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# Extremal problems on ray sensor configurations

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ABSTRACT

A sensor network is typically modeled as a collection of spatially distributed objects with the same shape. The *resilience* of a sensor network is the number of sensors that can be removed without disrupting the coverage which is the network's purpose. We introduce two new extremal problems for networks of one-dimensional sensors (lines, rays, and segments) in the two-dimensional plane, where network coverage means protecting locations from external intruders. (1) How well is it possible to simultaneously protect $k$ locations with $n$ (line/ray/segment)-shaped sensors from up to $k$ attackers? (2) How well is it possible to simultaneously protect $k$ locations with $n$ ray-shaped sensors from a single attacker? We show first that (1) and (2) are questions to be answered separately and provide complete answers for $k = 2$ locations for both questions. We also provide asymptotically tight answers for question (1) when $k = 3, 4$ and the locations are in convex position, and provide a general lower bound for question (1) that matches the specific asymptotic results.



## 1. Introduction

The physically distributed nature of wireless sensor networks, which consist of spatially distributed sensors deployed to monitor physical conditions in their environment, lends particular importance to the network's geometry.

A geometric model is chosen to represent the network's coverage: disks in the plane [5] can model the effective range of cell towers or broadcasting drones; line segments [1,8], rays [3,6], or infinite lines, might abstract the behavior of a laser beam or other barriers designed to detect or impede intruders. In each case the geometry of the network of sensors is a major factor in the range of possibilities for the coverage of such a network.

Previous work has examined the *resilience* of sensor networks [2–4,6,8], broadly defined as the number of sensors whose removal from a network does not diminish its coverage. Resilience is distinct from $k$-barrier coverage [7], which seeks $k$-fold coverage. Some results on resilience are algorithmic and describe disk networks [2] or ray networks [6], some results describe computational complexity [6], and others explore taxonomy of possible networks [3].

Motivated by previous work on the resilience of ray sensor networks, we seek in this paper to understand the following very natural situation: Suppose intruders approach a set of $k$ point locations, protected by a network of $n$ ray sensors, from afar and from an unknown direction. The quality of a network of sensors, then, can be measured in how many sensors must be disabled in order for intruders to reach their destinations. That is, how many of the rays must be removed before

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the target points are *exposed*, i.e. until each lies in an unbounded region. There are two natural and distinct notions of resilience for this kind of coverage, which lead to two natural extremal questions regarding how well a set of points can be protected:

**Question 1.** *Given $k$ designated target locations and allowing the placement of $n$ ray barriers, what number of those barriers is always sufficient and sometimes necessary to remove so that all target locations are exposed?*

With regard to this question we establish a general lower bound of the form $\frac{k}{k+1}n + O(1)$, and prove that such a bound is tight for $k = 1, 2, 3$, and for $k = 4$ provided the points are in convex position. This first question models a situation where there are as many intruders as target locations. It turns out that Question 1 has the same answer whether the network consists of line segments, rays, or infinite lines (see Proposition 3), but interestingly, this does not hold when there is only one intruder who must reach all locations, which is modeled by the following question:

**Question 2.** *Given $k$ designated target locations and allowing the placement of $n$ ray barriers, what number of those barriers is always sufficient and sometimes necessary to remove so that a **single** unbounded region contains all of the targets?*

We mention that these questions, while motivated by the topic of ray sensor networks, are natural questions in combinatorial geometry regarding how well a collection of rays can 'protect' a set of $k$ points. As mentioned above, the first question is agnostic to whether the barriers considered are rays, lines, or segments, but for the second question, rays are the only interesting type of barrier, in a sense.

We also find the answer to Question 2 quite interesting. While in the $k = 1$ case this is identical to the answer to Question 1 above (and hence, asymptotically is $\frac{n}{2}$), we show that for $k = 2$ the answer differs and is actually asymptotically $\frac{3}{4}n$ rather than $\frac{2}{3}n$.

The rest of this paper is organized as follows: in Section 2 we further formalize Question 1, give a general lower bound for any number of locations and sensors in general position, and give asymptotically tight results for $k = 1, 2, 3$ with locations in general position and for $k = 4$ locations in convex position; in Section 3 we address Question 2 and describe the cases when $k = 1$ and 2.

## 2. Exposing points

The first extremal question we examine asks how well a collection of locations can be protected from multiple intruders. This question is interesting and natural for rays, but it is also natural to define the question for lines and segments.

Let $p$ be a point in the plane, and consider a partition of the plane into connected regions. We say $p$ is ***exposed*** if the region containing $p$ is unbounded, and that the partition ***exposes*** $p$. Similarly, a set $P$ of points is exposed if every point in it is exposed.

Consider a finite set $P$ of points, and suppose the plane partition is induced by a set $R$ of rays, a set $L$ of lines, or a set $S$ of segments. A subset $X \subseteq R$ is an ***exposing set*** if the partition induced by $R \setminus X$ exposes $P$. The definition is the same for an exposing set of lines or of segments.

We define $\boldsymbol{r_k(n)}$ to be the smallest integer so that for any $k$ points, any set of $n$ rays has an exposing set of size $r_k(n)$. We call the analogous quantity when lines are used instead of rays $\boldsymbol{\ell_k(n)}$, and the quantity for segments we denote by $\boldsymbol{s_k(n)}$. When the points are in convex position, we call these quantities $\boldsymbol{r^c_k(n)}$, $\boldsymbol{\ell^c_k(n)}$ and $\boldsymbol{s^c_k(n)}$ respectively.

A ray emanating from a point $p$ is said to be anchored at $p$. A ***witness set*** is a set of $k$ rays, one anchored at each of the $k$ points.

**Observation 1.** *A witness set $W$ induces an exposing set $E(W)$ composed of those rays in $X$ (resp. lines or segments) its rays intersect.*

Thus, a witness set $W$ can be thought of as a set of routes of attack for $k$ intruders attacking the $k$ points that removing $E(W)$ exposes. Witness sets are naturally tied to exposing sets, in that for each exposing set there is a witness set that witnesses the fact that it exposes the target points.

**Lemma 2.** *Let $L$ be a set of $n$ lines and $P$ a set of $k$ points. Suppose $X \subseteq L$ is an exposing set of lines. Then there exists a witness set $W$ such that $E(W) \subseteq X$.*

**Proof.** Consider a point $p \in P$. It suffices to show that there is a ray anchored at $p$ completely contained in the unbounded region of the plane partition induced by $L \setminus X$. If $L = X$ then we may choose any ray, so suppose $X \neq L$.

The unbounded region containing $p$ is convex, as it is the intersection of half-planes, so its boundary contains at least one unbounded ray $r$. The ray anchored at $p$ parallel to $r$ does not intersect any element of $X$. $\square$

We call such a set $W$ a *witness set of* $X$, as its existence witnesses that $X$ is an exposing set. From Lemma 2 it is clear that finding an exposing set of line barriers is the same as finding a witness set. Note that Lemma 2 implies that if $X$ is an exposing set for a set of lines $L$, and $L'$ is another set of lines, then $X \cup L'$ is an exposing set for $L \cup L'$.

**Proposition 3.** $r_k(n) = \ell_k(n) = s_k(n) \geq r_k^c(n) = \ell_k^c(n) = s_k^c(n)$.

**Proof.** Fix a collection $P$ of $k$ points, and let $L$ be a set of $n$ lines. Choose a set of rays $R$ so that each ray in $R$ has a distinct line of $L$ as its supporting line, and so that each ray intersects all lines intersected by its supporting line. Choose a set of segments $S$ supported by lines in $L$ similarly.

Suppose $L' \subseteq L$ is an exposing set of minimal size. Let $R' \subseteq R$ be the set of rays whose supporting lines are in $L'$, and let $S' \subseteq S$ be similar for segments. Then $P$ is also exposed by $R \setminus R'$ and $S \setminus S'$, because each point of $P$ is in an unbounded region according to the partition induced by $L \setminus L'$, which is a refinement of the partitions according to $R \setminus R'$ and $S \setminus S'$.

On the other hand, the supporting lines of an exposing set of rays (or of segments) are witnessed by the same witness set that witnesses the rays. Since $P$ was arbitrary, this means the minimal size of an exposing set for any collection of $k$ points is the same whether we are using line barriers, ray barriers, or segment barriers. $\square$

**Remark.** Proposition 3 allows us to address the value of $r_k(n)$ by examining only line barriers, since the arguments using these seem to be more tractable.

We note in the next lemma that for fixed $k$, adding one barrier to any set of barriers can increase the size of the smallest exposing set by at most one, and of course cannot reduce the size of the smallest exposing set.

**Lemma 4.** $0 \leq r_k(n+1) - r_k(n) \leq 1$

Next we prove an asymptotic lower bound for $r_k(n)$ for all $k$. We then prove asymptotically matching upper bounds for $k = 1, 2, 3$, and 4. The upper bound proofs have a common flavor, in that we choose a small collection of canonical witness sets $W$ depending on the set of points $P$, show that each $E(W)$ is always an exposing set for $P$, and moreover that its size is small enough. In spite of this common flavor, generalizing these upper bounds as with the lower bounds has proven elusive.

**Theorem 5.** $r_k(n) \geq r_k^c(n) \geq \left\lceil \frac{k}{k+1} n \right\rceil - k$

**Proof.** Let $P$ be any set of $k$ points in the plane, convex or not. We construct a set of lines so that any exposing set has the desired size.

First suppose that $n = a(k+1)$. Choose $a$ different slopes that are not the slope between any pair of points in $P$. For each chosen slope $t$, place $k+1$ line barriers so that each pair of these barriers closest to each other has a single point between them (see Fig. 1 for an example with $a = 2$ and $k = 6$). These slope $t$ barriers we will call $t$-barriers.

Consider an exposing set $X$ of minimal size, and let $W$ be a witness set as guaranteed by Lemma 2. We proceed to lower-bound $|E(W)|$, and hence $|X|$.

Fix a slope $t$ among the chosen slopes and let $k'(t)$ denote the number of rays in $W$ of that slope. Note that $t$ induces an order on $P$ and the $t$-barriers, and each ray anchored at $p \in P$ not parallel to $t$ intersects all $t$-barriers either above or below $p$.

From this it is easy to see that rays in $W$ cross at least $k - k'(t)$ $t$-barriers.

Thus

$$|X| \geq |E(W)| \geq \sum_t (k - k'(t)) = \sum_t k - \sum_t k'(t) \geq ak - k = \frac{k}{k+1} n - k.$$

If instead $n = a(k+1)+b$, just add the additional $b < k+1$ barriers in a group of parallel barriers with slope different from the rest of the groups. The argument above applies to this group as well, except that we only have $b$ total barriers instead of $k+1$, so the barriers from this group crossed by witness rays is now $b - k'(t)$. This implies

$$|E(W)| \geq ak + b - k = \frac{k}{k+1}(n-b) + b - k \geq \left\lceil \frac{k}{k+1} n \right\rceil - k. \quad \square$$

Because the proof of Theorem 5 did not assume convexity of the $k$ points, it holds as a lower bound both for $r_k(n)$ and $r_k^c(n)$.

**Lemma 6.** *Suppose $P$ is a set of any number of points and $L$ a set of lines such that at most one line intersects the convex hull of $P$ (and no line of $L$ contains a point of $P$). Then there is an exposing set in $L$ of size at most $\lfloor (|L| - 1)/2 \rfloor$.*

**Proof.** Suppose there is no line through the convex hull of $P$. Then all points lie in the same convex region of the partition. Pick an arbitrary point $p$ in that region, and line $\ell \in L$. Consider the rays from $p$ parallel to $\ell$. One of these rays witnesses an exposing set of size at most $\left\lfloor \frac{1}{2}(|L| - 1) \right\rfloor$ by the pigeonhole principle.

If there is a line $\ell \in L$ through the convex hull, the points in $P$ are split into two regions. The proof proceeds similarly, taking rays starting at two arbitrary points, one in each of the two regions, parallel to $\ell$. $\square$

Lemma 6 allows to compute an exact bound when $k = 1$.

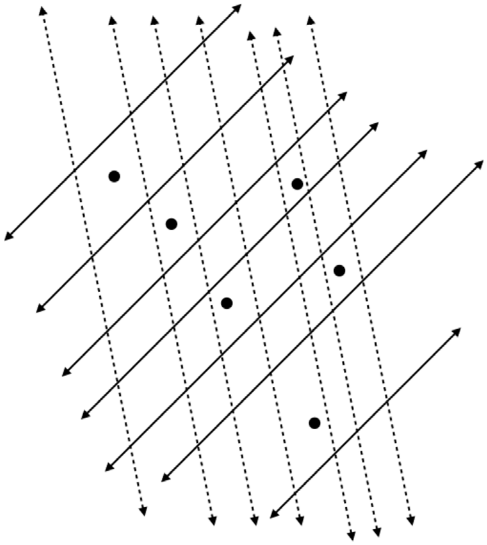

**Fig. 1.** Two sets of 7 $t$-barriers for a set of 6 points.

**Corollary 7.** $r_1(n) = \lfloor \frac{1}{2}(n-1) \rfloor$.

**Proof.** Let $L$ be any set of $n$ lines not containing some target point $p$. By Lemma 6, there is an exposing set of size at most $\lfloor \frac{1}{2}(n-1) \rfloor$. Thus, $r_1(n) \leq \lfloor \frac{1}{2}(n-1) \rfloor$. Furthermore, Theorem 5, with $k = 1$, implies that $r_1(n) \geq \lfloor \frac{1}{2}(n-1) \rfloor$. □

**Theorem 8.** $r_2(n) = \frac{2}{3}n - O(1)$.

**Proof.** Let $L = L_I \cup L_E$ be any set of $n$ lines not containing points $p$ and $q$, where $L_I$ are the lines that pass between $p$ and $q$ (the "interior" lines), and $L_E$ are the rest (the "exterior" lines). Let $|L| = n$ and $|L_I| = i$, so that $|L_E| = n - i$.

$L_E$ is clearly an exposing set of size $n - i$, witnessed by the rays along the line through $p$ and $q$ pointing outward. On the other hand, another exposing set of size at most $i + \frac{n-i}{2}$ is obtained by removing all elements of $L_I$, and at most half of the elements of $L_E$ (per Lemma 6).

$$r_2(n) \leq \max_i \{n - i, i + (n-i)/2\}$$

This minimum occurs when the two bounds are equal (as one bound is increasing and the other decreasing in $i$), and hence when $i = n/3$, from which we obtain $r_2(n) \leq 2n/3$ as desired. □

A more careful argument, keeping track for the floors and negative-one term in Lemma 6, implies $r_2(n) = \lfloor \frac{2}{3}(n-1) \rfloor$ exactly but, although not difficult, the proof of the tighter lower bound is more complicated than illuminating, so we provide the details in Appendix 5.

**Theorem 9.** $r_3(n) = \frac{3}{4}n - O(1)$.

**Proof.** Let $a$, $b$, and $c$ be any three points, listed in clockwise order. Consider the lines through each pair of these, with the rays at their ends labeled $a_1, a_2, b_1, b_2, c_1$, and $c_2$, in clockwise order according to the nearest point, as in Fig. 2.

Let $L$ be any set of $n$ lines not containing $a$, $b$, or $c$, and separate $L$ into "interior" and "exterior" lines $L_I$ and $L_E$ as before, according to whether they intersect the convex hull of $\{a, b, c\}$ or not, respectively. As before, let $|L| = n$ and $|L_I| = i$, so that $|L_E| = n - i$.

Let $A_1$ be those lines of $L_I$ intersecting $a_1$, $B_2$ be lines of $L_I$ intersecting $b_2$, and so on. For lines of $L_I$ that do not intersect any of these, include them with $A_1$ as well.

Letting $L_1 := A_1 \cup B_1 \cup C_1$ and $L_2 := A_2 \cup B_2 \cup C_2$, we have that $L_I$ is the disjoint union of $L_1$ and $L_2$, since a line intersecting any two of these labeled ends does not separate any of $a$, $b$, or $c$. Then either $|L_1| \leq \frac{i}{2}$ or $|L_2| \leq \frac{i}{2}$; without loss of generality, assume $|L_1| \leq \frac{i}{2}$.

$L_E \cup L_1$ is an exposing set, as witnessed by $W_1 = \{a_1, b_1, c_1\}$, and has size at most $n - \frac{i}{2}$.

For a second bound, we again note that $L_E \cup \{\ell\}$ satisfies the conditions of Lemma 6 for any $\ell \in L_I$, and so there is an exposing set for $L_E \cup \{\ell\} = L \setminus (L_I \setminus \{\ell\})$ of size at most $\lfloor \frac{1}{2}(n-i) \rfloor$, which together with the rest of $L_I$ yields an exposing set of size at most $i + \frac{1}{2}(n-i)$.

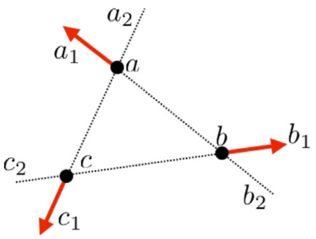


**Fig. 2.** A canonical witness set of three rays for a set of three points.

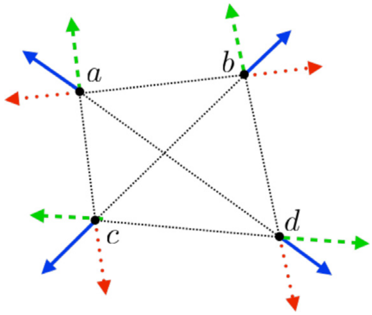


**Fig. 3.** The three canonical witness sets for 4 points in convex position; witness sets are grouped by dash type.

As before, the minimum of these two upper bounds is maximized when they are equal (again, as one bound is increasing and the other decreasing), and hence when $i_* = \frac{n}{2}$. Thus $r_3(n) \le n - i_*/2 = \frac{3}{4}n$.

From Theorem 5 we have that $\lceil \frac{3}{4}n \rceil - 3 \le r_3(n)$, so the result follows. $\square$

**Theorem 10.** $r_4^c(n) = \left\lceil \frac{4}{5}n \right\rceil - O(1)$.

**Proof.** Let $a$, $b$, $c$, and $d$ be points in convex position, and suppose $L$ is a set of $n$ lines containing none of the four points. Let $L_I$ and $L_E$ be the set of lines in $L$ which cross the convex hull of the points ("internal") and the set which do not ("external"), respectively. Again, let $|L| = n$ and $|L_I| = i$, so that $|L_E| = n - i$.

As before, using Lemma 6 yields an exposing set for $L_E \cup \{\ell\}$ (where $\ell \in L_I$) which together with the rest of $L_I$ is an exposing set of size at most $i - 1 + \lfloor \frac{1}{2}(n-i) \rfloor$.

Consider the diagram in Fig. 3. Group the rays anchored at the points and pointing outward along the line between each pair of points, as in the figure. Every line of $L_I$ avoids intersecting one of these groups entirely, so let $R$, $G$, and $B$ be the sets of lines that avoid the red, blue, and green rays respectively (the dotted, dashed, and solid rays). If a line does not intersect any, include it along with $R$.

Since $L_I = R \cup G \cup B$, one of $R$, $G$, or $B$ (say, $R$) must contain at least $\frac{1}{3}i$ lines; therefore, $G \cup B$ contains at most $\frac{2}{3}i$ lines. But $L_E \cup G \cup B$ is an exposing set, witnessed by the red rays, and thus $r_4^c(n) \le n - i + \frac{2}{3}i = n - \frac{1}{3}i$.

Together these bounds give $r_4^c(n) \le \max_i \min \begin{cases} i - 1 + \lfloor \frac{1}{2}(n-i) \rfloor \\ n - \frac{1}{3}i \end{cases}$.

As before, the minimum of these upper bounds is maximized when they are equal, and hence when $i_* = \frac{3}{5}(n+2)$.

This implies $r_4^c(n) \le \lfloor n - \frac{1}{3}(\frac{3}{5}(n+2)) \rfloor = \lfloor \frac{2}{5}(2n-1) \rfloor \le \lfloor \frac{4}{5}n \rfloor$. Together with the result of Theorem 5, we have $\left\lceil \frac{4}{5}n \right\rceil - 4 \le r_4^c(n) \le \lfloor \frac{4}{5}n \rfloor$, so that $r_4^c(n) = \left\lceil \frac{4}{5}n \right\rceil - O(1)$. $\square$

The case for $k \ge 5$, unfortunately, only gets more complicated. While we suspect that Theorem 5 provides the proper asymptotics, for the case $k = 5$ the proof balloons in difficulty even when the points are assumed to be in convex position. Indeed, already in the case $k = 5$, the types of 'pairing' strategies used when $k = 3, 4$, where a number of candidate exposing sets are compared and the best taken, fails when run naively as there are more potential 'escape routes' from each of the $k = 5$ points. For $k = 5$, it actually is possible to recover the result that $r_5^c(n) = \frac{5}{6}n + O(1)$ by recording all possible 'escape routes', and running a linear program but this does not easily generalize to $k \ge 6$ nor is the proof it yields enlightening. Furthermore, while we have no construction to suggest that the convex and general positions differ, we have no proof of this for $k \ge 4$.

We conjecture that $r_k^c(n) = \frac{k}{k+1}n + O(1)$ and suspect that this may remain the case for points in general position, but it seems a different approach will be required to prove this for general $k$.

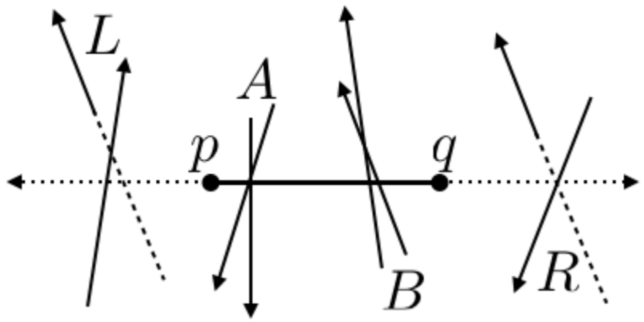


**Fig. 4.** The partition of rays in Theorem 13.

## 3. Simultaneously exposing points

**Definition 11.** Let $P$ be a set of points in the plane, and consider a partition of the plane into connected regions. We say the points of $P$ are ***jointly exposed***, or that the partition ***jointly exposes them***, if they are contained in the same region and that region is unbounded.

A ***joint exposing set*** for a set of rays (resp. segments) $R$ is a subset $X \subseteq R$ such that the partition induced by $R \setminus X$ jointly exposes the points of $P$. Similarly, a ***joint witness set*** is a set of rays, one anchored at each point of $P$, with a common direction.

$\boldsymbol{R_k(n)}$ is the smallest integer so that for any $k$ points, any set of $n$ rays has a joint exposing set of size $R_k(n)$.

**Observation 12.** *A joint witness set $W$ induces a joint exposing set $J(W)$ composed of those rays it intersects.*

Previously, when studying $r_k(n)$, we observed that the quantity was the same regardless of whether we considered ray barriers, line barriers, or segment barriers. Interestingly, this is not the case when studying $R_k(n)$, where it turns out that only ray barriers are interesting.

Indeed, if a finite number of segments is used to partition the plane, then there is only one unbounded region, and hence jointly exposing a set of points is the same as exposing the set of points. If lines are used, then placing all lines between two points forces one to remove all of them to put the points in the same unbounded region. If rays are used, however, the extremal problems for exposing a set of points and for jointly exposing them are genuinely different.

Our lower bound argument will still take the form of a construction: an arrangement of rays which requires at least a particular fraction of them to be part of any joint exposing set. We provide and argue bounds for such a construction for $R_2(n)$ in the remainder of this paper, but the (asymptotically) matching upper bound is relatively simple and, as before, relies on finding canonical witness sets, as we argue in the next theorem.

In arguing each bound we will make use of some results from [6] about an auxiliary graph, called the ***barrier graph***, used for the case of finding a path between two points by removing some of a set of ray barriers. The barrier graph for a pair of points and set of ray barriers is the graph whose vertices are the barriers and which has an edge for any pair of barriers which alone separates the two points. A vertex cover of a barrier graph, then, is a set of barriers whose removal leaves the two points in the same region, bounded or unbounded.

This notion is crucial to the problem of finding a joint exposing set for two points. Indeed, removal of any joint exposing set yields an unobstructed path between the two points, and hence its rays must form a vertex cover of the associated barrier graph (although typically not a minimal one).

**Theorem 13.** $R_2(n) \leq \frac{3}{4}n$

**Proof.** Let $p$ and $q$ be points in the plane, and $X$ any collection of $n$ rays not containing either point. Without loss of generality, suppose $p$ and $q$ lie on a common horizontal line $\ell$ and $p$ is left of $q$. Partition $X$ into four subsets (see Fig. 4): $L$, those rays whose supporting line intersects $\ell$ left of $p$; $R$, those rays whose supporting line intersects $\ell$ right of $q$; $A$, those rays themselves intersecting the segment $\overline{pq}$ from above; and $B$, those rays themselves intersecting $\overline{pq}$ from below. Include rays parallel to $\overline{pq}$ along with $L$.

As argued by [6], $X$ separates the points $p$ and $q$ if and only if some pair of rays in it does. Moreover, two rays from $A$ cannot be in such a pair, nor can two rays from $B$ or two rays from $L \cup R$; in addition, if $a \in A$, $b \in B$, then $a$ and $b$ separate $p$ from $q$ if and only if they intersect, while if $c \in A \cup B$ and $d \in L \cup R$, then $c$ and $d$ separate the two points if and only if they intersect on the same side of $\ell$ as the anchor of $c$.

Clearly then, each of $A, B, L$, and $R$ is on its own insufficient to separate $p$ from $q$ without a ray from one of the other sets, and the plane partition induced by any one of them jointly exposes $p$ and $q$. Thus, each of $(X \setminus A)$, $(X \setminus B)$, $(X \setminus L)$, and $(X \setminus R)$ is a joint exposing set for $p$ and $q$. One of $|A|$, $|B|$, $|L|$ or $|R|$ has size at least $n/4$, and the result follows. □

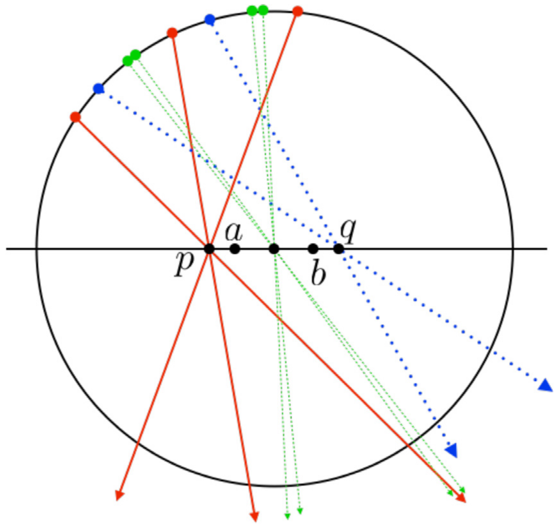


**Fig. 5.** The relative arrangement of some of the rays in $P$ (solid), $Q$ (dotted), and $M$ (dashed).

We now give a construction yielding a lower bound for $R_2(n)$ that asymptotically matches the upper bound of Theorem 13. While the construction has some similarities to the case of line barriers, the ray barriers involved make analysis significantly more difficult. In particular our analysis will heavily use the notion of barrier graphs introduced above.

**Theorem 14.** $R_2(n) = \frac{3}{4}n - O(1)$.

To prove Theorem 14 we will provide a general construction of a set $X$ of $n$ rays around a pair of points for the case when $n = 8m$, which only has large joint exposing sets of size at least $6m - O(1) = \frac{3}{4}n - O(1)$.

Fix points $p, q$ at $(-\delta, 0)$ and $(\delta, 0)$, and fix points $a$ and $b$ at $(-\delta', 0)$ and $(\delta', 0)$ for some $0 < \delta' < \delta < 1$ to be precisely determined later.

Note that a subset of $X$ is a joint exposing set for $a$ and $b$ if and only if there is some point $t$ that is jointly exposed along with $a$ and $b$, which is in an unbounded region of the plane partition due to $X$ (so that $t$ is initially exposed without removing any ray in $X$). This point can be thought of as marking a target region in which to jointly expose $a$ and $b$.

The anchors of all rays in $X$ will be located on the unit circle, as follows (see Fig. 5):

- $8m$ anchor locations are chosen by placing $4m$ of them uniformly around the top open half of the unit circle and $4m$ uniformly around the bottom open half and each half indexed $1, 2, \ldots, 4m$ from left to right. (Using open halves just avoids anchoring a ray on the line between $p$ and $q$.)
- $P$, a set of $2m$ rays intersecting the point $p$, where $m$ are anchored uniformly around the top of the unit circle in locations $1, 4, 7, \ldots,$ and $m$ are anchored uniformly around the bottom of the unit circle antipodal to these (so, in locations $4m, 4m-3, \ldots$).
- $Q$, a set of $2m$ rays similarly placed intersecting $q$, in locations $2, 5, 8, \ldots,$ on the top of the unit circle and $4m-1, 4m-4, \ldots,$ around the bottom.
- $M$, a set of $4m$ rays that intersect the origin. These anchors are placed in pairs, where the distance between members of a pair is $\delta''$, and so that the midpoints of the pairs are at locations $3, 6, 9, \ldots,$ on the top of the unit circle and $4m-2, 4m-5, \ldots$ around the bottom. Note that this placement gives antipodal rays of $M$ the same supporting line; if one perturbs the anchors around the unit circle a small enough amount, the argument will still follow.

The rays of $P$, $Q$, and $M$ are placed so that no anchor is on the $x$-axis, and so that no two rays are parallel.

Our choice of $\delta$, which is the distance from $p$ or $q$ to the origin, determines a circular permutation of the anchors in $X$ and their intersections with the unit circle (two per ray, one at the anchor and one not). There is some value of $\delta$ such that any smaller positive value induces the same permutation, so choose this or any smaller positive value for $\delta$.

In particular, this choice ensures that as one traverses the top of the unit circle in a clockwise fashion, one witnesses the tail of a ray in $P$ followed by an anchor of a ray from $P$, then a ray's anchor from $Q$ followed by the tail of a ray from $Q$, and then a pair of ray tails from $M$ and a pair of anchors from $M$ (the tails each contain an anchor), after which the pattern repeats. The same pattern is observed traversing the bottom of the unit circle in a clockwise fashion, except the order of consecutive anchors and tails from $P$ or $Q$ is reversed. This fact is key to understanding neighborhoods in $G$ and thus how vertex covers of $G$ are formed.

After fixing such a $\delta$, there is some circle centered at the origin that intersects the interior of each of the unbounded regions of the plane as partitioned by $X$, and without loss of generality we can always choose $t$ to be on this circle, in any of these regions' interiors.

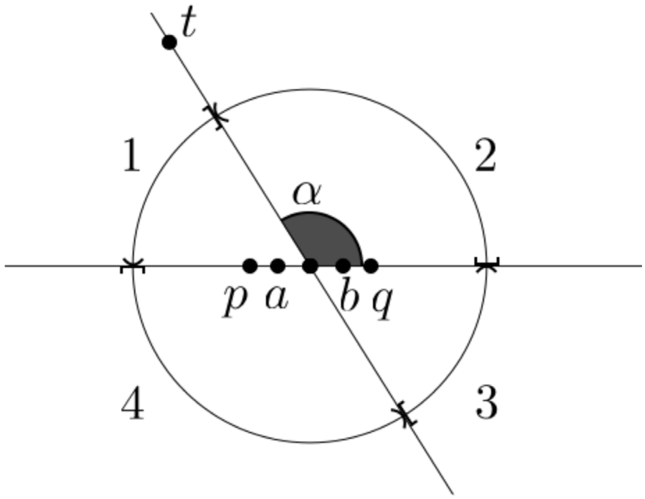


**Fig. 6.** The angle $\alpha$ from the positive $x$-axis to the line through $t$ and the midpoint of $\overline{ab}$ and the partition of the unit circle into half-open arcs which group the rays anchored on it.

Choose $0 < \delta' < \delta$, which controls the distance from $a$ and $b$ to the origin, so that the triangle with vertices $a$, $b$, and $t$ contains at most one anchor from a ray of $X$. Finally, choose $\delta''$ so that each pair anchored near the same designated location intersects exactly the same set of other rays in $X$. Ideally, we would place these paired rays in the exact same location, but general position forbids this, so we use this $\delta''$ to get as close as is needed.

Since $a$, $b$, and $t$ must have unobstructed paths to each other when jointly exposed, the corresponding joint exposing set forms a vertex cover of each of the barrier graphs $G_{a,b}$ and $G_{t,b}$ for the point pairs $(a, b)$ and $(t, b)$, respectively. Note that it also forms a vertex cover for the barrier graph $G_{t,a}$, however as any joint vertex cover of the first two graphs has this property, we will not explicitly exploit it. This connection to joint exposure of $a$ and $b$ yields the following lemma.

**Lemma 15.** *If $C$ is a minimum vertex cover of $G$, then $|C| \le R_2(n)$.*

Our goal, then, is to show that all vertex covers of $G := G_{a,b} \cup G_{t,b}$ contain at least $6m - O(1)$ vertices, for any choice of $t$.

Fix some point $t$ in an unbounded region of the partition according to $X$, and suppose without loss of generality that $t$ is in the top half plane. The line through the origin and $t$ forms an angle $\alpha$ with the positive $x$-axis, normalized to be between 0 and 1, where $\alpha = 1$ corresponds with $\pi$ radians (See Fig. 6). This line and the $x$-axis together divide the unit circle into four regions, labeled 1 through 4 as in the figure.

These regions partition the rays in $P$, $Q$, and $M$ further into four subsets each, according to the region containing the ray's anchor. For the rays anchored in region $i$, call these subsets $P^i(\alpha)$, $Q^i(\alpha)$, and $M^i(\alpha)$, or $P^i$, $Q^i$, and $M^i$ when $\alpha$ is clear from context.

We create an auxiliary graph whose vertices are the sets $P^i$, $Q^i$, and $M^i$, which we will call $G_\alpha = (V_\alpha, E_\alpha)$. $G_\alpha$ has an edge between a pair $A, B \in V_\alpha$ if and only if $G$ has an edge in $A \times B$. A vertex cover of $G_\alpha$ then corresponds to a vertex cover of $G$ by simply taking the union of the vertices of $G_\alpha$.

We will refer to the cardinalities of the vertices in $V_\alpha$ (which are sets of rays) as their *weights*. The weights of these vertices are as follows, where the $-O(1)$ terms account for rounding.

$$|P^1| = |P^3| = |Q^1| = |Q^3| = \alpha m - O(1)$$
$$|P^2| = |P^4| = |Q^2| = |Q^4| = (1-\alpha)m - O(1)$$
$$|M^1| = |M^3| = 2\alpha m - O(1)$$
$$|M^2| = |M^4| = 2(1-\alpha)m - O(1)$$

First, notice that each $A \in V_\alpha$ is an independent set of $G$, and moreover that many pairs $A, B \in V_\alpha$ do not share an edge, recorded as dots in the modified adjacency matrix in Fig. 7.

Inspecting the edges in $E_\alpha$, we see that they correspond to specific types of subgraphs of $G$. Recall that a *half graph* with $2x$ vertices is a bipartite graph with vertices $u_1, \ldots, u_x$ and $v_1, \ldots, v_x$ that has an edge $(u_i, v_j)$ whenever $i \le j$. A *doubled half graph* is the result of doubling the vertices on one side of a half graph, including adjacencies. Refer to Fig. 9 to see all of $G_\alpha$ with these edge types labeled.

**Lemma 16.** *If $(A, B) \in E_\alpha$, then the edges $(r_1, r_2) \in A \times B$ of $G$, induce a subgraph of $G$ that is either a complete bipartite graph, a half graph, or a doubled half graph.*

| | $P^1$ | $P^2$ | $P^3$ | $P^4$ | $Q^1$ | $Q^2$ | $Q^3$ | $Q^4$ | $M^1$ | $M^2$ | $M^3$ | $M^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P^1$ | · | · | · | c | h | · | · | · | h | · | h | c |
| $P^2$ | · | · | · | c | c | h | · | h | c | h | · | c |
| $P^3$ | · | · | · | c | h | · | h | · | h | · | h | c |
| $P^4$ | c | c | c | · | · | h | c | h | c | c | c | h |
| $Q^1$ | h | c | h | · | · | · | c | · | h | c | c | · |
| $Q^2$ | · | h | · | h | · | · | c | · | · | h | c | h |
| $Q^3$ | · | · | h | c | c | c | · | c | c | c | h | c |
| $Q^4$ | · | h | · | h | · | · | c | · | · | h | c | h |
| $M^1$ | h | c | h | c | h | · | c | · | · | · | c | c |
| $M^2$ | · | h | · | c | c | h | c | h | · | · | c | c |
| $M^3$ | h | · | h | c | c | c | h | c | c | c | · | c |
| $M^4$ | c | c | c | h | · | h | c | h | c | c | c | · |

**Fig. 7.** The modified adjacencies of $G_\alpha$, where $\cdot$ represents a non-edge, c represents a complete edge, and h represents a half edge.

**Proof.** We highlight the main idea; Fig. 7 records all complete, half (and doubled half), and empty pairs. Complete bipartite graphs occur when all pairs of rays from the respective regions form barriers; an example of this is $M^1$ and $Q^3$ – all pairs of such rays are separating $b$ from $t$.

Half edges occur when the position of the rays around the circle determine whether or not the corresponding pair forms a block. As an example, consider $P^1$ and $Q^1$ – here, depending on whether rays have an intersection below the pq line or not, they may form a barrier or not. That this forms a half graph is easily seen, since if a ray in $Q^1$ forms a barrier with a $P^1$ ray, it will also form a barrier (from $t$) with all steeper $P^1$ rays. Doubled half graphs occur similarly, with the $M$ rays (as these are double in the construction.) $\square$

With Lemma 16 in hand, we will refer to edges of $G_\alpha$ as either *complete edges* (for complete bipartite subgraphs of $G$) or *half edges* (for both half subgraphs and doubled half subgraphs of $G$), and we will refer to the subgraph of $G_\alpha$ induced by the complete edges as $G_\alpha^c$, while the subgraph induced by the half edges is $G_\alpha^h$.

Fig. 7 shows a modified adjacency matrix of $G_\alpha$, indicating complete edges with a c, half edges with an h, and non-edges with a dot.

With the next lemma we connect these edge labelings to vertex covers; in any vertex cover of our barrier graph $G$, at least one end of each complete edge of $G_\alpha$ is included as a set in the cover. What this means is that vertex covers of $G$ can be viewed as extensions of vertex covers of the auxiliary graph $G_\alpha^c$ by taking the union of the sets in the cover of $G_\alpha^c$ and adding rays appearing in the ends of edges of $G_\alpha^h$.

**Lemma 17.** *Let $C$ be a vertex cover of $G$, and let $(A, B) \in E_\alpha$ be a complete edge. Then either $A \subseteq C$ or $B \subseteq C$.*

**Proof.** Suppose $A \nsubseteq C$. Then $\exists x \in A \setminus C$. $x$ is adjacent to every element of $B$ since $(A, B)$ is complete, so $B \subseteq C$, because $C$ is a vertex cover of the edges of $G$. $\square$

In light of Lemma 17, we can bound the size of all vertex covers of $G$ below by beginning with vertex covers of $G_\alpha^c$ and showing that any extension to a vertex cover of $G$ includes at least $6m - O(1)$ rays in total. Moreover, we need only show that this is the case for minimum vertex covers of $G$.

The only vertex covers of $G$ that do not immediately satisfy the target lower bound of $\frac{3}{4}n - O(1) = 6m - O(1)$ are thus those that extend vertex covers of $G_\alpha^c$ with weight less than $6m - O(1)$; without loss of generality, we may extend minimal covers of $G_\alpha^c$. For such a minimal cover $C^c$ of $G_\alpha^c$, the rays we add to get a minimum vertex cover of $G$ are in the complement $V_\alpha^c \setminus C^c$, which is a maximal independent set in $G_\alpha^c$.

For any $\alpha$ there are 5 maximal independent sets in $G_\alpha^c$ with weight more than $2m + O(1)$, which means their complements, vertex covers of $G_\alpha^c$, have weight less than $6m - O(1)$. This was verified by an exhaustive enumeration of the vertex covers of this graph using the igraph package in *R*. Code is available at https://github.com/Kirkules/extremal_problems_ray_sensors. These independent sets of $G_\alpha^c$ are pictured in Fig. 10.

In the next lemma we show that a minimum vertex cover of $G$ cannot take too few vertices from both sides of a half edge (see Fig. 9).

**Lemma 18.** *Fix $t$ (and thus $\alpha$) and a vertex cover $C$ of the barrier graph $G$. If $(M^t, B)$ is a half edge of $G_\alpha$ (for any $B \in V_\alpha$), then $|B \cap C| \geq (1 - \epsilon)|B| - O(1)$, where $\epsilon$ depends on $B$.*

**Proof.** The doubled half graph between $M^t$ and $B$ labels $M^t$ as $\{m_1, \dots, m_{|M^t|}\}$, where $i < j \implies \mathcal{N}_B(m_i) \subseteq \mathcal{N}_B(m_j)$. (In our notation, $\mathcal{N}_B(m_i)$ is the neighborhood of $m_i$ in $B$.)

Let $r$ be the largest index so that $m_r \notin C$. Since the rays in $M^t$ are paired so that pairs have the same neighborhood, i.e. $\mathcal{N}_B(m_{2i-1}) = \mathcal{N}_B(m_{2i})$ for each $i$.

Because each consecutive pair has a neighborhood larger by 1 than the last, and because $r$ may be the smaller index of its pair, $\deg_B(m_r) \geq \frac{r}{2} - 1$. (We subtract 1 because the first pair, $m_1$ and $m_2$, may have no neighbors in $B$, depending on $\alpha$.)

Since $C$ is a cover of all edges in $G$ and $m_r \notin C$, we have that $\mathcal{N}_B(m_r) \subseteq C$. This means that $|B \cap C| \geq \deg_B(m_r) \geq \frac{r}{2} - 1$, and we may write:

$$
\begin{aligned}
|B \cap C| &\geq \frac{r}{2} - 1 \\
&= \frac{1}{2}(r+1) - \frac{3}{2} \\
&= \frac{|M^t|}{2} - \frac{|M^t|}{2} + \frac{1}{2}(r+1) - \frac{3}{2} \\
&= |B| - \frac{|M^t|}{2}\left(\frac{|M^t| - (r+1)}{|M^t|}\right) - \frac{3}{2} \\
&= |B| - |B|\epsilon - \frac{3}{2} \\
&= (1-\epsilon)|B| - O(1),
\end{aligned}
$$

where $\epsilon = \left(\frac{|M^t|-(r+1)}{|M^t|}\right)$, and where we have made use of the fact that $|B| = \frac{|M^t|}{2}$ since the only half edges adjacent to $M^t$ connect it to sets with exactly half as many rays. $\square$

For a vertex cover $C$ of $G$, this $\epsilon$ may be thought of as the fraction of $M^t$ included in $C$ due to this particular half edge. What this means is that $|M^t \cap C| \geq \epsilon|M^t| - 1$.

Moreover, applying the lemma to an $M^t$ adjacent to two half edges could yield both an $\epsilon_1$ and an $\epsilon_2$, marking fractions of $M^t$ included in the cover $C$ from each side of the ordered list of rays in $M^t$ (there are only two such orderings of the rays because the order comes from viewing the anchors of rays clockwise or counterclockwise around the unit circle). In other words, when $M^t$ is adjacent to two half edges, $|M^t \cap C| \geq (\epsilon_1 + \epsilon_2)|M^t| - 2$.

Lemma 18 is the final tool we need to prove Theorem 14.

**Proof of Theorem 14** ($R_2(n) = \frac{3}{4}n - O(1)$)**.**

The only vertex covers of $G^c$ with total weight below $\frac{3}{4}n - O(1)$ are the complements in $G$ of the five subgraphs of $G^h_\alpha$ pictured in Fig. 10, as discussed after Lemma 17. So, if a vertex cover of $G$ is to have weight smaller than $\frac{3}{4}n - O(1)$, it must extend one of these subgraphs' complements by adding enough vertices of the subgraph itself to become a cover. We therefore show that each such extension leading to a minimum vertex cover must still have total weight at least $\frac{3}{4}n - O(1) = 6m - O(1)$.

These subgraphs are labeled $S_{(a)}, S_{(b)}, S_{(c)}, S_{(d)}$, and $S_{(e)}$, where each $S_{(\cdot)} = (V_{(\cdot)}, E_{(\cdot)})$, and we address extending in each case separately. However, arguments for all cases are structured similarly: first we note the total weight of the complement of $S_{(\cdot)}$ (i.e. the rays already in the cover of $G$ before extending), and then we use Lemma 18 to bound below the number of rays that must be added from $S_{(\cdot)}$ to make a vertex cover of $G$, based on the edges appearing in $S_{(\cdot)}$.

$S_{(a)}$ : $V_\alpha \setminus V_{(a)}$ is a vertex cover of $G^c_\alpha$ with weight $|V_\alpha \setminus V_{(a)}| = |P^4| + |Q^1| + |Q^3| + |M^1| + |M^3| + |M^4| = m\,[(1-\alpha) + \alpha + \alpha + 2\alpha + 2\alpha + 2(1-\alpha)] = (3+3\alpha)m - O(1)$.

Any vertex cover of $G$ with no rays in $M^2$ must include all rays in $Q^2, P^2$, and $Q^4$ except for up to 3, since each of these sets is the neighborhood of some ray of $M^2$ (except at most one ray each). In this case, the weight of the extended vertex cover is $(3+3\alpha)m + |Q^2| + |P^2| + |Q^4| = (3+3\alpha)m + 3(1-\alpha)m - O(1) = 6m - O(1)$.

If instead $C$ is a vertex cover of $G$ with at least one ray in $M^2$, then as in Fig. 8, those rays in $M^2$ anchored further clockwise have more neighbors in $Q^2$ and $Q^4$, and simultaneously have *fewer* neighbors in $P^2$. So the ordering of $M^2$ in the half edge with $P^2$ is opposite the ordering in the half edges with $Q^2$ and $Q^4$.

By Lemma 18, then, there are $0 \leq \epsilon_1, \epsilon_2 \leq 1$ so that $|M^2 \cap C| \geq (\epsilon_1 + \epsilon_2)|M^2| - 2$ and $|\left(P^2 \cup Q^2 \cup Q^4\right) \cap C| \geq (1-\epsilon_1)|P^2| + (1-\epsilon_2)(|Q^2| + |Q^4|)$.

Therefore, the number of additional rays required to get a vertex cover of $G$ is at least

$$
\begin{aligned}
&|(M^2 \cup P^2 \cup Q^2 \cup Q^4) \cap C| \\
&\geq (\epsilon_1 + \epsilon_2)|M^2| - 2 + (1-\epsilon_1)|P^2| + (1-\epsilon_2)(|Q^2| + |Q^4|) \\
&\geq (\epsilon_1 + \epsilon_2) \cdot 2(1-\alpha)m + (1-\epsilon_1) \cdot (1-\alpha)m + (1-\epsilon_2) \cdot 2(1-\alpha)m - O(1) \\
&\geq (3+\epsilon_1)(1-\alpha)m - O(1) \\
&\geq (3-3\alpha)m - O(1),
\end{aligned}
$$

and so the size of a vertex cover extending $V_\alpha \setminus V_{(a)}$ is at least $(3+3\alpha)m + (3-3\alpha)m - O(1) = 6m - O(1)$.

$S_{(b)}$ : The complement of $S_{(b)}$ is a vertex cover of $G^c_\alpha$ satisfying:

$$|V_\alpha \setminus V_{(b)}| = |P^4| + |Q^1| + |Q^2| + |Q^4| + |M^1| + |M^2| + |M^4|$$
$$= (7 - 4\alpha)m - O(1),$$

Note that if $\alpha \leq 1/4$, then $|V_\alpha \setminus V_{(b)}| \geq 6m - O(1)$ and this case is done.
A vertex cover of $G$ with no rays from $M^3$ must include all rays in $P^1$, $P^3$, and $Q^3$, which together have weight $3\alpha m \geq (4\alpha - 1)m$ since $\alpha \leq 1$. In this case, the extended vertex cover has weight at least $6m - O(1)$.
So suppose $C$ is a vertex cover of $G$ taking at least one ray of $M^3$. Rays of $M^3$ anchored more counterclockwise have more neighbors in $Q^3$ and fewer in $P^1$ and $P^3$. By Lemma 18, there are $0 \leq \epsilon_1, \epsilon_2 \leq 1$ so that $|M^3 \cap C| \geq (\epsilon_1+\epsilon_2)|M^3|$ while $|(Q^3 \cup P^1 \cup P^3) \cap C| \geq (1 - \epsilon_1)|Q^3| + (1 - \epsilon_2)(|P^1| + |P^3|)$.

$$\begin{aligned}
&|(M^3 \cup P^3 \cup Q^3 \cup P^1) \cap C| \\
&\geq (\epsilon_\ell + \epsilon_r)|M^3| + (1 - \epsilon_\ell)(|P^1| + |P^3|) + (1 - \epsilon_r)|Q^3| \\
&= [(\epsilon_\ell + \epsilon_r)(2\alpha) + (1 - \epsilon_\ell)(\alpha + \alpha) + (1 - \epsilon_r)\alpha]m - O(1) \\
&= (3 + \epsilon_r)\alpha m - O(1) \\
&\geq (4\alpha - 1)m - O(1),
\end{aligned}$$

which again means the extended vertex cover has weight at least $(7 - 4\alpha)m + (4\alpha - 1)m - O(1) = 6m - O(1)$.

$S_{(c)}$ : $|V_\alpha \setminus V_{(c)}| = |P^2|+|P^4|+|Q^1|+|M^2|+|M^3|+|M^4| = [(1-\alpha)+(1-\alpha)+\alpha+2(1-\alpha)+2\alpha+2(1-\alpha)]m = (6-3\alpha)m-O(1)$, so the goal is to show any vertex cover $C$ of $G$ extending the complement of $S_{(c)}$ takes more than $3\alpha m - O(1)$ vertices from $S_{(c)}$, because $(6 - 3\alpha)m - 3\alpha m - O(1) = 6m - O(1)$.
A vertex cover of $G$ with no rays from $M^1$ must take all of $P^1$, $P^3$, and $Q^1$, totaling at least $3\alpha m - O(1)$ additional rays.
If instead (again applying Lemma 18) at least $(\epsilon_1 + \epsilon_2)|M^1| - 2$ rays are taken from $M^1$, then at least $(1 - \epsilon_1)|Q^1| + (1 - \epsilon_2)(|P^1| + |P^3|)$ rays are taken from $Q^1$, $P^1$, and $P^3$, giving that

$$\begin{aligned}
&|(M^1 \cup P^1 \cup P^3 \cup Q^1) \cap C| \\
&\geq (\epsilon_\ell + \epsilon_r)|M^1| + (1 - \epsilon_\ell)(|P^1| + |P^3|) + (1 - \epsilon_r)|Q^1| \\
&= [(\epsilon_\ell + \epsilon_r)(2\alpha) + (1 - \epsilon_\ell)(\alpha + \alpha) + (1 - \epsilon_r)(\alpha)]m - O(1) \\
&= (3 + \epsilon_r)\alpha m - O(1) \\
&\geq 3\alpha m - O(1),
\end{aligned}$$

and so the extended cover contains at least $(6 - 3\alpha)m + 3\alpha m - O(1) = 6m - O(1)$ rays.

$S_{(d)}$ : $|V_\alpha \setminus V_{(d)}| = |P^2| + |P^4| + |Q^1| + |Q^3| + |M^3| + |M^4| = 4m - O(1)$.
Since $S_{(d)}$ has two connected components with any edges (one component with $M^1$ and one with $M^2$), we can address them independently.
A cover with no rays from $M^1$ must include all rays in both $P^1$ and $P^3$, totaling $2\alpha m-O(1)$. If instead it takes $\epsilon|M^1|-1$ rays from $M^1$ as in Lemma 18, then the contribution from this connected component is $\epsilon|M^1|+(1-\epsilon)(|P^1|+|P^3|) = 2\epsilon\alpha m + (1 - \epsilon)(\alpha + \alpha)m - O(1) = 2\alpha m - O(1)$. Hence, vertex covers must take at least $2\alpha m - O(1)$ rays from this component.
Similarly, a cover with no rays from $M^2$ must include both $Q^2$ and $Q^4$, totaling $2(1 - \alpha)m - O(1)$. Instead taking $\epsilon|M^2| - 1$ from $M^2$ means the contribution from this component is $\epsilon|M^2| + (1 - \epsilon)(|Q^2| + |Q^4|) = 2(1 - \alpha)m - O(1)$, and the contribution to any vertex cover from this component is at least $2(1 - \alpha)m - O(1)$.
The two components together thus contribute at least $2m - O(1)$ rays to any vertex cover of $G$, and thus any cover extending $V_\alpha \setminus V_{(d)}$ contains at least $4m + 2m - O(1) = 6m - O(1)$ rays.

$S_{(e)}$ : $|V_\alpha \setminus V_{(e)}| = |P^1| + |P^2| + |P^3| + |Q^3| + |M^1| + |M^2| + |M^3| = (3 + 4\alpha)m - O(1)$.
A vertex cover of $G$ with nothing from $M^4$ takes all of $P^4$, $Q^2$, and $Q^4$, which have a combined weight of $(3-3\alpha)m \geq (3 - 4\alpha)m - O(1)$.
Taking instead $(\epsilon_1 + \epsilon_2)|M^4|$ rays from $M^4$ in the cover $C$ of $G$, we have that $|(M^4 \cup P^4 \cup Q^2 \cup Q^4) \cap C| \geq (\epsilon_1 + \epsilon_2)|M^4| + (1 - \epsilon_1)(|Q^2| + |Q^4|) + (1 - \epsilon_2)|P^4| = [(3 + \epsilon_2) - \epsilon_2\alpha]m$, which is at least $(3 - 4\alpha)m$ since $0 \leq \epsilon_2 \leq 1$.
But $(3 + 4\alpha)m + (3 - 4\alpha)m - O(1) \geq 6m - O(1)$.

Finally, because every vertex cover of $G$ extends a vertex cover of $G^c_\alpha$, and because (as we have just shown) every extension of such a cover to a cover of $G$ contains at least $6m - O(1)$ elements of $G$, it follows that every vertex cover of $G$ has at least $6m - O(1) = \frac{3}{4}n - O(1)$ vertices in it. Therefore, $\frac{3}{4}n - O(1) = R_2(n)$. □

## 4. Conclusions and future work

A set $P$ of $k$ points in the plane is "protected" by a set of $n$ rays (or lines, or segments), if every point of $P$ belongs to a bounded region. In this setting, we investigated two questions: (1) what number $r_k(n)$ of rays are always sufficient

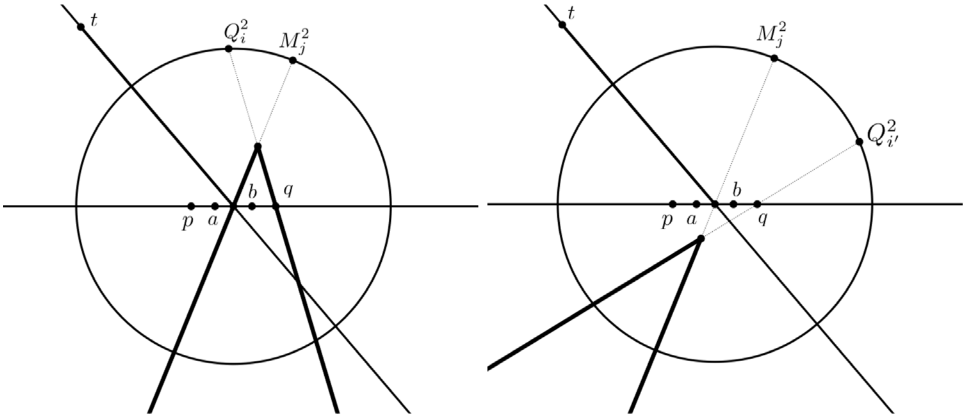


**Fig. 8.** Rays $Q_i^2 \in Q^2$ that are counterclockwise from $M_j^2 \in M^2$ form a barrier between $b$ and $t$ and between $a$ and $b$ (left), whereas rays $Q_{i'}^2$ that are clockwise from $M_j^2$ do not (right). This is what makes the edge $(Q^2, M^2)$ in $G_\alpha$ a half graph, which is doubled because each $M_j^2$ is paired with another ray with the exact same barrier graph adjacencies.

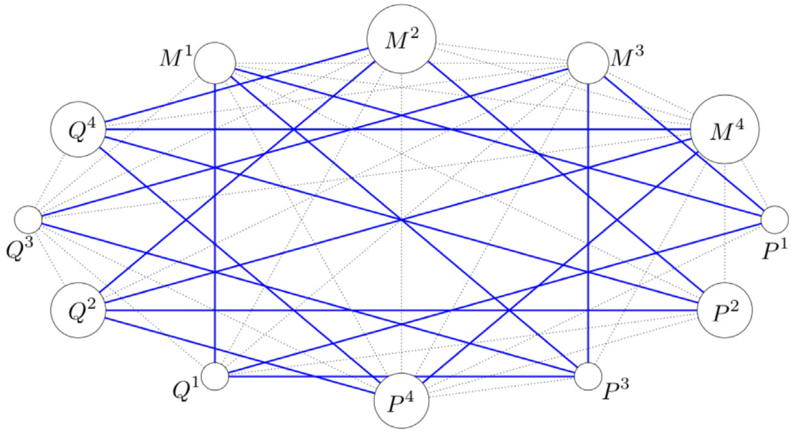


**Fig. 9.** $G_\alpha$ with half edges shown as solid blue lines, complete edges shown as dashed black lines, and relative vertex weights visualized by vertex radius.

to remove so that each of the $k$ points resides in an unbounded region, and (2) what number $R_k(n)$ of rays are always sufficient to remove so that all $k$ points reside in the same unbounded region. In our derivations, we also considered the cases when the rays are replaced by lines or by segments.

We provided complete answers for the case $k = 2$ for both questions, and provided asymptotically tight answers for Question 1 when $k = 3$ and for $k = 4$ when the points are in convex position.

The obvious next step in the study of $r_k(n)$ and $R_k(n)$ is to extend the results presented to larger values of $k$. Both of these present challenges that require new insights.

We conjecture that, for points in convex position and $k \geq 5$:

$$r_k^c(n) = \frac{k}{k+1} n + O(1)$$

We leave this, as well as the problem of deriving sharp bounds for $r_k(n)$ when $k \geq 4$ and for $R_k(n)$ when $k \geq 3$, as open problems worthy of further investigation.

## Appendix 5. The exact value of $r_2(n)$

Consider a set of lines $L$ and a point $p$ in the plane. The set of $n = |L|$ rays anchored at $p$ and each perpendicular to a different line of $L$ intersects any circle centered at $p$ in $n$ points. We call this set of locations around any circle centered at $p$ the *ray circle* for $p$ with respect $L$, and is written ⓟ since the set of line $L$ is usually understood from context.

The locations on ⓟ are identified with the rays themselves and with the corresponding lines, and also with the angle from some fixed direction pointing away from $p$, usually in the direction of the positive $x$-axis, because their relative locations are what is important, rather than their absolute locations.

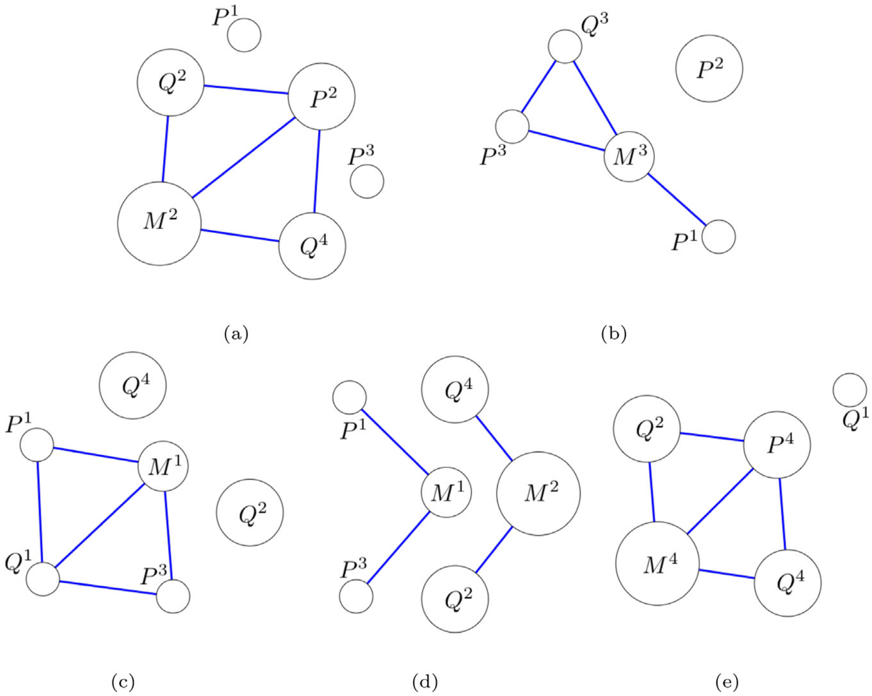


**Fig. 10.** The subgraphs of $G^h_\alpha$ whose vertices are maximal independent sets in $G^c_\alpha$, and that correspond to subgraphs of $G$ that need to be covered for the proof of Lemma 17. Call these subgraphs $S_{(a)}$, $S_{(b)}$, $S_{(c)}$, $S_{(d)}$, and $S_{(e)}$.

The location of a ray anchored at $p$ witnessing an exposing set $E(\{p\}) \subseteq L$ is the center of an open semicircle on ⓟ containing the locations of all lines in $E(\{p\})$, and thus we may refer to a witness arc or semicircle for $p$ rather than a witness ray.

**Theorem 19.** $r_2(n) = \left\lfloor \frac{2}{3}(n-1) \right\rfloor$

**Proof.** The upper bound provided by Theorem 8 can easily be tightened to match this; one merely sharpens the two upper bounds to $n-i$ and $i-1+\lfloor \frac{1}{2}(n-i) \rfloor$ instead of $n-i$ and $i+(n-i)/2$. For the matching lower bound, we provide a construction of $n = 3t+1$ lines with exposing sets no smaller than $2t$ when $t \geq 1$.

Fix $n = 3t+1$ for $t \geq 1$ and fix points $p$ and $q$ at $(-1, 0)$ and $(1, 0)$ respectively. Let $\alpha = \frac{\pi}{n+1}$. Place a set of $n$ lines $L = \{\ell_i\}_{i=1}^n$ as follows:

- The clockwise angle between $\ell_i$ and the negative $x$-axis is $\alpha i$
- If $i \equiv 0 \pmod 3$, then $\ell_i$ passes through $(-2, 0)$, which is to the left of $p$
- If $i \equiv 1 \pmod 3$, then $\ell_i$ passes through the origin
- If $i \equiv 2 \pmod 3$, then $\ell_i$ passes through $(2, 0)$, which is to the right of $q$

In other words, as the angles increase, we alternate placing $\ell_i$ between $p$ and $q$, to the right of $q$, and to the left of $p$.

Let $E = \{\ell_i \in L : i \equiv 0 \pmod 3 \text{ or } i \equiv 2 \pmod 3\}$, and $I = \{\ell_i \in L : i \equiv 1 \pmod 3\}$. $I$ is the set of *internal* lines, i.e. lines passing between $p$ and $q$, and $E = L \setminus I$ is the set of *external* lines. Note that the locations on ⓟ and ⓠ for external lines are the same, and for internal lines are opposite, because rays anchored at $p$ and $q$ point in opposite directions to be perpendicular to a line between them. See Fig. 11 to see an example the arrangement of locations on the ray circles ⓟ and ⓠ.

Let $X \subseteq L$ be an exposing set witnessed by arcs $A_p$ and $A_q$ on ⓟ and ⓠ, respectively. We will show that $|X| \geq 2t$ by showing that the smallest number of locations from $L$ on ⓟ and ⓠ contained in $A_p$ and $A_q$ (and thus the smallest number of witnessed lines) is $2t$.

First, note that the relative position of $p$ and $q$ means that all locations for $I$ are on the rightmost semicircle of ⓟ and on the leftmost semicircle of ⓠ. Consequently, together the left semicircle of ⓟ and the right semicircle of ⓠ witness no internal lines and all external lines, for a total of $2t$. Moreover, we may assume that $A_p$ covers at least as much of the left semicircle as $A_q$ does, since otherwise swapping their locations preserves the number of external locations they contain and may reduce the number of internal locations.

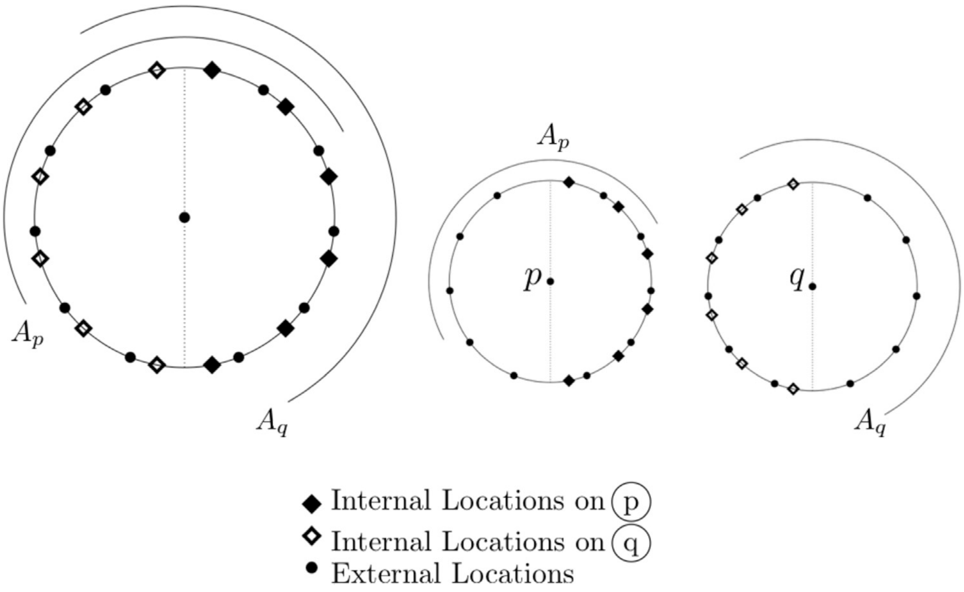


**Fig. 11.** The locations on ray circles for $p$ and $q$ (right), and how they look when overlaid (left).

If it is not just the left semicircle of ⓟ, $A_p$ must have one endpoint on the right and one endpoint on the left of ⓟ. Between the endpoint on the right and the nearest endpoint of the left semicircle are some number of interior locations, which is at least as large as the number of exterior locations in the same interval. This is due to the alternating construction and the fact that the locations on the right closest to the endpoints of the left semicircle are interior locations. So, if the left semicircle contains $m$ fewer internal locations than $A_p$, it contains no more than $m$ more external locations, and therefore contains at most as many locations as $A_p$. By a symmetrical argument, the right semicircle of ⓠ contains no more locations than $A_q$.

But this means that $A_p$ and $A_q$ witness no fewer lines than the left semicircle of ⓟ together with the right semicircle of ⓠ, which witness $2t$ lines (all the external ones).

Therefore every choice of arcs on ⓟ and ⓠ, and thus every set of witness rays anchored at $p$ and at $q$, witnesses at least $2t$ lines.

So, when $n = 3t + 1$ and $t \geq 1$ there exists a set of lines (our construction) with no exposing sets smaller than $2t$, and we have that $r_2(3t + 1) \geq 2t$ and also that $r_2(3t + 4) \geq 2t + 2$. Because these match our upper bound, in fact

$$r_2(3t + 1) = 2t \quad \text{and} \quad r_2(3t + 4) = 2t + 2 \tag{1}$$

Because $0 \leq r_2(i + 1) - r_2(i) \leq 1$ (Lemma 4), and since our upper bound gives that $r_2(3t + 2) \leq \left\lfloor \frac{2}{3}(3t + 2 - 1) \right\rfloor = 2t$, in fact

$$r_2(3t + 2) = 2t \tag{2}$$

as well.

But Eqs. (1) and (2) together with Lemma 4 mean that $r_2(3t + 3) = 2t + 1$, for otherwise the jump from $2t$ to $2t + 2$ would be too large either between $n = 3t + 2$ and $n = 3t + 3$, or between $n = 3t + 3$ and $n = 3t + 4$.

These observations together yield $r_2(n) = \left\lfloor \frac{2}{3}(n - 1) \right\rfloor$ for $n \geq 4$. □

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