Optimal linear-Vizing relationships for (total) domination in graphs

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Abstract

A total dominating set in a graph G is a set of vertices of G such that every vertex is adjacent to a vertex of the set. The total domination number $\gamma_t(G)$ is the minimum cardinality of a dominating set in G. In this paper, we study the following open problem posed by Yeo [J. Graph Theory 55 (2007), 325–337]. For each $\Delta \ge 3$, find the smallest value, r_Δ, such that for every connected graph G with each component of order at least 3, of order n, size m, total domination number γ_t, and bounded maximum degree Δ, satisfies $m \le \frac{1}{2}(\Delta + r_\Delta)(n - \gamma_t)$. The first author [J. Graph Theory 49 (2005), 285–290] showed that $r_\Delta \le \Delta$ for all $\Delta \ge 3$. Yeo [J. Graph Theory 55 (2007), 325–337] significantly improved this result and showed that $0.1 \ln(\Delta) < r_\Delta \le 2\sqrt{\Delta}$ for all $\Delta \ge 3$, and posed as an open problem to determine "whether r_Δ grows proportionally with $\ln(\Delta)$ or $\sqrt{\Delta}$ or some completely different function." In this paper, we determine the growth of r_Δ, and show that r_Δ is asymptotically $\ln(\Delta)$ and likewise determine the asymptotics of the analogous constant for standard domination.

Keywords: (Total) domination in graphs; Maximum degree; order; size
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1 Introduction

In this paper we continue the study of a linear Vizing-like relations relating the size (the number of edges) of a graph and the (total) domination number, the order (the number of vertices), and given bounded maximum degree.

A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S, where two vertices are neighbors in G if they are adjacent. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A *total dominating set*, abbreviated TD-set, of a graph G is a set S of vertices of G such that every vertex has a neighbor in S, where two vertices are neighbors in G if they are adjacent. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set in G. For fundamentals of total domination theory in graphs we refer the reader to the book [14]. A thorough treatise on domination in graphs can be found in [9, 10, 11].

For notation and graph theory terminology, we in general follow [9, 10, 11, 14]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of vertices adjacent to v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set $D \subseteq V(G)$, its *open neighborhood* is the set $N_G(D) = \cup_{v\in D} N_G(v)$, and its *closed neighborhood* is the set $N_G[D] = N_G(D) \cup D$. We denote the *degree* of v in G by $\deg_G(v) = |N_G(v)|$. The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *isolated vertex* in G is a vertex of degree 0 in G. A graph is *isolate-free* if it contains no isolated vertex. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$.

In this paper, we consider the following two open problems (that we will discuss in more detail in Section 2).

Problem 1 *For each $\Delta \geq 3$, find the smallest value, c_Δ, such that for every connected graph G of order n, size m, domination number $\gamma(G) = \gamma$, and bounded maximum degree $\Delta(G) \leq \Delta$,*

$$m \leq \left(\frac{\Delta + c_\Delta}{2}\right) n - \left(\frac{\Delta + c_\Delta + 2}{2}\right) \gamma.$$

Problem 2 *For each $\Delta \geq 3$, find the smallest value, r_Δ, such that for every connected graph G of order $n \geq 3$, size m, total domination number γ_t, and bounded maximum degree $\Delta(G) \leq \Delta$,*

$$m \leq \frac{1}{2}(\Delta + r_\Delta)(n - \gamma_t).$$

The best known lower and upper bounds (prior to this paper) to date on c_Δ and r_Δ for every $\Delta \geq 3$ are given by Rautenbach [18] and Yeo [24], and are summarized below.

Theorem 1 ([18, 24]) *For all $\Delta \geq 3$, we have $0.05 \ln(\Delta) < c_\Delta \leq \Delta$.*

Theorem 2 ([24]) *For all* $\Delta \geq 3$*, we have* $0.1 \ln(\Delta) < r_\Delta \leq 2\sqrt{\Delta}$*.*

As remarked by Yeo [24], "we do not have a guess if r_Δ actually grows proportionally with $\ln(\Delta)$ or $\sqrt{\Delta}$ or some completely different function."

In this paper we completely determine the asymptotic behavior of r_Δ and c_Δ.

Theorem 3

$$r_\Delta = (1 + \mathrm{o}(1)) \ln \Delta,$$

where the $\mathrm{o}(1)$ *term tends to zero as* $\Delta \to \infty$*.*

Theorem 4

$$c_\Delta = (1 + \mathrm{o}(1)) \ln \Delta,$$

where the $\mathrm{o}(1)$ *term tends to zero as* $\Delta \to \infty$*.*

The proof of these theorems is split into two parts. First we prove an upper bound for r_Δ; cf. Theorem 14 in Section 3. We observe that this, in turn, almost immediately provides an upper bound for c_Δ. The key ingredient in this direction is a new upper bound for the total domination number of a graph, in the vein of the classical probabilistic bound but relating the domination number to the *maximum degree* and size of a graph instead of the minimum degree.

Finally, in Section 4, we prove the lower bounds on c_Δ and r_Δ needed to prove the theorem. Here, the main ingredient is a lower bound for the domination number of random regular graphs. While the domination number of random regular graphs has been heavily studied, see eg. [6, 15], it seems that most of this effort has gone into regular graphs of small degree, especially cubic graphs, and in improving the upper bound. We could not find the precise result we needed – establishing the asymptotic sharpness of the classical probabilistic bound – in the literature, so we provide a complete proof here.

In the next section, we provide some more background and discussion along with a few important results we use, before turning to the proof.

2 Background and preliminaries

In 1965 Vizing [22] proved a classical result bounding the size m of a graph in terms of its order n and domination number γ.

Theorem 5 ([22]) *If* G *is a graph of order* n *and size* m *with domination number* $\gamma(G) = \gamma \geq 1$*, then* $m \leq \frac{1}{2}(n - \gamma)(n - \gamma + 2)$*.*

Vizing [22] constructed an infinite family of graphs achieving equality in the bound of Theorem 5. However, these graphs are disconnected and have maximum degree $\Delta = n - \gamma$. In 1991 Sanchis [19] improved the bound of Theorem 5 slightly, as did Fulman [7] in 1994. In a breakthrough paper in 1999 Rautenbach [18] showed that if the graph is isolate-free, then the square dependence on n and γ in Vizing's result in Theorem 5 turns into a linear dependence on n, γ, and bounded maximum degree Δ.

Theorem 6 ([18]) *If G is an isolate-free graph of order n, size m, domination number γ, and maximum degree $\Delta(G) \le \Delta$, where $\Delta \ge 3$, then*

$$m \le \Delta n - (\Delta + 1)\gamma.$$

Rautenbach's [18] result stated in Theorem 6 attracted considerable attention, and gave rise to the open problem which is formally stated as Problem 1 in Section 1. Theorem 6, due to Rautenbach, implies that $c_\Delta \le \Delta$, but it was initially suspected that c_Δ might be constant – perhaps even bounded by 3. Yeo [24], however, established the lower bound $c_\Delta > 0.05\ln(\Delta)$. These results of Rautenbach and Yeo, summarized earlier in Theorem 1, are the best known lower and upper bounds to date on c_Δ for every $\Delta \ge 3$.

Shifting our attention to Vizing-like relations between the size and the total domination number of a graph of given order, in 2004 Dankelmann, Domke, Goddard, Grobler, Hattingh, and Swart [5] established the following result.

Theorem 7 ([5]) *If G is an isolate-free graph having order n, size m, and total domination number $\gamma_t(G) = \gamma_t \ge 2$, then*

$$m \le \begin{cases} \binom{n-\gamma_t+2}{2} + \frac{\gamma_t}{2} - 1 & \text{if } \gamma_t \text{ is even,} \\ \binom{n-\gamma_t+1}{2} + \frac{\gamma_t}{2} + \frac{1}{2} & \text{if } \gamma_t \text{ is odd.} \end{cases}$$

As shown in [5], the bound in Theorem 7 is tight. However the extremal graphs that achieve equality in the bound have the property that they are disconnected and that their edges are very unevenly distributed in the sense that the minimum and maximum degrees differ greatly. In 2004 Sanchis [20] improved the bound of Theorem 7 slightly in the case when the graph G is connected and has total domination number at least 5.

Theorem 8 ([20]) *If G is a connected graph with order n, size m, and total domination number $\gamma_t(G) = \gamma_t \ge 5$, then*

$$m \le \binom{n - \gamma_t + 1}{2} + \left\lfloor \frac{\gamma_t}{2} \right\rfloor.$$

The degrees in graphs that achieve equality in the improved bound of Sanchis in Theorem 7 are also very unevenly distributed. In 2005 Henning [12] and in 2007 Shan, Kang, and

Henning [21] established a linear Vizing-like theorem relating the size of a graph and its order, total domination number, and bounded maximum degree. Their results showed that the square dependence on n and γ_t in Theorem 7 and Theorem 8 improves to a linear dependence on n, γ_t, and bounded maximum degree Δ.

Theorem 9 ([12, 21]) *Let G be a graph of order n, size m, total domination number γ_t, and bounded maximum degree $\Delta(G) \le \Delta$, where $\Delta \ge 3$. If every component of G has order at least 3, then $m \le \Delta(n - \gamma_t)$.*

The connected graphs that achieve equality in the upper bound of Theorem 9 were characterized in 2013 by Henning and Joubert [13]. As a consequence of their characterization, if the bounded maximum degree $\Delta \ge 4$, then we have strict inequality in the upper bound of Theorem 9. A natural problem is to improve the upper bound in Theorem 9 for bounded maximum degree $\Delta \ge 4$. This problem is formally stated as Problem 2 in Section 1.

By Theorem 9, we have $r_\Delta \le \Delta$ for all $\Delta \ge 3$. As shown in [12], $r_\Delta = \Delta$ in the special case when $\Delta = 3$, that is, $r_3 = 3$. However, the exact value of r_Δ has yet to be determined for any value of $\Delta \ge 4$. In 2007, Yeo [24] strengthened the upper bound $r_\Delta \le \Delta$ with an ingenious proof that uses the interplay between total domination in graphs and transversals in hypergraphs. Moreover, Yeo [24] established a non-trivial lower bound on r_Δ. These results of Yeo, which we summarize below, are the best known lower and upper bounds (prior to this paper) on r_Δ for every $\Delta \ge 3$.

Theorem 10 ([24]) *For all $\Delta \ge 3$,*

$$\frac{0.1\ln(\Delta)}{1 - \frac{0.1\ln(\Delta)}{\Delta}} < r_\Delta \le \max\{3, 2\sqrt{\Delta}\}.$$

As an immediate consequence of Theorem 9, we have the slightly weaker, but simpler, bound on r_Δ given in Theorem 2.

In order to prove Theorem 3 and 4, and completely settle the asymptotics of r_Δ and c_Δ, we recall a few other important ingredients.

First, we shall need the following upper bound on the domination number of an isolate-free graph due to Ore [17] in 1962, and the upper bound on the total domination of a connected graph of order at least 3 due to Cockayne, Dawes, and Hedetniemi [4] in 1980.

Theorem 11 ([17]) *If G is an isolate-free graph of order n, then $\gamma(G) \le \frac{1}{2}n$.*

Theorem 12 ([4]) *If G is a connected graph of order $n \ge 3$, then $\gamma_t(G) \le \frac{2}{3}n$.*

Finally, we recall the following classical bound on the domination number (found, eg. in Alon and Spencer [2].)

Theorem 13 ([2]) *If G is a graph of order n with minimum degree $\delta \geq 2$, then*

$$\gamma(G) \leq \left(\frac{1+\ln(\delta+1)}{\delta+1}\right) n.$$

While this bound is not an ingredient in our proofs per se, both it and the simple modification that

$$\gamma_t(G) \leq \left(\frac{1+\ln(\delta)}{\delta}\right) n,$$

are, in a sense, central to both our upper and lower bounds. Indeed, for the upper bound we prove (cf. Lemma 3 in Section 3) a modification that allows use of the *maximum degree* Δ instead of the minimum degree δ, that gains power when G has relatively close to $\Delta n/2$ edges.

The lower bounds for c_Δ and r_Δ rely on the asymptotic optimality of this bound. It has long been known – indeed, it is remarked upon in Alon and Spencer [2] – that this bound is asymptotically optimal as $\delta \to \infty$. Thematically it is related to, and even follows from, a lower bound on the transversal number of uniform hypergraphs established by Alon in [1]. In our case, it is especially important that there are *regular* graphs for which this is optimal. While the asymptotic optimality of this bound for random regular graphs is likely known by some experts in random graphs, it seems less known to experts in domination theory and does not seem to appear in the literature. For instance Yeo, in [24], used the results of [1] to construct regular bipartite graphs whose domination number is bounded below, to establish his lower bound in Theorem 10. His method is unable to obtain the optimal constant, however, necessitating the improvement in this paper.

3 Upper bounding r_Δ and c_Δ

In this section we prove the following upper bound on r_Δ.

Theorem 14 *For all $\Delta \geq 3$,*

$$r_\Delta \leq \frac{\Delta(6+\ln(\Delta))}{\Delta - \ln(\Delta) - 1} = (1+o(1))\ln(\Delta).$$

A simple observation shows that $c_\Delta \leq r_\Delta + 2$, and hence we have.

Theorem 15 *For all $\Delta \geq 3$, we have $c_\Delta \leq (1+o(1))\ln(\Delta)$.*

In order to prove Theorem 14, we first present three preliminary lemmas. We shall need the following convexity bound.

Lemma 1 *For real number a, where $0 < a < 1$, and real numbers x and y where $x \geq y$,*

$$a^x + a^y < a^{x+1} + a^{y-1}.$$

Proof. Since $0 < a < 1$ and $x \geq y$, we note that $1 - a > 0$ and $a^x - a^{y-1} < 0$. Hence,

$$a^x + a^y - a^{x+1} - a^{y-1} = a^x(1-a) - a^{y-1}(1-a) = (1-a)(a^x - a^{y-1}) < 0,$$

and so $a^x + a^y < a^{x+1} + a^{y-1}$, as desired. $\Box$

In order to apply Lemma 1 judiciously, we make the following observation.

Lemma 2 *If G is a connected graph of order $n \geq 3$, size m, and bounded maximum degree $\Delta(G) \leq \Delta$, then*

$$2m = \Delta n_\Delta + n_1 + \varepsilon d^*,$$

for some nonnegative integers n_1, n_Δ and d^* satisfying $n = n_1 + n_\Delta + \varepsilon$ and $2 \leq d^* \leq \Delta - 1$ and where $\varepsilon \in \{0, 1\}$.

Proof. This follows immediately starting with the degree sequence of G, thought of as a list of integers. So long as there are two numbers both less than Δ and greater than 1 in the sequence, increase the larger one by one and decrease the smaller. This process terminates with the desired sequence. $\Box$

As a remark, we highlight that the resulting sequence $(\Delta, \ldots, \Delta, d^*, 1, \ldots, 1)$ generated in the proof of Lemma 2 that contains n_Δ Δ's and n_1 1's, is not necessarily the degree sequence of G – indeed, it need not even be graphical.

We are now in a position to present our key lemma, which establishes an upper bound on the total domination of a graph in terms of the maximum degree and size.

Lemma 3 *If G is a connected graph of order $n \geq 3$, size m, and bounded maximum degree $\Delta(G) \leq \Delta$, then*

$$\gamma_t(G) \leq (n_1 + \varepsilon) + \left(\frac{1 + \ln(\Delta)}{\Delta}\right) n_\Delta,$$

where n_1, n_Δ and ε are nonnegative integers satisfying the statement of Lemma 2.

Proof. Let G be a connected graph of order $n \geq 3$, size m, and bounded maximum degree $\Delta(G) \leq \Delta$ and vertex set $V = V(G)$. Let R be a random subset of vertices of G, where a vertex is chosen to be in R with probability p and independently of the choice for any other vertex, where

$$p = \frac{\ln(\Delta)}{\Delta}.$$

We note that $0 < p < 1$ and $0 < 1 - p < 1$. Let S be the set of vertices in G that have no neighbor in R, that is,

$$S = \{v \in V : N_G(v) \cap R = \emptyset\}.$$

For each vertex $v \in S$, let v' be an arbitrary neighbor of v in G, and let

$$S' = \bigcup_{v \in S} \{v'\},$$

and so $|S'| \le |S|$. The set $R \cup S'$ is a TD-set of G. The expected value of $|R|$ is

$$E(|R|) = np = (n_1 + n_\Delta + \varepsilon)p.$$

The random variable $|S|$ can be written as the sum of n indicator random variables $X_v(S)$ for each $v \in V$, where $X_v(S) = 1$ if $v \in S$ and $X_v(S) = 0$ otherwise. For each vertex $v \in V$, the expected value of $X_v(S)$ is the probability that its neighbors are not in R; that is,

$$E(X_v(S)) = (1-p)^{\deg(v)}.$$

Let G have degree sequence $d_1, d_2 \ldots, d_n$, and so $1 \le d_i \le \Delta$ for all $i \in [n]$. By linearity of expectation, we have

$$E(|S|) \le \sum_{v \in V} E(X_v(S)) = \sum_{i=1}^{n} (1-p)^{d_i}.$$

As observed earlier, $0 \le 1 - p \le 1$. As shown in the proof of Lemma 2, we can write

$$\sum_{i=1}^{n} d_i = \Delta n_\Delta + n_1 + d^* \varepsilon,$$

where $n = n_1 + n_\Delta + \varepsilon$ and $1 \le d^* \le \Delta - 1$ and where $\varepsilon \in \{0, 1\}$. By repeated applications of Lemma 1, we have

$$\begin{aligned}
\sum_{i=1}^{n} (1-p)^{d_i} &\le n_\Delta (1-p)^\Delta + n_1(1-p) + \varepsilon(1-p)^{d^*} \\
&\le n_\Delta (1-p)^\Delta + n_1(1-p) + \varepsilon(1-p) \\
&= n_\Delta (1-p)^\Delta + (n_1 + \varepsilon)(1-p).
\end{aligned}$$

Using the inequality $1 - x \le e^{-x}$ for $x \in \mathbb{R}$, we therefore have by linearity of expectation that

$$\begin{aligned}
E(|R \cup S'|) &= E(|R|) + E(|S'|) \\
&\le E(|R|) + E(|S|) \\
&\le (n_1 + n_\Delta + \varepsilon)p + (n_1 + \varepsilon)(1-p) + n_\Delta (1-p)^\Delta \\
&= (n_1 + \varepsilon) + n_\Delta p + n_\Delta (1-p)^\Delta \\
&\le (n_1 + \varepsilon) + n_\Delta p + n_\Delta e^{-p\Delta} \\
&= (n_1 + \varepsilon) + \frac{n_\Delta \ln(\Delta)}{\Delta} + \frac{n_\Delta}{\Delta} \\
&= (n_1 + \varepsilon) + \left(\frac{1 + \ln(\Delta)}{\Delta} \right) n_\Delta.
\end{aligned}$$

Since expectation is an average value, there is a set R and an associated set S' such that $R \cup S'$ is a TD-set in G and

$$|R| + |S'| \le (n_1 + \varepsilon) + \left(\frac{1 + \ln(\Delta)}{\Delta} \right) n_\Delta.$$

Since $\gamma_t(G) \le |R| + |S'|$, this completes the proof of Lemma 3. $\Box$

We are now in a position to present a proof of our first main result, namely Theorem 14. Recall its statement.

Theorem 14 *For all* $\Delta \ge 3$,

$$r_\Delta \le \frac{\Delta(6 + \ln(\Delta))}{\Delta - \ln(\Delta) - 1} = (1 + \mathrm{o}(1)) \ln(\Delta).$$

Proof of Theorem 14. Let G be a connected graph of order $n \ge 3$, size m, and bounded maximum degree $\Delta(G) \le \Delta$ where $\Delta \ge 3$ is fixed, and let $\gamma_t(G) = \gamma_t$. By Lemma 2, $2m = \Delta n_\Delta + n_1 + \varepsilon d^*$ for some nonnegative integers n_1, n_Δ and d^* satisfying $n = n_1 + n_\Delta + \varepsilon$ and $2 \le d^* \le \Delta - 1$ and where $\varepsilon \in \{0, 1\}$. We note that $n_1 = n - n_\Delta - \varepsilon \le n - n_\Delta$. Moreover since $d^* < \Delta$ and $\varepsilon \in \{0, 1\}$, we can write $2m = \Delta n_\Delta + n_1 + \varepsilon d^* \le \Delta(n_\Delta + \epsilon) + n_1$. This yields the following claim.

Claim 1 *The following inequalities hold.*

(a) $2m \le \Delta(n_\Delta + \epsilon) + n_1$, *where* $n = n_1 + n_\Delta + \varepsilon$ *and* $\varepsilon \in \{0, 1\}$.
(b) $n_1 \le n - n_\Delta$.

We first consider the case when n_Δ is relatively large with respect to the order n. In what follows, let

$$\Phi(\Delta) = \frac{\Delta(6 + \ln(\Delta))}{\Delta - \ln(\Delta) - 1}.$$

Claim 2 *If* $n_\Delta \ge \frac{1}{3}n$, *then* $2m \le (\Delta + \Phi(\Delta))(n - \gamma_t)$ *holds.*

Proof. Suppose that $n_\Delta \ge \frac{1}{3}n$. By Claim 1(a), $2m \le \Delta(n_\Delta + \epsilon) + n_1$, where $n = n_1 + n_\Delta + \varepsilon$ and $\varepsilon \in \{0, 1\}$. Hence in this case when $n_\Delta \ge \frac{1}{3}n$, we note that $\Delta\varepsilon \le \Delta \le n \le 3n_\Delta$. Moreover by Claim 1(b), this case yields $n_1 \le n - n_\Delta \le 2n_\Delta$. By Lemma 3, we have

$$\begin{aligned}
(\Delta + \Phi(\Delta))(n - \gamma_t) &\ge (\Delta + \Phi(\Delta)) \left(n - (n_1 + \varepsilon) - \left(\tfrac{1 + \ln(\Delta)}{\Delta} \right) n_\Delta \right) \\
&= (\Delta + \Phi(\Delta)) \left(n_\Delta - \left(\tfrac{1 + \ln(\Delta)}{\Delta} \right) n_\Delta \right) \\
&= \Delta n_\Delta + \Phi(\Delta) n_\Delta - (1 + \ln(\Delta)) n_\Delta - \left(\tfrac{1 + \ln(\Delta)}{\Delta} \right) \Phi(\Delta) n_\Delta.
\end{aligned}$$

Therefore the following holds.

$$\begin{array}{rcl} 2m & \leq & (\Delta+\Phi(\Delta))(n-\gamma_t) \\ \Uparrow & & \\ \Delta(n_\Delta+\epsilon)+n_1 & \leq & \Delta n_\Delta + \Phi(\Delta)n_\Delta - (1+\ln(\Delta))n_\Delta - \left(\frac{1+\ln(\Delta)}{\Delta}\right)\Phi(\Delta)n_\Delta \\ \Updownarrow & & \\ \Delta\varepsilon + n_1 + (1+\ln(\Delta))n_\Delta & \leq & \left(\frac{\Delta-1-\ln(\Delta)}{\Delta}\right)\Phi(\Delta)n_\Delta \\ \Uparrow & & \\ 3n_\Delta + 2n_\Delta + (1+\ln(\Delta))n_\Delta & \leq & \left(\frac{\Delta-1-\ln(\Delta)}{\Delta}\right)\Phi(\Delta)n_\Delta \\ \Updownarrow & & \\ 5+(1+\ln(\Delta)) & \leq & \left(\frac{\Delta-1-\ln(\Delta)}{\Delta}\right)\Phi(\Delta) \\ \Updownarrow & & \\ \frac{\Delta(6+\ln(\Delta))}{\Delta-\ln(\Delta)-1} & \leq & \Phi(\Delta). \end{array}$$

Hence the inequality $2m \leq (\Delta+\Phi(\Delta))(n-\gamma_t)$ holds since by definition $\Phi(\Delta) = \frac{\Delta(6+\ln(\Delta))}{\Delta-\ln(\Delta)-1}$. This completes the proof of Claim 2. $(\Box)$

Next we consider the case when n_Δ is relatively small with respect to the order n.

Claim 3 *If $n_\Delta \leq \frac{1}{3}n$, then $2m \leq (\Delta+\Phi(\Delta))(n-\gamma_t)$ holds.*

Proof. Suppose that $n_\Delta \leq \frac{1}{3}n$. By Theorem 12, we have $\gamma_t \leq \frac{2}{3}n$, and so $n-\gamma_t \geq \frac{1}{3}n$. Thus in this case

$$(\Delta+\Phi(\Delta))(n-\gamma_t) \geq \frac{1}{3}\Delta n + \frac{1}{3}\Phi(\Delta)n.$$

By Claim 1(a), $2m \leq \Delta(n_\Delta+\epsilon)+n_1$, where $n = n_1+n_\Delta+\varepsilon$ and $\varepsilon \in \{0,1\}$. By supposition, $n_\Delta \leq \frac{1}{3}n$. Since $\varepsilon \leq 1$, we note that $\Delta\varepsilon \leq \Delta$. Moreover, trivially $n_1 \leq n$ and $\Delta \leq n$. These observations imply that $2m \leq \Delta(n_\Delta+\epsilon)+n_1 \leq \frac{1}{3}\Delta n + \Delta + n \leq \frac{1}{3}\Delta n + 2n$. Note that for $\Delta \geq 2$, we have $\Phi(\Delta) > 6+\ln(\Delta) > 6$, and so $\frac{1}{3}\Phi(\Delta)n > 2n$. Combining these, we have that

$$(\Delta+\Phi(\Delta))(n-\gamma_t) \geq \frac{1}{3}\Delta n + \frac{1}{3}\Phi(\Delta)n \geq \frac{1}{3}\Delta n + 2n \geq 2m,$$

as desired. $(\Box)$

By Claim 2 and Claim 3, the inequality $2m \leq (\Delta+\Phi(\Delta))(n-\gamma_t)$ holds, implying that $r_\Delta \leq \Phi(\Delta)$. This completes the proof of Theorem 14. $\Box$

Finally, we observe that our upper bound for r_Δ immediately implies a similar upper bound for c_Δ.

Lemma 4 *For all $\Delta \geq 3$, $c_\Delta \leq r_\Delta + 2$.*

Proof. Let G be a connected graph of order $n \geq 3$, size m, and bounded maximum degree $\Delta(G) \leq \Delta$ where $\Delta \geq 3$ is fixed. Let $\gamma(G) = \gamma$ and $\gamma_t(G) = \gamma_t$. By Ore's Theorem 11, we have $\gamma \leq \frac{1}{2}n$, and so $\gamma \leq n - \gamma$. By definition, $\gamma \leq \gamma_t$. By definition of r_Δ, we therefore have

$$\begin{aligned}
2m &\leq (\Delta + r_\Delta)(n - \gamma_t) \\
&\leq (\Delta + r_\Delta)(n - \gamma) \\
&= (\Delta + r_\Delta + 2 - 2)(n - \gamma) \\
&= (\Delta + r_\Delta + 2)(n - \gamma) - 2(n - \gamma) \\
&\leq (\Delta + r_\Delta + 2)(n - \gamma) - 2\gamma \\
&= (\Delta + r_\Delta + 2)n - (\Delta + r_\Delta + 4)\gamma.
\end{aligned}$$

Hence we have shown that

$$m \leq \left(\frac{\Delta + r_\Delta + 2}{2}\right) n - \left(\frac{\Delta + r_\Delta + 4}{2}\right) \gamma,$$

implying that $c_\Delta \leq r_\Delta + 2$. $\square$

4 Lower bounding r_Δ and c_Δ

In this section, we prove lower bounds for r_Δ and c_Δ. Combining this with the upper bounds obtained in Theorem 14 and Lemma 4, respectively, completes the proof of Theorems 3 and 4.

The key to the lower bounds is a lower bound for the domination number of random regular graphs. As commented above, this bound is similar to known bounds, but we could not find the version we need in the literature.

Theorem 16 *There exists a positive integer Δ_0 so that the following holds: Let Δ_0 be a positive integer, and let $\Delta \geq \Delta_0$ and let*

$$\epsilon = \frac{4\ln(\ln(\Delta))}{\ln(\Delta)}.$$

If G is a uniformly random Δ-regular graph of order n, then

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \Pr\left(\gamma(G) < \left\lfloor (1-\epsilon)\frac{\ln(\Delta)}{\Delta} n \right\rfloor\right) = 0.$$

Proof. While uniformly sampling Δ-regular graphs can be challenging, a number of simple models of random Δ-regular graphs are known to be contiguous to the uniform model, that is, events which hold with probability tending to 0 or 1 (as $n \to \infty$) in one do the same in the other. For fixed Δ on an even number of vertices, one of these models (see Jansen [16], and also Bollobás [3] and Wormald [23]) is to take the union of Δ independent perfect matchings and it is this which we use. Thus it suffices to prove the following claim.

Claim 4 *If G is a random graph of order n that is the union of Δ independently random chosen perfect matchings, then*

$$\gamma(G) \geq \left\lfloor (1-\epsilon)\frac{\ln(\Delta)}{\Delta} n \right\rfloor$$

with probability tending to 1.

Proof. Let $M_1, M_2, \ldots, M_\Delta$ be uniformly chosen perfect matchings on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ for some even n. Fix a set $X \subseteq V(G)$ of size

$$t = \left\lfloor (1-\epsilon)\frac{\ln(\Delta)}{\Delta} n \right\rfloor - 1.$$

We estimate the probability that X is a dominating set. Let $\mathcal{D}_i$ denote the event that vertex v_i is dominated by X. For a given matching M_j,

$$\Pr(v_i\text{'s neighbor in } M_j \text{ is not in } X) = \frac{(n-1)-t}{n}.$$

Since G is the union of Δ independent matchings, we infer that

$$\Pr(\mathcal{D}_i) = 1 - \left(\frac{(n-1)-t}{n}\right)^\Delta$$

if $v_i \notin X$.

Furthermore the events $\mathcal{D}_i$ are negatively correlated; and in particular,

$$\Pr(\mathcal{D}_i \mid \bigcap_{j<i} \mathcal{D}_j) \leq \Pr(\mathcal{D}_i).$$

This follows simply as if vertices $v_1, \ldots, v_{i-1}$ are already known to be dominated by X, there are fewer potential matching edges available to dominate v_i. All this implies that the probability that X is a dominating set is bounded by

$$
\begin{aligned}
\Pr\left(\bigcap_{i}^{n}\mathcal{D}_i\right) &= \prod_{i:v_i\notin X}\Pr\left(\mathcal{D}_i \Big| \bigcap_{j<i}\mathcal{D}_j\right)\\
&\leq \left(1-\left(1-\frac{t+1}{n}\right)^{\Delta}\right)^{n-t}\\
&\leq \left(1-\left(1-(1-\epsilon)\frac{\ln(\Delta)}{\Delta}\right)^{\Delta}\right)^{n-t}\\
&\leq \left(1-\exp\left(\frac{-(1-\epsilon)\ln(\Delta)}{1-(1-\epsilon)\frac{\ln(\Delta)}{\Delta}}\right)\right)^{n-t} \qquad (1)\\
&\leq \exp\left(-\left(\Delta^{-\frac{(1-\epsilon)}{1-(1-\epsilon)\ln(\Delta)/\Delta}}\right)(n-t)\right) \qquad (2)\\
&\leq \exp\left(-\frac{\Delta^{\epsilon-\mathrm{O}(\frac{\ln(\Delta)}{\Delta})}}{\Delta}(n-t)\right) \qquad (3)\\
&\leq \exp\left(-\frac{\Delta^{3\frac{\ln(\ln(\Delta))}{\ln(\Delta)}}}{\Delta}(n-t)\right) \qquad (4)\\
&= \exp\left(-\frac{(\ln(\Delta))^3}{\Delta}(n-t)\right).
\end{aligned}
$$

Here in the third inequality, namely Inequality (1), we use the real number inequality

$$1-x \geq e^{-\frac{x}{1-x}}$$

which is valid for $0<x<1$. In the fourth inequality, namely Inequality (2), we use the real number inequality $1+x\leq e^x$ which is valid for all real x, where in our case we take

$$x = \frac{-(1-\epsilon)}{1-(1-\epsilon)\left(\frac{\ln(\Delta)}{\Delta}\right)},$$

and so

$$\left(1-e^{x\ln(\Delta)}\right)^n = (1-\Delta^x)^n \leq \left(e^{-\Delta^x}\right)^n = e^{-(\Delta^x)n}.$$

In the second to last inequality, namely Inequality (3), we expand the geometric series

$$\frac{1}{1-(1-\epsilon)\left(\frac{\ln(\Delta)}{\Delta}\right)} = 1+\mathrm{O}\left(\frac{\ln(\Delta)}{\Delta}\right).$$

Finally, in the last inequality, namely Inequality (4), we use the fact that for Δ sufficiently large we can infer that the term

$$\mathrm{O}\left(\frac{\ln(\Delta)}{\Delta}\right) \leq \frac{\ln(\ln(\Delta))}{\ln(\Delta)},$$

and so, by our choice of $\epsilon = \frac{4\ln(\ln(\Delta))}{\ln(\Delta)}$ this yields

$$\epsilon - \mathrm{O}\left(\frac{\ln(\Delta)}{\Delta}\right) \geq \frac{4\ln(\ln(\Delta))}{\ln(\Delta)} - \frac{\ln(\ln(\Delta))}{\ln(\Delta)} = \frac{3\ln(\ln(\Delta))}{\ln(\Delta)}.$$

Using the entropy bound (see, eg. [8]),

$$\binom{n}{\alpha n} \leq \sum_{i \leq \alpha n} \binom{n}{i} \leq \exp\left(n\left(\alpha \ln\left(\frac{1}{\alpha}\right) + (1-\alpha)\ln\left(\frac{1}{1-\alpha}\right)\right)\right),$$

which is valid for $\alpha < \frac{1}{2}$, the expected number of dominating sets of size t, then, is bounded above by

$$\binom{n}{t}\exp\left(-\frac{(\ln(\Delta))^3}{\Delta}(n-t)\right) \leq \exp\left(n \times \Psi(\Delta)\right), \tag{5}$$

where

$$\Psi(\Delta) = \frac{\ln(\Delta)}{\Delta}\ln\left(\frac{\Delta}{\ln(\Delta)}\right) + \left(\frac{\Delta - \ln(\Delta)}{\Delta}\right)\ln\left(\frac{\Delta}{\Delta - \ln(\Delta)}\right) + \frac{(\ln\Delta)^3}{\Delta}\cdot\frac{t}{n} - \frac{(\ln\Delta)^3}{\Delta}. \tag{6}$$

Now, note that

$$\frac{\ln(\Delta)}{\Delta}\ln\left(\frac{\Delta}{\ln(\Delta)}\right) \leq \frac{(\ln(\Delta))^2}{\Delta}.$$

Likewise,

$$\ln\left(\frac{\Delta}{\Delta - \ln(\Delta)}\right) = \ln\left(1 + \frac{\ln(\Delta)}{\Delta - \ln(\Delta)}\right) \leq \frac{\ln(\Delta)}{\Delta - \ln(\Delta)},$$

so that

$$\left(\frac{\Delta - \ln(\Delta)}{\Delta}\right)\ln\left(\frac{\Delta}{\Delta - \ln(\Delta)}\right) \leq \frac{\ln(\Delta)}{\Delta}.$$

Finally note

$$\frac{(\ln\Delta)^3}{\Delta}\cdot\frac{t}{n} \leq \frac{(\ln\Delta)^4}{\Delta^2}.$$

The above inequalities yields the following upper bound on $\Psi(\Delta)$, as introduced in (6),

$$\Psi(\Delta) \leq \frac{(\ln(\Delta))^2}{\Delta} + \frac{\ln(\Delta)}{\Delta} + \frac{(\ln(\Delta))^4}{\Delta^2} - \frac{(\ln\Delta)^3}{\Delta}. \tag{7}$$

Therefore, by Inequality (5), we infer that for Δ sufficiently large the expected number of dominating sets of size t is bounded above by

$$\exp\left(n\left(\frac{(\ln(\Delta))^2}{\Delta} + \frac{\ln(\Delta)}{\Delta} + \frac{(\ln(\Delta))^4}{\Delta^2} - \frac{(\ln\Delta)^3}{\Delta}\right)\right). \tag{8}$$

For Δ sufficiently large, the exponent in (8) is negative, implying that the expected number of dominating sets of size t tends to 0 as n goes to infinity. As this expectation is an upper bound for the probability that there is a dominating set of size t, this implies that

$$\Pr\left(\gamma(G) \leq t\right) \rightarrow 0$$

as $n \to \infty$. Hence, asymptotically almost surely we have

$$\gamma(G) > t,$$

which completes the proof of Claim 4, and therefore of Theorem 16. □

Remark: The fact that Theorem 16 compares the domination number of a random regular number to $\frac{\ln \Delta}{\Delta}$ as opposed to $\frac{\ln(\Delta+1)}{\Delta+1}$ may seem surprising in light of the probabilistic upper bound; this is explained as

$$\left| \frac{\ln(\Delta)}{\Delta} - \frac{\ln(\Delta+1)}{\Delta+1} \right| = \mathrm{O}\left(\frac{\ln(\Delta)}{\Delta^2} \right),$$

so that the difference is absorbed into the error term, and we have expressed a preference for the cleaner statement.

From here, we finally establish the lower bounds for c_Δ and r_Δ.

Lemma 5

$$\begin{aligned} r_\Delta &\geq (1 - \mathrm{o}(1)) \ln(\Delta) \quad \textit{and} \\ c_\Delta &\geq (1 - \mathrm{o}(1)) \ln(\Delta). \end{aligned}$$

Proof. We prove the result for c_Δ. The result for r_Δ follows identically, noting that $\gamma_t(G) \geq \gamma(G)$ for every graph G. Computationally, the bound for r_Δ is technically slightly simpler because of the missing '+2' in the definition. For Δ sufficiently large and for $\epsilon = \frac{4\ln(\ln(\Delta))}{\ln(\Delta)}$, per Theorem 16 there exist Δ-regular graphs G of arbitrarily large order n, with

$$\gamma = \gamma(G) \geq (1 - \epsilon)\frac{\ln \Delta}{\Delta} n - 1 = (1 - \mathrm{o}(1))\frac{\ln \Delta}{\Delta} n.$$

By the definition of c_Δ,

$$\begin{aligned} 2m = \Delta n &\leq (\Delta + c_\Delta)n - (\Delta + c_\Delta + 2)\gamma. \\ &\leq \Delta n + c_\Delta n - (\Delta\gamma + c_\Delta\gamma + 2\gamma). \end{aligned}$$

Solving for c_Δ yields

$$c_\Delta \geq \frac{\Delta\gamma + 2\gamma}{n - \gamma}.$$

Now, observing that $\Delta\gamma = (1 - \mathrm{o}(1)) \ln(\Delta) n$, while $n - \gamma = (1 - \mathrm{o}(1))n$ and $2\gamma = \mathrm{o}(n)$ as $\Delta \to \infty$ finishes the result. □

5 Future Work

While we completely settled the asymptotics for these types of linear Vizing-like relations for the domination number and total domination number in this paper, some interesting questions remain. Perhaps the most interesting is the following: for other natural domination parameters γ', do there exist constants $c_\Delta^{\gamma'}$, independent of n, so that

$$2m \leq (\Delta + c_\Delta^{\gamma'})(n - \gamma')$$

for all n vertex graphs of size m, and if so what are the asymptotics? We note that in some cases, e.g. for connected versions of domination, the parameter need not be defined at all. For domination parameters which are more restrictive than the standard domination number and total domination number, one expects that $c_\Delta^{\gamma'}$ may be much larger than logarithmic. For instance, taking $\gamma' = i$ where $i = i(G)$ denotes the independent domination number of G (which is the minimum cardinality of a dominating set in G that is also independent), the example of $G = K_{\Delta,\Delta}$ of order n and $i(G) = \frac{n}{2}$ shows that in this case $c_\Delta^i \geq \Delta$.

More interesting, perhaps, are domination parameters that are weaker than standard domination; that is parameters so that $\gamma' \leq \gamma$. It was initially speculated that r_Δ and c_Δ might be bounded (perhaps even by 3) before logarithmic lower bounds were established by Yeo. Are there natural domination parameters where $c_\Delta^{\gamma'}$ is bounded?

One candidate is the distance 2-domination number, $\gamma_2(G)$, which is the domination number of the square of G. This is particularly interesting because the probabilistic upper bound (and likewise any natural analogue of Lemma 3) does not improve much for general graphs as the degrees in G^2 need not differ significantly from degrees in G. On the other hand, random-like graphs, as prove the sharpness of the upper bound for the standard domination number, expand locally in such a way that they do not provide sharpness examples for distance 2-domination. For a Δ-regular random graph, the degrees in G^2 are close to Δ^2 and hence the distance 2-domination number is $\mathrm{O}\big(\frac{\ln(\Delta)}{\Delta^2}\big)n$ which yields no non-trivial lower bound on $c_\Delta^{\gamma_2}$. Thus we end with the following open question that we have yet to settle.

Question 1 *Is it true that if $\gamma_2(G)$ denotes the distance 2-domination number of G, then*

$$c_\Delta^{\gamma_2} = \mathrm{O}(1)?$$

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