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ORIGINAL PAPER

On Rainbow-Cycle-Forbidding Edge Colorings of Finite Graphs

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Abstract

It is shown that whenever the edges of a connected simple graph on n vertices are colored with $n - 1$ colors appearing so that no cycle in G is rainbow, there must be a monochromatic edge cut in G. From this it follows that such colorings of G can be represented, or 'encoded,' by full binary trees with n leaves, with vertices labeled by subsets of $V(G)$, such that the leaf labels are singletons, the label of each non-leaf is the union of the labels of its children, and each label set induces a connected subgraph of G. It is also shown that $n - 1$ is the largest integer for which the main theorem holds, for each n, although for some graphs a certain strengthening of the hypothesis makes the theorem conclusion true with $n - 1$ replaced by $n - 2$.

Keywords Rainbow-cycle-forbidding · Edge coloring · Edge cut

Mathematics Subject Classification 05C151

1 Introduction

Given an edge-colored graph G, a subgraph of G is **rainbow** if no two different edges of the subgraph bear the same coloring. An edge coloring of G is **rainbow-cycle-forbidding** (RCF, for short) if no cycle in G is rainbow with respect to the coloring.

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Note that if G is an RCF-colored multigraph, then necessarily G is loopless, and for distinct vertices $u, v \in V(G)$, all edges with ends u and v have the same color.

A simple observation (noted, for instance, in [1]) is that if a coloring of a connected n-vertex graph G is RCF, then G contains at most $n-1$ colors. Indeed, if a coloring of G contains n colors, then the graph induced by taking one edge of each color necessarily contains a cycle, and this is certainly rainbow. More generally, the pigeonhole principle implies that an RCF edge colored graph G with c components contains at most $n - c$ colors. On the other hand, as we describe in the next section, it is not difficult to *construct* RCF colorings with this maximum number of colorings for any graph G. Following [1–3], we will call an RCF edge coloring of a graph G on n vertices with c components in which the maximum number, $n - c$, of colors appears a **JL-coloring**[1] of G.

When the edge-colored graph G is the complete graph K_n, as occurs frequently in the Ramsey/anti-Ramsey/mixed Ramsey hotbed where this study first arose, then RCF colorings are exactly the colorings which avoid rainbow K_3's (well known, but a proof appears in [2]). These RCF colorings of a complete graph are known as **Gallai colorings**, and JL-colorings of K_n are Gallai colorings with exactly $n - 1$ colors appearing.

JL-colorings of K_n turn out to be particularly nice. The color maximality forces a strong structural property on the colorings; they can all be formed by a recursive 'standard construction' which we describe in the next section. This, in turn, gives a bijection between JL-colorings of K_n and certain labeled full binary trees. These results can be derived by a characterization of Gallai colorings from [4], but a self-contained (and rather laborious) inductive proof appears in [2]. Adaptations of the proof from [2] show that similar structure theorems – in particular that every JL-coloring comes from the same 'standard construction' – also hold for complete bipartite graphs (cf. [3]) and complete multipartite graphs (cf. [1]).

The main result of this paper, Theorem 2, is that such a structure theorem for JL-colorings actually holds for *any* finite connected graph G. That is, *every* JL-coloring of *every* finite connected graph is constructed via the same 'standard construction' described in the next section. This result subsumes the results of [1–3], and moreover our proof of the more general result is significantly shorter than the proofs in these papers.

The key to the proof asserts the existence of a monochromatic cut in JL-colored graphs, or stated in the contrapositive, that an edge-colored graph using exactly $n - 1$ colors without a monochromatic cut cannot be rainbow cycle free. It is a natural question, regarding the strength of our result, whether similar results hold when fewer colors are used in the RCF coloring. This turns out not to be true even in the case of RCF colorings with $n - 2$ colors. We show, however, that in the case of the complete graph that the 'bad' examples, that is RCF colorings using $n - 2$ colors and without a monochromatic cut, can be completely characterized.

One may ask whether similar structural results may be possible with rainbow-structure free graphs. For instance, in [5] the authors studied the question of

[1] The reason for the terminology is that one of the co-authors was first introduced to the topic of RCF edge colorings by Jenö Lehel.

maximizing the number of colors in an edge coloring of a planar graph without any rainbow faces. While (in a non-tree) having a rainbow face implies a rainbow cycle the converse is not true, and they constructed graphs which require more than n colors to force a rainbow face, with the exact number depending on structural properties of the resulting graph. On the other hand, in some classes of planar graphs (such as chordal planar graphs, for instance) a rainbow cycle does imply the existence of a rainbow face, so our results apply directly. It would be an interesting question to study structural properties of rainbow-face free graphs.

One of the referees has called our attention to the resemblance between our Standard Construction of JL-colorings and the formation of Gomory-Hu trees. The tree structure that we generate, which captures monochromatic edge cuts in the graph, bears a passing similarity to Gomory-Hu trees which are of importance in combinatorial optimization. The Gomory-Hu trees capture, instead, the minimum weight edge cuts between source-destination pairs. Structurally, however, the trees are rather different (for instance in the Gomory-Hu trees, the edge weights are crucial, and the vertices of the tree are vertices of the graph – and in our case, the interior vertices of the trees are actually subsets of vertices) and the relationship seems to be a rather superficial one.

Finally, we note that this work ties nicely into a literature on bipartite decompositions of graphs. The structure theorems for JL colorings proved in [1–3] and the present work show that JL colorings provide a decomposition of the edges of graphs into bipartite graphs. Indeed, the main result of [2] implies that JL colorings of K_n are equivalent to certain decompositions of the edge set of K_n into $n-1$ complete bipartite graphs. Decomposing graphs into bipartite graphs has a long history. In [6], Graham and Pollak famously proved that at least $n-1$ complete bipartite graphs are required to decompose the edges of K_n. The JL-colorings of K_n form some (but, interestingly, not all) such decompositions. In more recent work, the authors of [7–9] have studied similar questions in random graphs. The results on JL colorings (including the main result of this paper), complement these results on decompositions into complete bipartite graphs, by showing that JL colorings provide particularly nice recursive decompositions of graphs into $n-1$ (not necessarily complete) bipartite graphs, whose iterative removal leaves connected subgraphs.

The remainder of this paper is organized as follows: in the next section we introduce some preliminaries and the 'standard construction' of JL-colorings. In Sect. 3 we prove the main theorem, and in Sect. 4 we prove our sharpness results.

2 Preliminaries and the Standard Construction

We begin with a brief note. Although our goal is a result about simple graphs, the statements continue to hold in the case of multigraphs and the proofs are easier in this setting. Thus, *throughout this paper 'graph' will mean 'multigraph'; loops and multiple edges are allowed.* It will be useful to remember that if a graph G has an RCF coloring, then G has no loops and multiple edges in G with the same end vertices must all bear the same color.

Now, we provide a bit of notation and a well known definition. If A, B are non-empty sets partitioning $V(G)$, then the set of edges of G with one end in A and one end in B, denoted $[A, B]$, is an **edge cut** in G.

Lemma 1 *Suppose the edges of G are colored,* $[A, B]$ *is a monochromatic edge cut in G with this coloring, and the coloring restricted to* $G[A]$ *and* $G[B]$*, the subgraphs of G induced by A and B, respectively, is RCF. Then the coloring on G is RCF.*

Proof There are no rainbow cycles in $G[A] \cup G[B]$, and any cycle with at least one vertex in A and at least one vertex in B must have at least two different edges in $[A, B]$. Therefore there are no rainbow cycles in G. □

The last result of this section complements our earlier observation and the construction contained in its proof is the main topic of this paper.

Theorem 1 *If G is a loopless connected graph on n vertices, then there is an* RCF *edge coloring of G with* $n - 1$ *colors appearing.*

Proof Let T be a spanning tree in G, and color the $n-1$ edges of T with $n-1$ different colors. These will be the colors appearing in the final coloring.

Choose an edge $uv \in E(T)$. Suppose that green is the color of uv. The vertices in G are naturally partitioned into vertex sets A and B (which we will call 'shores') by the choice of uv; that is, A is the set of vertices of G connected to u by a path in $T - uv$, and B is the set of vertices of G connected to v by a path in $T - uv$. Let every edge in the edge cut $[A, B]$ be colored green.

Since $T[A]$, $T[B]$ are spanning trees in $G[A]$, $G[B]$, respectively, both $G[A]$ and $G[B]$ are connected loopless graphs, and each already has a rainbow spanning tree installed.

If $T[A]$ has at least one edge (i.e. if $|A| > 1$), iterate the procedure just described with G, T replaced by $G[A]$, $T[A]$, and proceed similarly with B. Continue until it is impossible to continue. The assumption that G is loopless guarantees every edge will be colored.

By induction on n we can conclude that the resulting edge colorings of $G[A]$, $G[B]$ with $|A| - 1$, $|B| - 1$ colors, respectively, are RCF, and therefore the resulting edge coloring of G with $n - 1$ colors is RCF by Lemma 1. □

The role of the spanning tree T in the proof above makes it plain that $n - 1$ colors appear, and that $G[A]$ and $G[B]$ are connected. Once it is agreed that $V(G)$ can be partitioned into non-empty sets A and B such that $G[A]$ and $G[B]$ are connected (assuming G is connected), it is easy to see that every RCF edge coloring of a loopless connected graph G on n vertices, with $n-1$ colors appearing, obtained by the method in this proof is also achievable by the following procedure, which does not mention spanning trees; and, conversely, every instance of the following can be carried out using a spanning tree as in the proof of Theorem 1.

The **Standard Construction** of RCF edge colorings, with $n-1$ colors appearing, of a connected loopless multigraph G on n vertices is defined as follows:

1. If $n > 1$, find an edge cut $[A, B]$ in G such that $G[A]$ and $G[B]$ are connected. Color the edges of $[A, B]$ with a color that will not be used again.
2. If $|A| = 1$ there are no edges to color in $G[A]$. If $|A| > 1$, iterate step 1 on $G[A]$, and the same for $G[B]$ if $|B| > 1$. At each step, pick colors such that the color set to appear on $G[A]$ is disjoint from that on $G[B]$, and neither can contain the color on $[A, B]$. Continue until all edges are colored.

Every coloring obtained by the standard construction can be 'encoded' by a full binary tree with vertices labeled by subsets of $V(G)$, as described in the abstract. The root of the tree will be labeled $V(G)$ and its 'children' will be labeled A and B, the shore sets of the first edge cut. We note that for a general graph (unlike for K_n) the full binary tree in this encoding is not necessarily unique, either as a labeled or unlabeled object. For instance, for the path P_n the unique JL-coloring can be encoded by *every* full binary tree with n leaves, properly labeled. This can be seen by taking an ordered full binary tree with n leaves, labeling the leaves left-to-right with elements of the set $\{v_1, \ldots, v_n\}$, and taking the label of a parent to be the union of the labels of its children. On the other hand, for the star, $K_{1,n}$, while there is a unique isomorphism class of tree arising in such an encoding, there are $n!$ different labelings of this tree that encode the (again, unique) JL-coloring. In general, the labeled full binary tree representation will not be unique, as there may be more than one monochromatic edge cut.

We end this section with an example of such an encoding of a JL-coloring (Figs. 1, 2).

Example:

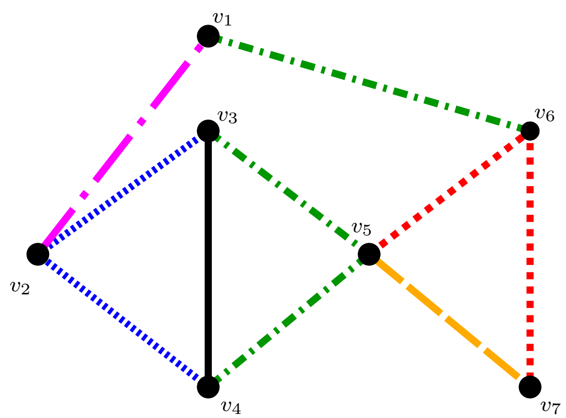

Fig. 1 An RCF coloring of a connected simple graph G on seven vertices with six different edge colors

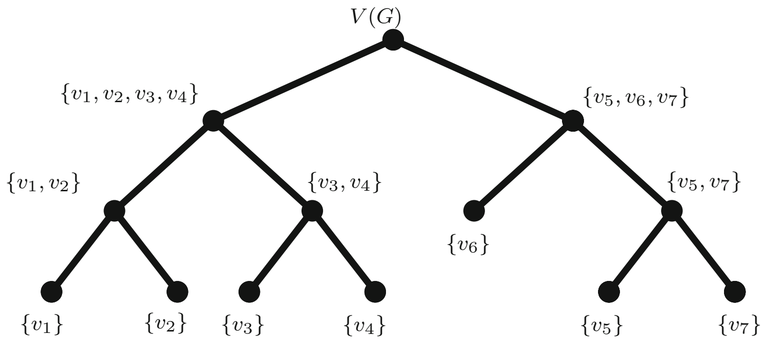

Fig. 2 An encoding of the coloring in Fig. 1

3 The Main Result

The main purpose of the present section is to prove the following theorem.

Theorem 2 *All JL-colorings of finite connected graphs are achievable by the standard construction.*

Suppose that G is a graph on n vertices and c components, and $f : E(G) \to \{1, \ldots, k\}$ is an edge coloring of G such that each color $1, \ldots, k$ appears (i.e., f is surjective). The **slack** of f is $s(f) = n - c - k$.

If $S \subseteq E(G)$ and $j \in \{1, \ldots, k\}$, we will say that the color j is **dedicated to** S if and only if $f^{-1}(j) \subseteq S$. When S is the set of edges incident to a vertex $v \in V(G)$, we will say that a color dedicated to S is **dedicated to** v.

Lemma 2 *Let G and f be as above.*

(i) *If f is a RCF coloring, then $s(f) \geq 0$, with equality if and only if f is a JL-coloring.*
(ii) *A color $j \in \{1, \ldots, k\}$ can be dedicated to at most two different vertices; further, j is dedicated to two different vertices if and only if j appears only on the edges between the two.*

Proof Claim (i) follows from our observation that G must be colored with at most $n - c$ colors in order to avoid rainbow cycles, and claim (ii) is straightforward. □

Lemma 3 *Suppose that G is a graph on n vertices with c components , f is an edge coloring of G, $S \subseteq E(G)$, $G' = G - S$ has $c' = c + x$ components, and S has d colors dedicated to it. If f restricted to G' is RCF, then $x \leq s(f) + d$.*

Proof Let k be the number of colors appearing on the edges of G. Then $k' = k - d$ colors appear on the edges of G'. Let s' denote the slack of the restriction of f to G'. Then by Lemma 2 (i), if f restricted to G' is RCF, we have $0 \leq s' = n - c' - k' = n - (c + x) - (k - d) = s(f) + d - x$. □

Corollary 1 *Suppose that G is a connected graph with a JL-coloring, and $[A, B]$ is an edge cut in G. Then there is at least one color dedicated to $[A, B]$.*

Proof Let $n = |V(G)|$, f be the JL coloring of $E(G)$, and $S = [A, B]$, then the terms x and $s(f)$ in Lemma 3 satisfy $s(f) = n - (n-1) - 1 = 0$ and $x \geq 1$, because S is an edge cut in a connected graph. Therefore, by Lemma 3, $d \geq x - s(f) \geq 1$. □

Corollary 2 *In every JL-coloring of a loopless connected graph G on $n > 1$ vertices, for each $v \in V(G)$ some color is dedicated to v.*

Proof Since $n > 1$, for each $v \in V(G)$ the set of edges incident to v is an edge cut, $[\{v\}, V(G)\setminus \{v\}]$. □

Corollary 3 *If G is a JL-colored connected simple graph on $n > 1$ vertices, then at least one color appears exactly once in G.*

Proof Each of the n vertices of G has a color dedicated to it and there are only $n-1$ colors appearing. Therefore some color must be dedicated to two different vertices. The conclusion follows from Lemma 2 and the assumption that G is simple. □

Theorem 3 *Suppose that G is a finite connected graph with a JL-coloring f which admits a monochromatic edge cut $[A, B]$. Then $G[A]$, $G[B]$ are connected, the restrictions of f to each of $G[A]$, $G[B]$ are JL-colorings, the color sets on $G[A]$, $G[B]$ are disjoint, and neither contains the single color on $[A, B]$.*

Proof Let $S = [A, B], n = |V(G)|$, and $G' = G - S$. Let c be the number of components of G, $c' = c + x$ be the number of components of G', and d be the number of colors dedicated to S. Then $c = 1$, because G is connected, and $c' = c + x \geq 2$ because S is an edge cut. Therefore, $x \geq 1$.

Since f is a JL-coloring, $s(f) = 0$. Therefore by Lemma 3, $1 \leq x \leq s(f) + d = d$. By the assumption that $[A, B]$ is monochromatic and Corollary 1, $d = 1$. By the inequality above, it follows that $x = 1$, so $c' = c + x = 1 + 1 = 2$. From this we conclude that $G[A]$ and $G[B]$ are connected.

Because the one color on S is dedicated to S, it does not appear in $G[A] \cup G[B]$. Let the sets of colors appearing in the restrictions of f to $G[A]$ and $G[B]$ be $C(A)$ and $C(B)$, respectively. Then $|C(A) \cup C(B)| = n - 2$, and $|C(A)| \leq |A| - 1$, $|C(B)| \leq |B| - 1$, because the colorings on $G[A]$ and $G[B]$ are RCF. Since $|A| + |B| = n$, from these facts we conclude that $|C(A)| = |A| - 1$, $|C(B)| = |B| - 1$, and $C(A) \cap C(B) = \emptyset$. Therefore, $G[A]$ and $G[B]$ are JL-colored, with disjoint color sets. □

Theorem 3 implies that if for every JL-coloring of every finite connected graph G, with $|V(G)| > 1$, there must be a monochromatic edge cut in G, then JL-colorings of such graphs are all obtainable by the standard construction, and are therefore representable by vertex-labeled binary trees, as described in the abstract.

Theorem 4 *If G is a finite connected graph on $n > 1$ vertices with a JL-coloring, then there is a monochromatic edge cut in G.*

Proof Let G be a counterexample to the claim of the theorem with minimum $n + |E(G)|$. So G has a JL-coloring which admits no monochromatic edge cut. Then

G must be loopless, because the coloring is RCF, and must have no multiple edges between vertices—otherwise, the simple graph obtained by collapsing multiple edges into simple edges with the color borne by those multiple edges would be a counterexample with the same number of vertices and fewer edges.

Since G is connected and simple on $n > 1$ vertices, and JL-colored, by Corollary 3 there is an edge $e = uv \in E(G)$ bearing a color—let us call it red—which appears only on e. Let $G^* = G/e$, the result of contracting e. (The edge e disappears and the vertices u and v merge into a new vertex w which is incident in G^* to any edge of G, except e, which was incident to either u or v.) Let each edge of G^* bear the color that it bore in G. The number of colors on G^* is one less than $n-1$, the number of colors on G, because the color red was dedicated to $\{e\}$. Also, $|V(G^*)| = n - 1$. Since we can suppose that $n > 2$, we have $n-1 > 1$. Clearly G^* is connected. If G^* is RCF, then G^* is JL-colored, connected on more than one vertex, and $|V(G^*)| + |E(G^*)| < |V(G)| + |E(G)|$. It would then follow that there is a monochromatic edge cut $[A^*, B^*]$ in G^*. But then there is a monochromatic cut $[A, B]$ in G: if, without loss of generality, $w \in A^*$, take $A = (A^* \setminus \{w\}) \cup \{u, v\}$ and $B = B^*$.

The proof will be over if we show that there are no rainbow cycles in G^*. Since there are no rainbow cycles in G, the only cycles in G^* that might be rainbow must contain the vertex w. Let C^* be a rainbow cycle in G^* containing w; let x, y be the neighbors of w on the cycle. The possibilities are:

(i) $x = y$ and C^* is a double edge arising from the edges xu, xv in G. But then $G[\{u, v, x\}]$ is a rainbow C_3 in G.
(ii) $x \neq y$ and the edges xw, yw on C^* arise from edges xu, yv (or xv, yu) in G. But then we have a rainbow cycle in G with all the edges of C^*, letting xu replace xw and yv replace yw, together with e.
(iii) $x \neq y$ and the edges xw, yw arise from edges xu, yu (or xv, yv) in G; then the edges of C^* define a rainbow cycle in G.

□

Proof of Theorem 2 This follows immediately by combining Theorem 3 and 4. □

Finally, we note that the contrapositive of Theorem 4 is interesting in its own right.

Corollary 4 *Suppose G is an $n-1$-edge colored graph on n vertices without a monochromatic cut. Then G contains a rainbow cycle.*

4 Sharpness Results

For the rest of this paper, it will be more convenient to think of the contrapositive of Theorem 4, stated as Corollary 4. If G is connected and edge colored with $|V(G)| - 1$ colors and there is no monochromatic edge cut, then there must be a rainbow cycle. The remainder of this paper deals with the sharpness of this result; if the number of colors appearing is less than $|V(G)| - 1$ might the conclusion hold, and, if not, what strengthening of the hypothesis will force the conclusion?

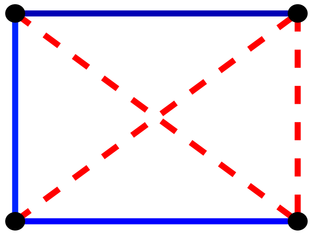

Fig. 3 A robustly colored K_4 using $|V(G)| - 2 = 2$ colors

With reference to a coloring of $E(G)$, we say that G is **robustly colored** if $V(G)$ cannot be partitioned into two non-empty parts $X, \bar{X}$ so that $[X, \bar{X}]$ is monochromatic.

We observe that K_4 can be robustly 2-colored without a rainbow cycle, by taking the two color classes to be edge disjoint spanning trees. Therefore we cannot relax the $|V(G)| - 1$ requirement without additional restrictions (Fig. 3).

First, we show that for all $n \geq 4$ there are robust $n-2$ colorings of K_n avoiding rainbow cycles. The main result of this section shows that all such colorings arise from this coloring of K_4 via a construction which we call the **cloning construction**.

Suppose that $n \geq 4$ and K_n is robustly colored with exactly $n-2$ colors appearing so that there are no rainbow cycles in K_n. Let K_{n+1} be formed as follows: pick any arbitrary vertex v and clone it—if we call the cloned vertex w, then for all vertices $x \in V(K_n)$ with $x \neq v$, give the edge wx the same color that appeared on vx. Give the edge vw a color not appearing in K_n.

Exactly $n - 2 + 1 = (n+1) - 2$ colors appear on K_{n+1}. The single edge bearing the new color cannot be an edge cut in K_{n+1}, because $n \geq 4$; therefore, if there is a monochromatic edge cut $[A, B]$ in K_{n+1}, v and w must be on the same side of the cut. Without loss of generality, let $v, w \in A$. But then $[A \setminus \{w\}, B]$ is a monochromatic edge cut in K_n. Therefore the edge coloring of K_{n+1} is robust.

Suppose C is a rainbow cycle in K_{n+1}. It would necessarily have to contain w. If $v \notin V(C)$, then $(V(C) \setminus \{w\}) \cup \{v\}$ is the vertex set of a rainbow cycle in K_n. So both v and w appear on C. Let $x \in V(K_n) \setminus \{v\}$ be a neighbor of w on C. If $vx \in E(C)$ then C is not rainbow. Therefore vx is a chord of C, of the same color as wx. It follows that one of the cycles in $C \cup vx$ is a rainbow cycle in K_n. Thus, no such C exists.

Next, we show that every robust edge coloring of K_n with $n-2$ colors that avoids rainbow cycles can be created by the cloning construction.

Theorem 5 *If K_n is edge colored with $n-2$ colors avoiding monochromatic cuts with $n > 4$ and each color appears at least twice, then there is a rainbow cycle.*

Proof Let $G = K_n$ be edge colored with $n-2$ colors, where each color appears at least twice so that there are no monochromatic edge cuts. Suppose also that there are no rainbow cycles.

We claim that there is no isolated edge in G of any color. That is, each edge is adjacent to another edge of the same color. Suppose, to the contrary, that $ww_1 \in E(G)$ is colored red, and neither w nor w_1 is incident to a red edge other than ww_1. Contract ww_1—let w_2 be the new vertex obtained by merging w and w_1—to obtain

$G' = G/ww_1$, a robustly edge colored graph on $n-1$ vertices, with $n-2$ colors appearing. If the coloring of G' forbids rainbow cycles then G' is JL-colored, which implies that the coloring of G' is not robust after all.

Suppose G' contains a rainbow cycle C'. If C' either contains no red edge, or does not pass through w_2, then there is a rainbow cycle in G. Therefore, C' contains w_2 and does contain a red edge, say xy, $x, y \in V(G)\setminus\{w, w_1\}$, by the assumption that ww_1 is isolated from other red edges. This also implies that the cycle C in G obtained by 'opening' w_2 into ww_1 is of length at least 4. Further, at least one of the edges wx, wy, w_1x, w_1y is a chord of C which creates, with C, two cycles, at least one of which is rainbow. This establishes that the edge ww_1 which is not adjacent to another edge of G bearing its color cannot exist.

We also note there exists an induced subgraph $H = K_{n-1}$ on $n-1$ vertices that has all $n-2$ colors appearing on the edges. If not, then for every vertex $y \in V(G)$ we have that there is at least one color dedicated to y. If any color is dedicated to both y_1 and y_2 for some $y_1, y_2 \in V(G)$ with $y_1 \neq y_2$, then that color appears on only one edge. Since we assume every colors appears at least twice, then all of these dedicated colors must be distinct. This is impossible since we have n vertices and $n-2$ colors.

Let v be the vertex in G missing from H. If H is not JL-colored, then there is a rainbow cycle in H and thus in G. So, we may assume H is JL-colored. By Corollary 3, we know that some color appears exactly once. Call this color green and say that it appears on the edge xy.

Then, without loss of generality, xv is green since there is no isolated edge. For every $x' \in V(G)\setminus\{v, x, y\}$ the edge from v to x' is either the same color as xx' or it is green; otherwise we have a rainbow K_3.

Moreover, there is a monochromatic cut $[X, Y]$ in H (by Theorem 4 since H is JL-colored) with x and y in the same part. If x and y were in different parts then the monochromatic cut would be green, but $|V(G)\setminus\{v\}| \geq 4$, and green supposedly appears only on xy, among edges of $G-v$. Say $x, y \in X$ and the edges in $[X, Y]$ are colored red. The edges from v to Y must be red or green (otherwise we have a rainbow K_3). Also, at least one edge must be green or $[X \cup \{v\}, Y]$ would be a monochromatic cut. Since no edge in the subgraph induced by Y is green or red by Theorem 3, this implies all edges from v to Y are green. This also implies all v to X edges are red or green. The proof is finished after the following observations.

If $|Y| = 1$, then $|X| \geq 3$. Then for $z \in X$ with $z \neq x, y$ we have vz is colored green. (If it were red, then vzx would be a rainbow K_3). This forces vy to be green; otherwise, because yz is neither red nor green, if vy were red then vyz would be a rainbow K_3. But then $[\{v\}, X \cup Y]$ is a monochromatic cut, because $z \in X\setminus\{x, y\}$ was arbitrary.

If $|Y| \geq 2$ then the subgraph induced by Y is JL-colored and therefore admits a color that appears only on one edge in Y; this color is neither red nor green. This color must appear on an edge incident to v, because there is no isolated edge in G of any color. But, this color cannot appear on an edge incident to v since all such edges are red or green. □

Corollary 5 *For $n > 4$, every robust edge coloring of K_n with $n-2$ colors that avoids rainbow cycles is created by the cloning construction.*

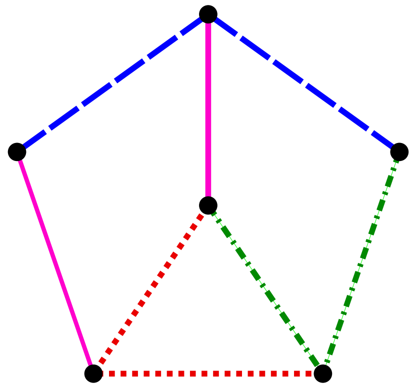

Fig. 4 A connected graph on 6 vertices with 4 colors in which every color appears at least twice, there is no monochromatic cut and there is no rainbow cycle

Proof By Theorem 5 we now know that in all robust edge colorings of K_n with $n > 4$ using $n-2$ colors in which we avoid rainbow cycles, at least one color appears exactly once. Pick some edge where its assigned color appears only on that edge; call that edge uv. For all $x \in V(K_n)$ with $x \neq u, v$, if the color on xu is different than the color on xv we have a rainbow K_3. So we can regard the coloring as arising from an RCF robust edge coloring of $K_n - u$ with $n-3$ colors by the cloning construction—here u is a 'clone' of v. □

One may extend the idea of the cloning construction to general graphs—begin with an RCF edge coloring of G with $n-2$ colors appearing created by the cloning construction. Let $x \in V(G)$. We may extend to a graph on $n+1$ vertices by partially cloning x: introduce a new vertex v and add the edge zv for some, but not necessarily all, vertices $z \in V(G) \setminus \{v\}$. Color each added edge zv with the color on zx, and let the new edge vx bear a new color. This creates a robustly colored graph on $n+1$ vertices, uses $n-1$ colors and avoids rainbow cycles. One may then ask if all robustly colored connected graphs on n vertices with $n \geq 4$ using $n-2$ colors and avoiding rainbow cycles are a result of this cloning construction; i.e., can the results of Theorem 5 be extended to all connected graphs G on n vertices? Unfortunately, such a hope is false and we end this paper with a counterexample given in Fig. 4.

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