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Routing number of dense and expanding graphs

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Consider a connected graph G, with a pebble placed on each vertex of G. The routing number, $rt(G)$, of G is the minimum number of steps needed to route any permutation on the vertices of G, where a step consists of selecting a matching in the graph and swapping the pebbles on the endpoints of each edge. Alon, Chung, and Graham [*SIAM J. Discrete Math.*, 7 (1994), pp. 516–530.] introduced this parameter, and (among other results) gave a bound based on the spectral gap for general graphs. The bound they obtain is polylogarithmic for graphs with a sufficiently strong spectral gap. In this paper, we use spectral properties and probablistic methods to investigate when this upper bound can be improved to be constant depending on the gap and the vertex degrees.

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1. Introduction

Let $G = (V, E)$ be a connected simple graph with n vertices. Alon, Chung and Graham introduced the notion of the *routing number* of G, behind which is the following simple process: imagine a pebble on each vertex of the graph labeled with the vertex it sits on, and let π be an arbitrary permutation in S_V. The goal, then, is to move the pebbles according to π; that is, to move the pebble labeled v to $\pi(v)$. In any given step, a (not necessarily maximal) matching is selected in the graph and the pebbles at the endpoints are interchanged. The routing number of G for the permutation π, denoted by $rt(G, \pi)$, is the minimum number of steps needed to route all of the pebbles to their desired vertex as determined by π. Finally, the routing number of the graph G is

$$rt(G) = \max_{\pi \in S_V} rt(G, \pi).$$

Classes of graphs for which the routing number is known include complete graphs, $K_{n,n}$, paths, cycles, and stars (see [1], [5]). In particular,

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$rt(K_n) = 2$, $rt(K_{n,n}) = 4$, and $rt(P_n) = n$, where P_n is an n-vertex path. Additionally, there are a number of other classes of graphs for which bounds on the routing number are known. In [1], Alon, Chung, and Graham gave preliminary bounds for trees, general complete bipartite graphs, Cartesian products, hypercubes, and grids. For any tree T, Zhang showed in [7] that $rt(T) \leq \frac{3n}{2} + O(\log n)$, confirming a conjecture made by Alon, Chung, and Graham. In [5], Li, Lu, and Yang improved the bounds on general complete bipartite graphs and hypercubes. Specifically, they showed that $n + 1 \leq rt(Q_n) \leq 2n - 2$ using a computer search to prove that $rt(Q_3) = 4$, then applying the bound of [1] for Cartesian products of graphs. Alon, Chung, and Graham conjectured that $rt(Q_n) \sim \alpha n$, and while the above bounds show that $\alpha \in [1, 2]$, they conjecture that the correct value of α is closer to 1 than to 2. However, finding more precise asymptotics for the routing number of the hypercube is still an open and interesting question.

The motivation for our paper is the following result of Alon, Chung, and Graham.

Theorem ([1]). *For a d-regular graph G, $rt(G) \leq O\left(\frac{1}{(1-\sigma)^2} \log^2 n\right)$, where $\sigma = \max_{i \neq 1} |1 - \lambda_i(\mathcal{L})|$ is the spectral gap of the normalized Laplacian.*

We briefly note that this result was originally stated in terms of the second eigenvalue of the adjacency matrix for so-called (n, d, λ)-graphs; that is d-regular graphs with second adjacency eigenvalue λ, for which $\sigma = \frac{\lambda}{d}$. We state the result in terms of σ, however, to give a clearer comparison to our own results that in some cases apply to irregular graphs, for which the normalized Laplacian is more appropriate.

Our main results improve the upper bound of this result of Alon, Chung, Graham in the case where σ is small. In particular, among other results, we prove the following.

Theorem 1. *For all $k > 0$ and $C > 0$, there exists $N_{k,C} \in \mathbb{N}$ such that for any regular graph on $n \geq N_{k,C}$ vertices with degree $d \geq \exp\left(\frac{C \log n}{\log \log n}\right)$, and $\sigma = k \cdot d^{-1/2} < \frac{1}{3}$,*

$$\log(rt(G)) = O\left(\frac{\log n}{\log d}\right).$$

This result improves the Alon, Chung and Graham result throughout its range on d. However, the improvement is clearest when d is polynomial in n, in which case it gives the following constant bound on the routing number and hence improves the Alon, Chung and Graham result by a factor of $\log^2(n)$.

Corollary 1. *For all $k > 0$ and $\epsilon > 0$, there exist $N_{k,\epsilon} \in \mathbb{N}$ and $C_{k,\epsilon} \in \mathbb{N}$ such that for any regular graph G on $n \geq N_{k,\epsilon}$ vertices with degree $d = n^{\epsilon}$ and $\sigma = kd^{-1/2} < \frac{1}{3}$, $rt(G) \leq C_{k,\epsilon}$.*

At their heart, the strategy of our proofs is similar to that of Alon, Chung, and Graham: we use the fact that permutations can be written as the product of two permutations of order two and build disjoint paths between vertices involved in a transposition to route pebbles along. However, instead of using random walks to find paths, we will build paths between vertices more directly using information about the spectrum of the normalized Laplacian. To accomplish this, we will use a random partitioning of the transpositions to select a collection of transpositions to be routed simultaneously, and Hall's theorem for hypergraphs [2] to select disjoint paths. As a result, we will get an upper bound for the routing number dependent upon the length of the paths and the number of partite sets.

Before we prove our main result, we begin with an easier case that will demonstrate the basis of our proof idea.

2. Warm-up: extremely dense graphs

As a starting point, we ask what bounds one can get on the routing number by density alone? If the minimum degree is at least half of the vertices, then between any two vertices there is some overlap in the neighborhoods of the vertices. More specifically, if we take any transposition in the decomposition of a permutation, their neighborhoods share a nonempty intersection, which allows us to route this transposition through a three-vertex path. The larger this minimum degree, the more of these transpositions we will be able to route simultaneously because this minimum overlap will be larger.

As noted in the introduction, any permutation in S_n can be split into a product of two permutations of order two; that is, two permutations where the cycle structure consists entirely of transpositions. This is implicit in the work of Alon, Chung, and Graham in showing that $rt(K_n) = 2$ and we observe that it can easily be directly verified for cyclic permutations, and follows immediately from this. When proving upper bounds for the routing number of a graph, it is common to rewrite the arbitrary permutation as this product of two permutations of order two.

Theorem 2. *Let $\epsilon > 0$. For a graph G with minimum degree $\delta = \left(\frac{1}{2} + \epsilon\right) n$, $rt(G) \leq \frac{9}{\epsilon}$.*

Remark: Up to the constant 9, $rt(G) = \Omega\left(\frac{1}{\epsilon}\right)$ is best possible. Consider a graph with vertex set $V = A \cup B \cup C$, where A and B have size $\left(\frac{1}{2} - \epsilon\right) n$

and C has size $2\epsilon n$, and every vertex in A is adjacent to each vertex in $A \cup C$, every vertex in B is adjacent to each vertex in $B \cup C$, and every vertex in C is also adjacent to each other vertex in C. Then the minimum degree of this graph is $\left(\frac{1}{2} + \epsilon\right) n$. However, if a permutation took each vertex in A and swapped it with a vertex in B, then at each step only $2\epsilon n$ vertices from A or B could be moved into C, which is necessary to route them to their target. Thus, $\Omega\left(\frac{1}{\epsilon}\right)$ steps are required to move all vertices in A and B through C.

Proof. Let G be a graph with minimum degree $\delta = \left(\frac{1}{2} + \epsilon\right) n$ for some $\epsilon > 0$ and let π be a permutation on the vertices. Then $\pi = \pi_2\pi_1$ for some permutations $\pi_1, \pi_2 \in S_V$ of order 2. To route the vertices according to π_1, write π_1 as the product of disjoint transpositions and order the transpositions arbitrarily. Now, select the first $\frac{\epsilon n}{3}$ of these transpositions. For each transposition (v, v'), v and v' have at least ϵn common neighbors, meaning that at least $\frac{\epsilon n}{3}$ of these common neighbors are not in any of the transpositions in this selection. Thus, for each transposition (v, v') we select a middle vertex x that is adjacent to both v and v', is not in any of the selected transpositions, and also has not been selected as the middle vertex for any other transposition in this selection. Hence, we can simultaneously route each of these ϵn transpositions through their corresponding selected vertex x, returning the pebble initially on x back to x, in three steps because vxv' is a path of length 3. Since π_1 has at most $\frac{n}{2}$ disjoint transpositions, we must repeat this process at most $\frac{n}{2} \div \frac{\epsilon n}{3} = \frac{3}{2\epsilon}$ times to route all of the vertices according to π_1. Consequently, we can route all vertices according to π_1 in $\frac{9}{2\epsilon}$ steps. Similarly, all vertices can be routed according to π_2 in $\frac{9}{2\epsilon}$ steps. Therefore, the pebbles on the vertices of G can be routed according to π in $\frac{9}{\epsilon}$ steps, so by the arbitrary selection of $\pi \in S_V$, $rt(G) \leq \frac{9}{\epsilon}$. $\square$

This is the best that one can obtain by minimum degree alone. Indeed once $\delta < \frac{n}{2}$, then the graph need not even be connected. Thus, such a naive approach is insufficient, in general, to ensure that the graph has constant routing number. Our techniques for this will involve techniques from spectral graph theory, which we introduce in the next section.

3. Spectral graph theory prerequisites

In order to guarantee a constant routing number for graphs with degree cn, where c is some constant less than $\frac{1}{2}$, we will need to impose additional conditions on our graph. To do this, we will use tools from spectral graph theory to help generalize our results.

The *normalized Laplacian* of a graph G is the $n \times n$ matrix given by

$$\mathcal{L}(u,v) = \begin{cases} 1 & \text{if } u = v \text{ and } \deg(v) > 0 \\ -\frac{1}{\sqrt{\deg(u)\deg(v)}} & \text{if } u \sim v \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, the normalized Laplacian can be defined as

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2},$$

where D is the diagonal degree matrix of G and A is the adjacency matrix of G. For more on the normalized Laplacian, see [3]. We will denote the eigenvalues of $\mathcal{L}$ by $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2$. Note that $\lambda_2 > 0$ iff G is connected and $\lambda_n = 2$ iff G is bipartite. The *spectral gap* of $\mathcal{L}$ is given by $\sigma = \max\{|1-\lambda_2|, |1-\lambda_n|\}$. Thus, $\sigma < 1$ if G is connected and non-bipartite.

The spectral gap, in a sense, measures the randomness of the edge distribution. In order to make this precise, we introduce the *volume* of a set of vertices. For $X \subset V(G)$, the volume of X is

$$\mathrm{Vol}(X) = \sum_{v \in X} \deg(v).$$

With this in hand, we recall the following fundamental result on the edge distribution of a graph with some spectral gap.

Lemma 1 (Expander Mixing Lemma, [3])**.** *Let $\lambda_1 \le \lambda_2 \le \cdots \lambda_n$ be the eigenvalues of the normalized Laplacian of G and let $\sigma = \max\{|1-\lambda_2|, |1-\lambda_n|\}$. Then for subsets of vertices X, Y, the number of edges $e(X,Y)$ with one vertex in X and the other vertex in Y is bounded by*

$$\left| e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)} \right| \le \sigma \sqrt{\mathrm{Vol}(X)\mathrm{Vol}(Y)}$$

and

$$\left| e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(G)} \right| \le \sigma \sqrt{\mathrm{Vol}(\overline{X})\mathrm{Vol}(\overline{Y})}.$$

As a corollary to the Expander Mixing Lemma, we quantify vertex expansion in a graph in the following sense.

Corollary 2. *Let G be a graph and let X be a subset of the vertices of G. If $N(X)$ denotes the set of vertices adjacent to at least one vertex of X, then* $\mathrm{Vol}(N(X)) \ge \min\left\{\frac{1}{2}\mathrm{Vol}(G), \frac{1}{4\sigma^2}\mathrm{Vol}(X)\right\}$.

Proof. Let G be a graph and let X be a subset of the vertices of G. By the Expander Mixing Lemma,

$$\mathrm{Vol}(X) = e(X, N(X)) \leq \frac{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}{\mathrm{Vol}(G)} + \sigma\sqrt{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}.$$

Then

$$\frac{1}{2}\mathrm{Vol}(X) \leq \frac{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}{\mathrm{Vol}(G)}$$

or

$$\frac{1}{2}\mathrm{Vol}(X) \leq \sigma\sqrt{\mathrm{Vol}(X)\mathrm{Vol}(N(X))}.$$

Therefore, $\mathrm{Vol}(N(X)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or $\mathrm{Vol}(N(X)) \geq \frac{1}{4\sigma^2}\mathrm{Vol}(X)$. Thus

$$\mathrm{Vol}(N(X)) \geq \min\left\{\frac{1}{2}\mathrm{Vol}(G), \frac{1}{4\sigma^2}\mathrm{Vol}(X)\right\}.$$

□

4. Graphs with linear degree

In order to guarantee a constant routing number for graphs with minimum degree cn, where c is some constant less than $\frac{1}{2}$, we will need to take a slightly different approach. Instead of relying on the neighborhoods of two vertices to overlap, we will use the Expander Mixing Lemma to guarantee that there are many edges between the neighborhoods of any two vertices. Notice that in the following theorem, in order to compensate for reducing the minimum degree, we need to add a condition on σ. This is a theme throughout this paper: in order to weaken the degree condition, we will need to strengthen the condition on σ, therefore bringing more structure to the graph.

Theorem 3. *Fix $0 < c < 1$. Let G be a graph with minimum degree at least $\delta = cn$, with $\sigma < c^2$. Then*

$$rt(G) \leq \frac{16}{c^2(c^2 - \sigma)}.$$

Proof. Let G be a graph with minimum degree $\delta = cn$ for some $c > 0$ and with $\sigma < c^2$. Let π be a permutation of $V(G)$. Then $\pi = \pi_2\pi_1$ for some $\pi_1, \pi_2 \in S_V$ of order 2, meaning that each of π_1 and π_2 can be written as a product of disjoint transpositions. Let (v, v') be a transposition in π_1 or π_2. Let $N(v)$ be the neighborhood of v and let $N(v')$ be the neighborhood of v'. Since $\deg(v) \geq cn$ and $\deg(v') \geq cn$, $|N(v)| \geq cn$ and $|N(v')| \geq cn$ and in turn, $\mathrm{Vol}(N(v)) \geq (cn)^2$ and $\mathrm{Vol}(N(v')) \geq (cn)^2$.

Let

$$f\left(\mathrm{Vol}(N(v))\right) = \frac{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))}{\mathrm{Vol}(G)} - \sigma\sqrt{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))}.$$

Then

$$\begin{aligned}
f'\left(\mathrm{Vol}(N(v))\right) &= \frac{\mathrm{Vol}(N(v'))}{\mathrm{Vol}(G)} - \frac{\sigma\sqrt{\mathrm{Vol}(N(v'))}}{2\sqrt{\mathrm{Vol}(N(v))}} \\
&= \sqrt{\mathrm{Vol}(N(v'))}\left(\frac{\sqrt{\mathrm{Vol}(N(v'))}}{\mathrm{Vol}(G)} - \frac{\sigma}{2\sqrt{\mathrm{Vol}(N(v))}}\right) \\
&> \sqrt{\mathrm{Vol}(N(v'))}\left(\frac{\sqrt{\mathrm{Vol}(N(v'))}}{\mathrm{Vol}(G)} - \frac{c^2n^2}{2\mathrm{Vol}(G)\sqrt{\mathrm{Vol}(N(v))}}\right) \\
&\geq \sqrt{\mathrm{Vol}(N(v'))}\left(\frac{\sqrt{\mathrm{Vol}(N(v'))}}{\mathrm{Vol}(G)} - \frac{\sqrt{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))}}{2\mathrm{Vol}(G)\sqrt{\mathrm{Vol}(N(v'))}}\right) \\
&> 0.
\end{aligned}$$

Thus, f is increasing as a function of $\mathrm{Vol}(N(v))$ and, by symmetry, as a function of $\mathrm{Vol}(N(v'))$. Hence, since $\mathrm{Vol}(N(v)) \geq (cn)^2$ and $\mathrm{Vol}(N(v')) \geq (cn)^2$,

$$\begin{aligned}
e(N(v), N(v')) &\geq \frac{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))}{\mathrm{Vol}(G)} - \sigma\sqrt{\mathrm{Vol}(N(v))\mathrm{Vol}(N(v'))} \\
&\geq \frac{(cn)^4}{\mathrm{Vol}(G)} - \sigma(cn)^2 \\
&= (cn)^2\left(\frac{(cn)^2}{\mathrm{Vol}(G)} - \sigma\right) \\
&\geq (cn)^2(c^2 - \sigma).
\end{aligned}$$

Let

$$\epsilon = \frac{c^2(c^2 - \sigma)}{4}.$$

Take a collection of the first ϵn transpositions in π_1. Then there are $2\epsilon n$ vertices in this collection. We will say that an edge is unused if one of its vertices is either in the collection or is incident to an edge that has already been assigned. Also, define $N(v)$ to be the set of vertices that are adjacent to v. Thus, for the first transposition (v_1, v_1') of the collection, there are at most $2\epsilon n^2 \leq \frac{1}{2}c^2n^2(c^2-\sigma)$ edges between $N(v_1)$ and $N(v_1')$ used of the at least $(cn)^2(c^2-\sigma)$ edges that must be present. Select one of the unused edges to pair with this transposition. For the next transposition (v_2, v_2'), there are at most $(2\epsilon n+2)n$ used edges between $N(v_2)$ and $N(v_2')$. Since $(2\epsilon n+2)n < 4\epsilon n^2 \leq (cn)^2(c^2-\sigma)$, an unused edge between $N(v_2)$ and $N(v_2')$ can be selected to pair with this transposition. Proceeding inductively, for each $i \leq \epsilon n$, there are at most $(2\epsilon n+2i-2)n < 4\epsilon n^2 \leq (cn)^2(c^2-\sigma)$ used edges between $N(v_i)$ and $N(v_i')$. Thus, an unused edge can be selected to pair with the transposition (v_i, v_i'). Since the selected paths between each v_i and v_i' are disjoint, we can route each of the transpositions (v_i, v_i') simultaneously in four steps, leaving the two middle vertices in each path back in their original positions.

Since there are at most $\frac{n}{2}$ transpositions in π_1, the above process must be repeated at most $\frac{1}{2\epsilon}$ times to route all of the transpositions in π_1. Since each collection of ϵn transpositions routes in four steps, it will take at most $\frac{2}{\epsilon}$ steps to route all of the vertices according to π_1. By performing the same process on π_2, it will also take at most $\frac{2}{\epsilon}$ steps to route all of the vertices according to π_2. Therefore,

$$rt(G) \leq \frac{4}{\epsilon} = \frac{16}{c^2(c^2-\sigma)}.$$

□

5. Graphs with sublinear degree

If we desire a constant routing number, our goal is to route a positive proportion of the transpositions simultaneously. Unless these transpositions are spread out, this will be impossible because there could be too much overlap in the neighborhoods of these transpositions that we are seeking to route. For example, if we attempted to route a collection of transpositions including a vertex and all of its neighbors simultaneously, we would not be able to. While in the previous proof, we ordered the transpositions arbitrarily, we will now need to select the collections of transpositions more carefully. In order to do this, we will require regularity of the graph in order to better control the iterated neighborhoods of a vertex.

5.1. Preliminaries

We will now partition the transpositions randomly. To do this, we will use Talagrand's inequality, which allows us to quantify the likelihood that a random variable is close to its mean given certain conditions.

Theorem ([6]). *Let $c > 0$, $r \geq 0$, and d be given and let the non-negative measurable function g on the product space $\Omega = \prod_i \Omega_i$ satisfy the following two conditions, for each $x \in \Omega$: (a) changing any coordinate x_j changes the value of $g(x)$ by at most c; and (b) if $g(x) = s$ then there is a set of at most $rs + d$ coordinates that certify that $g(x) \geq s$. Let $X_1, \ldots, X_n$ be independent random variables, where X_i takes values in Ω_i; let $X = (X_1, \ldots, X_n)$ and let $g(X)$ have mean μ. Then for each $t \geq 0$,*

$$\mathbb{P}(g(X) - \mu \geq t) \leq \exp\left(-\frac{t^2}{2c^2(r\mu + d + rt)}\right)$$

and

$$\mathbb{P}(g(X) - \mu \leq -t) \leq \exp\left(-\frac{t^2}{2c^2(r\mu + d + t/3c)}\right).$$

We use Talagrand's inequality in the lemma that follows to provide more structure to the interactions between the neighborhoods of each vertex and the partition of transpositions that is used to route a number of the transpositions simultaneously. Specifically, this lemma states that we can partition a collection of disjoint transpositions so that most of the vertices that have a path of length j from any fixed vertex are not in any single part of the partition.

Lemma 2. *Fix $C > 0$. There exists $N_C \in \mathbb{N}$ such that for all $n \geq N_C$, if G is a d-regular graph on n vertices with $d \geq \exp\left(C\frac{\log n}{\log\log n}\right)$, $\mathcal{T}$ is a collection of disjoint transpositions of the vertices, and $c \geq \exp\left(-\frac{C\log n}{2\log\log n}\right)$, then there exists a partition $X_1, \ldots, X_{4/c}$ of $\mathcal{T}$ so that both of the following hold.*

1. *$|X_i| \leq \frac{nc}{4}$ for all $i \in \{1, \ldots, \frac{4}{c}\}$.*
2. *Let $N_j(v) = \{u \in V(G) :$ there is a path of length j from u to $v\}$. For any $v \in V(G)$, $i \in \left\{1, \ldots, \frac{4}{c}\right\}$, and $j \in \{1, \ldots, n\}$, at most $c|N_j(v)|$ vertices in $N_j(v)$ are in transpositions of X_i.*

Remark: In the regime we care about, c will be significantly larger than the the minimum asserted here – in the (most important) case that the minimum degree is a polynomial in n, for instace, c is a constant not depending on n. Even when d is of the form $\exp\left(\frac{\log n}{\log\log n}\right)$, c will be poly-logarithmic in $1/\log n$.

Proof. Create a partition $X_1, \ldots, X_{4/c}$ of $\mathcal{T}$ by, for each transposition (τ_1, τ_2), placing it in a part from $X_1, \ldots, X_{4/c}$ uniformly at random. Then $\mathbb{E}(|X_i|) =$

$|\mathcal{T}| \cdot \frac{c}{4} \leq \frac{nc}{8}$. Since $|X_i|$ has a binomial distribution with $p = \frac{c}{4}$, $\sigma(|X_i|) = \sqrt{\frac{n}{2} \cdot \frac{c}{4}\left(1 - \frac{c}{4}\right)} = \sqrt{\frac{nc(4-c)}{32}}$. Thus, by Hoeffding's inequality, if $|\mathcal{T}| = \frac{n}{2}$,

$$\begin{aligned}\mathbb{P}\left(|X_i| \geq \frac{nc}{4}\right) &= \mathbb{P}\left(|X_i| \geq \left(\frac{c}{4} + \frac{c}{4}\right)\frac{n}{2}\right) \\ &\leq \exp\left(-2\left(\frac{c}{4}\right)^2 \cdot \frac{n}{2}\right) \\ &= \exp\left(-\frac{c^2 n}{16}\right).\end{aligned}$$

If $|\mathcal{T}| < \frac{n}{2}$, then Hoeffding's inequality [4] would give a smaller upper bound. Thus, $\mathbb{P}\left(|X_i| \geq \frac{nc}{4}\right) \leq \exp\left(-\frac{c^2 n}{16}\right)$ for each $i \in \{1, \dots, 4/c\}$.

Now, fix $j \in \{1, \dots, n\}$. Define $h_j(v, X_i)$ to be the number of vertices in $N_j(v)$ that are also in transpositions of X_i. First, note that by changing the placement of a single transposition, $h_j(v, X_i)$ changes by at most 2. Second, h_j is 1-certifiable because if part X_i is selected for s transpositions containing a neighbor of v, $h_j(v, X_i) \geq s$. Third, note that $\mathbb{E}(h_j(v, X_i)) \leq \frac{c \cdot |N_j(v)|}{4}$. Hence, by Talagrand's inequality,

$$\begin{aligned}\mathbb{P}\left(h_j(v, X_i) \leq c \cdot |N_j(v)|\right) &= \mathbb{P}\left(h_j(v, X_i) \geq \frac{3c|N_j(v)|}{4} + \frac{c|N_j(v)|}{4}\right) \\ &\leq \mathbb{P}\left(h_j(v, X_i) - \mathbb{E}(h_j(v, X_i) \geq \frac{3c|N_j(v)|}{4}\right) \\ &\leq \exp\left(-\frac{\left(\frac{3c|N_j(v)|}{4}\right)^2}{2(2)^2\left(\mathbb{E}(h_j(v, X_i)) + \frac{3c|N_j(v)|}{4}\right)}\right) \\ &\leq \exp\left(-\frac{9}{128}c|N_j(v)|\right).\end{aligned}$$

Note that $|N_j(v)| \geq d - 1$ for any $v \in V(G)$ and any $j \in \{1, \dots, n\}$. The probability that $|X_i| \geq \frac{nc}{4}$ or $h(v, X_i) \geq cd$ for any $v \in V(G)$ and any $i \in \{1, \dots, \frac{4}{c}\}$ is at most

$$\sum_{i=1}^{4/c} \mathbb{P}\left(|X_i| \geq \frac{nc}{4}\right) + \sum_{\substack{v \in V(G) \\ i \in \{1, \dots, 4/c\} \\ j \in \{1, \dots, n\}}} \mathbb{P}(h_j(v, X_i) \geq c|N_j(v)|)$$

$$
\begin{aligned}
&= \sum_{i=1}^{4/c} \exp\left(\frac{c^2 n}{16}\right) + \sum_{\substack{v\in V(G)\\ i\in\{1,\dots,4/c\}\\ j\in\{1,\dots,q\}}} \exp\left(-\frac{9}{128}c|N_j(v)|\right)\\
&\le \frac{4}{c}\exp\left(-\frac{c^2 n}{16}\right) + \frac{4n^2}{c}\exp\left(-\frac{9}{128}c(d-1)\right)\\
&< 1
\end{aligned}
$$

for sufficiently large n, where here we use the fact that our bounds on c and d to ensure that the exponent in the second exponential is tending to negative infinity. Therefore, there exists such a partition $X_1, \dots, X_{4/c}$ of $\mathcal{T}$. $\square$

Once we have this partition, our goal will be to build paths between the vertices of the transpositions. For each part of the partition, we want to find a collection of disjoint paths through which we will be able to route all of the transpositions simultaneously. In order to do that, we will use Hall's theorem for hypergraphs, stated below.

Theorem ([2])**.** *Let $\mathcal{A}$ be a family of n-uniform hypergraphs. A sufficient condition for the existence of a system of disjoint representatives of $\mathcal{A}$ is that for every $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$.*

First, we use Lemma 2 to partition the disjoint collection of transpositions that comprise π_1 into parts (X_i) satisfying the conclusions of the lemma. Our goal is to route the transpositions of a given part X_i simultaneously. In this direction, we select a positive integer z sufficiently large to guarantee many paths. For a particular i and for each transposition $(v_j, v'_j) \in X_i$, build a hypergraph $\Gamma_{(v_j,v'_j)}$ with vertex set $V(G)$, where there exists a hyperedge $\{u_1, \dots, u_{z-2}\} \in E\left(\Gamma_{(v_j,v'_j)}\right)$ if and only if $v_j, u_1, \dots, u_{z-2}, v'_j$ is a path from v_j to v'_j and none of $u_1, \dots, u_{z-2}$ are in any transposition of X_i. This yields that $\Gamma_{(v_j,v'_j)}$ is a $(z-2)$-uniform hypergraph for each $(v_j, v'_j) \in X_i$.

Our goal is to find a system of disjoint representatives for $\mathcal{A} = \{\Gamma_{(v_j,v'_j)} : v_j, v'_j \in X_i\}$, because this would give us a collection of z-vertex paths through which we can simultaneously route each transposition of X_i. By Hall's theorem for hypergraphs, there exists such a system if for each $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $(z-2)(|\mathcal{B}| - 1)$. Verifying this condition is equivalent to fixing a subset T of transpositions, then finding a collection of vertex-disjoint paths joining the vertices of a transposition in T with size $(z-2)(|T| - 1)$.

5.2. Proofs of results

We begin with a theorem whose proof has a similar flavor to our main theorem in that it uses the random partition of transpositions and Hall's theorem for hypergraphs as described above, but has a stronger degree condition, which in turn will give us a better bound. This degree condition also allows us to use paths of length four through which to route the transpositions of our permutation.

Theorem 4. *Fix $0 < c < \frac{1}{6}$. Then there exist an $N_c \in \mathbb{N}$ so that the following holds: Let G be a d-regular graph on $n \geq N_c$ vertices. Suppose $\sigma < \frac{d(1-6c)^2}{n}$. Then $rt(G) \leq \frac{32}{c}$.*

Remark: The c here can technically depend in a mild way on n (as per the statement of Lemma 2), however it cannot be too small – the point is that if σ is too large, then we lose sufficient control on the (iterated) neighborhoods to apply our techniques. In general, σ being small yields the best results, and in general σ is of order at least $\frac{1}{\sqrt{d}}$. The requirement in *this* result is in terms of $\sigma = O(\frac{d}{n})$ – and this becomes problematic once $d = o(n^{2/3})$, and hence this result is really interesting only for graphs with degree $d = n^{\epsilon}$ for some $\epsilon \geq \frac{2}{3}$.

Proof. Let G be a d-regular graph where $d = n^{\epsilon}$. Consider a permutation π of the vertices. Then $\pi = \pi_2\pi_1$ for some $\pi_1, \pi_2 \in S_V$ of order two. Thus, π_1 and π_2 can each be written as a product of disjoint transpositions.

Let $\mathcal{T} = \{(v, v') \in \pi_1\}$, the collection of all transpositions in π_1. By Lemma 2, there exists a partition $X_1, \ldots, X_{4/c}$ in which each part X_i has size at most $\frac{nc}{4}$ and no vertex v has more than cd of its neighbors in X_i. To route the transpositions of X_i simultaneously, we will show that we can find disjoint paths between τ_1 and τ_2 for each $(\tau_1, \tau_2) \in X_i$.

For a particular i and for each transposition $(v_j, v_j') \in X_i$, build a hypergraph $\Gamma_{(v_j,v_j')}$ with vertex set $V(G)$, where there exists a hyperedge $\{u_1, u_2\} \in E\left(\Gamma_{(v_j,v_j')}\right)$ if and only if v_j, u_1, u_2, v_j' is a path from v_j to v_j' and neither u_1 nor u_2 are vertices in transpositions of X_i. This yields that $\Gamma_{(v_j,v_j')}$ is a 2-uniform hypergraph for each $(v_j, v_j') \in X_i$. While a 2-uniform hypergraph is, of course, simply a graph, we state $\Gamma_{(v_j,v_j')}$ as a hypergraph to more easily use Hall's theorem for hypergraphs. Our goal is to find a system of disjoint representatives for $\mathcal{A} = \{\Gamma_{(v_j,v_j')} : v_j, v_j' \in X_i\}$, because this would give us a collection of disjoint paths of length four (including v_j and v_j', the vertices in the transposition) through which we can simultaneously route each transposition of X_i. By Hall's theorem for hypergraphs, there exists such a system if for each $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $2(|\mathcal{B}| - 1)$. Verifying this condition is equivalent to fixing a subset T of transpositions, then finding a collection with size $2(|T| - 1)$ of vertex-disjoint paths, where each path joins the vertices of some transposition in T.

Let $T \subseteq X_i$, let $t = |T|$, and let $N(T) = \bigcup_{v \in (v,v') \in T} N(v)$. Fix maximum matching in $\bigcup_{(v,v') \in T} \Gamma_{(v,v')}$. Hall's condition is satisfied for this T unless, this matching has cardinality less than $2t$; we assume, by way of contradiction, that the matching has size less than $2t$. Then this matching saturates fewer than $4t$ vertices. For convenience when counting, we will say that a vertex u is used if u is in this maximum matching or if there exists u' such that $(u, u') \in X_i$. Recall that for each vertex v in a transposition of T, there are at most cd neighbors in X_i, meaning that $|N(T) \cap X_i| \leq 2tcd$. Furthermore, each of the $4t$ vertices in the matching is adjacent to at most cd vertices in transpositions of X_i. Hence, the total number of unused vertices in $N(T)$ must be at least $2td - 2tcd - 4tcd = 2td - 6tcd$. Consequently, the average number of unused neighbors per transposition of T is at least $2d - 6cd$. Hence, there exists some transposition $(v, v') \in T$ such that the total unused neighbors of v and v' is at least $2d - 6cd$.

Since v has at most d unused vertices in its neighborhood and the sum of unused neighbors of v and the unused neighbors of v' is at least $2d - 6cd$, v' has at least $d - 6cd$ unused neighbors. Similarly, v must also have at least $d - 6cd$ unused neighbors. Thus, if V is the set of unused neighbors of v and V' is the set of unused neighbors of v', $\mathrm{Vol}(V) \geq d(d - 6cd)$ and $\mathrm{Vol}(V') \geq d(d - 6cd)$. Hence, by the Expander Mixing Lemma,

$$\begin{aligned} e(V, V') &\geq \frac{\mathrm{Vol}(V)\mathrm{Vol}(V')}{\mathrm{Vol}(G)} - \sigma\sqrt{\mathrm{Vol}(V)\mathrm{Vol}(V')} \\ &\geq \frac{d^2(d - 6cd)^2}{nd} - \sigma d^2 \\ &= d^2\left(\frac{d(1 - 6c)^2}{n} - \sigma\right). \end{aligned}$$

Now, since $\sigma < \frac{d(1-6c)^2}{n}$, $e(V, V') > 0$, which implies that there is an edge between an unused neighbor of v and an unused neighbor of v'. Consequently, there exists an edge in $\Gamma_{(v,v')}$ that is not in the matching. Therefore, this contradicts the maximality of the matching.

As a result, there exists a matching of size at least $2t$, meaning that by Hall's theorem for hypergraphs we can select a collection of disjoint edges $\{\{u_1, u_2\} \in E(G)\}$ so that for each transposition (v, v') in X_i, there exists an edge $\{u_1, u_2\}$ in this collection such that v, u_1, u_2, v' is a path. This implies we can route all of the transpositions of X_i through these disjoint paths simultaneously in 4 steps, returning the pebbles on u_1 and u_2 to their prior positions.

Since there are $\frac{4}{c}$ cells in this partition of $\mathcal{T}$, it will take at most $\frac{16}{c}$ steps to route all transpositions of the permutation π_1. By subsequently repeating this process for the transpositions of π_2, it will take at most $\frac{32}{c}$ steps to route all of the vertices according to the permutation π. Therefore, $rt(G) \leq \frac{32}{c}$. $\square$

Notice that in this proof, the paths that we built between v and v' for a transposition $(v, v') \in X_i$ only contained four vertices. By extending these paths, we can weaken the restriction on the degree of the graph. However, this gives us a weaker result on the routing number, as the paths through which the transpositions are routed will be longer.

Theorem 1. *For all $k > 0$, $C > 0$, there exists $N_{k,C} \in \mathbb{N}$ such that for any regular graph G on $n \geq N_{k,C}$ vertices with degree $d \geq \exp\left(\frac{C \log n}{\log\log n}\right)$ and $\sigma = kd^{-1/2} < \frac{1}{3}$, $rt(G) \leq (8z^5 + 8z^2)(2k)^z$, where z is the least even integer such that*

$$z \geq \frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2.$$

Remark: In the introduction, this result was stated as $\log(rt(G)) = O\left(\frac{\log n}{\log d}\right)$. Note that if $rt(G) \leq (8z^5 + 8z^2)(2k)^z$, then for some constant C,

$$\begin{aligned}
\log(rt(G)) &= \log(8z^5 + 8z^2) + Cz \\
&= O(z) \\
&= O\left(\frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2\right) \\
&= O\left(\frac{\log n}{\log d}\right).
\end{aligned}$$

Corollary 1. *For all $k > 0$ and $\epsilon > 0$, there exist $N_{k,\epsilon} \in \mathbb{N}$ and $C_{k,\epsilon} \in \mathbb{N}$ such that for any regular graph G on $n \geq N_{k,\epsilon}$ vertices with degree $d = n^\epsilon$ and $\sigma = kd^{-1/2} < \frac{1}{3}$, $rt(G) \leq C_{k,\epsilon}$.*

Since $\log(rt(G)) = O\left(\frac{\log n}{\log d}\right)$ by Theorem 1, $\log(rt(G)) = O\left(\frac{1}{\epsilon}\right)$ when $d = n^\epsilon$.

Proof of Theorem 1. Let G be a d-regular graph with $d > \exp\left(\frac{C \log n}{\log\log n}\right)$ and $\sigma = kd^{-1/2} < \frac{1}{3}$. Consider a permutation π of the vertices. Then $\pi = \pi_2\pi_1$

for some $\pi_1, \pi_2 \in S_V$ of order two. Thus, π_1 and π_2 can each be written as a product of disjoint transpositions. Let z be the least even integer such that

$$z \geq \frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2$$

and let

$$c = \frac{1}{\lceil (z-1)(1+z^3)(4k^2)^{z/2} \rceil}.$$

We note that c here is (at least) polylogarithmic in $\frac{1}{\log n}$ – this follows from the computation in the remark above and our assumption that $d \geq \exp\left(\frac{C\log n}{\log\log n}\right)$. In particular, it satisfies the necessary lower bound for c in Lemma 2.

Let $\mathcal{T} = \{(v, v') \in \pi_1\}$, the collection of transpositions in π_1. Then by Lemma 2, there exists a partition $X_1, \ldots, X_{4/c}$ in which each part X_i has size at most $\frac{nc}{4}$ and for each $j \in \{1, \ldots, z\}$, no vertex $x \in V(G)$ has more than cd^j vertices in its jth neighborhood that are also in transpositions of X_i for any i. Fix $i \in \{1, \ldots, 4/c\}$. For each transposition $(v_j, v'_j) \in X_i$, build a hypergraph $\Gamma_{(v_j, v'_j)}$ with vertex set $V(G)$, where there exists a hyperedge $\{u_1, \ldots, u_{z-2}\} \in E\left(\Gamma_{(v_j,v'_j)}\right)$ if and only if $v_j, u_1, \ldots, u_{z-2}, v'_j$ is a path from v_j to v'_j and u_k is not in a transposition of X_i for all $k \in \{1, \ldots, z-2\}$. This yields that $\Gamma_{(v_j,v'_j)}$ is a $(z-2)$-uniform hypergraph for each $(v_j, v'_j) \in X_i$.

Our goal is to find a system of disjoint representatives for $\mathcal{A} = \{\Gamma_{(v_j,v'_j)} : v_j, v'_j \in X_i\}$, because this would give us a collection of disjoint z-vertex paths through which we can simultaneously route each transposition of X_i. By Hall's theorem for hypergraphs, there exists such a system if for each $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $(z-2)(|\mathcal{B}|-1)$. Verifying this condition is equivalent to fixing a subset T of transpositions, then finding a collection of vertex-disjoint paths with size $(z-2)(|T|-1)$, each of which join the vertices of a transposition in T.

Let $T \subseteq X_i$ and let $t = |T|$. Fix a maximum matching in $\bigcup_{(v,v')\in T} \Gamma_{(v,v')}$. Hall's condition is satisfied for this T unless this matching has size less than zt; we assume, by way of contradiction, that the matching has size less than zt. Give each vertex a distance j away from a vertex in any transposition of T a weight of d^{z-j}. To count the weight used by the paths in this matching, first note that there are fewer than z^2t vertices in the matching. For each vertex x in the matching and for each $j \in \{1, \ldots, z\}$, there are at most cd^j paths of length j connecting x to a vertex in a transposition of X_i. From each of these paths, x gets weight cd^{z-j}. Thus, even if all of these paths connected

x to a vertex in a transposition in T, x would get weight at most $(cd^j)(cd^{z-j})$ from being in the jth neighborhood of vertices in transpositions of T. Thus, summing over all $j \in \{1, \ldots, z\}$, each vertex in the matching has weight at most zc^2d^z. Hence, the total weight used by vertices in the matching is at most $z^3tc^2d^z$. Therefore, there exists a permutation $(v, v') \in T$ that uses weight at most $z^3c^2d^z$.

For notational purposes, define $N(v)$ to be the neighborhood of v and define $N^*(v) \subseteq N(v)$ to be the set of all unused vertices in $N(v)$. Then, define $N_2(v)$ to be the neighborhood of $N^*(v)$ and define $N_2^*(v) \subseteq N_2(v)$ to be the set of all unused vertices in $N_2(v)$. Proceed inductively in this way, defining $N_m(v)$ to be the neighborhood of $N_{m-1}^*(v)$ and defining $N_m^*(v) \subseteq N_m(v)$ to be the set of all unused vertices of $N_m(v)$.

To prove that there exists a path of unused vertices joining the vertices of a transposition in T, thus contradicting the maximality of the matching, we will prove the following lemma.

Lemma 3. *In this case,*

$$\begin{aligned}&\mathrm{Vol}(N_m^*(v))\\ &\quad\geq \min\left\{\frac{d^{m+1}\left(1-(c+z^3c^2)\sum\limits_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}, \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G)\right\}\end{aligned}$$

for all $m \leq \frac{z}{2}$.

We leave the inductive proof of this lemma until after the proof of the main theorem. The crux of this lemma is that it implies by regularity that $|N_{z/2-1}^*(v)| \geq \left(\frac{1}{2}-\frac{z^2}{c}\right)n$ or

$$|N_{z/2-1}^*(v)| \geq \frac{d^{z/2-1}\left(1-(c+z^3c^2)\sum\limits_{i=1}^{z/2-1}(4k^2)^{i-1}\right)}{(4k^2)^{z/2-2}}.$$

In the latter case, since $c < \frac{1}{(z-1)(1+z^3)(4k^2)^{(z-1)/2}}$,

$$1-(c+z^3c^2)\sum_{i=1}^{z/2-1}(4k^2)^{i-1} \geq 1-(c+z^3c^2)\left(\frac{z}{2}-1\right)(4k^2)^{z/2-1} \geq \frac{1}{2}.$$

Furthermore, since $z \geq \frac{2\log\left(\frac{n}{4k^2}\right)}{\log\left(\frac{d}{4k^2}\right)} + 2$,

$$\frac{z-2}{2}\log\left(\frac{d}{4k^2}\right) \geq \log\left(\frac{n}{4k^2}\right),$$

meaning that

$$\left(\frac{d}{4k^2}\right)^{z/2-1} \geq \frac{n}{4k^2},$$

which finally implies that

$$\frac{d^{z/2-1}}{2(4k^2)^{z/2-2}} \geq \frac{n}{2}.$$

Thus,

$$\begin{aligned}|N^*_{z/2-1}(v)| &\geq \frac{d^{z/2-1}\left(1-(c+z^3c^2)\sum_{i=1}^{z/2-1}(4k^2)^{i-1}\right)}{(4k^2)^{z/2-2}}\\ &\geq \frac{d^{z/2-1}}{2(4k^2)^{z/2-2}}\\ &\geq \frac{n}{2}.\end{aligned}$$

Therefore, in either case $|N^*_{z/2-1}(v)| \geq \left(\frac{1}{2} - \frac{z^2}{c}\right)n$. By an identical argument, the same is true for $N^*_{z/2-1}(v')$.

Note that this implies that

$$\mathrm{Vol}\left(\overline{N^*_{z/2-1}(v)}\right) \leq \left(\frac{1}{2} - \frac{z^2c}{8}\right)\mathrm{Vol}(G) < \frac{1}{2}\mathrm{Vol}(G)$$

and

$$\mathrm{Vol}\left(\overline{N^*_{z/2-1}(v')}\right) < \frac{1}{2}\mathrm{Vol}(G).$$

By the Expander Mixing Lemma, then,

$$\begin{aligned}e\left(N^*_{z/2-1}(v), N^*_{z/2-1}(v')\right) \geq{}& \frac{\mathrm{Vol}\left(N^*_{z/2-1}(v)\right)\mathrm{Vol}\left(N^*_{z/2-1}(v')\right)}{\mathrm{Vol}(G)}\\ &- \sigma\sqrt{\mathrm{Vol}\left(\overline{N^*_{z/2-1}(v)}\right)\mathrm{Vol}\left(\overline{N^*_{z/2-1}(v')}\right)}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{4}\mathrm{Vol}(G) - \sigma\sqrt{\frac{1}{4}[\mathrm{Vol}(G)]^2}\\
&= \left(\frac{1}{4} - \frac{1}{2}\sigma\right)\mathrm{Vol}(G)\\
&> 0
\end{aligned}
$$

because $\sigma < \frac{1}{3}$. This implies that there is an edge between $N^*_{z/2-1}(v)$ and $N^*_{z/2-1}(v')$. Since these two sets are constructed by building paths of unused vertices in each iterated neighborhood of v and v', respectively, this means that there exists a $(z-2)$-vertex path of unused vertices that can be extended to a path between v and v', which contradicts the maximality of the matching on T. Therefore, there exists a matching that saturates X_i.

Since there is a matching that saturates X_i, there exist disjoint z-vertex paths such that for each transposition $(v, v') \in X_i$, one of these paths connects v and v'. Because these paths are all disjoint, each transposition can be routed along these paths simultaneously, returning all pebbles not on v or v' to their prior location, in z steps. Since there are $\frac{4}{c}$ parts of the partition, the permutation, π_1 can be routed in $\frac{4z}{c}$ steps. By repeating this process for π_2, we can route the permutation π on G in $\frac{8z}{c}$ steps. Therefore, by the arbitrary selection of π,

$$
\begin{aligned}
rt(G) &\leq \frac{8z}{c}\\
&= 8z\left\lceil (z-1)(1+z^3)(4k^2)^{z/2}\right\rceil\\
&\leq (8z^5 + 8z^2)(4k^2)^{z/2}.
\end{aligned}
$$

□

We now return to prove the lemma that we omitted from the main proof.

Lemma 3. *For all $m \leq \frac{z}{2}$,*

$$
\mathrm{Vol}(N^*_m(v)) \geq \min\left\{\frac{d^{m+1}\left(1-(c+z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}, \left(\frac{1}{2} - \frac{z^2c}{8}\right)\mathrm{Vol}(G)\right\}.
$$

Proof of Lemma 3. We will prove this by induction. For $m = 1$, note that $|N_1(v)| = d$. By construction of X_i, there are at most cd vertices in $N(v)$

that are also in transpositions of X_i. Furthermore, the total used weight of the transposition (v, v') is at most $z^3c^2d^z$, meaning that there must be used weight at most $z^3c^2d^z$ in $N(v)$. However, each vertex in $N(v)$ that has positive weight must have weight at least d^{z-1}. Thus, there must be at most z^3c^2d vertices of $N(v)$ used by the paths already in the matching. Hence, there are at least $d(1-(c+z^3c^2))$ unused vertices in $N(v)$, which implies that $\mathrm{Vol}(N_1^*(v)) \geq d^2(1-(c+z^3c^2))$. This proves the base case.

Now suppose as an induction hypothesis that

$$\mathrm{Vol}(N_{m-1}^*(v)) \geq \min\left\{\frac{d^m\left(1-(c+z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-2}}, \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G)\right\}.$$

We will prove the induction through the following series of three claims.

Claim 1. *If* $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$, *then* $\mathrm{Vol}(N_m^*(v)) \geq \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G)$.

Proof of Claim 1. Since each path contains z vertices and the maximum matching in question contains less than zt such paths, there are at most z^2t vertices in the matching. Thus, since $t = |T|$, where $T \subseteq X_i$ and $|X_i| \leq \frac{cn}{8}$, there are at most $\frac{z^2cn}{8}$ used vertices in $|X_i|$. Hence, because $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ implies that $|N_m(v)| \geq \frac{1}{2}n$, we get that $|N_m^*(v)| \geq \left(\frac{1}{2}-\frac{z^2c}{8}\right)n$. Therefore,

$$\mathrm{Vol}(N_m^*(v)) \geq \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G).$$

□

Claim 2. *If* $\mathrm{Vol}(N_{m-1}^*(v)) \geq \left(\frac{1}{2}-\frac{z^2c}{8}\right)\mathrm{Vol}(G)$, *then* $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$.

Proof of Claim 2. By Lemma 2, $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or $\mathrm{Vol}(N_m(v)) \geq \frac{\mathrm{Vol}(N_{m-1}^*(v))}{4\sigma^2}$. However, note that since $c = \frac{1}{\lceil (z-1)(1+z^3)(4k^2)^{z/2}\rceil} < \frac{4-8\sigma}{z^2}$ as $\sigma < \frac{1}{3}$,

$$\frac{\mathrm{Vol}(N_{m-1}^*(v))}{4\sigma^2} \geq \frac{\frac{1}{2}-\frac{z^2c}{8}}{4\sigma^2}\mathrm{Vol}(G)$$

$$
\begin{aligned}
&> \frac{\frac{1}{2} - \frac{z^2}{8}\frac{4-8\sigma}{z^2}}{4\sigma^2}\mathrm{Vol}(G) \\
&= \frac{1}{4\sigma}\mathrm{Vol}(G) \\
&> \frac{3}{4}\mathrm{Vol}(G).
\end{aligned}
$$

As a result, $\mathrm{Vol}(N_m(v))) \geq \frac{1}{2}\mathrm{Vol}(G)$ in either case. □

Claim 3. *If*

$$
\mathrm{Vol}(N^*_{m-1}(v)) \geq \frac{d^m\left(1-(c+z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-2}},
$$

then $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ *or*

$$
\mathrm{Vol}(N^*_{m-1}(v)) \geq \frac{d^{m+1}\left(1-(c+z^3c^2)\sum_{i=1}^{m}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}
$$

Proof of Claim 3. By Lemma 2, $\mathrm{Vol}(N_m(v)) \geq \frac{1}{2}\mathrm{Vol}(G)$ or

$$
\begin{aligned}
\mathrm{Vol}(N_m(v)) &\geq \frac{\mathrm{Vol}(N^*_{m-1}(v_1))}{4\sigma^2} \\
&\geq \frac{d^m\left(1-(c+z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-2}4\sigma^2} \\
&= \frac{d^{m+1}\left(1-(c+z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}}.
\end{aligned}
$$

By the construction of X_i, there are at most cd^m vertices in $N_m(v)$ that are also in transpositions of X_i. Furthermore, there must be used weight at most $z^3c^2d^z$ in $N_m(v)$. However, each vertex in $N_m(v)$ that has positive weight must have weight at least d^{z-m}. Thus, there must be at most $z^3c^2d^m$ vertices of $N_m(v)$ used by paths already in the matching. Hence,

$$
|N^*_m(v_1)| \geq \frac{d^m\left(1-(c+z^3c^2)\sum_{i=1}^{m-1}(4k^2)^{i-1}\right)}{(4k^2)^{m-1}} - z^3cd^m - cd^m
$$

$$= \frac{d^m \left(1 - (c + z^3c^2) \sum_{i=1}^{m} (4k^2)^{i-1}\right)}{(4k^2)^{m-1}}.$$

Therefore,

$$\mathrm{Vol}(N_m^*(v)) \geq \frac{d^{m+1} \left(1 - (c + z^3c^2) \sum_{i=1}^{m} (4k^2)^{i-1}\right)}{(4k^2)^{m-1}}.$$

□

As a result of these three claims, we have shown that for all $m \leq \frac{z}{2}$,

$$\mathrm{Vol}(N_m^*(v)) \geq \min \left\{ \frac{d^{m+1} \left(1 - (c + z^3c^2) \sum_{i=1}^{m} (4k^2)^{i-1}\right)}{(4k^2)^{m-1}}, \left(\frac{1}{2} - \frac{z^2c}{8}\right) \mathrm{Vol}(G) \right\}.$$

□

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