma_2(f)(x) &= \frac{1}{2}\Delta\Gamma(f) - \Gamma(f,\Delta f) \\
&= \frac{1}{2}\sum_{y\sim x}[\Gamma(f)(y) - \Gamma(f)(x)] - \frac{1}{2}\sum_{y\sim x}(f(y)-f(x))(\Delta f(y) - \Delta f(x)) \\
&= \frac{1}{4}\sum_{y_1\sim x}\left[\sum_{z\sim y_1}(f(z)-f(y_1))^2 - \sum_{y_2\sim x}(f(y_2)-f(x))^2\right] \\
&\qquad - \frac{1}{2}\sum_{y_1\sim x}(f(y_1)-f(x))\left[\sum_{z\sim y_1}(f(z)-f(y)) - \sum_{y_2\sim x}(f(y_2)-f(x))\right] \\
&= \frac{1}{4}\sum_{y\sim x}\sum_{z\sim y}(f(z)-f(y))^2 - \frac{1}{4}\sum_{y_1\sim x}\sum_{y_2\sim x}(f(y_2)-f(x))^2 \\
&\qquad - \frac{1}{2}\sum_{y_1\sim x}\sum_{z\sim y_1}(f(y_1)-f(x))(f(z)-f(y_1)) + \frac{1}{2}\sum_{y_1\sim x}\sum_{y_2\sim x}(f(y_1)-f(x))(f(y_2)-f(x)) \\
&= \frac{1}{4}\sum_{y\sim x}\sum_{z\sim y}(f(z)-f(y))^2 - \frac{\mathrm{d}(x)}{4}\sum_{y\sim x}(f(y)-f(x))^2 \\
&\qquad - \frac{1}{2}\sum_{y_1\sim x}\sum_{z\sim y_1}(f(y_1)-f(x))(f(z)-f(y_1)) + \frac{1}{2}\sum_{y_1\sim x}\sum_{y_2\sim x}(f(y_1)-f(x))(f(y_2)-f(x)).
\end{aligned}
$$

To this sum, the edges between N_1 and N_2 contribute

$$
\sum_{z\in N_2}\sum_{\substack{y\sim z\\ y\in N_1}}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right].
$$

To the sum, the edges between two vertices in N_1 contribute

$$
\begin{aligned}
&\sum_{\substack{y,y'\in N_1\\ \{y,y'\}\in E(G)}}\left[\frac{1}{2}(f(y)-f(y'))^2 - \frac{1}{2}(f(y)-f(x))(f(y')-f(y)) - \frac{1}{2}(f(y')-f(x))(f(y)-f(y'))\right] \\
&\qquad = \sum_{\substack{y,y'\in N_1(x)\\ \{y,y'\}\in E(G)}}(f(y)-f(y'))^2.
\end{aligned}
$$

Lastly, the edges between x and N_1 contribute

$$
\begin{aligned}
&\sum_{y\sim x}\left[\frac{1}{4}(f(x)-f(y))^2 - \frac{\mathrm{d}(x)}{4}(f(y)-f(x))^2 + \frac{1}{2}(f(y)-f(x))^2 + \frac{1}{2}(f(y)-f(x))\Delta f(x)\right] \\
&\qquad = \frac{3-\mathrm{d}(x)}{4}\sum_{y\sim x}(f(x)-f(y))^2 + \frac{1}{2}\Delta f(x)\sum_{y\sim x}(f(y)-f(x)) \\
&\qquad = \frac{3-\mathrm{d}(x)}{2}\Gamma(f)(x) + \frac{1}{2}(\Delta f(x))^2.
\end{aligned}
$$

Substituting this rewritten Γ_2 into Definition 1 we see $CD(\infty, K)$ is satisfied at a vertex x if and only if for all functions f

$$
\begin{aligned}
&\sum_{z\in N_2}\sum_{\substack{y\sim z\\ y\in N_1}}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right] + \sum_{\substack{y,y'\in N_1\\ \{y,y'\}\in E(G)}}(f(y)-f(y'))^2 \\
&\qquad \geq \left(\frac{2K+\mathrm{d}(x)-3}{2}\right)\Gamma(f)(x) - \frac{1}{2}\Delta(f(x))^2.
\end{aligned}
$$

□

A similar combinatorial reworking of the curvature dimension inequality was established by Klartag, et al. in [15]. While their presentation of the curvature dimension inequality is useful in many applications, ours proved more fruitful for the types of inequalities that we derive below.

3 Discrepancy Inequalities

Spectral graph theory, and in particular the spectral theory of Laplace operators on graphs, is particularly useful in giving 'cheap' certifications of graph properties; for instance, despite the question of graph Hamiltonicity being NP-complete, polynomial time checks of regularity and spectral gap conditions may be used to certify Hamiltonicity [16]. One of the most important properties that the spectra of a graph certifies is the pseudo-randomness of the edge set, through what is commonly known as the Expander Mixing Lemma, which we state now in its form for regular graphs.

Lemma 2 (Expander Mixing Lemma)**.** [1] Suppose $G = (V, E)$ is a d-regular n-vertex graph. Denote by λ the second largest eigenvalue in absolute value of the adjacency matrix. Then $\forall S, T \subseteq V$,

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|},$$

where $e(S, T)$ denotes the number of edges between S and T.

This simple proposition implies that graphs with a large spectral gap behave like a random graph with respect to its edge distribution. Having global control on the edge distribution is extremely useful when it occurs, but in some cases it is too much to hope for. For example, many real world graphs display clustering where neighbors of a vertex are more likely to, themselves, be connected. Furthermore, as most degrees tend to be small in comparison to the size of the entire network, much of the 'action' in the graph occurs locally.

For these types of graphs, Bakry-Émery Curvature can be particularly useful. The Bakry-Émery Curvature at a vertex x is the largest K such that the matrix $\Gamma_2 - K\Gamma$ is positive semi definite. The order of this matrix depends only on the size of second neighborhood of x, as opposed to the number of vertices in the graph n, and hence for a graph with max degree $\Delta(G)$, the curvature at a point x may be calculated with high accuracy in time complexity $O(\Delta(G)^6 \log(\Delta(G)))$ as documented in [6]. Hence, for a graph of bounded degree a global curvature lower bound can be computed in linear time in the order of the graph, with the implied constant depending on the maximum degree.

This makes using curvature to certify certain graph properties efficient computationally. Moreover, curvature lower bounds certify similar properties to those of the graph spectrum. For instance, curvature bounds imply bounds on mixing of random walks, on the diameter of a graph, and (through Buser's inequality) regarding cut properties of the graph.

In this section, we show that curvature actually implies a type of 'local' discrepancy inequality, which implies that at vertices with a lower curvature bound the edge distribution within the first and second neighborhood behaves, in a sense, pseudo-randomly. We now present our main result below.

Theorem 3. Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$. Fix $X \subseteq N_1(x)$ with $\overline{X} = N_1(x) \setminus X$. If $|X| = \alpha|N_1(x)|$, then

$$\sum_{z \in N_2} \left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)} \right) \leq$$
$$\frac{3}{4}[\alpha^2 \cdot e(\overline{X}, N_2) + (1-\alpha)^2 \cdot e(X, N_2)] + e(X, \overline{X}) - \left(\frac{2K + \mathrm{d}(x) - 3}{2} \right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

The sum on the lefthand side of the inequality of Theorem 3 measures how 'randomly' edges are distributed from $N_1(x)$ to $N_2(x)$ in the following sense. If the edges were distributed randomly, and a set $X \subseteq N_1(x)$ were fixed we would expect a proportion close to $|X|/|N_1(x)|$ of the edges to have an endpoint in X – moreover, we would expect this to (roughly) be true for the edges incident to every vertex in $N_2(x)$. The sum on the left hand side exactly measures this deviation from 'random' in the aggregate – note that if the edges were precisely distributed in this way then this discrepancy term would be zero. Of course, this is too much to expect even if the edges in the neighborhood are truly distributed randomly!

Then, what we obtain is an upper bound on this deviation in such a way that the larger the curvature is the more 'random-like' the behavior is. Note that this mimics the theme of the Expander Mixing Lemma – a graph parameter is certifying a random like behavior of the edge set – only now locally, between the first and second neighborhoods. Hence, this theorem acts as a *local* discrepancy inequality.

Proof. Fix $x \in V(G)$. We proceed by defining an explicit function f and interpreting the curvature dimension inequality, per Proposition 1. Define $f(x) = 0$. Let $X \subseteq N_1(x)$ such that $|X| = \alpha|N_1(x)|$ and let $\overline{X} = N_1(x) \setminus X$.

For each $y \in X$, let $f(y) = 1 - \alpha$ and for each $y \in \overline{X}$, let $f(y) = -\alpha$. Note that for this function $\Delta f(x) = 0$ and

$$\begin{aligned}
\Gamma(f)(x) &= \frac{1}{2}[\alpha(1-\alpha)^2 + (1-\alpha)(\alpha)^2]\mathrm{d}(x) \\
&= \frac{1}{2}(\alpha - 2\alpha^2 + \alpha^3 + \alpha^2 - \alpha^3)\mathrm{d}(x) \\
&= \frac{1}{2}(\alpha - \alpha^2)\mathrm{d}(x) \\
&= \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).
\end{aligned}$$

Our goal is, for each $z \in N_2(x)$, to select $f(z)$ in order to minimize the left side of the inequality derived from $CD(\infty, K)$. Thus, fix $z \in N_2(x)$. Then

$$\begin{aligned}
&\sum_{\substack{y\sim z \\ y\in N_1}} \left[\frac{1}{4}(f(z) - f(y))^2 - \frac{1}{2}(f(z) - f(y))(f(y) - f(x))\right] \\
&= \mathrm{d}_X(z)\left[\frac{1}{4}(f(z) - (1-\alpha))^2 - \frac{1}{2}(f(z) - (1-\alpha))(1-\alpha)\right] + \mathrm{d}_{\overline{X}}(z)\left[\frac{1}{4}(f(z)+\alpha)^2 + \frac{1}{2}(f(z)+\alpha)\alpha\right] \\
&= \frac{1}{4}\mathrm{d}_{N_1}(z)f(z)^2 + \left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right] f(z) + \frac{3}{4}[\alpha^2 \mathrm{d}_{\overline{X}}(z) + (1-\alpha)^2 \mathrm{d}_X(z)].
\end{aligned}$$

By taking a formal derivative with respect to $f(z)$, we see that the sum is minimized when

$$f(z) = -\frac{2\left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right]}{\mathrm{d}_{N_1}(z)}.$$

For this selection of $f(z)$,

$$\begin{aligned}
&\sum_{\substack{y\sim z \\ y\in N_1}} \left[\frac{1}{4}(f(z) - f(y))^2 - \frac{1}{2}(f(z) - f(y))(f(y) - f(x))\right] \\
&= -\frac{\left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right]^2}{\mathrm{d}_{N_1}(z)} + \frac{3}{4}\left[\alpha^2 \mathrm{d}_{\overline{X}}(z) + (1-\alpha)^2 \mathrm{d}_X(z)\right].
\end{aligned}$$

Summing over $z \in N_2$, we find

$$\begin{aligned}
&\sum_{z\in N_2}\sum_{\substack{y\sim z \\ y\in N_1}} \left[\frac{1}{4}(f(z) - f(y))^2 - \frac{1}{2}(f(z) - f(y))(f(y) - f(x))\right] \\
&= \sum_{z\in N_2}\left(-\frac{\left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right]^2}{\mathrm{d}_{N_1}(z)} + \frac{3}{4}\left[\alpha^2 \mathrm{d}_{\overline{X}}(z) + (1-\alpha)^2 \mathrm{d}_X(z)\right]\right) \\
&= \sum_{z\in N_2}\left(-\frac{\left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right]^2}{\mathrm{d}_{N_1}(z)}\right) + \frac{3}{4}[\alpha^2 \cdot e(\overline{X}, N_2) + (1-\alpha)^2 \cdot e(X, N_2)].
\end{aligned}$$

As a result, satisfying $CD(\infty, K)$ implies that

$$\begin{aligned}
-\sum_{z\in N_2}\left(\frac{\left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right]^2}{\mathrm{d}_{N_1}(z)}\right) + \frac{3}{4}[\alpha^2 \cdot e(\overline{X}, N_2) + (1-\alpha)^2 \cdot e(X, N_2)] + e(X, \overline{X}) \geq \\
\left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).
\end{aligned}$$

Solving for this first term yields

$$\begin{aligned}
\sum_{z\in N_2}\left(\frac{\left[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)\right]^2}{\mathrm{d}_{N_1}(z)}\right) \leq \\
\frac{3}{4}[\alpha^2 \cdot e(\overline{X}, N_2) + (1-\alpha)^2 \cdot e(X, N_2)] + e(X, \overline{X}) - \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).
\end{aligned}$$

□

Corollary 4. Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$. Let $X \subseteq N_1(x)$ with $\overline{X} = N_1(x) \setminus X$. If $|X| = \alpha|N_1(x)|$, then

$$\frac{\left(\sum_{z\in N_2} |\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)|\right)^2}{e(N_1, N_2)} \leq$$
$$\frac{3}{4}\left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, \overline{X}) - \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

Proof. We use Cauchy-Schwarz to simplify the sum

$$\sum_{z\in N_2} \left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z) + (1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)}\right).$$

Using Cauchy-Schwarz in the form of

$$\sum a_i^2 \geq \frac{(\sum a_i b_i)^2}{\sum b_i^2},$$

with

$$a_i = \frac{\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)}{\sqrt{\mathrm{d}_{N_1}(z)}} \text{ and } b_i = \sqrt{\mathrm{d}_{N_1}(z)}$$

we obtain

$$\sum_{z\in N_2} \left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)}\right) \geq \frac{\left(\sum_{z\in N_2} [\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]\right)^2}{\sum_{z\in N_2} \mathrm{d}_{N_1}(z)}$$
$$= \frac{\left(\sum_{z\in N_2} [\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]\right)^2}{e(N_1, N_2)}.$$

As a result, satisfying $CD(\infty, K)$ implies that

$$-\frac{(\sum |\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)|)^2}{e(N_1, N_2)} + \frac{3}{4}\left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, \overline{X}) \geq$$
$$\left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

□

This corollary gives us a much cleaner version of the above theorem. However, in some situations, this corollary gives away too much in its use of Cauchy-Schwarz. While we typically think of applying these results to bound the discrepancy of a graph whose curvature is known they can also be applied to bound the curvature in cases where the local discrepancy is understood. For example, our theorem gives a sharp bound on curvature for the graph $\mathbb{Z}^d$, while the corollary only gives us an asymptotic upper bound of d on the curvature. We will more fully explore similar examples later in Section 4.

We highlight a few special instances of these results.

Corollary 5. Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$. For any $X \subseteq N_1(x)$ with $|X| = \frac{1}{2}|N_1(x)|$, if $\overline{X} = N_1(x) \setminus X$, then

$$\frac{\left(\sum_{z\in N_2} |\mathrm{d}_{\overline{X}}(z) - \mathrm{d}_X(z)|\right)^2}{e(N_1, N_2)} \leq \frac{3}{4}e(N_1, N_2) + 4e(X, \overline{X}) - \frac{2K + \mathrm{d}(x) - 3}{4}\mathrm{d}(x).$$

Proof. Taking $\alpha = \frac{1}{2}$ in the above theorem directly gives the corollary. □

Corollary 6. Suppose G is a triangle-free graph satisfying $CD(\infty, K)$ at $x \in V(G)$. Let $X \subseteq N_1(x)$ with $\overline{X} = N_1(x) \setminus X$. If $|X| = \alpha|N_1(x)|$, then

$$\frac{\left(\sum_{z\in N_2} |\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)|\right)^2}{e(N_1, N_2)} \leq$$
$$\frac{3}{4}\left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] - \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

Proof. If $e(X, \overline{X}) > 0$, then there exist adjacent vertices $y, y' \in N_1$. This edge then creates a triangle with x. Since G is triangle-free, then, $e(X, \overline{X}) = 0$. □

3.1 Local Buser Inequality

The isoperimetric problem in a graph – how sparse is the sparsest normalized cut? – is well-known to be closely related to the spectrum of the normalized Laplace operator via the Cheeger and Buser inequalities. It concerns the isoperimetric constant

$$\Phi(G) = \min_{S \subseteq V(G)} \frac{e(S, \overline{S})}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))}.$$

The 'standard' form of the Cheeger inequality for graphs states that

$$\Phi(G)^2/2 \leq \lambda_2(\mathcal{L}) \leq 2\Phi(G).$$

Here the lower bound is the analogue of Cheeger's inequality from Riemannian geometry, while the upper bound is roughly an analogue of Buser's inequality. We remark here that the upper bound can be improved under a curvature assumption; such a result depending on curvature is the original Buser's inequality from manifolds, and is known for graphs satisfying $CD(N, K)$ [15].

The simple upper bound $\lambda_2(\mathcal{L}) \leq 2\Phi(G)$ on the first non-trivial eigenvalue of the Laplacian proceeds by fixing a set S minimizing the isoperimetric constant $\frac{e(S,\overline{S})}{\mathrm{vol}(S)}$. Then one explicitly defines a vector φ based on this cut and computes via the Rayleigh quotient that

$$\lambda_2(\mathcal{L}) \leq \frac{e(S, \overline{S})\mathrm{vol}(V)}{\mathrm{vol}(S)\mathrm{vol}(\overline{S})} \leq 2\Phi(G).$$

This inequality is tremendously useful in practice, as it implies that for all sets S

$$e(S, \overline{S}) \geq \lambda_2(\mathcal{L})\frac{\mathrm{vol}(S)\mathrm{vol}(\overline{S})}{\mathrm{vol}(G)},$$

which when restricted to the d-regular case states

$$e(S, \overline{S}) \geq \frac{d\lambda_2(\mathcal{L})}{n}|S||\overline{S}|. \tag{1}$$

We remark that local variations of Cheeger's inequality have been the basis of several successful local graph partitioning algorithms. While Cheeger's inequality is a statement about all subsets of G, of which there are exponentially many, most proofs of Cheeger's inequality (and its variants) follow by considering a much smaller number of cuts defined by some process – eg. via (classically) an eigenvector sweep, or a sweep of the distribution of a random walk after some number of steps. The underlying ideas have, then, been used to develop a number of fast algorithms for finding sparse cuts in a graph. For example, one such algorithm given by Spielman and Teng in 2004 [21] considers subsets based off of a rapid mixing result for random walks given by Lovász and Simonovits [17, 18]. Through a modified Cheeger's inequality, this graph partitioning algorithm runs in time proportional to the size of the output – independent of the size of G. This approach and others like it have been particularly useful in solving problems involving massive networks.

It turns out that, just as curvature provides a 'local' version of a discrepancy inequality it also provides a local version of the Buser inequality in the form of (1). We explore this in a variant of Theorem 3 and in Section 4 we use it to prove two connectivity results, one local and the other global. Both proofs rely on the connectivity of the punctured ball of radius two, which we define as $\mathring{B}_2(x)$, the subgraph of G containing all vertices with distance 1 or 2 from x, and all edges between these vertices, except those edges between vertices of distance 2 from x. Theorem 3 and the corollaries that followed, show how a curvature lower bound ensures that most vertices have a close to proportionate edge count between bipartitions of $N_1(x)$. We use this to speak to edge counts between sets in a variant of our main result, Theorem 8. This allows us to show, for example, Corrolary 9: Suppose G satisfies $CD(\infty, K)$ at x, and x has degree d, for all partitions of $\mathring{B}_2(x)$ into S and $\overline{S}$,

$$e(S, \overline{S}) \geq \frac{2K + d - 3}{4d}|N(x) \cap S| \cdot |N_1(x) \cap \overline{S}|.$$

Notice the similarity between this inequality and the Cheeger inequality on regular graphs, equation (1). To prove Theorem 8 we first present the following lemma which gives us a reworking of the discrepancy term of Theorem 3.

Lemma 7. Suppose $x \in V(G)$ and let $X, \overline{X}$ partition $N_1(x)$ and $A, \overline{A}$ partition $N_2(x)$. Let $\alpha = \frac{|X|}{|N_1(x)|}$. Then

$$\sum_{z \in N_2(x)} \left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)} \right)$$

$$= \Big[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\Big] - e(X, \overline{A}) - e(\overline{X}, A) + \left(\sum_{z \in \overline{A}} \frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)} \right) + \left(\sum_{z \in A} \frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)} \right).$$

Proof. Equality follows from a series of algebraic manipulations. First, we split the sum over $N_2(x)$ of the discrepancy term into a sum over A and a sum over $\overline{A}$,

$$\sum_{z\in\overline{A}}\left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z)-(1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)}\right)+\sum_{z\in A}\left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z)-(1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)}\right).$$

The squares are rewritten in terms of edges to N_1 to find

$$\sum_{z\in\overline{A}}\left(\frac{[\alpha \mathrm{d}_{N_1}(z)-\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)}\right)+\sum_{z\in A}\left(\frac{[(1-\alpha)\mathrm{d}_{N_1}(z)-\mathrm{d}_{\overline{X}}(z)]^2}{\mathrm{d}_{N_1}(z)}\right).$$

We expand the squares to find

$$\sum_{z\in\overline{A}}\left(\alpha^2\mathrm{d}_{N_1}(z)-2\alpha\mathrm{d}_X(z)+\frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right)+\sum_{z\in A}\left((1-\alpha)^2\mathrm{d}_{N_1}(z)-2(1-\alpha)\mathrm{d}_{\overline{X}}(z)+\frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right).$$

Summing over z in A and $\overline{A}$ yields

$$\begin{aligned}&\alpha^2e(\overline{X},\overline{A})+\alpha^2e(X,\overline{A})-2\alpha e(X,\overline{A})+\left(\sum_{z\in\overline{A}}\frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right)\\&\quad+(1-\alpha)^2e(X,A)+(1-\alpha)^2e(\overline{X},A)-2(1-\alpha)e(\overline{X},A)+\left(\sum_{z\in A}\frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right)\end{aligned}$$

which can be seen as

$$\begin{aligned}&\alpha^2e(\overline{X},N_2)-\alpha^2e(\overline{X},A)+\alpha^2e(X,\overline{A})-2\alpha e(X,\overline{A})+\left(\sum_{z\in\overline{A}}\frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right)\\&\quad+(1-\alpha)^2e(X,N_2)-(1-\alpha)^2e(X,\overline{A})+(1-\alpha)^2e(\overline{X},A)-2(1-\alpha)e(\overline{X},A)+\left(\sum_{z\in A}\frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right).\end{aligned}$$

The discrepancy term is then rewritten as to more easily compare it against other terms in the $CD(\infty,K)$ inequality

$$\begin{aligned}&\Big[\alpha^2e(\overline{X},N_2)+(1-\alpha)^2e(X,N_2)\Big]-2\alpha e(X,\overline{A})-(1-\alpha)^2e(X,\overline{A})+\alpha^2e(X,\overline{A})\\&\qquad-2(1-\alpha)e(\overline{X},A)-\alpha^2e(\overline{X},A)+(1-\alpha)^2e(\overline{X},A)+\left(\sum_{z\in\overline{A}}\frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right)+\left(\sum_{z\in A}\frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right)\\&\quad=\Big[\alpha^2e(\overline{X},N_2)+(1-\alpha)^2e(X,N_2)\Big]-\Big[2\alpha+(1-\alpha)^2-\alpha^2\Big]e(X,\overline{A})\\&\qquad-\Big[2(1-\alpha)+\alpha^2-(1-\alpha)^2\Big]e(\overline{X},A)+\left(\sum_{z\in\overline{A}}\frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right)+\left(\sum_{z\in A}\frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right)\\&\quad=\Big[\alpha^2e(\overline{X},N_2)+(1-\alpha)^2e(X,N_2)\Big]-e(X,\overline{A})-e(\overline{X},A)+\left(\sum_{z\in\overline{A}}\frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right)+\left(\sum_{z\in A}\frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right).\end{aligned}$$

□

In Lemma 7, we see that the discrepancy term may be rewritten to provide information on the distribution of edges between all bipartitions of $\mathring{B}_2(x)$. We may use Lemma 7 to relate this distribution to graph curvature in Theorem 8.

Theorem 8. Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$. Let $X, \overline{X}$ partition $N_1(x)$ and $A, \overline{A}$ partition $N_2(x)$. It follows that

$$e(X, \overline{A}) + e(\overline{X}, A) + e(X, \overline{X}) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{4}\right) \frac{|X| \cdot |\overline{X}|}{\mathrm{d}(x)}.$$

Proof. Using Theorem 3 , let $\alpha = \frac{|X|}{\mathrm{d}(x)}$. We have the inequality,

$$\sum_{z \in N_2(x)} \left(- \frac{[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]^2}{\mathrm{d}_{N_1}(z)} \right) + \frac{3}{4}\left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, \overline{X}) \geq$$

$$\left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

Through Lemma 7 we use the partition of $N_2(x)$ into A and $\overline{A}$ and substitute this into the inequality to find

$$- \left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, \overline{A}) + e(\overline{X}, A) - \left(\sum_{z \in \overline{A}} \frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right) - \left(\sum_{z \in A} \frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right)$$
$$+ \frac{3}{4}\left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, \overline{X}) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x)$$

which simplifies to

$$- \frac{1}{4}\left[\alpha^2 e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)\right] + e(X, \overline{A}) + e(\overline{X}, A) - \left(\sum_{z \in \overline{A}} \frac{\mathrm{d}_X(z)^2}{\mathrm{d}_{N_1}(z)}\right) - \left(\sum_{z \in A} \frac{\mathrm{d}_{\overline{X}}(z)^2}{\mathrm{d}_{N_1}(z)}\right) + e(X, \overline{X})$$
$$\geq \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

Most notably,

$$e(X, \overline{A}) + e(\overline{X}, A) + e(X, \overline{X}) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{2}\right) \cdot \frac{1}{2}\alpha(1-\alpha)\mathrm{d}(x).$$

With the substitution $\alpha = \frac{|X|}{\mathrm{d}(x)}$ and $1 - \alpha = \frac{|\overline{X}|}{\mathrm{d}(x)}$ we have

$$e(X, \overline{A}) + e(\overline{X}, A) + e(X, \overline{X}) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{4}\right) \cdot \frac{|X| \cdot |\overline{X}|}{\mathrm{d}(x)}.$$

□

This immediately gives us the following corollary.

Corollary 9. Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$. Let the degree of x be d. For all partitions of $\mathring{B}_2(x)$ into S and $\overline{S}$,

$$e(S, \overline{S}) \geq \frac{2K + d - 3}{4d} |N_1(x) \cap S| \cdot |N_1(x) \cap \overline{S}|.$$

Proof. Given some partition of $\mathring{B}_2(x)$ into S and $\overline{S}$ we use Theorem 8 and let $X = S \cap N_1(x)$, $A = S \cap N_2(x)$, $\overline{X} = \overline{S} \cap N_1(x)$ and $\overline{A} = \overline{S} \cap N_2(x)$. □

3.2 Discrepancy of the neighbors of a single vertex

In our application of the curvature dimension inequality in Theorem 3, we interpreted the results of applying the curvature dimension inequality to a carefully chosen function. While the $CD(\infty, K)$ inequality deals with functions from the second neighborhood of a vertex, this turned out to be slightly illusory – after fixing a function on the first neighborhood, the value of the function on the second neighborhood that gives the 'best' result is essentially a calculus problem. Hence, by choosing a function that partitioned the first neighborhood we were able to use this minimization to tell us about the edge distribution between the first and second neighborhood with respect to this partition.

One of the advantages of knowing that a graph satisfies $CD(\infty, K)$ is that it actually satisfies this inequality for *all* functions. In Theorem 3, we looked at a function that revealed how edges behave with respect to a partition. Here we show another application of this idea, by choosing a function that highlights the importance of a particular edge xy; then the interplay of edges between the second and first neighborhoods of the central vertex x allows us to understand the involvement of xy in short cycles.

Theorem 10. Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$. Suppose $y \in N_1(x)$, and let $A = \{y' \in N_1(x) : y' \neq y\}$. Then

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} \frac{1}{\mathrm{d}_A(z)+1} \leq \mathrm{d}_{N_1(x)}(y) + \frac{3}{4}\mathrm{d}_{N_2(x)}(y) - \frac{1}{4}\mathrm{d}(x) - \frac{1}{2}K + \frac{5}{4}.$$

We saw in the previous section that a curvature lower bound ensures edges from N_2 to N_1 are well distributed. We may also note that for all y in $N_1(x)$, each edges between $N_2(x) \cap N_1(y)$ and $N_1(x) \setminus y$ corresponds to a unique 4 cycle containing the edge xy. We first evaluate the curvature inequality at a function which allows us to bound the edges of this type, and use this to bound the number of four-cycles.

Proof. Fix $x \in V(G)$ and define $f(x) = 0$. Then fix $y \in N_1(x)$ and define $f(y) = 1$. Let $A = \{y' \in N_1(x) : y' \neq y\}$ and for all $y' \in A$, define $f(y') = 0$. For all $z \in N_2(x)$ such that $z \not\sim y$, let $f(z) = 0$. Here we get that $\Delta f(x) = 1$ and $\Gamma(f)(x) = \frac{1}{2}$.

Then for each $z \in N_2(x)$ that is adjacent to y,

$$\begin{aligned}\sum_{y \sim z}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right] &= \frac{1}{4}(f(z)-1)^2 - \frac{1}{2}(f(z)-1) + \frac{1}{4}\mathrm{d}_A(z)f(z)^2 \\ &= \frac{1}{4}(\mathrm{d}_A(z)+1)f(z)^2 - f(z) + \frac{3}{4}.\end{aligned}$$

This term is maximized when $f(z) = \frac{2}{\mathrm{d}_A(z)+1}$, which yields that

$$\sum_{y \sim z}\left[\frac{1}{4}(f(z)-f(y))^2 - \frac{1}{2}(f(z)-f(y))(f(y)-f(x))\right] = -\frac{1}{\mathrm{d}_A(z)+1} + \frac{3}{4}.$$

Therefore, $CD(\infty, K)$ implies that

$$\sum_{\substack{z \in N(y) \\ d(x,z)=2}} -\frac{1}{\mathrm{d}_A(z)+1} + \frac{3}{4}\mathrm{d}_{N_2}(y) + \mathrm{d}_{N_1}(y) \geq \frac{2K + \mathrm{d}(x) - 3}{4} - \frac{1}{2}.$$

Rearranging this inequality yields the result of the theorem. □

4 Applications and Examples

4.1 Connectivity

Thus far we have seen several ways in which curvature of a graph can be used to certify similar properties to spectral properties, only locally. Eigenvalues of the Laplacian are well known to certify notions of connectivity of a graph in various forms: through Fielder's Theorem[8], Kirchhoff's matrix tree theorem [14], and Cheeger's inequality, for example.

In this section we present two results relating the curvature to connectivity. One, like before, is a 'local' connectivity result. The other, however, actually provides a truly global connectivity bound. We highlight here that a curvature bound alone cannot imply that a graph is connected. If one takes two disjoint copies of a single graph, curvature can never see the existence of the other copy – a lower curvature bound for one copy implies a lower curvature bound for both. None the less, a curvature lower bound *does* imply a strong bound on the connectivity of a connected graph. We note that a lower bound on connectivity for positively curved graphs can also be obtained by combining results of Chung, Lin and Yau [4] and Fielder [8], however, we find a result which applies even when the curvature lower bound for a graph is not too negative.

We say G is l-connected if after the removal of any $l-1$ vertices G remains connected. We define $\kappa(G)$ as the largest l for which G is l-connected. We see through Corollary 9 that even moderate curvature conditions ensure balls of radius 2 to be well stitched together, we exploit the manner in which these balls overlap to provide a global connectivity bound.

Theorem 11 (Global Connectivity)**.** Suppose G is a connected graph which satisfies $CD(\infty, K)$ with minimum degree δ. Then

$$\kappa(G) \geq \frac{2K + \delta + 5}{8}.$$

Remark: Unlike most theorems in this paper, Theorem 11 requires the graph to satisfy $CD(\infty, K)$ at *all* $x \in V(G)$.

Proof. Suppose $G = (V, E)$ is a connected graph with vertex connectivity $\kappa(G)$. Then there exists $U = \{x_1, \ldots, x_{\kappa(G)}\} \subseteq V$ such that $G \setminus U$ is not connected. Label the components of $V \setminus U$ as $A_1, A_2, \cdots, A_k$. Let $S = A_1$ and $\overline{S} = \cup_{i=2}^{k} A_i$. Choose U_S and $U_{\overline{S}}$ to partition U in such a way as to minimize $e(S \cup U_S, \overline{S} \cup U_{\overline{S}})$. Notice that as a consequence of the minimization of $e(S \cup U_S, \overline{S} \cup U_{\overline{S}})$, for all $x_i \in U_S$, it is the case that $\mathrm{d}_{S \cup U_S}(x_i) \geq \mathrm{d}_{\overline{S} \cup U_{\overline{S}}}(x_i)$ and for $x_i \in U_{\overline{S}}$, it is the case that $\mathrm{d}_{\overline{S} \cup U_{\overline{S}}}(x_i) \geq \mathrm{d}_{S \cup U_S}(x_i)$. Fix $x \in U$ which maximizes $\min\{\mathrm{d}_S(x_i), \mathrm{d}_{\overline{S}}(x_i)\}$. Without loss of generality, assume $x \in U_S$. We now apply Corollary 9 with $X = N_1(x) \cap (S \cup U_S)$, $\overline{X} = N_1(x) \cap (\overline{S} \cup U_{\overline{S}})$, $A = N_2(x) \cap (S \cup U_S)$, and $\overline{A} = N_2(x) \cap (\overline{S} \cup U_{\overline{S}})$. From this we see,

$$e(N_{1,2}(x) \cap (S \cup U_S), N_{1,2}(x) \cap (\overline{S} \cup U_{\overline{S}})) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{4}\right) \frac{|X| \cdot |\overline{X}|}{\mathrm{d}(x)}.$$

Notice that every edge passing from $S \cup U_S$ to $\overline{S} \cup U_{\overline{S}}$ is adjacent to a member of U, and by our choice of x, each member of U is incident to at most $|\overline{X}|$ of such edges. This allows us to bound the number of edges in $N_{1,2}(x)$ which pass from $S \cup U_S$ to $\overline{S} \cup U_{\overline{S}}$ above by $(\kappa(G) - 1)|\overline{X}|$,

$$(\kappa(G) - 1)|\overline{X}| \geq \left(\frac{2K + \mathrm{d}(x) - 3}{4}\right) \frac{|X| \cdot |\overline{X}|}{\mathrm{d}(x)}.$$

Recalling that $\frac{|X|}{\mathrm{d}(x)} \geq \frac{1}{2}$, we see

$$(\kappa(G) - 1) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{8}\right).$$

Therefore,

$$\kappa(G) \geq \frac{2K + \mathrm{d}(x) + 5}{8}.$$

□

We remark that such a curvature based lower bound can, in theory at least, provide an efficient way of bounding the connectivity of a graph. Classical algorithms for determining vertex connectivity run in $O(|V(G)|^3|E(G)|)$ time ($O(|V| \cdot |E|)$ for determining the number of vertex disjoint paths between any fixed pair of vertices). Meanwhile, as briefly discussed above in Section 3, for a bounded degree graph the curvature at a given point can be computed in constant (depending on the degree bound) time and hence a global curvature bound can be computed in linear time see [6] – a reasonably accurate curvature calculation at vertex requires a binary search on potential curvatures, taking $\log(\Delta)$ checks, and each check requires an eigenvalue calculation of a matrix of size $O(\Delta^2)$ (serving as an upper bound on the size of the second neighborhood of x) which takes at most $O(\Delta^6)$ time. Therefore, a curvature computation of graph with maximum degree Δ takes at most time $O(\Delta^6 \log(\Delta)|V(G)|))$. In practice, we expect to see degrees much smaller than the size of the network, making a curvature calculation in linear in V potentially more efficient than a more classical computation.

Finally, we present a slight improvement of a result in [7]. In [7], Cushing et al. showed that, with very few exceptions, if the curvature at x is positive then $\mathring{B}_2(x)$ is connected. Through Corollary 9 we show that with more moderate curvature assumptions, allowing for negative curvature, this remains true.

Theorem 12 (Local Connectivity)**.** Suppose G is a graph satisfying $CD(\infty, K)$ at $x \in V(G)$ and $2K + \mathrm{d}(x) > 3$, then $\mathring{B}_2(x)$ is connected.

Proof. Let $S, \overline{S}$ be nonempty and partition $\mathring{B}_2(x)$. Suppose $N_1 \subseteq \overline{S}$. Since S is nonempty it contains a member of N_2 and we see $e(S, \overline{S}) > 0$. A symmetric argument covers the case when $N_1 \subset S$. Supposing that $N_1 \cap S$ and $N_1 \cap \overline{S}$ are both nonempty, we use the criss cross lemma letting $X = N_1 \cap S$, $A = N_2 \cap S$, $\overline{X} = N_1 \cap \overline{S}$ and $\overline{A} = N_2 \cap \overline{S}$ to find $e(S, \overline{S}) \geq \left(\frac{2K + \mathrm{d}(x) - 3}{4}\right) \cdot \frac{|X| \cdot |\overline{X}|}{\mathrm{d}(x)}$ which by assumption is greater than 0. Therefore, for all $S, \overline{S}$ nonempty and partitions of $\mathring{B}_2(x)$, $e(S, \overline{S}) > 0$ and therefore $\mathring{B}_2(x)$ is connected. $\square$

Sharpness Example for Theorem 12: The two graphs in Figure 1 show where Theorem 12 is sharp. The graph on the right has curvature -1.5 and $d(x) = 6$ so $2K + d(x) = 3$ and we see that $\mathring{B}_2(x)$ is disconnected. On the other hand, the graph on the left has curvature $-.5$, $d(x) = 4$, we also have $2K + d(x) = 3$ but in this case $\mathring{B}_2(x)$ is connected. Following the string of inequalities leading to Corollary 9, we observe that the inequality in Theorem 12 only needs to be made strict when $N_2(x)$ is empty.

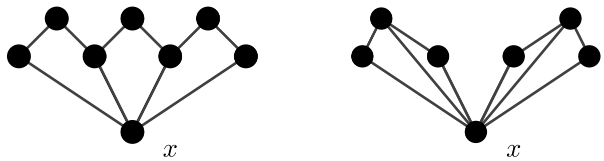

Figure 1: In the graph on the left, $\mathring{B}_2(x)$ is connected and $N_2(x)$ is nonempty, while in the graph on the right $\mathring{B}_2(x)$ is disconnected and $N_2(x)$ is empty.

4.2 Disjoint Cycles

Many of the most classical problems in extremal combinatorics deal with problems of subgraph containment: under what conditions are various subgraphs forced to exist. Likewise, in the study of complex networks, the study of motifs found in large graphs has attracted heavy study.

In a classical paper in this vein, Corradi and Hajnal [5] found that a sufficiently large graph with minimum degree δ contains at least $\frac{\delta}{2}$ disjoint cycles. This has inspired dozens of generalizations all with the theme of certifying the existence of many (typically small) disjoint subgraphs of graphs.

In this section we show that curvature can be used, in addition to degree information, to provide stronger bounds on the existence of small disjoint cycles in graphs. Curvature most naturally gives a vehicle to study cycles of length 3 and 4, as the curvature of a graph only gives us information within the first two neighborhoods of any given vertex. However, the local discrepancy inequalities above can be used to quantify the number of these short cycles that an edge must be contained in.

A consequence of $\mathring{B}_2(x)$ being connected is that every edge from x is in either a triangle or a 4-cycle. Therefore, Theorem 12 or the result of Cushing gives us the following corollary regarding small cycles.

Corollary 13. Suppose G is a graph satisfying $CD(\infty, 0)$ at $x \in V(G)$ with $d(x) \geq 4$. Then for all $y \in N_1(x)$, there exists a 3-cycle or a 4-cycle containing the edge xy.

Under a slightly more stringent degree requirement, Theorem 10 can be used to improve this guarantee as follows.

Theorem 14. Suppose G is a graph satisfying $CD(\infty, 0)$ at $x \in V(G)$ with $d(x) \geq 6$. Then for every $y \in N_1(x)$, there exists a 3-cycle containing the edge xy or at least $\frac{1}{2}\mathrm{d}(y)$ 4-cycles containing the edge xy.

Before proving this statement, we need the following simple lemma.

Lemma 15. Suppose $a_1, \ldots, a_n$ are non-negative integers satisfying $\sum_{i=1}^{n} a_i \leq \frac{1}{2}n$. Then

$$\sum_{i=1}^{n} \frac{1}{a_i + 1} \geq \frac{3}{4}n.$$

Proof. Consider a sequence of nonnegative integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i \leq \frac{1}{2}n$ that minimizes $\sum_{i=1}^{n} \frac{1}{a_i+1}$. Then there exists $j \in \{1, \ldots, n\}$ such that $a_j = 0$. If $a_k \geq 2$ for some k, then replacing a_j and a_k with ones would preserve or decrease $\sum_{i=1}^{n} a_i$. However, this replacement would also reduce $\sum_{i=1}^{n} \frac{1}{a_i+1}$ because $\frac{1}{2} + \frac{1}{2} < 1 + \frac{1}{a_k+1}$, contradicting the minimality of the sequence. Thus, $a_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$.

Since $a_1,\dots,a_n$ must be a $\{0,1\}$-sequence, $\sum_{i=1}^n \frac{1}{a_i+1}$ decreases as the number of ones increases. Thus, the $\{0,1\}$ sequence that minimizes this sum must contain $\lfloor \frac{1}{2}n \rfloor$ ones, the maximum allowable subject to the constraint $\sum_{i=1}^n a_i \le \frac{1}{2}n$. For this minimizing sequence, with n even (n odd produces a slightly larger sum), we have

$$\sum_{i=1}^n \frac{1}{a_i+1} = \frac{3}{4}n.$$

□

Proof of Theorem 14. Fix two adjacent vertices $x, y \in V(G)$. By Theorem 10, we have that

$$\sum_{\substack{z\in N(y)\\ d(x,z)=2}} \frac{1}{\mathrm{d}_A(z)+1} \le -\frac{1}{4}\mathrm{d}(x) + \mathrm{d}_{N_1}(y) + \frac{3}{4}\mathrm{d}_{N_2}(y) + \frac{5}{4}.$$

Suppose that there are no triangles involving edge xy. Then $\mathrm{d}_{N_1}(y) = 0$ as every edge within N_1 forms a triangle with x. Furthermore, if there exists an edge between a vertex in $z \in N(y)$ with $d(x,z) = 2$ and $y' \in A$, then $xyzy'$ forms a four-cycle including the edge xy.

Suppose by way of contradiction that

$$\sum_{\substack{z\in N(y)\\ d(x,z)=2}} \mathrm{d}_A(z) \le \frac{1}{2}\mathrm{d}_{N_2}(y).$$

By the above lemma, this implies that

$$\sum_{\substack{z\in N(y)\\ d(x,z)=2}} \frac{1}{\mathrm{d}_A(z)+1} \ge \frac{3}{4}\mathrm{d}_{N_2}(y).$$

Combining this inequality with the inequality derived from Theorem 10 yields

$$\frac{3}{4}\mathrm{d}_{N_2}(y) \le -\frac{1}{4}\mathrm{d}(x) + \frac{3}{4}\mathrm{d}_{N_2}(y) + \frac{5}{4}.$$

This is equivalent to $\mathrm{d}(x) \le 5$, which contradicts the assumption that $\delta(G) \ge 6$. Thus,

$$\sum_{\substack{z\in N(y)\\ d(x,z)=2}} \mathrm{d}_A(z) \ge \frac{1}{2}\mathrm{d}_{N_2}(y),$$

meaning that G must contain at least $\frac{1}{2}\mathrm{d}(y)$ 4-cycles containing the edge xy. □

In this calculation, it is important that $\mathrm{d}_A(z)$ is an integer. In Lemma 15, if $a_1,\dots,a_n$ were nonnegative real numbers, the lower bound on $\sum_{i=1}^n \frac{1}{a_i+1}$ would instead by $\frac{2}{3}n$ by Jensen's inequality. Furthermore, such a lower bound could not be improved, as taking $a_i = \frac{1}{2}$ for all i realizes this lower bound. However, due to our application on graphs, a_i must be an integer, giving us this coefficient of $\frac{3}{4}$. This lower bound is ideal, as the left side cancels out the $\frac{3}{4}\mathrm{d}_{N_2}(y)$ on the right side. Furthermore, this lower bound is realized when $\mathrm{d}_A(z) = 1$ for half of the neighbors of y with $d(x,z) = 2$ and $\mathrm{d}_A(z) = 0$ for the other such neighbors. With this partition,

$$\sum_{\substack{z\in N(y)\\ d(x,z)}} \frac{1}{\mathrm{d}_A(z)+1} = \frac{3}{4}\mathrm{d}_{N_2}(y).$$

If this sum were any smaller, as it would be if more than half of the neighbors of y had an edge to A, then we would get a result comparing $\mathrm{d}(x)$ and $\mathrm{d}(y)$, which would in turn force a regularity-type condition on the graph to obtain a similar result.

This local result about the number of 4-cycles containing an edge can be transformed into global results about edge-disjoint and vertex-disjoint 4-cycles.

Corollary 16. Suppose G is a d-regular, triangle-free graph with $d \ge 6$ satisfying $CD(\infty, 0)$. Then G contains at least $\frac{n}{64}$ edge-disjoint 4-cycles.

Proof. By Theorem 14, every edge is in at least $\frac{1}{2}d$ 4-cycles. Since there are $\frac{nd}{2}$ edges, this yields at least $\frac{d^2n}{16}$ total 4-cycles in G. Note that any edge can appear in at most d^2 4-cycles, so each 4-cycle shares an edge with at most $4d^2$ 4-cycles. Therefore, the graph must contain at least $\frac{n}{64}$ edge-disjoint 4-cycles. □

In the above proof, we examined the number of 4-cycles in which a given edge can appear in order to get a lower bound on the number of edge-disjoint 4-cycles. Similarly, examining the number of 4-cycles in which a given vertex can appear produces a lower bound on the number of vertex-disjoint 4-cycles in a graph.

Corollary 17. Suppose G is a d-regular, triangle-free graph with $d \geq 6$ satisfying $CD(\infty, 0)$. Then G contains at least $\frac{n}{32d}$ vertex-disjoint 4-cycles.

Proof. By Theorem 14, every edge is in at least $\frac{1}{2}d$ 4-cycles. Since there are $\frac{nd}{2}$ edges, this yields at least $\frac{d^2 n}{16}$ total 4-cycles in G. Fix a vertex x. Since a 4-cycle containing x must have two neighbors of x and a second vertex adjacent to each of these neighbors, x is contained in at most $\binom{d}{2}(d-1) \leq \frac{d^3}{2}$ 4-cycles. Thus, every 4-cycle shares a vertex with at most $2d^3$ other 4-cycles. Dividing the total number of 4-cycles in G by this maximum number of 4-cycles that share a vertex yields a total of at least $\frac{n}{32d}$ vertex-disjoint 4-cycles. □

Unfortunately, this result decreases in d, which does not mesh with our intuition. Ultimately, as the degree of a regular graph increases, so too should the number of disjoint cycles. For graphs with large degree, this bound is quite bad. For example, this bound gives only 1 disjoint cycle in a complete graph. For graphs with small degree, however, this bound is much closer to correct. For example, the hypercube Q_d can be decomposed into $\frac{n}{4}$ 4-cycles while the above bound gives at least $\Omega(\frac{n}{\log n})$ vertex-disjoint 4-cycles. We believe that the above technique can be improved to guarantee $\Omega(n)$ disjoint 4-cycles.

Despite its flaws, this result does give us an improvement on the result by Corradi and Hajnal if d is small. In fact, these two results can be combined to give at least $\Omega(\sqrt{n})$ disjoint cycles on a sufficiently large graph.

Corollary 18. Suppose G is a d-regular, triangle-free graph on at least $\frac{3}{2}d$ vertices with $d \geq 6$ satisfying $CD(\infty, K)$. Then G has at least $\Omega(\sqrt{n})$ disjoint cycles.

Proof. If $d \geq \Omega(\sqrt{n})$, then the above result of Corradi and Hajnal states that G has at least $\frac{d}{2} \geq \Omega(\sqrt{n})$ disjoint cycles. If $d \leq O(\sqrt{n})$, then our above result states that G contains at least $\frac{n}{32d} \geq \Omega(\sqrt{n})$ disjoint 4-cycles. □

4.3 Examples

Finally, we present a few examples to illustrate that the inequality presented in Theorem 3 can actually be tight. This implies, in turn, that the functions considered actually witness the limiting curvature for some family of graphs. This, in turn, means that the discrepancy properties of graph neighborhoods can, at times, imply sharp upper bounds on the curvature for graph families.

Example 1: Regular Trees We find that Corollary 5 gives a sharp upper bound on graph curvature of regular trees. Suppose that G is a d-regular tree. In any tree, every vertex in N_2 must be adjacent to exactly one vertex in N_1, as any vertex $z \in N_2$ with $\mathrm{d}_{N_1}(z)$ would form a 4-cycle with x and its neighbors in N_1. Thus, we have that

$$\sum_{z \in N_2} |\mathrm{d}_{\overline{X}}(z) - \mathrm{d}_X(z)| = \sum_{z \in N_2} 1 = d(d-1).$$

Furthermore, every tree is triangle-free, which implies that $e(X, \overline{X}) = 0$. Our theorem, in this case, states that

$$d(d-1) \leq \frac{3}{4}d(d-1) - \frac{2K + d - 3}{4}d.$$

Solving for K here yields that $K \leq 2 - d$, and as shown in [15], the curvature of a tree is exactly $K = 2 - d$. Therefore, our theorem gives a sharp upper bound on the curvature of a d-regular tree.

Example 2: $\mathbb{Z}^d$ In this example, we find Theorem 3 provides a sharp upper bound on the curvature of $\mathbb{Z}^d$. Consider the graph $\mathbb{Z}^d$. Let $x = (0, \ldots, 0)$. Then $N_1(x) = \{(y_1, \ldots, y_d) : \sum |y_i| = 1\}$ and $N_2(x) = \{(z_1, \ldots, z_d) : \sum |z_i| = 2\}$. Define $X \subseteq N_1(x)$ to be the points $(y_1, \ldots, y_d)$ where $y_i = 1$ for some $i \in \{1, \ldots, d\}$ and $y_j = 0$ for all $j \neq i$. Then $\overline{X} \subseteq N_1(x)$ is the set of points $(y_1, \ldots, y_d)$ where $y_i = -1$ for some $i \in \{1, \ldots, d\}$ and $y_j = 0$ for all $j \neq i$. Note here that $\alpha = \frac{1}{2}$, as N_1 is split evenly according to this partition.

Let $z \in N_2(x)$. We will analyze how each possible z contributes to the sum on the left side.

- If z contains two 1s, then $|\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)| = 1$ and $\mathrm{d}_{N_1}(z) = 2$. Thus, the contribution of z to the sum is $\frac{1}{2}$. There are $\binom{d}{2}$ such vertices.
- If z contains two -1s, then $|\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)| = 1$ and $\mathrm{d}_{N_1}(z) = 2$. Thus, the contribution of z to the sum is $\frac{1}{2}$. There are again $\binom{d}{2}$ such vertices.

- If z contains a 2, then $|\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)| = \frac{1}{2}$ and $\mathrm{d}_{N_1}(z) = 1$. Thus, the contribution of z to the sum is $\frac{1}{4}$. There are d such vertices.
- If z contains a -2, then $|\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)| = \frac{1}{2}$ and $\mathrm{d}_{N_1}(z) = 1$. Thus, the contribution of z to the sum is $\frac{1}{4}$. There are d such vertices.
- If z contains a 1 and a -1, then $|\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)| = 0$ and $\mathrm{d}_{N_1}(z) = 2$. Thus, the contribution of z to the sum is 0. There are $2\binom{d}{2}$ such vertices.

Thus, the sum over all z yields

$$\sum_{z \in N_2} \left(\frac{[\alpha \mathrm{d}_{\overline{X}}(z) - (1-\alpha)\mathrm{d}_X(z)]}{\mathrm{d}_{N_1}(z)} \right) = \frac{1}{2}\binom{d}{2} + \frac{1}{2}\binom{d}{2} + \frac{1}{4}d + \frac{1}{4}d + 0 \cdot 2\binom{d}{2} = \frac{1}{2}d^2.$$

On the right side, $\alpha = \frac{1}{2}$. Thus,

$$\frac{3}{4}[\alpha^2 \cdot e(\overline{X}, N_2) + (1-\alpha)^2 e(X, N_2)] = \frac{3}{16} e(N_1, N_2).$$

To compute $e(N_1, N_2)$, every vertex in N_2 with two nonzero coordinates has two neighbors in N_1 and there are $4\binom{d}{2}$ of these vertices. Also, every vertex in N_2 with one nonzero coordinate has one neighbor in N_1 and there are $2d$ of these vertices. Thus, $e(N_1, N_2) = 8\binom{d}{2} + 2d = 4d^2 - 2d$. Since $\mathbb{Z}^d$ is triangle-free, we have that $e(X, \overline{X}) = 0$. Finally, the curvature term yields $\frac{d(2K+2d-3)}{8}$.

Therefore, our discrepancy inequality yields that

$$\frac{1}{2}d^2 \leq \frac{3}{16}(4d^2 - 2d) - \frac{d(2K + 2d - 3)}{8}.$$

Solving for K yields that $K \leq 0$, which again is a tight upper bound as [15] gives that $K = 0$.

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