

TOPOLOGICAL RAMSEY SPACES, ASSOCIATED
ULTRAFILTERS, AND THEIR APPLICATIONS TO THE
TUKEY THEORY OF ULTRAFILTERS AND
DEDEKIND CUTS OF NONSTANDARD ARITHMETIC.

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Abstract

This dissertation makes contributions to the areas of combinatorial set theory, the model theory of arithmetic, and the Tukey theory of ultrafilters. The main results are broken into three parts.

In the first part, we identify some new partition relations among finite trees and use them to answer an open question of Dobrinen; namely, "for $n < \omega$, are the notions of Ramsey for \mathcal{R}_n and selective for \mathcal{R}_n equivalent?" We show that for each $n < \omega$, it is consistent with ZFC that there exists a selective for \mathcal{R}_n ultrafilter which is not Ramsey for \mathcal{R}_n .

In the second part, we extend results of Blass concerning Dedekind cuts associated to ultrafilter mappings from p-point and weakly-Ramsey ultrafilters to ultrafilter mappings from Ramsey for \mathcal{R}_1 ultrafilters. Blass associates to each ultrafilter \mathcal{U} on a countable set X and each function g with domain X a Dedekind cut in the model of arithmetic given by the ultrapower $\omega^{\text{ran}(g)}/g(\mathcal{U})$. We characterize, under the continuum hypothesis, the cuts obtainable from an ultrafilter mapping from a Ramsey for \mathcal{R}_1 ultrafilter. We also show that the only cut obtainable for ultrafilter mappings between p-points, which are Tukey reducible to a given Ramsey for \mathcal{R}_1 ultrafilter, is the standard cut consisting of equivalence classes of constant sequences. These

results imply new existence theorems for various special kinds of ultrafilters.

In final part of the dissertation, we extend results of Dobrinen and Todorčević concerning the canonical Ramsey theory of \mathcal{R}_1 to the space \mathcal{H}^2 given by forming the product of the space \mathcal{R}_1 with itself. These results imply new existence theorems for initial Tukey structures of nonprincipal ultrafilters. These results shed light on the following open question of Dobrinen concerning the Tukey theory of ultrafilters, "what are the possible initial Tukey structures for ultrafilters on a countable base set?" In particular, we show for the first time that it is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure.

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Introduction

This dissertation makes contributions to the areas of topological Ramsey theory, canonical Ramsey theory, the Tukey theory of ultrafilters, and the model theory of arithmetic. We apply the methods of axiomatic set theory and tools of mathematical logic such as the method of ultrapowers, and the theory of forcing. We use model-theoretic, measure-theoretic, and topological notions to characterize and apply infinite-dimensional Ramsey properties. We also make use of the methods of combinatorial set theory such as the theory of combinatorial forcing and the method of mixing and separating. The first two chapters give the background and setting for the remaining three chapters of the dissertation. In Chapter 3, we show that there are topological Ramsey spaces that support ultrafilters which are Ramsey but not selective for the space. In Chapter 4, we characterize the Dedekind cuts which arise from Ramsey ultrafilters for certain topological Ramsey spaces. In Chapter 5, we develop a canonical Ramsey theory for a product topological Ramsey space and apply it to the Tukey theory of ultrafilters.

An ultrafilter on a set X , by definition, forms a subset of the powerset of X . As such an ultrafilter \mathcal{U} can be identified with the partial order (\mathcal{U}, \supseteq) . Tukey theory gives a method for classifying an ultrafilter \mathcal{U} based on the partial order (\mathcal{U}, \supseteq) . Tukey theory, in general, provides a means for classifying partially ordered sets. Re-

cently, Dobrinen, Mijares, Todorčević, and the author have applied infinite dimensional canonical Ramsey theory to the Tukey theory of ultrafilters (see [41],[17], [18], and [15]). When restricted to ultrafilters, we say that the ultrafilter \mathcal{U} is Tukey reducible to the ultrafilter \mathcal{V} if there is a function from \mathcal{U} to \mathcal{V} with the property that any filter base of \mathcal{U} is mapped to a filter base of \mathcal{V} . This implies, among other things, that the Tukey reducibility relation is a weakening of the extensively studied Rudin-Keisler reducibility relation (see [3], [5], [31], [4], and [33] for more on the Rudin-Keisler ordering of ultrafilters). In Chapter 1, we give an introduction to the infinite dimensional canonical Ramsey theory. In the final section of the first chapter we introduce the Tukey theory of ultrafilters and examine some applications of the canonical Ramsey theory. The results from this section form the prototype arguments for the results we obtain in Chapter 5.

Chapter 1 begins with an introduction to the finite dimensional Ramsey Theorem and its infinite dimensional analogues; such as the Nash-Williams Theorem, the Galvin Lemma, the Galvin-Prikry Theorem, the Silver Theorem, and the Ellentuck Theorem. The Ellentuck Theorem is considered the first result of topological Ramsey theory. In Section 1.2, we introduce the abstract Ellentuck Theorem and the notion of a topological Ramsey space. Topological Ramsey spaces play a central role throughout this dissertation. In Chapter 2, we give a general definition that can support many examples of topological Ramsey spaces arising from certain trees of finite height on ω . Included among these types of spaces are the important topological Ramsey spaces \mathcal{R}_n , for $n < \omega$, introduced by Dobrinen and Todorčević in [17] and [18]. Dobrinen and Todorčević have proven canonical Ramsey theorems for these spaces and applied these results to the Tukey theory of ultrafilters. In Chapter 5, using similar arguments to Dobrinen and Todorčević in [17] we new prove

canonical Ramsey theorems for a space which can be seen as product of \mathcal{R}_1 with itself, and apply them to obtain new results in the Tukey theory of ultrafilters. In particular, we show for the first time that it is consistent ZFC that the four-element Boolean algebra appears as an initial Tukey structure, where initial Tukey structure refers to a collection of nonprincipal ultrafilters closed under the Tukey reducibility relation.

Each topological Ramsey space supports a notion of selective and a notion of Ramsey ultrafilter (see [35]). Mijares in [35] has shown that each topological Ramsey space can be considered as a forcing notion which adjoins a Ramsey ultrafilter for the space. Moreover, Mijares has shown that every Ramsey ultrafilter for the space is a selective ultrafilter for the space. Thus, it is consistent with ZFC that for a given topological Ramsey space, there exists a Ramsey and selective ultrafilter for the space. In Section 2.2, we introduce these notions in the context of topological Ramsey spaces generated by trees on ω . In Section 2.3, we provide some properties for trees on ω that guarantee that the topological Ramsey space arising from the tree is forcing equivalent to one of the spaces \mathcal{R}_n , for $n < \omega$. The final two sections of Chapter 2 include a study of the properties of these ultrafilters restricted to spaces generated from trees on ω . We state and prove equivalent formulations of the notions of Ramsey and selective in this context. In particular, we show that Ramsey ultrafilters for the space satisfy a localized version of the Ellentuck Theorem. The topological Ramsey spaces generated by trees on ω form the setting for the main results of Chapters 3, 4, and 5. Thus we use the results of Chapter 2 throughout the final three chapters of this manuscript. The graph in Figure 1 provides a layout of how the sections of Chapter 1 are related to one another and the other chapters of the dissertation. In particular, Chapters 3, 4, and 5 can be read in any order.

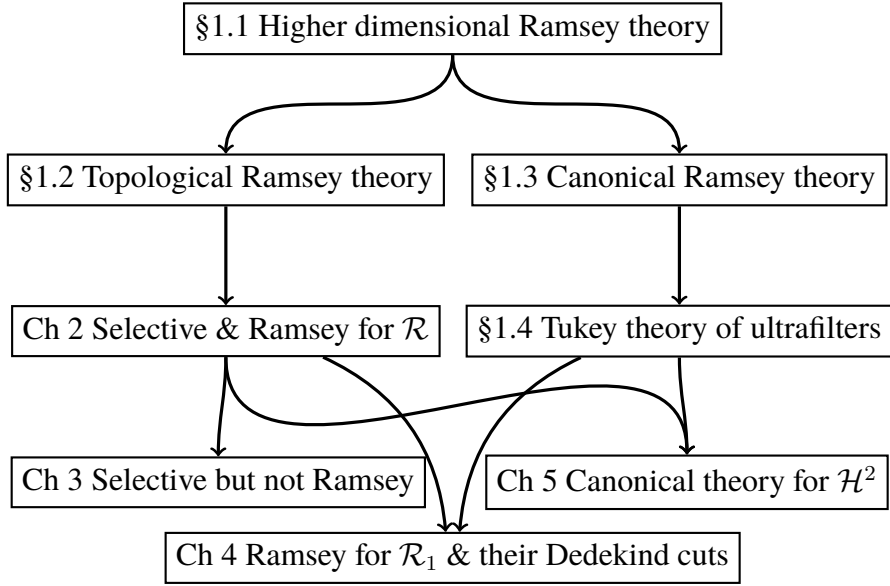


Figure 1: Interdependences of sections of Chapter 1 and Chapters 2-5

The results of Chapters 3, 4, and 5 comprise the main results of this dissertation. Initially this work was motivated by the theorems and proofs of Blass in [4]. The contents of Chapter 4 are generalizations of theorems in [4] to Ramsey and selective ultrafilters for the topological Ramsey space \mathcal{R}_1 . Blass in [4] associates to each ultrafilter and each ultrafilter mapping a Dedekind cut in a nonstandard model of arithmetic. An ultrafilter mapping is simply the name given to a function witnessing the Rudin-Kiesler reducibility of two ultrafilters. If \mathcal{U} is an ultrafilter on X and $g : X \rightarrow Y$ is an ultrafilter mapping then the associated Dedekind cut is taken in the ultrapower, $\omega^Y/g(\mathcal{U})$. Blass in [4] has characterized, assuming the continuum hypothesis, the Dedekind cuts obtainable when \mathcal{U} is taken to be a Ramsey ultrafilter, a p-point ultrafilter, or a weakly-Ramsey ultrafilter on ω . Ramsey and selective for \mathcal{R}_1 ultrafilters, introduced in Chapter 2, are examples of weakly-Ramsey and p-point ultrafilters, respectively. In Chapter 4, we show that, assuming the continuum hypothesis, there are p-points which are neither weakly-Ramsey nor Ramsey

for \mathcal{R}_1 ultrafilters and there are weakly-Ramsey ultrafilters which are not Ramsey for \mathcal{R}_1 ultrafilters. These results are obtained by characterizing, under the continuum hypothesis, the cuts obtainable when \mathcal{U} is a Ramsey for \mathcal{R}_1 ultrafilter and g is taken to be any ultrafilter mapping. In Section 4.4, we use the results of Dobrinen and Todorčević in [17] to show that if \mathcal{U} is a Ramsey for \mathcal{R}_1 ultrafilter obtained from forcing with a particular partial order, and g is an ultrafilter mapping to some p -point Tukey reducible to \mathcal{U} , then the cut associated to \mathcal{U} and g is the standard cut, where the standard cut is the cut whose lower half consists of exactly the equivalence classes of constant sequences. Chapter 3 concludes with some open questions related to these Dedekind cuts and some corollaries of the main results concerning cuts obtained from ultrafilters for \mathcal{R}_1 . Before obtaining the results of Chapter 4, the author investigated extensions of the theorems of [4] to more general ultrafilters satisfying weak-partition properties. After hitting a dead end with these weak-partition properties, and following a suggestion of Dobrinen, the author investigated the ultrafilters obtained by forcing from the partial order (\mathbb{P}_n, \leq_n) studied by Laflamme in [31]. The work of Laflamme in [31] both motivated and inspired the work of Dobrinen and Todorčević in [17] and [18]. Thus it was natural to consider Dedekind cuts obtained from the ultrafilter mappings of Ramsey for \mathcal{R}_1 ultrafilters.

During the same time, Dobrinen introduced the author to the topological Ramsey space \mathcal{H}^2 , which can be seen as a product of \mathcal{R}_1 with itself, and also as a space generated by a tree on ω^2 . The space \mathcal{H}^2 forms the setting for Chapter 5 where we develop a canonical Ramsey theory for \mathcal{H}^2 similar to the canonical theory developed for \mathcal{R}_1 by Dobrinen and Todorčević in [17]. Following Dobrinen and Todorčević we apply these results to the Tukey theory of ultrafilters. In the first section of Chapter 5, we give some background on the space \mathcal{H}^2 and prove that \mathcal{H}^2 forms a topological

Ramsey space. The primary result, which follows from the work of Sokić in [44], needed to obtain these results is a generalization of the finite Ramsey theorem to finite products of finite sets. In the second section of Chapter 5, we develop an extension of the Erdős-Rado theorem to finite products of finite sets. Following [17] as a guide, we use the theory of mixing and separating to generalize the Erdős-Rado theorem to a canonical Ramsey theorem for \mathcal{H}^2 . In the final section of Chapter 5, we apply the canonical Ramsey theory in conjunction with basic Tukey reductions for Ramsey ultrafilters for \mathcal{H}^2 to show that it is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure.

Chapter 3 does not require any canonical Ramsey theory and answers a question that arose during the study of the results in Chapters 4 and 5. The prototype example of a topological Ramsey space is the Ellentuck space. The fact that the Ellentuck space is a topological Ramsey space is equivalent to the Ellentuck theorem (see Section 1.2). When restricting the notions of Ramsey and selective ultrafilter for an arbitrary topological Ramsey space to the Ellentuck space, one obtains the more familiar notions of a Ramsey and selective ultrafilter on ω , respectively. Thus, by a well-known result of Kunen, the notions of Ramsey and selective for the Ellentuck space are equivalent. This has led Dobrinen to ask the following question: For a given topological Ramsey space \mathcal{R} , are the notions of selective for \mathcal{R} and Ramsey for \mathcal{R} equivalent? In the first section of Chapter 3, we show that for the space \mathcal{R}_1 the two notions are not equivalent. We show that it is consistent with ZFC that there exists a selective for \mathcal{R}_1 ultrafilter which is not Ramsey. In Section 3.1, we extend this result to the spaces \mathcal{R}_n for $n < \omega$. These results are obtained by constructing closely related spaces \mathcal{R}_n^* under which forcing with the appropriate partial order gives an ultrafilter which is selective but not Ramsey for \mathcal{R}_n . In Section 3.3, we

extend these results to finite products of the spaces \mathcal{R}_n for $n < \omega$.

Chapter 1

Infinite-dimensional Ramsey theory

The primary purpose of this chapter is to outline the background material needed to understand the work in the other chapters of this dissertation. In Section 1.1, we use the finite Ramsey Theorem to introduce the notion of dimension and the main results of higher-dimensional Ramsey theory. We include higher-dimensional extensions of the Ramsey Theorem due to Nash-Williams, Silver, Galvin, Prikry, and Ellentuck. Further, we include an example of Baumgartner showing that the unrestricted infinite-dimensional Ramsey Theorem fails. In Section 1.2, we present the details of an abstract method for extending finite-dimensional Ramsey-like results to their higher-dimensional analogues. The abstract theory, introduced by Carlson and Simpson in [7], is best described topologically via the notion of a topological Ramsey space. Building on their work, Todorčević in [46] has isolated four axioms which guarantee that a space forms a topological Ramsey space. We present the definition of a topological Ramsey space and these axioms in Section 1.2. As we shall see, the Ramsey Theorem can be stated in terms of partition properties where only partitions into finitely many parts are considered. In Section 1.3, we look at two

extensions, due to Erdős and Rado, and Pudlák and Rödl, of the Ramsey Theorem to partitions of any size. Theorems extending Ramsey-like results to unrestricted partition size are called canonical Ramsey theorems, and their study is what we refer to as canonical Ramsey theory. Later in Chapter 5, we generalize these canonical Ramsey theorems to another topological Ramsey space introduced in Chapter 2. Our interest in canonical Ramsey theory stems from its application to the Tukey theory of ultrafilters. In Section 1.4, we introduce the basics of the Tukey theory of ultrafilters and outline an application of Todorčević of the canonical Ramsey theory to the Tukey theory of ultrafilters. In Chapter 5, we prove similar theorems by applying the canonical Ramsey theorems developed in Section 5.3 for a topological Ramsey space introduced in Chapter 2.

1.1 Higher-dimensional Ramsey theory

The pigeonhole principle states that, "If $n + 1$ pigeons are placed in n pigeonholes then one pigeonhole contains at least two pigeons." The principle can be reformulated in terms of partitions. If $\{0, 1, 2, \dots, n\}$ is partitioned into n parts then there is a two-element subset of $\{0, 1, 2, \dots, n\}$ such that all of its elements lie in one piece of the partition. Following the standard conventions, we let ω be the first infinite ordinal consisting of all the of finite ordinals. In particular, $n < \omega$ if and only if n is a natural number. Furthermore, for each $n < \omega$, n consists of the set of all smaller ordinals, that is, $n = \{0, 1, \dots, n - 1\}$.

Theorem 1.1.1 (Finite Pigeonhole Principle). *For each $l, n < \omega$, there exists $m < \omega$ such that for each partition of m into l parts there exists an n -element subset of m such that all of its elements lie in one piece of the partition.*

This principle can be generalized to infinite sets provided that we only partition the set into finitely many parts.

Theorem 1.1.2 (Infinite Pigeonhole Principle). *For each partition of ω into finitely many parts there exists an infinite subset of ω such that all of its elements lie in one piece of the partition.*

Although it is not apparent, this principle can be seen as the one-dimensional analogue of the celebrated Ramsey theorem. Following standard conventions, for each $n < \omega$ and each set A we let $[A]^n$ denote the set of all n -element subsets of A and $[A]^\omega$ denote the collection of infinite subsets of A .

Theorem 1.1.3 (The Infinite Ramsey Theorem, [42]). *Let $k < \omega$. For each partition of $[\omega]^k$ into finitely many parts there exists an infinite subset of ω all of whose k -element subsets lie in one piece of the partition.*

If one considers k to be the *dimension* in the preceding theorem then the infinite pigeonhole principle is a one-dimensional version of the higher-dimensional infinite Ramsey theorem. By a standard compactness argument, a finite version of the infinite Ramsey theorem holds, and has the property that the finite pigeonhole principle is equivalent to a one-dimensional finite Ramsey theorem.

Theorem 1.1.4 (The finite Ramsey Theorem, [42]). *Let $k < \omega$. For each $l, n < \omega$ with $k \leq n$, there exists $m < \omega$ such that for each partition of $[m]^k$ into l parts there exists an n -element subset of m all of whose k -element subsets lie in one piece of the partition.*

The partition calculus, introduced by Erdős and Rado in [21], succinctly describes the theorems of this section and many other similar statements. The central concept of the partition calculus is the partition relation.

Definition 1.1.5. If α, β, γ and δ are cardinal numbers, then the partition relation

$$\alpha \rightarrow (\beta)_\delta^\gamma \tag{1.1.1}$$

means that for each partition of the γ -element subsets of α into δ parts there exists a subset of α of size β all of whose γ -element subsets lie in one piece of the partition. If $\delta = 2$ then we write $\alpha \rightarrow (\beta)^\gamma$.

For example, the infinite Ramsey Theorem states for each pair of natural numbers k and l , $\omega \rightarrow (\omega)_l^k$. It is natural to ask the following question:

Question 1.1.6. Does an infinite-dimensional analogue of the infinite Ramsey Theorem hold? Naively one can simply ask, does the infinite Ramsey Theorem hold if $k = \omega$? That is, if l is a positive integer, does $\omega \rightarrow (\omega)_l^\omega$ hold?

Assuming ZFC, the answer is no, there exists a partition of $[\omega]^\omega$ into finitely many parts such that no infinite subset of ω has all of its infinite subsets in one piece of the partition. The first example of a set which fails the conclusion of the unrestricted infinite-dimensional Ramsey Theorem was given by Rado in [22] using a well-ordering of the continuum. In [34], Mathias studies the existence of sets that fail the conclusion of the unrestricted infinite-dimensional Ramsey Theorem in ZF without the axioms of choice. In [32], Mathias shows that the existence of q -point ultrafilter leads to a set which fails the conclusion of the unrestricted infinite-dimensional Ramsey Theorem. In order to give an example of such a set, we introduce the notion of an ultrafilter and state some of their basic properties. The terms "filter" and "ultrafilter" were first introduced in 1937 by Cartan in [9] and [8]. However, the concept of an ultrafilter arose in the earlier work of Stone in [45] on the topological

representation of Boolean algebras. The notion of a "filterbase" was used by Vietoris and Carathéodory, under a different name, as early as 1913 (see [6] and [50]).

Definition 1.1.7. Let X be a nonempty set. A *filter* \mathcal{F} on X is a nonempty subset of the powerset of X which does not contain the empty set ($\emptyset \notin \mathcal{F}$), is closed under finite intersections ($A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$), and is closed upward ($A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$). An *ultrafilter* \mathcal{U} on X is a filter on X with the property that every subset of X or its complement in X is an element of \mathcal{U} ($\forall A, A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$). A subset \mathcal{B} of \mathcal{F} is a *filterbase*, if for each $X \in \mathcal{F}$ there exists $Y \in \mathcal{B}$ such that $Y \subseteq X$.

Example 1.1.8. For each natural number n , $\mathcal{U}_n = \{X \subseteq \omega : n \in X\}$ forms an ultrafilter on ω . A subset of ω is *co-finite* if its complement in ω is finite. The collection of all cofinite subsets of ω forms a filter on ω which is known as the *Fréchet filter*.

Definition 1.1.9. An ultrafilter \mathcal{U} on X is *principal* if there exists $x \in X$ such that $\mathcal{U} = \{Y \subseteq X : x \in Y\}$. We call x the *principal element of \mathcal{U}* since \mathcal{U} is the only ultrafilter containing $\{x\}$.

The next lemma outlines some of the immediate consequences of the definition of an ultrafilter.

Lemma 1.1.10. *Let \mathcal{U} be an ultrafilter on X .*

1. *If k a positive integer and $X_0 \cup X_1 \cup \dots \cup X_{k-1} \in \mathcal{U}$ then there exists $j < k$ such that $X_j \in \mathcal{U}$.*
2. *If \mathcal{U} contains a finite set then \mathcal{U} is principal. Equivalently, every nonprincipal ultrafilter contains the Fréchet filter.*

3. If \mathcal{F} is a filter on X and $\mathcal{U} \subseteq \mathcal{F}$, then $\mathcal{U} = \mathcal{F}$.

The final item in the previous lemma admits a converse which can be used in conjunction with Zorn's lemma to prove that the Fréchet filter can be extended to a nonprincipal ultrafilter.

Lemma 1.1.11. *Let \mathcal{F} be a filter on X . \mathcal{F} is an ultrafilter on X if and only if for all filters \mathcal{F}' on X with $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{F} = \mathcal{F}'$.*

Theorem 1.1.12. *There exists a nonprincipal ultrafilter on ω .*

The next definition is needed to give an example of a set which fails the conclusion of the unrestricted infinite-dimensional Ramsey Theorem.

Definition 1.1.13. Let \mathcal{U} be a nonprincipal ultrafilter on ω and let $(x_i)_{i < \omega}$ be a sequence in a topological space X . We say that $(x_i)_{i < \omega}$ converges to x along \mathcal{U} and write $\lim_{i \rightarrow \mathcal{U}} x_i = x$, if for each neighborhood U of x , the set $\{i \in \omega : x_i \in U\} \in \mathcal{U}$.

The partition defined in the next theorem verifies that $\omega \not\rightarrow (\omega)^\omega$. Thus, the unrestricted infinite-dimensional analogue of the Ramsey Theorem fails in ZFC.

Theorem 1.1.14 (Baumgartner, [32]). *Given a nonprincipal ultrafilter \mathcal{U} on ω define the partition $\{\Pi_0, \Pi_1\}$ of $[\omega]^\omega$ given by letting*

$$\Pi_0 = \{X \in [\omega]^\omega : \lim_{n \rightarrow \mathcal{U}} (-1)^{|X \cap n|} = -1\}$$

$$\Pi_1 = \{X \in [\omega]^\omega : \lim_{n \rightarrow \mathcal{U}} (-1)^{|X \cap n|} = 1\}.$$

Then there is no infinite subset of ω such that all of its infinite subsets lie in one piece of the partition.

Leaving behind the case of $k = \omega$ for the moment, it is possible to generalize the notion of dimension to the notion of rank for families of finite subsets of ω , which satisfy similar properties to the families $[\omega]^k$ for k a positive integer. The work of Nash-Williams provides such a generalization extending the infinite Ramsey Theorem to other families of finite subsets of ω . For each family \mathcal{F} of finite subsets of ω and each subset X of ω , we let $\mathcal{F}|X = \{x \in \mathcal{F} : x \subseteq X\}$. For example, for each natural number n and each subset X of ω , $[\omega]^n|X = [X]^n$. In the following we let $[\omega]^{<\omega}$ denote the collection of all finite subsets of ω . If $s \in [\omega]^{<\omega}$ and X is a subset of ω then we write $s \sqsubseteq X$, if there exists $x \in X$ such that $s = \{y \in X : y < x\}$.

Definition 1.1.15. Let \mathcal{F} be a subset of $[\omega]^{<\omega}$. The family \mathcal{F} is *Ramsey*, if for each infinite subset X of ω and each partition of \mathcal{F} into finitely many parts there exists an infinite set $Y \subseteq X$ such that $\mathcal{F}|Y$ is completely contained in one piece of the partition.

By the infinite Ramsey Theorem, for each natural number n , $[\omega]^n$ is a Ramsey family. The next property will be needed to extend the infinite Ramsey Theorem to families of finite subsets of the natural numbers.

Definition 1.1.16. Let \mathcal{F} be a subset of $[\omega]^{<\omega}$. The family \mathcal{F} is *Nash-Williams*, if for each $s, t \in \mathcal{F}$, $s \sqsubseteq t \Rightarrow s = t$.

For each natural number n , $[\omega]^n$ is a Nash-Williams family. The next lemma shows that any Ramsey family can be restricted to a Nash-Williams family.

Lemma 1.1.17 ([46]). *Suppose that \mathcal{F} is a Ramsey family of finite subsets of ω . For each infinite subset X of ω there exists an infinite set $Y \subseteq X$ such that $\mathcal{F}|Y$ is Nash-Williams.*

The next theorem provides a type of converse for the previous lemma. Its original proof due to Nash-Williams was the first use of the method of combinatorial forcing. We outline the proof from [46] as we will use variants of the combinatorial forcing in later chapters.

Theorem 1.1.18 (The Nash-Williams Theorem, [38]). *Every Nash-Williams family is Ramsey.*

Sketch of proof. The following is a sketch of the proof of this theorem from [46]. For all of the lemmas and definitions within this proof we fix a Nash-Williams family \mathcal{F} of finite subsets of ω . Notice that it is enough to prove the theorem for partitions of \mathcal{F} into two parts. Let $\{\mathcal{F}_0, \mathcal{F}_1\}$ be a fixed partition of \mathcal{F} into two parts.

Definition 1.1.19 ([46]). Suppose that s and t are finite subsets of ω and X is an infinite subset of ω . s and t are \sqsubseteq -comparable, if $s \sqsubseteq t$ or $t \sqsubseteq s$. We say that X *accepts* s if s is \sqsubseteq -comparable to some $t \in \mathcal{F}_0$ and $t \subseteq s \cup X$. On the other hand, X *rejects* s if there is no $t \subseteq s \cup X$ in \mathcal{F}_0 such that t is \sqsubseteq -comparable to s . We say that X *strongly accepts* s if for each infinite subset Y of X , Y *accepts* s . If X *strongly accepts* s or X *rejects* s then we say that X *decides* s .

The next three lemmas outline some of the basic facts about these notions.

Lemma 1.1.20 ([46]). *For each infinite set X and each finite set s there exists an infinite set $Y \subseteq X$ such that Y decides s .*

Lemma 1.1.21 ([46]). *Let X be an infinite subsets of ω and s be a finite subset of ω . If X strongly accepts (rejects) s and Y is an infinite subset of X , then Y strongly accepts (rejects) s .*

Lemma 1.1.22 ([46]). *For each infinite set X there exists an infinite set $Y \subseteq X$ such that Y decides all of its finite subsets.*

Lemma 1.1.23 ([46]). *If the infinite set X strongly accepts the finite set s then there are only finitely many $x \in X$ such that $x > \max(s)$ and X rejects $s \cup \{x\}$.*

Lemma 1.1.24 ([46]). *There exists an infinite set X such that either X rejects \emptyset or X strongly accepts all of its finite subsets.*

Let X be a infinite set satisfying the previous lemma. If X rejects \emptyset then $\mathcal{F}_0|X = \emptyset$ and $\mathcal{F}|X \subseteq \mathcal{F}_1|X$. On the other hand, if X strongly accepts all of its finite subsets then it accepts all the elements of $\mathcal{F}|X$. So for each $s \in \mathcal{F}|X$, there exists $t \in \mathcal{F}_0$ such that t is \sqsubseteq -comparable to s . Since \mathcal{F} is Nash-Williams $s = t$ and $s \in \mathcal{F}_0|X$. So $\mathcal{F}|X \subseteq \mathcal{F}_0$. Therefore \mathcal{F} is a Ramsey family. \square

Corollary 1.1.25 (The Ramsey Theorem). *For each positive integer k , $[\omega]^k$ is a Ramsey family.*

Proof. If $s, t \in [\omega]^k$ and $s \subseteq t$, then $s = t$ since $|s| = |t|$. In particular, $[\omega]^k$ is Nash-Williams. By the Nash-Williams Theorem it is a Ramsey family. \square

Example 1.1.26. The *Schreier family* is the set

$$\mathcal{SB} = \{s \in [\omega]^{<\omega} : \min(s) = |s|\}.$$

If $s, t \in \mathcal{SB}$ and $s \subseteq t$, then $s = t$ since $|s| = |t|$. In particular, \mathcal{SB} is Nash-Williams. By the Nash-Williams Theorem \mathcal{SB} is a Ramsey family.

Lemma 1.1.27 ([46]). *Suppose that $\mathcal{F}_1, \mathcal{F}_2, \dots$ is a countable collection of Nash-Williams families then $\{s \in [\omega]^\omega : s \in \mathcal{F}_{\min(s)}\}$ is a Nash-Williams family.*

The notion of rank for a Nash-Williams family can be viewed as a generalization of the dimension of the finite families $[\omega]^n$ for n a positive integer.

Definition 1.1.28 ([46]). Suppose that $\mathcal{F} \subseteq [\omega]^{<\omega}$ and

$$T(\mathcal{F}) = \{s \in [\omega]^\omega : (\exists t \in \mathcal{F}) s \sqsubseteq t\}.$$

$T(\mathcal{F})$ is a tree without infinite branches ordered by the relation \sqsubseteq . Let $\rho_{T(\mathcal{F})}$ be the recursively defined function from $T(\mathcal{F})$ into the ordinals given by the rule:

$$\rho_{T(\mathcal{F})}(s) = \sup\{\rho_{T(\mathcal{F})}(t) + 1 : t \in T(\mathcal{F}) \ \& \ t \sqsubset s\}$$

The *rank of \mathcal{F}* is the ordinal $\rho_{T(\mathcal{F})}(\emptyset)$.

Example 1.1.29. For each natural number n , the rank of $[\omega]^n$ is n . The rank of the Schreier family is ω . Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a countable collection of Nash-Williams families with ranks $\rho_0, \rho_1, \rho_2, \dots$ respectively. Then $\{s \in [\omega]^\omega : s \in \mathcal{F}_{\min(s)}\}$ has rank $\sup\{\rho_i + 1 : i < \omega\}$.

The Nash-Williams Theorem has been applied to many diverse areas of mathematics. The theorem was originally discovered in the context of the theory of well-quasi-orderings. However, it also has applications to Banach space geometry (see [1]). The theorem facilitates the understanding of the important notions of fronts, barriers and Sperner families. We will use the concepts of fronts and barriers in the next section to describe the canonical Ramsey theory.

Definition 1.1.30 ([46]). Let \mathcal{F} be a subset of $[\omega]^{<\omega}$. The family \mathcal{F} is *Sperner*, if for each $s, t \in \mathcal{F}$, $s \subseteq t \Rightarrow s = t$.

Definition 1.1.31 ([46]). Let \mathcal{F} be a subset of $[\omega]^{<\omega}$ and X be an infinite subset of ω .

1. \mathcal{F} is a *front on X* if \mathcal{F} is Nash-Williams and every infinite subset of X has an initial segment in \mathcal{F} .
2. \mathcal{F} is a *barrier on X* if \mathcal{F} is Sperner and every infinite subset of X has an initial segment in \mathcal{F} .

Example 1.1.32. For each positive integer k , $[\omega]^k$ is a barrier on ω . The Schreier family $\mathcal{SB} = \{s \in [\omega]^{<\omega} : \min(s) = |s|\}$ is a barrier on ω which we will now refer to as the *Schreier barrier*.

The Nash-Williams Theorem provides a way of relating fronts to barriers.

Corollary 1.1.33 ([46]). *Let X be an infinite subset of ω and \mathcal{F} be a collection of finite subsets of ω . If \mathcal{F} is a front on X then there is an infinite set $Y \subseteq X$ such that $\mathcal{F}|Y = \{s \in \mathcal{F} : s \subseteq Y\}$ is a barrier on Y .*

We introduce both fronts and barriers since many recursive constructions lead to fronts and not barriers. For some applications of the theory of fronts and barriers see [46] and [2]. Next, we look at a generalization of the Nash-Williams Theorem originally due to Galvin. The result relates arbitrary families of finite subsets of ω to barriers.

Theorem 1.1.34 (The Galvin Lemma, [24]). *Let \mathcal{F} be an arbitrary family of finite subsets of ω and \mathcal{F}_0 be the collection of all \subseteq -minimal elements of \mathcal{F} . Then either there is an infinite subset X of ω such that $\mathcal{F}|X = \emptyset$ or $\mathcal{F}_0|X$ is a barrier on X .*

Although there are partitions which fail the conclusion of the infinite-dimensional Ramsey Theorem, there are also partitions of $[\omega]^\omega$ which satisfy the conclusion of

the infinite-dimensional Ramsey Theorem. The Galvin Lemma can be rephrased as a statement that certain simply defined subsets of $[\omega]^\omega$ satisfy the conclusion of the infinite-dimensional Ramsey Theorem. In order to be more precise, we introduce the notion of the Ramsey and Ramsey null properties. Instead of working with partitions of $[\omega]^\omega$ into two parts we work with subsets of $[\omega]^\omega$.

Definition 1.1.35 (Galvin and Prikry, [25]). For each infinite set A and each finite set a we let

$$[a, A] = \{B \in [\omega]^\omega : a \sqsubseteq B \subseteq A\}.$$

A subset \mathcal{X} of $[\omega]^\omega$ is *Ramsey* if for each non-empty $[a, A]$ there exists an infinite set $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$ and *Ramsey null* if for each non-empty $[a, A]$ there exists $B \in [a, A]$, $[a, B] \cap \mathcal{X} = \emptyset$.

Example 1.1.36. Every nonprincipal ultrafilter \mathcal{U} on ω is an example of a Ramsey null set. Let $[a, A]$ be a nonempty set, then either $A \in \mathcal{U}$ or $A \notin \mathcal{U}$. If $A \notin \mathcal{U}$ then $[a, A] \cap \mathcal{U} = \emptyset$ since \mathcal{U} is closed upwards. If $A \in \mathcal{U}$ and $\{a_0, a_1, a_2, \dots\}$ is its increasing enumeration, then either $\{a_0, a_2, a_4, \dots\} \notin \mathcal{U}$ or $\{a_1, a_3, a_5, \dots\} \notin \mathcal{U}$. Let B denote whichever is not in \mathcal{U} union the finite set a . Then $[a, B] \cap \mathcal{U} = \emptyset$ since \mathcal{U} is closed upwards.

If the infinite subsets of ω are identified with their characteristic functions in 2^ω then $[\omega]^\omega$ is identified with a subspace of 2^ω endowed with the product topology. Since this topology arises from a metric we refer to it as *the metric topology on $[\omega]^\omega$* . The Galvin Lemma is equivalent to the statement that all metric open subsets of $[\omega]^\omega$ are Ramsey (see [25]). For the remainder of this section, we state some theorems showing that certain simply defined subsets of $[\omega]^\omega$ are Ramsey.

Theorem 1.1.37 (Galvin and Prikry, [25]). *The collection of Ramsey subsets of $[\omega]^\omega$ is a σ -algebra.*

Theorem 1.1.38 (Galvin and Prikry, [25]). *There exists a subset of $[\omega]^\omega$ which has the Baire property with respect to the metric topology but is not Ramsey.*

The next theorem is one of the main results of the paper [25].

Theorem 1.1.39 (The Galvin-Prikry Theorem, [25]). *Every Borel subset of $[\omega]^\omega$ with respect to the metric topology is Ramsey.*

The original proof of the next theorem by Silver in [43] used metamathematical concepts such as absoluteness and forcing. Since there are analytic sets which are not Borel, the theorem shows that the Borel σ -algebra and the σ -algebra of Ramsey sets do not coincide.

Theorem 1.1.40 (The Silver Theorem, [43]). *Every analytic subset of $[\omega]^\omega$ with respect to the metric topology is Ramsey.*

The first result of topological Ramsey theory was the infinite dimensional extension of the Ramsey Theorem known as the Ellentuck Theorem (see [19]). Ellentuck proved this theorem in order to give a simpler proof of the previous theorem of Silver. In order to state the Ellentuck Theorem it is necessary to introduce the Ellentuck space.

The *Ellentuck space* is the set $[\omega]^\omega$ of all infinite subsets of ω with the topology generated by the basic open sets,

$$\{[a, A] : a \in [\omega]^{<\omega} \ \& \ B \in [\omega]^\omega\}. \quad (1.1.2)$$

Recall that a subset of a topological space is *nowhere dense* if its closure has empty interior and *meager* if it is the countable union of nowhere dense sets. A subset \mathcal{X} of a topological space has the *Baire property* if and only if $\mathcal{X} = \mathcal{O} \Delta \mathcal{M}$ for some open set \mathcal{O} and some meager set \mathcal{M} . The next theorem is the infinite-dimensional analogue of the finite dimensional Ramsey Theorem.

Theorem 1.1.41 (The Ellentuck Theorem, [19]). *Every subset of the Ellentuck space with the Baire property is Ramsey and every meager subset is Ramsey null.*

In the next section, we present a topological method for extending pigeonhole-like results to infinite dimensional Ellentuck-like theorems.

1.2 Topological Ramsey theory

A topological Ramsey space \mathcal{R} , by definition, is a space that satisfies an abstract version of the Ellentuck Theorem. In order to state an abstract version of the Ramsey property for \mathcal{R} it is necessary to have an abstract notion of the partial order " \sqsubseteq " and an abstraction notion of the restriction map " r " used to define \sqsubseteq . To this end, we consider triples (\mathcal{R}, \leq, r) where \mathcal{R} is a nonempty set, \leq is a quasi-ordering on \mathcal{R} , and $r : \mathcal{R} \times \omega \rightarrow \mathcal{AR}$. For each such triple we can define an abstract notion of the Ramsey property and endow \mathcal{R} with a topology similar to the Ellentuck space.

Definition 1.2.1 ([46]). Let (\mathcal{R}, \leq, r) be a triple such that \mathcal{R} is a nonempty set, \leq is a quasi-ordering on \mathcal{R} and $r : \mathcal{R} \times \omega \rightarrow \mathcal{AR}$ is surjective. For each $a \in \mathcal{AR}$ and each $B \in \mathcal{R}$, let

$$[a, B] = \{A \in \mathcal{R} : A \leq B \ \& \ (\exists n)r_n(A) = a\}. \quad (1.2.1)$$

The *Ellentuck topology* on \mathcal{R} is the topology generated by the sets $[a, B]$ where $a \in \mathcal{AR}$ and $B \in \mathcal{R}$.

A subset \mathcal{X} of \mathcal{R} is *Ramsey* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. A subset \mathcal{X} of \mathcal{R} is *Ramsey null* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

A triple (\mathcal{R}, \leq, r) with its Ellentuck topology is a *topological Ramsey space* if every subset of \mathcal{R} with the Baire property is Ramsey and if every meager subset of \mathcal{R} is Ramsey null.

We follow the presentation of the abstract Ellentuck theorem given by Todorćević in [46], rather than the earlier reference [7]. In particular, we introduce four axioms about triples (\mathcal{R}, \leq, r) sufficient for proving an abstract version of the Ellentuck Theorem. The first axiom we consider tells us that \mathcal{R} is collection of infinite sequences of objects and \mathcal{AR} is collection of finite sequences approximating these infinite sequences.

A.1 For each $A, B \in \mathcal{R}$,

- (a) $r_0(A) = \emptyset$.
- (b) $A \neq B$ implies $r_i(A) \neq r_i(B)$ for some i .
- (c) $r_i(A) = r_j(B)$ implies $i = j$ and $r_k(A) = r_k(B)$ for all $k < i$.

On the basis of this axiom, \mathcal{R} can be identified with a subset of \mathcal{AR}^ω by associating $A \in \mathcal{R}$ with the sequence $(r_i(A))_{i < \omega}$. Similarly, $a \in \mathcal{AR}$ can be identified with $(r_i(A))_{i < j}$ where j is the unique natural number such that $a = r_j(A)$ for some $A \in \mathcal{R}$. For each $a \in \mathcal{AR}$, let $|a|$ equal the natural numbers i for which $a = r_i(a)$. For $a, b \in \mathcal{AR}$, $a \sqsubseteq b$ if and only if $a = r_i(b)$ for some $i \leq |b|$. $a \sqsubset b$ if and only if $a = r_i(b)$ for some $i < |b|$.

A.2 *There is a quasi-ordering \leq_{fin} on \mathcal{AR} such that*

- (a) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ *is finite for all $b \in \mathcal{AR}$,*
- (b) $A \leq B$ *iff $(\forall i)(\exists j) r_i(A) \leq_{\text{fin}} r_j(B)$,*
- (c) $\forall a, b, c \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\text{fin}} c \rightarrow \exists d \sqsubseteq c a \leq_{\text{fin}} d]$.

For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, $\text{depth}_B(a)$ is the least i , if it exists, such that $a \leq_{\text{fin}} r_i(B)$. If such an i does not exist, then we write $\text{depth}_B(a) = \infty$. If $\text{depth}_B(a) = i < \infty$, then $[\text{depth}_B(a), B]$ denotes $[r_i(a), B]$.

A.3 *For each $A, B \in \mathcal{R}$ and each $a \in \mathcal{AR}$,*

- (a) *If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.*
- (b) $A \leq B$ and $[a, A] \neq \emptyset$ *imply that there is an $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.*

If $n > |a|$, then $r_n[a, A]$ denotes the collection of all $b \in \mathcal{AR}_n$ such that $a \sqsubset b$ and $b \leq_{\text{fin}} A$.

A.4 *For each $B \in \mathcal{R}$ and each $a \in \mathcal{AR}$, if $\text{depth}_B(a) < \infty$ and $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\text{depth}_B(a), B]$ such that*

$$r_{|a|+1}[a, A] \subseteq \mathcal{O} \text{ or } r_{|a|+1}[a, A] \subseteq \mathcal{O}^c. \quad (1.2.2)$$

The next result, using a slightly different set of axioms, is a theorem of Carlson and Simpson in [7]. The version using **A.1-A.4** can be found as Theorem 5.4 in [46].

Theorem 1.2.2 (The Abstract Ellentuck Theorem, [46]). *If (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1, A.2, A.3** and **A.4** then (\mathcal{R}, \leq, r) forms a topological Ramsey space.*

Corollary 1.2.3 (The Abstract Galvin-Prikry Theorem, [46]). *Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. Every Borel subset of \mathcal{R} with respect to the metric topology is Ramsey.*

Corollary 1.2.4 (The Abstract Silver Theorem, [46]). *Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. Every analytic subset of \mathcal{R} with respect to the metric topology is Ramsey.*

Corollary 1.2.5 ([46]). *Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. Every meager subset of \mathcal{R} is nowhere dense.*

Various subsets of the collection of finite approximations \mathcal{AR} of a given topological Ramsey space have the Ramsey property. If \mathcal{G} is any subset of \mathcal{AR} and $X \in \mathcal{R}$, we let $\mathcal{G}|X$ denote the collection of all $s \in \mathcal{G}$ such that $s \leq_{\text{fin}} X$.

Definition 1.2.6 ([46]). A family $\mathcal{F} \subseteq \mathcal{AR}$ of finite approximations is

- (1) *Nash-Williams* if $a \not\sqsubseteq b$ for all $a \neq b \in \mathcal{F}$;
- (2) *Sperner* if $a \not\leq_{\text{fin}} b$ for all $a \neq b \in \mathcal{F}$;
- (3) *Ramsey* if for every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ and every $X \in \mathcal{R}$, there are $Y \leq X$ and $i \in \{0, 1\}$ such that $\mathcal{F}_i|Y = \emptyset$.

The following result can be found as Theorem 5.17 in [46].

Theorem 1.2.7 (The Abstract Nash-Williams Theorem, [46]). *Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. Then every Nash-Williams family of finite approximations is Ramsey.*

Definition 1.2.8 ([46]). Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. A family $\mathcal{F} \subseteq \mathcal{AR}$ is a *front (barrier)* on $[\emptyset, X]$ if

- (1) For each $Y \in [\emptyset, X]$, there is an $a \in \mathcal{F}$ such that $a \sqsubset Y$, and
- (2) \mathcal{F} is Nash-Williams (Sperner).

Remark 1.2.9. Any front on a topological Ramsey space is Nash-Williams; hence, Ramsey, by Theorem 1.2.7.

Corollary 1.2.10 (The Abstract Ramsey Theorem). *Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. Then for each $n < \omega$, the family of finite approximations \mathcal{AR}_n is Ramsey.*

Theorem 1.2.11 (The Abstract Galvin Lemma, [46]). *Suppose (\mathcal{R}, \leq, r) is a closed subspace of \mathcal{AR}^ω and satisfies **A.1**, **A.2**, **A.3** and **A.4**. Let \mathcal{F} be an arbitrary family of finite approximation and \mathcal{F}_0 be the collection of all \leq -minimal elements of \mathcal{F} . Then either there exists $X \in \mathcal{R}$ such that $\mathcal{F}|X = \emptyset$ or $\mathcal{F}_0|X$ is a barrier on X .*

In the next section, we look at extensions of the Ramsey Theorem where partitions into infinitely many parts are allowed.

1.3 Canonical Ramsey theory

In 1950, Erdős and Rado extended the infinite Ramsey Theorem, Theorem 1.1.3, allowing for partitions into infinitely many parts. For each positive integer k and each partition P of $[\omega]^k$, there is an equivalence relation E_P on $[\omega]^k$ defined by aE_Pb if and only if a and b are in the same piece of the partition. Thus, the equivalence classes of E_P form the pieces of the partition P of $[\omega]^k$. On the other hand, if E is an equivalence relation on $[\omega]^k$ then the equivalence classes form a partition which we denote by P_E . These two constructions are inverses to one another, in the

following sense, if P is a partition of $[\omega]^k$ and E is an equivalence relation on $[\omega]^k$, then $P_{E_P} = P$ and $E_{P_E} = E$.

Example 1.3.1. Let E denote the equality relation on ω . The partition P_E is the partition of ω into singleton sets. Notice there is no infinite set X such that all of its elements are contained in one piece of P_E . Thus, $\omega \not\rightarrow (\omega)_\omega^1$. In fact, any partition P of ω into finite parts will also have this property. On the other hand, for any partition P of ω into finite parts, there exists an infinite set X such that $E_P \upharpoonright X$ is the equality relation on X .

The previous example shows that the conclusion of the one-dimensional Ramsey Theorem for arbitrary partitions fails. However, the example also implies that for any equivalence relation E on ω and infinite set $Y \subseteq \omega$ there exists an infinite $X \subseteq Y$ such that $E \upharpoonright X$ is either the equality relation on X or the trivial relation on X (The trivial relation on X denotes the relation with exactly one equivalence class). It is possible to prove similar results for higher dimensions. The equality and trivial relations are examples of canonical equivalence relations. In general, for dimension k we can identify finitely many canonical relations on $[\omega]^k$ such that any equivalence relation can be restricted to some canonical relation on $[\omega]^k$.

Definition 1.3.2. Let k be a positive integer. Each $I \subseteq \{0, 1, \dots, k-1\}$ gives rise to a *canonical equivalence relation* on $[\omega]^k$ given by:

$$\{x_0, x_1, \dots, x_{k-1}\} E_I \{y_0, y_1, \dots, y_{k-1}\} \text{ if and only if } (\forall i \in I) x_i = y_i, \quad (1.3.1)$$

where the k -element sets $\{x_0, x_1, \dots, x_{k-1}\}$ and $\{y_0, y_1, \dots, y_{k-1}\}$ are taken to be in increasing order.

Theorem 1.3.3 (The Erdős-Rado Theorem, [20]). *For every $k \geq 1$, every equivalence relation E on $[\omega]^k$ and every infinite set Y , there is an infinite subset X of Y and an index set $I \subseteq \{0, 1, \dots, k-1\}$ such that $E \upharpoonright [X]^k = E_I \upharpoonright [X]^k$.*

Remark 1.3.4. Let k be a positive integer. The k -dimensional Ramsey Theorem follows from the Erdős-Rado Theorem since the only canonical relation with finitely many equivalence classes is the trivial relation. In other words, if P is a partition of $[\omega]^k$ into finitely many pieces, then for any finite set Y there is an infinite set $X \subseteq Y$ such that $[Y]^k$ is contained in one piece of the partition, *i.e.* $E_p \upharpoonright [Y]^k$ is the trivial relation on $[Y]^k$.

In 1982, Pudlák and Rödl proved a generalization of the Erdős-Rado theorem, Theorem 1.3.3. The result extends Theorem 1.3.3 from equivalence relations on the k -element subsets of ω to equivalence relations on general barriers.

Definition 1.3.5. If \mathcal{F} is a front, a mapping $\varphi : \mathcal{F} \rightarrow \mathbb{N}$ is called *irreducible* if it is (a) *inner*, meaning that $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$, and (b) *Nash-Williams*, meaning that for each $a, b \in \mathcal{F}$, $\varphi(a) \not\subseteq \varphi(b)$. If E is an equivalence relation on \mathcal{F} and X in an infinite subset of ω , we say that E is *represented by φ on $\mathcal{F}|X$* if and only if for all $s, t \in \mathcal{F}|X$,

$$sEt \Leftrightarrow \varphi(s) = \varphi(t). \quad (1.3.2)$$

Theorem 1.3.6 (The Pudlák-Rödl Theorem, [40]). *For every barrier \mathcal{F} on ω , every equivalence relation E on \mathcal{F} , and every infinite set Y , there is an infinite $X \subseteq Y$ such that E is represented by an irreducible mapping on $\mathcal{F}|X$.*

Remark 1.3.7. The Erdős-Rado Theorem and the Nash-Williams Theorem follow from the Pudlák-Rödl Theorem. The Erdős-Rado Theorem follows since for each

positive integer k , the only equivalence relations on $[\omega]^k$ representable by irreducible mappings are the canonical relations. The Nash-Williams Theorem follows since the only equivalence relation representable by an irreducible mapping with only finitely many equivalence classes is the trivial relation.

In Chapter 5, we extend both the Erdős-Rado Theorem and the Pudlak-Rödl Theorem to a topological Ramsey space we define in the next chapter. In the next section, we look at an application of the Pudlak-Rödl Theorem to the Tukey theory of ultrafilters.

1.4 An application to the Tukey theory of ultrafilters

Research into the Tukey theory of ultrafilters has made use of results from set theory, topology and Ramsey theory. For example, see [16], [12], [26], [37], [41], [17], [18] and [15]. For a detailed introduction to this area of research see the recent survey paper [13] of Dobrinen.

In this section, we state an application of the Pudlak-Rödl Theorem to the Tukey theory of ultrafilters due to Todorčević. First we introduce the basic concepts of the Tukey theory of ultrafilters. Suppose that \mathcal{U} and \mathcal{V} are ultrafilters on the base sets X and Y respectively. A function f from \mathcal{U} to \mathcal{V} is *cofinal* if every cofinal subset of (\mathcal{U}, \supseteq) is mapped by f to a cofinal subset of (\mathcal{V}, \supseteq) . In other words, f maps filter bases of \mathcal{U} to filter bases of \mathcal{V} . We say that \mathcal{V} is *Tukey reducible to \mathcal{U}* and write $\mathcal{V} \leq_T \mathcal{U}$ if there exists a cofinal map $f : \mathcal{U} \rightarrow \mathcal{V}$. If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ then we write $\mathcal{V} \equiv_T \mathcal{U}$ and say that \mathcal{U} and \mathcal{V} are *Tukey equivalent*. The relation \equiv_T is an equivalence relation and \leq_T is a partial order on its equivalence classes. The equivalence classes are also called *Tukey types*.

When restricted to ultrafilters, the Tukey reducibility relation is a coarsening of the Rudin-Keisler reducibility relation. We say that \mathcal{V} is *Rudin-Keisler reducible to* \mathcal{U} and write $\mathcal{V} \leq_{RK} \mathcal{U}$ if there exists a map $f : X \rightarrow Y$ such that $\mathcal{V} = \{Z \subseteq Y : f^{-1}(Z) \in \mathcal{U}\}$. We denote the ultrafilter $\{Z \subseteq Y : f^{-1}(Z) \in \mathcal{U}\}$ by $f(\mathcal{U})$. If $h(\mathcal{U}) = \mathcal{V}$ then the map sending $X \in \mathcal{U}$ to $h''X \in \mathcal{V}$ witnesses Tukey reducibility. Thus, if $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{V} \leq_T \mathcal{U}$. This leads to the following question:

Question 1.4.1. For a given ultrafilter \mathcal{U} , what is the structure of the Rudin-Keisler ordering of the ultrafilters Tukey reducible to \mathcal{U} ?

For certain ultrafilters \mathcal{U} , the notion of a basic Tukey reduction can be used in conjunction with the canonical Ramsey theory to identify the Rudin-Keisler structure of the ultrafilters Tukey reducible to \mathcal{U} . We introduce some notation in order to describe the concept of a basic Tukey reduction. Note that the family of characteristic functions of elements of $[\omega]^\omega$ is closed in the subspace topology it inherits from 2^ω . A sequence $(X_n)_{n < \omega}$ of elements of $[\omega]^\omega$ converges to an element $X \in [\omega]^\omega$ if and only if for each $k < \omega$ there is an $N < \omega$ such that for each $n \geq N$, $r_k(X_n) = r_k(X)$. A function $f : [\omega]^\omega \rightarrow \mathcal{P}(\omega)$ is *continuous* if and only if for each convergent sequence $(X_n)_{n < \omega}$ in $[\omega]^\omega$ with $X_n \rightarrow X$, we also have $f(X_n) \rightarrow f(X)$ in the topology obtained by identifying $\mathcal{P}(\omega)$ with 2^ω . A function $f : \mathcal{U} \rightarrow \mathcal{V}$ is said to be *continuous* if it is continuous with respect to the topologies on \mathcal{U} and \mathcal{V} taken as subspaces of 2^ω .

Definition 1.4.2 (Dobrinen, [12]). Assume that \mathcal{U} a nonprincipal ultrafilter on ω . \mathcal{U} has *basic Tukey reductions* if whenever \mathcal{V} is a nonprincipal ultrafilter on ω and $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map, there is an $X \in \mathcal{U}$, a continuous monotone map $f' : \mathcal{U} \rightarrow \mathcal{V}$ and a function $\check{f} : [\omega]^{<\omega} \rightarrow [\omega]^{<\omega}$ such that

1. $f \upharpoonright (\mathcal{U} \upharpoonright X)$ is continuous.
2. f' extends $f \upharpoonright (\mathcal{U} \upharpoonright X)$ to \mathcal{U} .
3. (a) For each $k < \omega$ and each $s \in [\omega]^{<\omega}$, if $\max(s) \leq k$ then $\check{f}(s) \subseteq k$,
 (b) $s \sqsubseteq t \in [\omega]^{<\omega}$ implies that $\check{f}(s) \sqsubseteq \check{f}(t)$,
 (c) For each $Y \in \mathcal{U}$, $f'(Y) = \bigcup_{k < \omega} \check{f}(r_k(Y))$, and
 (d) \check{f} is monotonic, that is, if $s, t \in [\omega]^{<\omega}$ with $s \subseteq t$, then $\check{f}(s) \subseteq \check{f}(t)$.

The next theorem of Dobrinen and Todorćević from [16] has become an important tool in the study of the Tukey structure of ultrafilters Tukey reducible to some p-point ultrafilter (see Section 4.1 for the definition of a p-point).

Theorem 1.4.3 ([16]). *If \mathcal{U} is a p-point ultrafilter on ω then \mathcal{U} has basic Tukey reductions.*

Definition 1.4.4. Let \mathcal{F} be a subset of $[\omega]^{<\omega}$ and \mathcal{U} be a subset of $[\omega]^\omega$. \mathcal{F} is a *front on \mathcal{U}* , if \mathcal{F} is Nash-Williams and for each $X \in \mathcal{U}$, there exists $s \in \mathcal{F}$ such that $s \sqsubseteq X$.

The next theorem, an adaptation of Proposition 5.5 of Dobrinen and Todorćević from [17] to selective ultrafilters, exhibits a connection between basic Tukey reductions and canonical Ramsey theory. If \mathcal{U} is an ultrafilter and \mathcal{F} is a front on \mathcal{U} then we let $\mathcal{U} \upharpoonright \mathcal{F}$ denote the set $\{\mathcal{F}|X : X \in \mathcal{U}\}$. In the next proof sketch, we tactically assume that if \mathcal{U} is a selective ultrafilter and \mathcal{F} is a front on \mathcal{U} then $\mathcal{U} \upharpoonright \mathcal{F}$ generates an ultrafilter on \mathcal{F} which we denote by $\langle \mathcal{U} \upharpoonright \mathcal{F} \rangle$.

Theorem 1.4.5 ([17]). *Suppose that \mathcal{U} is a selective ultrafilter and has basic Tukey reductions. If \mathcal{V} is a nonprincipal ultrafilter on ω such that $\mathcal{V} \leq_T \mathcal{U}$, then there is front \mathcal{F} on \mathcal{U} and function $g : \mathcal{F} \rightarrow \omega$ such that $\mathcal{V} = g(\langle \mathcal{U} \upharpoonright \mathcal{F} \rangle)$.*

Proof Sketch. If $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map then there is a monotone function $\check{f} : [\omega]^{<\omega} \rightarrow [\omega]^\omega$ and a continuous monotone cofinal map $f' : \mathcal{U} \rightarrow \mathcal{V}$ satisfying (1)-(3) in the definition of basic Tukey reduction. Let \mathcal{F} be the set of all $r_n(X)$ such that $X \in \mathcal{U}$ and n is minimal such that $\check{g}(r_n(X)) \neq \emptyset$. Define $g : \mathcal{F} \rightarrow \omega$ by $g(s) = \min(\check{f}(s))$, for $s \in \mathcal{F}$. Since $\mathcal{U} \upharpoonright \mathcal{F}$ forms an ultrafilter on \mathcal{F} , $g(\langle \mathcal{U} \upharpoonright \mathcal{F} \rangle)$ is an ultrafilter on ω . Since for each $X \in \mathcal{U}$, $g(\mathcal{F}|X) \subseteq f'(X)$ and f' is cofinal, we have $\mathcal{V} = g(\langle \mathcal{U} \upharpoonright \mathcal{F} \rangle)$. \square

The previous theorem gives an idea of one way that canonical Ramsey theory can be applied to the Tukey theory of ultrafilters; namely, analyzing the nonprincipal ultrafilters Tukey reducible to some selective ultrafilter \mathcal{U} is equivalent to analyzing infinite partitions of fronts on \mathcal{U} .

The first application of canonical Ramsey theory to the Tukey-types of ultrafilters was given by Todorćević in [41]. In [41], Todorćević uses the Pudlák-Rödl Theorem (see Theorem 1.3.6) to prove the next result.

Theorem 1.4.6 ([41]). *If \mathcal{U} is a selective ultrafilter and $\mathcal{V} \leq_T \mathcal{U}$, then \mathcal{V} must be Rudin-Keisler equivalent to some Fubini power of the ultrafilter \mathcal{U} .*

Remark 1.4.7. In terms of initial Tukey structures, this means that it is consistent with ZFC that the one-element partial order appears as an initial Tukey structure.

Dobrinen and Todorćević in [17] have also applied the canonical Ramsey theory to the Tukey theory of ultrafilters. Dobrinen and Todorćević in [17] develop a new canonical theory for the topological Ramsey space \mathcal{R}_1 (see Example 2.2.3 for definition of \mathcal{R}_1). Assuming the continuum hypothesis, Martin's axiom, or forcing with \mathcal{R}_1 using a certain partial order, it is shown that there is a subset $\mathcal{C} \subseteq \mathcal{R}_1$ such that

\mathcal{C} generates a p-point ultrafilter \mathcal{U}_1 on a countable base, there is a selective ultrafilter \mathcal{U}_0 on ω such that $\mathcal{U}_1 >_T \mathcal{U}_0$, and whenever \mathcal{V} is a nonprincipal ultrafilter on ω Tukey reducible to \mathcal{U}_1 then either $\mathcal{V} \equiv_T \mathcal{U}_1$ or $\mathcal{V} \equiv_T \mathcal{U}_0$. Thus, the Tukey structure of the nonprincipal ultrafilters Tukey reducible to \mathcal{U}_1 is isomorphic to the two-element Boolean algebra. Therefore, assuming the continuum hypothesis, Martin's axiom, or forcing with \mathcal{R}_1 using a certain partial order, the two element Boolean algebra appears as an initial structure in Tukey types of ultrafilters. Furthermore, it is proved in [17] that the Rudin-Keisler structure of the p-points Tukey reducible to \mathcal{U}_1 is isomorphic to the partial order (ω, \leq) . Example 5.4.10 provides a detailed outline of how the canonical Ramsey theory for \mathcal{R}_1 can be used in conjunction with basic Tukey reductions to obtain these results. This example forms the prototype for arguments and results in Chapter 5.

Chapter 2

Selective & Ramsey ultrafilters for topological Ramsey spaces generated by trees

In this chapter, we investigate generalizations of the notions of selective and Ramsey ultrafilters on ω to topological Ramsey spaces constructed from trees on ω . These generalizations, in the context of any topological Ramsey space, were introduced by Mijares in [35] and studied by the author in [48]. In this chapter, we derive various equivalent characterizations of the generalizations of selective and Ramsey ultrafilters. In the next chapter, we use these characterizations to give examples of topological Ramsey spaces constructed from trees which support ultrafilters that are selective but not Ramsey for the topological Ramsey space. In the final chapter of this thesis, we use the generalization of the notion of Ramsey to show that it is consistent with ZFC that the four element Boolean algebra appears as an initial structure in the Tukey type of p-point ultrafilters.

In Section 2.1, we introduce the notions of Ramsey and selective ultrafilters on ω . In Section 2.2, we provide a framework for constructing topological Ramsey spaces from certain trees on ω . In the same section, we reproduce the definitions of Dobrinen and Todorćević from [17] and [18] of the important examples \mathcal{R}_n , for $n < \omega$. In Section 2.3, we introduce the generalizations of selective and Ramsey ultrafilters for spaces constructed from trees. In Section 2.4, we explore the question of the existence of Ramsey ultrafilters for spaces constructed from trees. In Section 2.4, we prove that for a certain family of trees on ω , forcing with the space using almost-reduction is equivalent to forcing with \mathcal{R}_n using almost-reduction for some $n < \omega$. In the final two sections of this chapter, we introduce and identify equivalent formulations of the generalizations of the notions of selective and Ramsey ultrafilters introduced by Mijares in [35] restricted to the spaces from Section 2.2.

2.1 Selective and Ramsey ultrafilters on ω

A nonprincipal ultrafilter on ω is Ramsey if it satisfies a localized version of the infinite Ramsey Theorem. According to Booth (see [5] p. 23), the existence of Ramsey ultrafilters, assuming the continuum hypothesis, was first shown by Galvin.

Definition 2.1.1. Let \mathcal{U} be a nonprincipal ultrafilter on ω . We say that \mathcal{U} is *Ramsey* if for each positive integer k and each partition of the k -element subsets of ω into finitely many parts, there is a set $X \in \mathcal{U}$ all of whose k -element subsets lie in one part of the partition.

A nonprincipal ultrafilter on ω is selective if every countable collection of sets can be diagonalized in \mathcal{U} . Booth introduced the notion of a selective ultrafilter in [5].

Definition 2.1.2. Let \mathcal{U} be a nonprincipal ultrafilter on ω . If $i < \omega$ and $A = \{a_0, a_1, a_2, \dots\}$ is an infinite subset of ω enumerated in increasing order, then we let

$$A/i = A \setminus \{a_0, a_1, \dots, a_{i-1}\}. \quad (2.1.1)$$

\mathcal{U} is *selective*, if for each decreasing sequence $A_0 \supseteq A_1 \supseteq \dots$ of members of \mathcal{U} there exists $A = \{a_0, a_1, \dots\} \in \mathcal{U}$ enumerated in increasing order such that for all $i < \omega$,

$$A/i \subseteq A_i. \quad (2.1.2)$$

The next theorem, due to Kunen, shows that the notions of selective and Ramsey are equivalent. Item 3 below is equivalent to \mathcal{U} being minimal with respect to the Rudin-Keisler ordering.

Theorem 2.1.3 (Kunen, [5]). *Let \mathcal{U} be an ultrafilter on ω . The following conditions are equivalent:*

1. \mathcal{U} is selective.
2. \mathcal{U} is Ramsey.
3. Every function on ω is constant or one-to-one on some set in \mathcal{U} .
4. For each decreasing sequence $A_0 \supseteq A_1 \supseteq \dots$ of members of \mathcal{U} there exists $A = \{a_0, a_1, \dots\} \in \mathcal{U}$ enumerated in increasing order such that for all $i < \omega$, $A/i \subseteq A_{a_i}$.

Generalizations of the previous theorem have been studied in many contexts. For example, the notions of selective coideal (see [33]) and semiselective coideals (see [23]) have been shown to also satisfy similar Ramsey properties. In [35], Mijares

generalizes the notion of selective ultrafilter on ω to a notion of selective ultrafilter on an arbitrary topological Ramsey space \mathcal{R} . Mijares also generalizes the notion of Ramsey ultrafilter on ω to a notion of Ramsey ultrafilter for \mathcal{R} and shows that if an ultrafilter is Ramsey for \mathcal{R} then it is also selective for \mathcal{R} . If one takes \mathcal{R} to be the Ellentuck space then the two generalizations reduce to the concepts of selective and Ramsey ultrafilter on ω . The theorem of Kunen above shows that the notions of selective for the Ellentuck space and Ramsey for the Ellentuck space are equivalent. This leads to the following question asked by Dobrinen about the generalizations from selective and Ramsey to arbitrary topological Ramsey spaces.

Question 2.1.4. For any given topological Ramsey space \mathcal{R} , are the notions of selective for \mathcal{R} and Ramsey for \mathcal{R} equivalent?

In the next chapter, we show that there are topological spaces \mathcal{R} such that, assuming continuum hypothesis, Martin's axiom, or forcing with \mathcal{R} using almost reduction, there are selective for \mathcal{R} ultrafilters which are not Ramsey for \mathcal{R} .

2.2 Topological Ramsey spaces generated by trees

In this section, in order to avoid repeating similar definitions, we introduce a framework for constructing topological Ramsey spaces from certain trees on ω . For each set X , $X^{<\omega}$ denotes the collection of all finite sequences of elements of X . For each finite sequence s , we let $|s|$ denote the length of s . For each $i < |s|$, $\pi_i(s)$ denotes the sequence of the first $i + 1$ elements of s and s_i denotes the i^{th} element of the sequence. For each pair of sequences s and t , we say that s is an *initial segment* of t and write $s \sqsubseteq t$ if there exists $i \leq |t|$ such that $s = \pi_i(t)$.

The *closure* of $T \subseteq X^{<\omega}$ (denoted by $cl(T)$) is the set of all initial segments of elements of T . A subset T of $X^{<\omega}$ is a *tree* on X , if $cl(T) = T$. A *maximal node* of T , is a sequence s in T such that for each $t \in T$, $s \sqsubseteq t \Rightarrow s = t$. The *body* of T (denoted by $[T]$) is the set of all maximal nodes of T . The *height* of T is the smallest ordinal greater than or equal to the length of each element of T .

Let k be a positive integer. The *lexicographical order* of $(\omega^k)^{<\omega}$ is defined as follows: s is lexicographically less than or equal to t if and only if $s \sqsubseteq t$ or $|s| = |t|$ and the least i on which s and t disagree, $s_i \leq t_i$ where \leq is taken to be the product order on ω^k . If S and T are trees on ω^k , then S is *isomorphic* to T if there exists a bijection $h : S \rightarrow T$ which preserves the lexicographic ordering. A *subtree* of T is a tree S such that $S \subseteq T$. Given two trees S and T on ω^k , we let $\binom{T}{S}$ denote the set of all subtrees of T that are isomorphic to S . If S is a subtree of T then we will write $S \leq T$. Figure 2.1 gives an example of two trees on ω that are not isomorphic. Notice that the two trees in Figure 2.1 are drawn in lexicographically increasing order from bottom-to-top and from left-to-right.

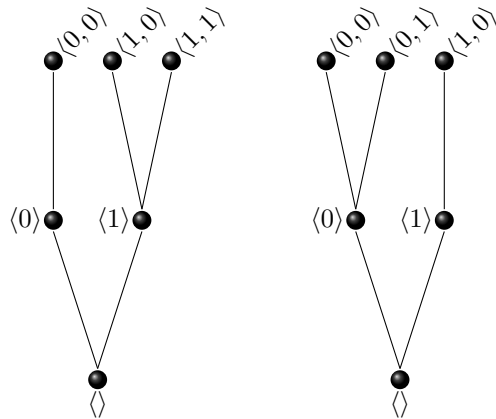


Figure 2.1: Two nonisomorphic trees on ω

For each positive integer k and trees S, T and U on ω^k , the partition relation

$$T \rightarrow (S)^U \quad (2.2.1)$$

means that for each partition of $\binom{T}{U}$ into two parts there exists $V \in \binom{T}{S}$ such that $\binom{V}{U}$ lies in one part of the partition. Figure 2.2 gives an example of trees which satisfy a partition relation; it is equivalent to the relation, $6 \rightarrow (3)^2$.

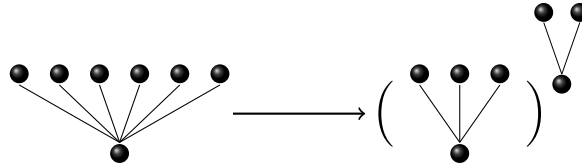


Figure 2.2: Example of a partition relation among trees

Definition 2.2.1 ($(\mathcal{R}(T), \leq, r)$). Suppose that k is a positive integer. Let T be a tree on ω^k , *i.e.* T is a collection of sequences of k -tuples of natural numbers closed under initial segments. Assume that T has the following properties:

1. For all $s, t \in [T]$, $|s| = |t|$ and
2. $\pi_0''[T] = \{\langle (n, \dots, n) \rangle \in \omega^k : n < \omega\}$.

Let $\mathcal{R}(T)$ denote the set of all subtrees of T isomorphic to T , *i.e.* $\binom{T}{T}$.

Since each $S \in \mathcal{R}(T)$ is isomorphic to T there exists a strictly increasing sequence $(k_i^S)_{i < \omega}$ such that

$$\pi_0''[S] = \{\langle (k_i^S, \dots, k_i^S) \rangle : i < \omega\}.$$

For each $i < \omega$, let

$$S(i) = cl(\{s \in [S] : \pi_0(s) = \langle (k_i^S, \dots, k_i^S) \rangle\}) \quad (2.2.2)$$

$$r_i(S) = \bigcup_{j < i} S(j). \quad (2.2.3)$$

Let $\mathcal{AR}(T) = \bigcup_{i < \omega} \{r_i(S) : S \in \mathcal{R}(T)\}$ and define $r : \omega \times \mathcal{R}(T) \rightarrow \mathcal{AR}(T)$ by letting $r(i, S) = \bigcup_{j < i} S(j)$. For $S, S' \in \mathcal{R}(T)$, $S \leq S'$ if and only if S is subtree of S' . For $S, S' \in \mathcal{R}(T)$ *almost-reduction* is defined as follows: $S \leq^* S'$ if and only if there exists $i < \omega$ such that $S \setminus r_i(S) \subseteq S'$.

Example 2.2.2. Let $T_0 = \{\langle \rangle, \langle n \rangle : n < \omega\}$. Then T_0 is a tree on ω and for all $s, t \in [T_0]$, $|s| = |t| = 1$ and $\pi_0''[T_0] = \{\langle n \rangle : n < \omega\}$. The space $(\mathcal{R}(T_0), \leq, r)$ is identical to the Ellentuck space. Figure 2.3 gives a graph of the tree T_0 .

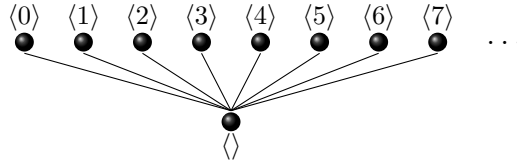


Figure 2.3: Graph of T_0

Example 2.2.3. In this example we review the definition of the triple (\mathcal{R}_1, \leq, r) defined by Dobrinen and Todorćević in [17]. Technically, the space \mathcal{R}_1 is defined differently but is equivalent to the definition given here. The construction of \mathcal{R}_1 in [17] is a thinned version of a forcing of Laflamme in [31] which uses forcing to adjoin a weakly-Ramsey ultrafilter satisfying complete combinatorics over $\text{HOD}(\mathbb{R})$. Dobrinen and Todorćević introduced the space in order to develop its canonical Ramsey

theory and apply the canonical theory to the Tukey theory of ultrafilters. For each $i < \omega$, let

$$T_1(i) = \left\{ \langle \rangle, \langle i \rangle, \langle i, j \rangle : \frac{i(i+1)}{2} \leq j < \frac{(i+1)(i+2)}{2} \right\} \quad (2.2.4)$$

Let $T_1 = \bigcup_{i < \omega} T_1(i)$. Note that T_1 is a tree on ω and for all $s, t \in [T_1]$, $|s| = |t| = 2$ and $\pi_0''[T_1] = \{\langle n \rangle : n < \omega\}$. Let (\mathcal{R}_1, \leq, r) denote the triple $(\mathcal{R}(T_1), \leq, r)$. Figure 3.1 includes a graph of the tree T_1 . Theorem 3.9 of Dobrinen and Todorćević in [17] shows that \mathcal{R}_1 forms a topological Ramsey space.

Theorem 2.2.4 ([17]). *(\mathcal{R}_1, \leq, r) satisfies A.1-A.4 and forms a topological Ramsey space.*

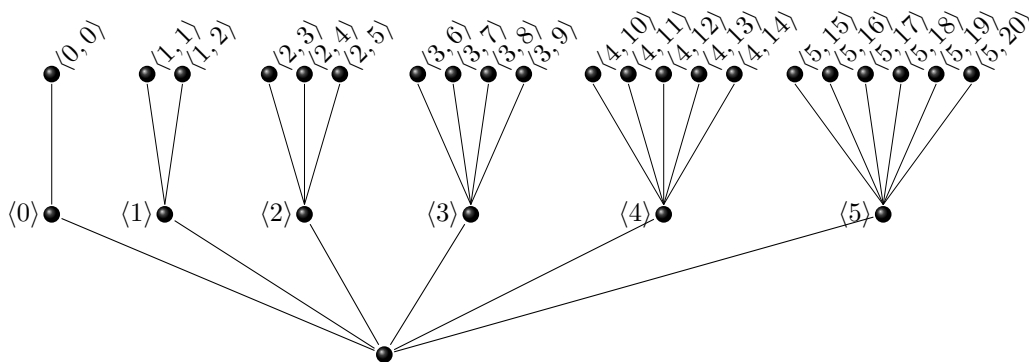


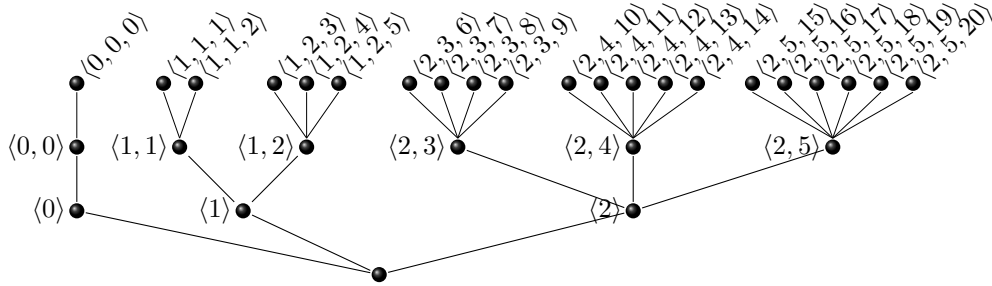
Figure 2.4: Graph of T_1

Example 2.2.5. In this example we define the triples (\mathcal{R}_n, \leq, r) for $1 < n < \omega$. These spaces were first defined by Dobrinen and Todorćević in [18]. Technically, the space \mathcal{R}_n is defined differently but is equivalent to the definition given here. The construction of \mathcal{R}_n in [17] is a thinned down version of a forcing of Laflamme in [31] which uses forcing to adjoin a $(n + 1)$ -Ramsey ultrafilter having exactly

n Rudin-Keisler predecessors, a linearly ordered chain of p -points. Dobrinen and Todorčević introduced the space in order to develop its canonical Ramsey theory and apply the canonical theory to the Tukey theory of ultrafilters. Assume n is a positive integer and T_1, T_2, \dots, T_n have been defined. For each $i < \omega$, let

$$T_{n+1}(i) = \left\{ \langle \rangle, \langle i \rangle, \langle i \rangle \frown s : s \in T_n(j) \ \& \ \frac{i(i+1)}{2} \leq j < \frac{(i+1)(i+2)}{2} \right\}. \quad (2.2.5)$$

Let $T_{n+1} = \bigcup_{i < \omega} T_{n+1}(i)$ and $(\mathcal{R}_{n+1}, \leq, r)$ denote the triple $(\mathcal{R}(T_{n+1}), \leq, r)$. Figure 2.5 includes a graph of the tree T_2 . The next result is Theorem 3.23 of Dobrinen



We use the next result in Chapter 3 to construct more trees on ω which give rise to topological Ramsey spaces.

Corollary 2.2.8. *Let l be a positive integer. For each pair of positive integers k and n with $k \leq n$ there exists $m < \omega$ such that*

$$r_m(T_l) \rightarrow (r_n(T_l))^{r_k(T_l)}. \quad (2.2.6)$$

Example 2.2.9 $((\mathcal{H}^2, \leq, r))$. Dobrinen constructed the space \mathcal{H}^2 to investigate classification theorems similar to those in [17] and [18] for \mathcal{H}^2 . The space was inspired by the topological Ramsey space \mathcal{R}_1 constructed in [17], Theorem 9 of Blass in [3] which shows that Martin's axiom implies that there is a p-point with two Rudin-Keisler incomparable predecessors, and Theorem 57 of Dobrinen from the joint paper [16] with Todorćević which shows that Martin's axiom implies that there is a p-point with two Tukey incomparable predecessors. In particular, the new construction modifies \mathcal{R}_1 in such a way that for each element X in the resulting space and each $n < \omega$, X contains an n -square (by n -square we mean subset of $\omega \times \omega$ of the form $X \times Y$ where X and Y are both n -element subsets of ω) as in the proof of Theorem 9 in [3]. Let $T_1 \otimes T_1$ (see Section 3.3 for the definition of \otimes) denote the following infinite tree on ω^2 of height 2.

$$T_1 \otimes T_1 = cl\left(\bigcup_{n < \omega} \left\{ \langle (n, n), (i, j) \rangle : \frac{n(n+1)}{2} \leq i, j < \frac{(n+1)(n+2)}{2} \right\}\right). \quad (2.2.7)$$

Let (\mathcal{H}^2, \leq, r) denote the triple $(\mathcal{R}(T_1 \otimes T_1), \leq, r)$. In Section 5.1, we prove that (\mathcal{H}^2, \leq, r) forms a topological Ramsey space. Figure 2.6 includes a graph of the tree $T_1 \otimes T_1$ used to generate \mathcal{H}^2 with the final rows of each subtree labeled.

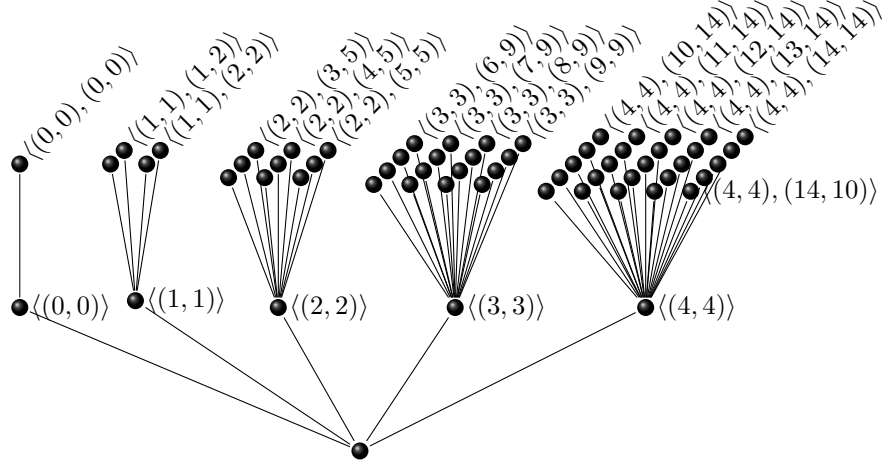


Figure 2.6: Graph of the tree $T_1 \otimes T_1$

2.3 Selective and Ramsey ultrafilters for $\mathcal{R}(T)$

In this section, following Dobrinen and Todorćević in [17] and [18], we introduce a generalization of the notion of selective ultrafilter to triples built from trees on ω . This generalization is equivalent to the notion of selective for \mathcal{R} introduced by Mijares in [35] when restricted to spaces of the form $\mathcal{R}(T)$. The generalization of Ramsey we use is exactly the generalization of Ramsey for topological Ramsey spaces, introduced by Mijares in [35], restricted to triples built from trees on ω . These ultrafilters will be the primary objects of study in chapters 3, 4 and 5.

Definition 2.3.1. Let k be a positive integer and T be a tree on ω^k , i.e. T is a collection of finite sequences of k -tuples of natural numbers. Suppose that $(\mathcal{R}(T), \leq, r)$ satisfies A.1-A.4 and forms a topological Ramsey space. Let \mathcal{U} be an ultrafilter on $[T]$.

1. We say that \mathcal{U} is *generated* by $\mathcal{G} \subseteq \mathcal{R}(T)$, if $\{[S] : S \in \mathcal{G}\}$ is cofinal in (\mathcal{U}, \supseteq) .

2. An ultrafilter \mathcal{U} generated by $\mathcal{G} \subseteq \mathcal{R}(T)$ is *selective for $\mathcal{R}(T)$* if and only if for each decreasing sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ of elements of \mathcal{G} , there exists another $S \in \mathcal{G}$ such that for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$.
3. An ultrafilter \mathcal{U} generated by $\mathcal{G} \subseteq \mathcal{R}(T)$ is *Ramsey for $\mathcal{R}(T)$* if and only if for each $i < \omega$ and each partition of $\binom{T}{r_i(T)}$ into two parts there exists $S \in \mathcal{G}$ such that $\binom{S}{r_i(T)}$ lies in one part of the partition.

Example 2.3.2. Recall that if $T_0 = \{\langle \rangle, \langle n \rangle : n < \omega\}$ then the space $(\mathcal{R}(T_0), \leq, r)$ is identical to the Ellentuck space. The maximal nodes of T_0 are in one-to-one correspondence with the natural numbers via the map $\langle n \rangle \mapsto n$, $n < \omega$. Thus the ultrafilters on $[T_0]$ are in one-to-one correspondence with the ultrafilters on ω . For a given \mathcal{U} on $[T_0]$, let $\hat{\mathcal{U}}$ denote the corresponding ultrafilter on ω .

It is straightforward to show that, \mathcal{U} is a selective for $\mathcal{R}(T_0)$ if and only if for each decreasing sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ of elements of $\hat{\mathcal{U}}$ there exists $A = \{a_0, a_1, a_2, \dots\} \in \hat{\mathcal{U}}$ enumerated in increasing order such that for each $n < \omega$,

$$A \setminus \{a_0, a_1, \dots, a_{n-1}\} \subseteq A_n.$$

In other words, \mathcal{U} is a selective for $\mathcal{R}(T_0)$ if and only if $\hat{\mathcal{U}}$ is a selective ultrafilter.

Along similar lines one can also show that \mathcal{U} is a Ramsey for $\mathcal{R}(T_0)$ if and only if $\hat{\mathcal{U}}$ is a Ramsey ultrafilter. Thus, by Theorem 2.1.3, \mathcal{U} is a selective for $\mathcal{R}(T_0)$ if and only if \mathcal{U} is a Ramsey for $\mathcal{R}(T_0)$.

In the next section, we show that if $\mathcal{R}(T)$ generates a topological Ramsey space then it is consistent with ZFC that a Ramsey for $\mathcal{R}(T)$ ultrafilter on $[T]$ exists. In the next chapter, we prove that for certain trees T , it is consistent with ZFC that that a selective for $\mathcal{R}(T)$ ultrafilter exists which is not Ramsey for $\mathcal{R}(T)$.

2.4 Forcing with $\mathcal{R}(T)$ using almost-reduction

Before stating the main results of this section, we introduce the methods and basic facts surrounding the set-theoretic technique of forcing. Forcing was introduced by Cohen in [10] and [11] in order to prove that the continuum hypothesis was independent of the axioms of ZFC. For a more thorough introduction to the technique of forcing and some of its applications see the texts, [30],[27] and [28].

The primary purpose of forcing is to extend a set theoretic universe V , the *ground model*, by adjoining a generic set G . Generally the generic extension, denoted by $V[G]$, is used to show that some statement in the language of ZFC is independent of the axioms of ZFC. In the theory of forcing, partially ordered sets (P, \leq) are referred to as *notions of forcing*.

Notation 2.4.1. Let (P, \leq) be a fixed forcing notion. We call the elements of P forcing conditions. If p and q are forcing conditions then we say q is stronger than p if $q \leq p$.

Definition 2.4.2 ([27]). Let (P, \leq) be a fixed forcing notion. A set Δ of conditions is *dense* if for each $p \in P$ there exists $q \in \Delta$ such that $q \leq p$.

A set G of conditions is *generic* if the following two conditions hold,

1. G is a *filter* on (P, \leq) ; that is,
 - (a) if $q \in G$ and $q \leq p$ then $p \in G$,
 - (b) if $p, q \in G$ then there exist $r \in G$ such that $r \leq p, q$.
2. For every dense set $\Delta \subseteq P$ in V , $G \cap \Delta \neq \emptyset$.

A model V of ZFC is *transitive* if $x \in V$ implies $x \subseteq V$. For the reader unfamiliar with forcing we state the two main theorems about generic extensions. In all

of our applications of forcing we only make use of part (1) and part (2) of Theorem 2.4.3.

Theorem 2.4.3 (The generic model theorem, [27]). *Let V be a transitive model of ZFC and let (P, \leq) be a forcing notion in V . If G is a set of generic conditions, then there exists a transitive model $V[G]$ such that:*

- (1) $V[G]$ is a model of ZFC.
- (2) $V \subseteq V[G]$.
- (3) $G \in V[G]$.
- (4) If V' is a transitive model of ZF such that $V \subseteq V'$ and $G \in V'$, then $V[G] \subseteq V'$.

The model $V[G]$ is known as a *generic extension* of V .

For each forcing notion there is an associated *forcing language*. The forcing language has names for each element of $V[G]$. We follow the convention of denoting the name of x in $V[G]$ by \dot{x} in V . For example, \dot{G} denotes the name for G in V . The forcing relation \Vdash is a relation defined in V between conditions and sentences of the forcing language. If $p \Vdash \sigma$ then we say that p *forces* σ . The next theorem describes the connection between truth in $V[G]$ and the forcing relation in V .

Theorem 2.4.4 (The forcing theorem, [27]). *Let (P, \leq) be a notion of forcing in V . If σ is a sentence in the forcing language, then for every generic G , $V[G] \models \sigma$ if and only if there exists $p \in G$, $p \Vdash \sigma$.*

The next result addresses the existence of Ramsey ultrafilters for $\mathcal{R}(T)$. We omit its proof as it follows by applying Lemma 3.3 of Mijares in [35] to topological Ramsey spaces of the form $\mathcal{R}(T)$.

Theorem 2.4.5 (Mijares, [35]). *Let k be a positive integer and T be a tree on ω^k . Suppose that $(\mathcal{R}(T), \leq, r)$ satisfies **A.1-A.4** and forms a topological Ramsey space. Forcing with $(\mathcal{R}(T), \leq^*)$ adjoins no new elements of $(\mathcal{AR}(T))^\omega$, and if \mathcal{G} is a $(\mathcal{R}(T), \leq^*)$ -generic filter over some ground model V , then \mathcal{G} generates a Ramsey for $\mathcal{R}(T)$ ultrafilter in $V[\mathcal{G}]$.*

If two forcing notions (P, \leq_P) and (Q, \leq_Q) produce the same generic models then we say that (P, \leq_P) is *forcing equivalent* to (Q, \leq_Q) . More precisely, (P, \leq_P) is forcing equivalent to (Q, \leq_Q) , if for every $G \subseteq P$ which is P -generic over V , there exists $H \subseteq Q$ which is Q -generic over V such that $V[G] = V[H]$ and for every Q -generic H there is a P -generic G such that $V[H] = V[G]$. The next theorem is main result of this section.

Theorem 2.4.6. *Suppose that T is a tree on ω such that*

- (1) $||[T(0)]|| = 1$,
- (2) For all $s, t \in [T]$, $|s| = |t|$,
- (3) $\pi_0''[T] = \{\langle n \rangle : n < \omega\}$,
- (4) $(\mathcal{R}(T), \leq, r)$ satisfies **A.1-A.4** and forms a topological Ramsey space.

Then there exists $k < \omega$ such that $(\mathcal{R}(T), \leq^)$ is equivalent to (\mathcal{R}_k, \leq^*) . Figure 2.7 gives a partial graph of a typical tree on ω satisfying (1)-(4).*

Proof. Suppose that T is a tree on ω satisfying (1)-(4). The notion of a dense embedding provides a method for showing that two forcing notions are equivalent. We say that $p, q \in P$ are *incompatible* and write $p \perp q$ if there is no $r \in P$ such that $r \leq p$ and $r \leq q$.

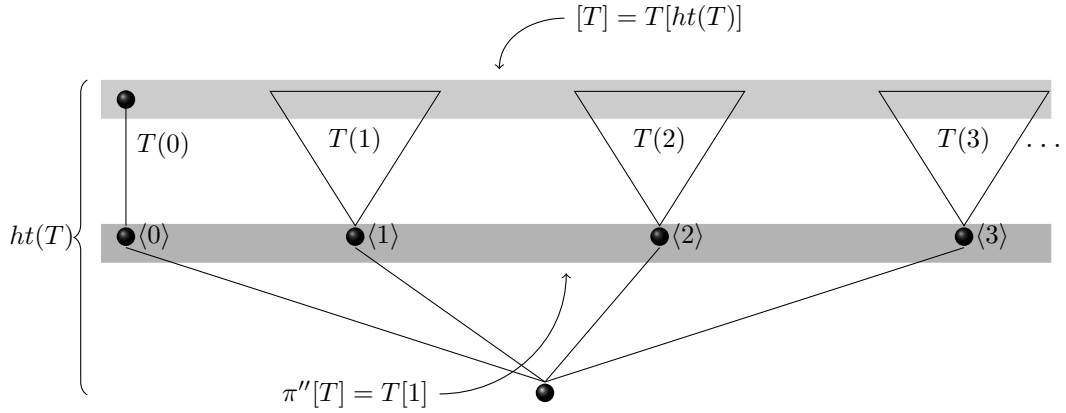


Figure 2.7: Graph of a typical tree satisfying (1)-(4)

Definition 2.4.7. Let (P, \leq_P) and (Q, \leq_Q) be two partial orders. A function $h : P \rightarrow Q$ is called a *dense embedding* if it satisfies the following:

1. For each $p, q \in P$, if $p \leq_P q$ then $h(p) \leq_Q h(q)$.
2. For each $p, q \in P$, if $p \perp_P q$ then $h(p) \perp_Q h(q)$.
3. $h''P$ is *dense* in (Q, \leq_Q) . That is, for each $q \in Q$ there exists $p \in P$ such that $h(p) \leq_Q q$.

Fact 2.4.8 ([27]). Let (P, \leq_P) and (Q, \leq_Q) be two partial orders. If there exists a dense embedding $h : P \rightarrow Q$, then (P, \leq_P) and (Q, \leq_Q) are equivalent.

Thus, it is enough to show that there exists $k < \omega$ and a dense embedding of $(\mathcal{R}(T), \leq^*)$ into (\mathcal{R}_k, \leq^*) .

Notation 2.4.9. For each tree S on ω satisfying (1)-(4) in the statement of Theorem 2.4.6 and each $i \leq ht(S)$ we let $S[i] = \{s \in S : |s| = i\}$. We associate to each tree S a finite sequence of functions $(f_1^S, f_2^S, \dots, f_{ht(S)-1}^S)$ such that for each

$0 < i < ht(S)$, $f_i^S : S[i] \rightarrow \omega$ and for each $s \in S[i]$,

$$f_i^S(s) = |S[i+1] \cap \pi_{i-1}^{-1}\{s\}|.$$

In other words, $f_i^S(s)$ denotes the number of immediate successors of the node s in the tree S .

The next three claims will be used to construct the dense embedding needed to complete the proof.

Definition 2.4.10. Suppose $0 < i < ht(T)$. We say that f_i^T is bounded on $T[i]$ if there exists $n < \omega$ such that for all $s \in T[i]$, $f_i^T(s) \leq n$. If f is not bounded on $T[i]$ then we say that f_i^T is unbounded on $T[i]$.

Claim 1. For each $0 < i < ht(T)$, either for all $s \in T[i]$, $f_i^T(s) = 1$ or f_i^T is unbounded on $T[i]$.

Proof. Let $0 < i < ht(T)$ be given and suppose that f_i^T is bounded on $T[i]$. Then there exists a $n < \omega$ such that for all $s \in T[i]$, $f_i^T(s) \leq n$. For each $s \in T[i]$, let $\{t_0^s, t_1^s, \dots, t_{f_i^T(s)-1}^s\}$ denote the lexicographically increasing enumeration of $T[i+1] \cap \pi_{i-1}^{-1}\{s\}$. For each $j < n$, let

$$\Pi_j = \{t \in \mathcal{AR}_1(T) : (\exists s \in T[i]) \pi_i(t) = t_j^s\}.$$

Since f_i^T is bounded by n on $T[i]$, it follows that $\{\Pi_0, \Pi_1, \dots, \Pi_{n-1}\}$ is a partition of $\mathcal{AR}_1(T)$. Recall that we are assuming that $\mathcal{R}(T)$ forms a topological Ramsey space; thus, there exists $S \in \mathcal{R}(T)$ and there exists $j < n$ such that $\mathcal{AR}_1(T) \upharpoonright S \subseteq \Pi_j$.

Toward a contradiction, suppose that there exists $s \in T[i]$ such that $f_i^T(s) > 1$.

Then there exists $s' \in S[i]$ such that $f_i^S(s') > 1$. Thus there exists $t, t' \in \mathcal{AR}_1(T) \upharpoonright S$ such that either $t \notin \Pi_j$ or $t' \notin \Pi_j$, which is a contradiction. Therefore for each $s \in T[i]$, $f_i^T(s) \leq 1$. Since T satisfies (2), $f_i^T(s) \neq 0$. Hence, either f_i^T is unbounded on $T[i]$ or for all $s \in T[i]$, $f_i^T(s) = 1$. \square

Claim 2. *There is a tree T' on ω satisfying (1)-(4) in the statement of Theorem 2.4.6 such that for each $0 < i < ht(T')$, $f_i^{T'}$ is unbounded on $T'[i]$ and $(\mathcal{R}(T), \leq^*)$ is isomorphic to $(\mathcal{R}(T'), \leq^*)$.*

Proof. Let $I = \{i < ht(T) : i > 0 \text{ \& } f_i^T \text{ is unbounded on } T[i]\}$. By Claim 1, for each positive integer i in $ht(T) \setminus I$ and each $s \in T[i]$, $f_i^T(s) = 1$. Let $k = |I|$ and let $\{i_0, i_1, \dots, i_{k-1}\}$ be the increasing enumeration of I . Define T' by letting

$$T' = cl(\{(t_0, t_{i_0}, t_{i_1}, \dots, t_{i_{k-1}}) \in \omega^{<\omega} : (t_0, t_1, \dots, t_{ht(T)-1}) \in [T]\}).$$

In other words, T' is obtained from T by removing the i^{th} levels of T where $f_i^T = 1$. Thus, for all $i < ht(T')$, $f_i^{T'}$ is unbounded on $T'[i]$. By the definition of T' , we find that T' satisfies (1)-(3). Hence, $(\mathcal{R}(T'), \leq, r)$ is well-defined.

Define the map $h : \mathcal{R}(T) \rightarrow \mathcal{R}(T')$ as follows: for each $S \in \mathcal{R}(T) \rightarrow \mathcal{R}(T')$ let

$$h(S) = cl(\{(t_0, t_{i_0}, t_{i_1}, \dots, t_{i_{k-1}}) \in \omega^{<\omega} : (t_0, t_1, \dots, t_{ht(T)-1}) \in [S]\}).$$

For example, $h(T) = T'$. The map h is both \leq and \leq^* order-preserving and bijective. The map h can be extended to any subtree of T and its extension preserves the approximations maps, that is, for all $S \in \mathcal{R}(T)$ and $n < \omega$, $h(r_n(S)) = r_n(h(S))$. Therefore $(\mathcal{R}(T'), \leq, r)$ satisfies **A.1-A.4** and forms a closed subspace of $(\mathcal{AR}(T))^\omega$.

By the abstract Ellentuck theorem, $\mathcal{R}(T')$ satisfies (4). Moreover, the map h is an order-isomorphism between $(\mathcal{R}(T), \leq^*)$ and $(\mathcal{R}(T'), \leq^*)$. \square

Claim 3. *Suppose that S is a tree on ω with $ht(S) > 1$ and satisfying (1)-(4) in the statement of Theorem 2.4.6. If for each $0 < i < ht(S)$, f_i^S is unbounded on $S[i]$, then $\binom{S}{T_{ht(S)-1}} \neq \emptyset$ and $\binom{T_{ht(S)-1}}{S} \neq \emptyset$.*

Proof. First, we proceed by induction on $ht(S)$ to show that $\binom{S}{T_{ht(S)-1}} \neq \emptyset$. For the base case we assume that $ht(S) = 2$. Since f_1^S is unbounded on $T[1]$, there exists a strictly increasing sequence $\{n_0, n_1, n_2, \dots\}$ such that for each $i < \omega$, $f_1^S(\langle n_i \rangle) \geq i$. Thus, there exists another strictly increasing sequence $\{k_0, k_1, \dots\}$ such that for each $i < \omega$, $\binom{S(k_i)}{T_1(i)} \neq \emptyset$. For each $i < \omega$, let U_i be an element of $\binom{S(k_i)}{T_1(i)}$ and let $U = \bigcup_{i < \omega} U_i$. By construction, we have $U \in \binom{S}{T_1}$. Hence, $\binom{S}{T_{ht(S)-1}} \neq \emptyset$ when $ht(S) = 2$.

Proceeding with the inductive step, we suppose that $k < \omega$, and for trees S with $1 < ht(S) < k$ satisfying (1)-(4) in the statement of Theorem 2.4.6, if for each $0 < i < ht(S)$, f_i^S is unbounded on $S[i]$, then $\binom{S}{T_{ht(S)-1}} \neq \emptyset$. Let S be a tree satisfying (1)-(4) in the statement of Theorem 2.4.6 of height k . For each $t \in \mathcal{AR}_2(S)$, let t_0 and t_1 denote the lexicographically least elements of $[t(0)]$ and $[t(1)]$, respectively (see Figure 2.8). Let

$$\Pi_0 = \{t \in \mathcal{AR}_2(S) : f_{k-1}^S(\pi_{k-2}(t_0)) \geq f_{k-1}^S(\pi_{k-2}(t_1))\}$$

and

$$\Pi_1 = \{t \in \mathcal{AR}_2(S) : f_{k-1}^S(\pi_{k-2}(t_0)) < f_{k-1}^S(\pi_{k-2}(t_1))\}.$$

By the Ramsey theorem for \mathcal{R} , there exists $U \in \mathcal{R}(S)$ and $j < 2$ such that $\mathcal{AR}_2(S) \upharpoonright U$

$U \subseteq \Pi_j$. If $j = 0$ then f_{k-1}^S is bounded on $U[k-1]$ by $f_{k-1}^S(\pi_{k-2}(t))$ where t is the unique element in $[U(0)]$. Thus, $j = 0$ implies that f_{k-1}^U is bounded on $U[k-1]$. Since U is isomorphic to S and π_{k-1}^S is unbounded on $S[k-1]$, it must be the case that $j \neq 0$. By a similar argument, there exists $V \in \mathcal{R}$ such that $V \leq U$ and for all $u, v \in [V]$ with $\pi_0(u) < \pi_0(v)$, there exists $t \in \mathcal{AR}_2(S) \upharpoonright U$ such that $t_0 = u$ and $t_1 = v$. Since $j = 1$ we have, for all $u, v \in [V]$, $\pi_0(u) < \pi_1(v)$ implies that $f_{k-1}^S(\pi_{k-2}(u)) < f_{k-1}^S(\pi_{k-2}(v))$. In particular, for each $n < \omega$, f_{k-1}^S is greater than n on $(V \setminus r_n(V))[k-1]$.

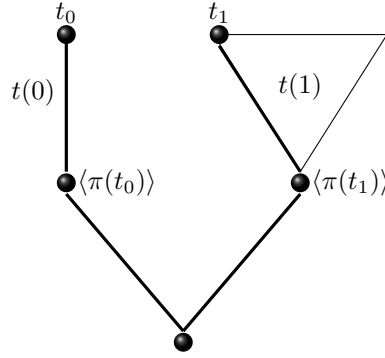


Figure 2.8: Graph of $t \in \mathcal{AR}_2(T)$ with lexicographically least elements of $[t(0)]$ and $[t(1)]$ labeled.

Let $\hat{V} = V \setminus [V]$. The tree \hat{V} satisfies (1)-(4) in the statement of Theorem 2.4.6 and has the property that for each $0 < i < ht(\hat{V})$, $f_i^{\hat{V}}$ is unbounded on $\hat{V}[i]$. By the inductive hypothesis, $\binom{\hat{V}}{T_{k-2}} \neq \emptyset$ since $ht(\hat{V}) < k$. Let W be some element of $\binom{\hat{V}}{T_{k-2}}$. For each $n < \omega$, f_{k-1}^S is greater than n on $[W(n+1)] = \{s \in W(n) : |s| = k-1\}$. Hence, there is a strictly increasing sequence $\{k_0, k_1, \dots\}$ of natural numbers such that for all $n < \omega$, $\binom{V^{(k_n)}}{T_{k-1}(n)} \neq \emptyset$. For each $n < \omega$, let $V'_n \in \binom{V^{(k_n)}}{T_{k-1}(n)}$ and $V' = \bigcup_{n < \omega} V'_n$. By its construction, $V' \in \binom{S}{T_{k-1}}$. Thus the inductive step is complete and we may conclude that, for each tree S on ω with $ht(S) < \omega$ satisfying (1)-(4) in the statement of Theorem 2.4.6, if for each $0 < i < ht(S)$, f_i^S is unbounded on $S[i]$,

then $\binom{S}{T_{ht(S)-1}} \neq \emptyset$.

Next, we show that $\binom{T_{ht(S)-1}}{S} \neq \emptyset$. Let S be a tree on ω such that $ht(S) < \omega$ satisfies (1)-(4) in the statement of Theorem 2.4.6 and for each $0 < i < ht(S)$, f_i^S is unbounded on $S[i]$. There exists a strictly increasing sequence $\{k'_0, k'_1, k'_2, \dots\}$ such that for each $i < \omega$, $\binom{T_{ht(S)-1}(k'_i)}{S(i)} \neq \emptyset$ since for each $n < \omega$ and each $0 < i < ht(S)$, $f_i^{T_{ht(S)-1}}$ is greater than n on $T_{ht(S)-1}(n)[i]$. Thus for each $l < \omega$, $\binom{T_{ht(S)-1}(k'_l)}{S(l)} \neq \emptyset$. For each $l < \omega$, let $U_l \in \binom{T_{ht(S)-1}(k'_l)}{S(l)}$ and $U = \bigcup_{l < \omega} U_l$. By its construction, $U \in \binom{T_{ht(S)-1}}{S}$. Therefore $\binom{T_{ht(S)-1}}{S} \neq \emptyset$. \square

By Claim 2 there is a tree T' and a bijective \leq^* -ordering preserving map $h : \mathcal{R}(T) \rightarrow \mathcal{R}(T')$ satisfying the assumptions of Claim 3. By Claim 3, $\binom{T_{ht(T')-1}}{T'}$ and $\binom{T'}{T_{ht(T')-1}}$ are both nonempty. Let W be an element of $\binom{T'}{T_{ht(T')-1}}$. Since W is isomorphic to $T_{ht(T')-1}$ there is a bijective \leq^* -order preserving map $g : \binom{W}{W} \rightarrow \mathcal{R}_{ht(T')-1}$. We define $f : \mathcal{R}(T') \rightarrow \binom{W}{W}$ as follows: for each $S \in \mathcal{R}(T')$ we let $f(S)$ be the projection to the subtree of S isomorphic to $T_{ht(T')-1}$ with the same lexicographical position in S as W within T' .

Let $\Gamma : \mathcal{R}(T) \rightarrow \mathcal{R}_{ht(T')-1}$ denote the function $g \circ f \circ h$. Since f is a projection and g and h are order-preserving bijections, we find that for all $U, V \in \mathcal{R}(T)$, $U \leq^* V \Rightarrow \Gamma(U) \leq^* \Gamma(V)$.

Suppose $U, V \in \mathcal{R}(T)$ and $U \perp V$. By way of contradiction suppose that there exists $X \in \mathcal{R}_{ht(T')-1}$ such that $X \leq^* \Gamma(U)$ and $X \leq^* \Gamma(V)$. Hence, $g^{-1}(X) \leq^* f \circ h(U)$ and $g^{-1}(X) \leq^* f \circ h(V)$. Since $g^{-1}(X)$ is isomorphic to $T_{ht(T')-1}$ and $\binom{T_{ht(T')-1}}{T'} \neq \emptyset$, there exists Y in $\binom{g^{-1}(X)}{T'}$. Additionally, since $g^{-1}(X) \leq W \leq T'$, it follows that $Y \in \mathcal{R}(T')$. By its construction, $h^{-1}(Y) \leq^* U$ and $h^{-1}(Y) \leq^* V$ which is a contradiction since $h^{-1}(Y) \in \mathcal{R}(T)$. Thus, $\Gamma(U) \perp \Gamma(V)$.

Suppose that $U \in \mathcal{R}_{ht(T')-1}$, then $g^{-1}(U) \in \binom{T'}{T_{ht(T')-1}}$. Since $\binom{T_{ht(T')-1}}{T'}$ is nonempty, we find that there exists a tree $Y \in \binom{g^{-1}(U)}{T'}$. Then $h^{-1}(Y) \in \mathcal{R}(T)$ and $\Gamma(h^{-1}(Y)) = g \circ f \circ h(h^{-1}(Y)) = g \circ f(Y)$. Since $f(Y) \leq Y$, we find that $\Gamma(h^{-1}(Y)) = g \circ f(Y) \leq g(Y) \leq g(g^{-1}(U)) = U$. Therefore, $\Gamma''\mathcal{R}(T)$ is dense in $\mathcal{R}_{ht(T')-1}$.

The previous three paragraphs, taken together, verify that Γ is a dense embedding of $(\mathcal{R}(T), \leq^*)$ into (\mathcal{R}_k, \leq^*) where $k = ht(T') - 1$. Therefore, $(\mathcal{R}(T), \leq^*)$ is equivalent to (\mathcal{R}_k, \leq^*) . \square

2.5 Equivalents of selective for $\mathcal{R}(T)$

In this section we prove a theorem providing equivalent formulations of the notion of selective for $\mathcal{R}(T)$ ultrafilter where T is a tree as in Definition 2.2.1. Let $\pi : [T] \rightarrow \omega$, denote the map given by letting $\pi(s) = \text{depth}_T(s)$, for $s \in [T]$. Notice that for all $s \in [T]$, $\pi_0(s) = \langle \pi(s) \rangle$.

Theorem 2.5.1. *Let k be a positive integer and T be a tree on ω^k . Suppose that*

- (1) $||[T(0)]|| = 1$,
- (2) For all $s, t \in [T]$, $|s| = |t|$,
- (3) $\pi_0''[T] = \{\langle (n, \dots, n) \rangle : n < \omega\}$,
- (4) $(\mathcal{R}(T), \leq, r)$ satisfies **A.1-A.4** and forms a topological Ramsey space.

Suppose that $\mathcal{C} \subseteq \mathcal{R}(T)$ generates an ultrafilter \mathcal{U} on the base set $[T]$. The following statements are equivalent:

1. \mathcal{U} is selective for $\mathcal{R}(T)$.

2. \mathcal{U} is a p -point and $\pi(\mathcal{U})$ is selective.

3. For each decreasing sequence $S_0 \geq S_1 \geq S_2 \geq \dots$ of members of \mathcal{C} , there is another $S \in \mathcal{C}$ such that for each $n < \omega$, $S \setminus r_n(S) \subseteq S_{\text{depth}_T(r_n(S))}$.

Proof. First we show that 1 \Rightarrow 2. Suppose that \mathcal{U} is selective for $\mathcal{R}(T)$. To show that \mathcal{U} is a p -point consider a sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ of elements of \mathcal{U} . Since \mathcal{U} is generated by \mathcal{C} , there exists a sequence $S_0 \geq S_1 \geq S_2 \geq \dots$ of elements of \mathcal{C} such that for all $i < \omega$, $[S_i] \subseteq A_i$. Since \mathcal{U} is selective for $\mathcal{R}(T)$, there exists $S \in \mathcal{C}$ such that for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$. As each $r_i(S)$ is finite for each $i < \omega$, $[S] \subseteq^* [S_i] \subseteq A_i$. Hence \mathcal{U} is a p -point ultrafilter on $[T]$.

To show that $\pi(\mathcal{U})$ is selective consider a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ of elements of $\pi(\mathcal{U})$. Since \mathcal{U} is generated by \mathcal{C} , there exists a sequence $S_0 \geq S_1 \geq S_2 \geq \dots$ of elements of \mathcal{C} such that for all $i < \omega$, $\pi''[S_i] \subseteq X_i$. Since \mathcal{U} is selective for $\mathcal{R}(T)$, there exists $S \in \mathcal{C}$ such that for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$. Let $\{x_0, x_1, \dots\}$ be the increasing enumeration of $\pi''[S]$. Then for each $i < \omega$, $\pi''[S] \setminus \{x_0, x_1, \dots, x_{i-1}\} = \pi''([S] \setminus [r_i(S)]) \subseteq \pi''[S_i] \subseteq X_i$. Thus, $\pi(\mathcal{U})$ is selective.

Next we show that 2 \Rightarrow 3. Suppose that \mathcal{U} is a p -point and $\pi(\mathcal{U})$ is selective. To show that \mathcal{U} satisfies condition 3, consider an arbitrary decreasing sequence $S_0 \geq S_1 \geq S_2 \geq \dots$ of members of \mathcal{C} . There is an $S \in \mathcal{C}$ such that for each $n < \omega$, $[S] \subseteq^* [S_n]$. Let $(k_i)_{i < \omega}$ be the strictly increasing sequence such that for all $i < \omega$, $S(i) \subseteq T(k_i)$. Define $(k'_n)_{n < \omega}$ recursively by letting,

$$\begin{cases} k'_0 = k_0, \\ k'_{i+1} \text{ is the smallest } k_j > k'_i \text{ such that } \{x \in [S] : \pi(x) > k_j\} \subseteq [S_{k'_i}]. \end{cases} \quad (2.5.1)$$

Now define $g : \omega \rightarrow \omega$ by letting

$$g(n) = i \text{ if } k'_i \leq n < k'_{i+1}. \quad (2.5.2)$$

Note that g can not be constant mod $\pi(\mathcal{U})$ as $\pi(\mathcal{U})$ is nonprincipal. Since $\pi(\mathcal{U})$ is selective, it must be the case that there is a $Y \subseteq \pi''[S]$ such that g is increasing on Y and $Y \in \pi(\mathcal{U})$. Enumerate Y in increasing order, as $\{y_0, y_1, \dots\}$. Then either $\{y_0, y_2, y_4, \dots\}$ or $\{y_1, y_3, y_5, \dots\}$ is a member of $\pi(\mathcal{U})$. Let $Z = \{z_0, z_1, \dots\}$ denote which ever is in $\pi(\mathcal{U})$. By construction, we find that for each pair $i < j$ of natural numbers there exists $k'_{l+1} < \omega$ such that $z_i < k'_{l+1} < z_j$.

Since $\pi^{-1}(Z) \in \mathcal{U}$ and $[S] \in \mathcal{U}$, there is a $S' \in \mathcal{C}$ such that $[S'] \subseteq \pi^{-1}(Z) \cap [S]$. Let $(k''_i)_{i < \omega}$ be the strictly increasing sequence such that for all $i < \omega$, $S'(i) \subseteq T(k''_i)$. Since $\pi''[S'] \subseteq Z$, we find that for each $n < \omega$, each $m > n$ and each $s \in [S'(m)]$, there exists k'_{l+1} such that $\pi(s) = k''_m > k'_{l+1} > k''_n$. By definition of the sequence $(k'_i)_{i < \omega}$, it follows that $s \in [S_{k'_l}]$. On the other hand, $k'_{l+1} > k''_n$ implies that $S_{k'_l} \subseteq S_{k''_n}$. So $s \in [S_{k''_n}]$ and $S' \setminus r_n(S') \subseteq S_{k''_n} = S_{\text{depth}_T(r_n(S'))}$. Thus, $2 \Rightarrow 3$ holds.

Next note that for each $S \in \mathcal{R}(T)$, $\text{depth}_T(r_n(S)) \geq n$. Thus, trivially, $3 \Rightarrow 1$ holds. \square

2.6 Equivalents of Ramsey for $\mathcal{R}(T)$

In this section we prove theorems providing equivalent formulations of the notion of Ramsey for $\mathcal{R}(T)$ ultrafilter where T is a tree as in Definition 2.2.1.

Notation 2.6.1. *Let T be a tree satisfying Definition 2.2.1. For $S \in \mathcal{R}(T)$, $s \in$*

$\mathcal{AR}(T)$ and $n < \omega$ we let

$$\mathcal{R}(T)(n) \upharpoonright S = \{S'(n) : S' \in \mathcal{R}(T) \ \& \ S' \leq S\} \quad (2.6.1)$$

$$\mathcal{R}(T)(n) \upharpoonright S/s = \{S'(n) \in \mathcal{R}(T)(n) \upharpoonright S : s \cup S'(n) \in \mathcal{AR}(T)|S\}. \quad (2.6.2)$$

Theorem 2.6.2. *Let k be a positive integer and T be a tree on ω^k . Suppose that*

- (1) $||[T(0)]|| = 1$,
- (2) *For all $s, t \in [T]$, $|s| = |t|$,*
- (3) $\pi_0''[T] = \{\langle (n, \dots, n) \rangle : n < \omega\}$,
- (4) $(\mathcal{R}(T), \leq, r)$ *satisfies **A.1-A.4** and forms a topological Ramsey space.*

Let \mathcal{U} be an ultrafilter on $[T]$ generated by some subset $\mathcal{C} \subseteq \mathcal{R}(T)$. \mathcal{U} is Ramsey for $\mathcal{R}(T)$ if and only if \mathcal{U} is selective for $\mathcal{R}(T)$ and for each $n < \omega$, $\mathcal{U}|_{\mathcal{R}(T)(n)} = \{\mathcal{R}(T)(n) \upharpoonright A : A \in \mathcal{C}\}$ generates an ultrafilter on $\mathcal{R}(T)(n)$.

Proof. (\Rightarrow) By Lemma 3.8 of Mijares in [35], every Ramsey for $\mathcal{R}(T)$ ultrafilter is selective for $\mathcal{R}(T)$. One the other hand, if \mathcal{U} is Ramsey for $\mathcal{R}(T)$ then for each $n < \omega$, $\mathcal{U}|_{\mathcal{R}(T)(n)} = \{\mathcal{R}(T)(n) \upharpoonright A : A \in \mathcal{C}\}$ forms an ultrafilter on $\mathcal{R}(T)(n)$.

(\Leftarrow) Suppose \mathcal{U} is selective for $\mathcal{R}(T)$ and for each $n < \omega$, $\mathcal{U}|_{\mathcal{R}(T)(n)} = \{\mathcal{R}(T)(n) \upharpoonright S : S \in \mathcal{C}\}$ forms an ultrafilter on $\mathcal{R}(T)(n)$. Since $\mathcal{AR}_1 = \mathcal{R}(T)(0)$ and $\mathcal{U}|_{\mathcal{R}(T)(0)}$ forms an ultrafilter on $\mathcal{R}(T)(0)$, it follows that if $i = 0$ then every partition of \mathcal{AR}_i into two parts there exists $S \in \mathcal{C}$ such that $\mathcal{AR}_i|S$ lies in one part of the partition. We proceed by induction on i to show that \mathcal{U} is Ramsey for $\mathcal{R}(T)$. The previous remarks show that the base case of the induction holds.

Let i be a natural number and suppose that every partition of \mathcal{AR}_i into two parts there exists $S \in \mathcal{C}$ such that $\mathcal{AR}_i|S$ lies in one part of the partition. Let $\{\Pi_0, \Pi_1\}$ be a partition of \mathcal{AR}_{i+1} . We show that there exists $S \in \mathcal{C}$ such that $\mathcal{AR}_{i+1}|S$ lies in one part of the partition.

For each $s \in \mathcal{AR}_i$, let $A_s = \{p \in \mathcal{R}(T)(i) : s \cup p \in \Pi_0\}$. Let

$$\Pi'_0 = \{s \in \mathcal{AR}_i : A_s \in \mathcal{U}|\mathcal{R}(T)(i)\}$$

$$\Pi'_1 = \{s \in \mathcal{AR}_i : \mathcal{R}(T)(i) \setminus A_s \in \mathcal{U}|\mathcal{R}(T)(i)\}.$$

Since $\mathcal{U}|\mathcal{R}(T)(i)$ forms an ultrafilter on $\mathcal{R}(T)(i)$ it follows that $\{\Pi'_0, \Pi'_1\}$ is a partition of \mathcal{AR}_i . By the inductive hypothesis, there exists $S \in \mathcal{C}$ and $j < 2$ such that $\mathcal{R}(T)(i)|S \subseteq \Pi'_j$.

We first consider the case when $j = 0$. In particular, for each $s \in \mathcal{AR}_i|S$, $A_s \in \mathcal{U}|\mathcal{R}(T)(i)$. For each $n < \omega$, let $B_n = \bigcap_{\text{depth}_T(s) \leq n} A_s \in \mathcal{U}|\mathcal{R}(T)$. Hence there exists a sequence $\{S_n : n < \omega\}$ of elements of \mathcal{C} such that $S_0 \geq S_1 \geq S_2 \geq \dots$ and for each $n < \omega$, $\mathcal{R}(T)(i)|S_n \subseteq B_n$. By Theorem 2.5.1 there exist $S \in \mathcal{C}$ such that for each $n < \omega$, $S \setminus r_n(S) \subseteq S_{\text{depth}_T(r_n(S))}$.

Suppose that $t \in \mathcal{AR}_{i+1}|S$. Then $r_i(t) \in \mathcal{AR}_i|A$ and $t(i) \in \mathcal{R}(T)(i)$. If $k = \text{depth}_S(r_i(t))$ then $t(i) \in \mathcal{R}(T)(i)|(S \setminus r_k(S)) \subseteq \mathcal{R}(T)(i)|S_{\text{depth}_T(r_k(S))} \subseteq A_{r_i(t)}$. Hence, $r_i(t) \cup t(i) \in \Pi_0$. So in the case when $j = 0$, $\mathcal{AR}_{i+1}|S \subseteq \Pi_j$. By an identical argument in the case when $j = 1$, there exists $S \in \mathcal{C}$ such that $\mathcal{AR}_{i+1}|S \subseteq \Pi_j$.

By induction we find that for each $i < \omega$ and each partition of \mathcal{AR}_i into two parts there exists $S \in \mathcal{C}$ such that $\mathcal{AR}_i|S = \{s \in \mathcal{AR}_i : s \subseteq S\}$ lies in one part of the partition. In other words, \mathcal{U} is a Ramsey for $\mathcal{R}(T)$ ultrafilter on $[T]$. \square

Ramsey for $\mathcal{R}(T)$ ultrafilters also satisfy a localized version of the abstract El-

lentuck theorem. For the remainder of this section, we let k be a positive integer and T be a tree on ω^k . Suppose that T has the following properties

- (1) $||[T(0)]|| = 1$,
- (2) For all $s, t \in [T]$, $|s| = |t|$,
- (3) $\pi_0''[T] = \{\langle (n, \dots, n) \rangle : n < \omega\}$,
- (4) $(\mathcal{R}(T), \leq, r)$ satisfies **A.1-A.4** and forms a topological Ramsey space.

We also let \mathcal{U} denote an ultrafilter on $[T]$ generated by $\mathcal{C} \subseteq \mathcal{R}(T)$. To avoid repeating certain phrases, in this section only, we shall let the variables A, B, C, \dots run over elements of \mathcal{C} , the variables s, t, u, \dots over elements of $\mathcal{AR}(T)$ and the variables p, q, r, \dots over elements of $\bigcup_{i < \omega} \mathcal{R}(T)(i)$.

Lemma 2.6.3. *Let \mathcal{U} be a selective for $\mathcal{R}(T)$ ultrafilter on the base set $[T]$ generated by $\mathcal{C} \subseteq \mathcal{R}(T)$. Suppose that $P(\cdot, \cdot)$ is a property such that:*

1. *If $P(A, s)$ holds and $B/s \subseteq A$ then $P(B, s)$ holds.*
2. *For each A and each $s \in \mathcal{AR}|A$ there exists $B \leq A$ such that $P(B, s)$ holds.*

Then for each A there exists $B \leq A$ such that for all $s \in \mathcal{AR}|B$, $P(B, s)$ holds.

Proof. For each $i < \omega$, let $H_i = \{s \in \mathcal{AR} : \max(\pi''s) \leq i \text{ or } s = \emptyset\}$. If $A \in \mathcal{U}$ then the second hypothesis of the claim implies that there exists $B_0 \leq A$ such that for all $s \in H_0$, $P(B_0, s)$ holds.

Suppose that $B_0 \geq B_1 \geq \dots \geq B_{n-1}$ are given and for each $i < n$ and each $s \in H_i$, $P(B_i, s)$ holds. Note that for each $i < \omega$, H_i is finite. Applying the second hypothesis of the claim finitely many times there exists $B_n \leq B_{n-1}$ such that for all $s \in H_n$, $P(B_n, s)$ holds.

The previous two paragraphs recursively define a sequence $\{B_i\}_{i < \omega}$ that is a \leq -decreasing sequence of members of \mathcal{U} . Since \mathcal{U} is a selective for $\mathcal{R}(T)$ ultrafilter, there exists $B \in \mathcal{C}$ such that $B \leq A$ and for all $i < \omega$, $B/r_i(B) \subseteq B_{\text{depth}_T(r_i(B))}$. If $s \in \mathcal{AR}|B$ and $\text{depth}_B(s) = j$, then $B/s = B/r_j(B) \subseteq B_{\text{depth}_T(r_j(B))}$. From the definition of $B_{\text{depth}_T(r_j(B))}$ it follows that $P(B_{\text{depth}_T(r_j(B))}, s)$ holds since $\max(\pi''s) = \text{depth}_T(r_j(B))$. The first hypothesis of the claim implies that $P(B, s)$ holds. Therefore, for all $s \in \mathcal{AR}|B$, $P(B, s)$ holds. \square

Lemma 2.6.4. *Let \mathcal{U} be a selective for $\mathcal{R}(T)$ ultrafilter on the base set $[T]$ generated by $\mathcal{C} \subseteq \mathcal{R}(T)$. Suppose that $P(\cdot, \cdot)$ is a property such that:*

1. *If $P(A, s)$ holds and $B/s \subseteq A$ then $P(B, s)$ holds.*
2. *If $P(A, s)$ holds then there exists $B \leq A$ such that*

for all $p \in \mathcal{R}(T)(|s|) \upharpoonright B/s$, $P(B, s \cup p)$ holds.

If $P(A, \emptyset)$ holds then there exists $B \leq A$ such that for all $s \in \mathcal{AR}|B$, $P(B, s)$ holds.

Proof. Suppose that $P(A, \emptyset)$ holds and $P(\cdot, \cdot)$ satisfies the two hypotheses of the Lemma. First we will show by induction that, for each $k < \omega$, A can be refined to a set $B \in \mathcal{C}$ such that $P(B, s)$ holds for all $s \in \mathcal{AR}|B$ with length less than or equal to k .

Since \emptyset is the only element of length 0, the base case of the induction holds. Let $k < \omega$, and suppose that the inductive hypothesis, there exists $B \leq A$ such that $P(B, s)$ holds for all $s \in \mathcal{AR}|B$ with length less than or equal to k , is true.

Next, let $Q_k(A, s)$ be the statement: for all $p \in \mathcal{R}(T)(k) \upharpoonright A/s$, $P(A, r_k(s) \cup p)$

holds. By the inductive hypothesis, we find there exists $B \leq A$ such that for each $s \in \mathcal{AR}|B$, $P(B, r_{k-1}(s))$ holds. By the second hypothesis of the Lemma, there exists $B' \leq B$ such that for all $p \in \mathcal{R}(T)(k) \upharpoonright B'/r_{k-1}(s)$, $P(B', r_{k-1}(s) \cup p)$ holds, *i.e.*, $Q_k(B', s)$ holds. Therefore the second hypothesis of Lemma 2.6.3 holds for $Q_k(\cdot, \cdot)$.

Suppose that $s \in \mathcal{AR}|A$, $B/s \subseteq A$ and $Q_k(A, s)$ holds. Then for all $p \in \mathcal{R}(T)(k) \upharpoonright B/s$, $P(A, r_{k-1}(s) \cup p)$ since $B/s \subseteq A/s$. By the first hypothesis of this Lemma, we find that $P(B, r_{k-1}(s) \cup p)$ for each $p \in \mathcal{R}(T)(k) \upharpoonright B/s$, *i.e.* $Q_k(B, s)$ holds. Thus, $Q_k(\cdot, \cdot)$ satisfies the first hypothesis of Lemma 2.6.3.

Applying Lemma 2.6.3 to $Q_k(\cdot, \cdot)$, there exists $C \leq B$ such that for all $s \in \mathcal{AR}|C$, $Q_k(C, s)$ holds. If t is a finite approximation of C of length $k + 1$, then for all $p \in \mathcal{R}(T)(k) \upharpoonright C/r_k(t)$, $P(C, r_k(t) \cup p)$ holds. Since $t(k) \in \mathcal{R}(T)(k) \upharpoonright C/r_k(t)$ we find that $P(C, t)$ holds. Thus, there exists $C \leq A$ such that $P(C, s)$ holds for all $s \in \mathcal{AR}|C$ with length less than or equal to $k + 1$.

The previous induction argument shows that we can construct a decreasing sequence $B_0 \geq B_1 \geq B_2 \cdots$ of members of \mathcal{C} such that for each $k < \omega$, $B_k \leq A$ and $P(B_k, s)$ holds for all s with length less than or equal to k .

Since \mathcal{U} is selective for $\mathcal{R}(T)$, there exists $B \leq A$ such that for each $n < \omega$, $B/r_n(B) \subseteq B_{\text{depth}_T(r_n(B))}$. Let s be a finite approximation of B such that $\max(\pi''s) = \text{depth}_T(r_n(B))$. Then

$$B/s = B/r_n(B) \subseteq B_{\text{depth}_T(r_n(B))} \text{ and } P(B_{\text{depth}_T(r_n(B))}, s)$$

holds since $|s| \leq n$. By the first assumption of the hypothesis, $P(B, s)$ holds. Since s was arbitrary, we find that for all $s \in \mathcal{AR}|B$, $P(B, s)$ holds. \square

Definition 2.6.5 (J. Mijares, [35]). A subset \mathcal{X} of $\mathcal{R}(T)$ is \mathcal{C} -Ramsey if for every nonempty $[s, A]$ with $A \in \mathcal{C}$ there exists $B \leq [s, A] \cap \mathcal{C}$ such that either $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}(T)$ is \mathcal{C} -Ramsey null if for every nonempty $[s, A]$ with $A \in \mathcal{C}$ there is a $B \in [s, A] \cap \mathcal{C}$ such that $[s, A] \cap \mathcal{X} = \emptyset$.

Definition 2.6.6. We say that \mathcal{C} satisfies the abstract Ellentuck theorem if and only if every property of Baire subset of $\mathcal{R}(T)$ is \mathcal{C} -Ramsey and if every meager subset of $\mathcal{R}(T)$ is \mathcal{C} -Ramsey null.

Theorem 2.6.7 (Localized Ellentuck theorem for $\mathcal{R}(T)$). *Suppose that $\mathcal{C} \subseteq \mathcal{R}(T)$ generates an ultrafilter on $[T]$. The following statements are equivalent:*

1. \mathcal{C} satisfies the abstract Ellentuck theorem.
2. \mathcal{U} is Ramsey for $\mathcal{R}(T)$.

Proof. It is clear that 3 \Rightarrow 1. We show that 2 \Rightarrow 1. Suppose \mathcal{U} is Ramsey for $\mathcal{R}(T)$. By Theorem 2.6.2, \mathcal{U} is a selective for $\mathcal{R}(T)$ ultrafilter generated by \mathcal{C} and for each $m < \omega$, $\mathcal{C} \upharpoonright \mathcal{R}(T)(m) = \{\mathcal{R}(T)(m) \upharpoonright A : A \in \mathcal{C}\}$ generates an ultrafilter on $\mathcal{R}(T)(m)$. The following notion of combinatorial forcing is based on a version of combinatorial forcing used by Todorćević, originally due to Nash-Williams in [38], to prove the Ellentuck Theorem in the book [46]. For the purposes of the next definition, we fix a subset \mathcal{X} of $\mathcal{R}(T)$.

Definition 2.6.8. We say that A accepts s if $\{B \in \mathcal{C} : s \sqsubseteq B \ \& \ B/s \subseteq A\} \subseteq \mathcal{X}$. We say that A rejects s if for every B such that $B/s \subseteq A$, B does not accept s . Finally, we say that A decides s if A accepts s or A rejects s .

Claim 4. 1. *If A accepts (rejects) s and $B/s \subseteq A$ then B accepts (rejects) s .*

2. For every A and s , there exists $B \leq A$ such that B decides s .

Proof. 1. Let A and s be given. Suppose that A accepts s and $B/s \subseteq A$. Then

$$\{C \in \mathcal{R}(T) : s \sqsubseteq C \ \& \ C/s \subseteq B\} \subseteq \{C \in \mathcal{R}(T) : s \sqsubseteq C \ \& \ C/s \subseteq A\} \subseteq \mathcal{X}. \quad (2.6.3)$$

Therefore B accepts s .

2. Suppose that A rejects s and $B/s \subseteq A$. For each C such that $C/s \subseteq B$, $C/s \subseteq A$. Thus, C does not accept s . In other words, B rejects s .

If A does not reject s then there exists B such that $B/s \subseteq A$ and B accepts s . Since $[B/s] \in \mathcal{U}$ we find that there exists a $C \leq B$ such that $[C] \subseteq [B/s]$. Therefore $C \leq A$ and C accepts s . Hence there exists $C \leq A$ such that C decides s (if A rejects s , let $C = A$).

□

Claim 5. Every set in \mathcal{U} can be refined to a set in \mathcal{U} that decides all of its finite approximations.

Proof. Let $P(A, s)$ be the statement: A decides s . By Claim 4, the hypotheses of Claim 2.6.3 for $P(\cdot, \cdot)$ hold. So Claim 2.6.3 implies that for each A there exists $B \leq A$ such that for all $s \in \mathcal{AR}|B$, $P(B, s)$. In other words, B decides all of its finite approximations. □

Claim 6. Assume that A decides all of its finite approximations. If A rejects some s , then there exists $B \leq A$ such that

$$\text{for all } p \in \mathcal{R}(T)(|s|) \upharpoonright B/s, B \text{ rejects } s \cup p.$$

Proof. Consider the set,

$$\mathcal{Y} = \{p \in \mathcal{R}(T)(n) : s \cup p \in \mathcal{AR}_{n+1} \text{ \& } A \text{ rejects } s \cup p\}.$$

Since for each $m < \omega$, $\mathcal{C}|\mathcal{R}(T)(m) = \{\mathcal{R}(T)(m) \upharpoonright A : A \in \mathcal{C}\}$ generates an ultrafilter on $\mathcal{R}_1(m)$, there exists $B \leq A$ such that either $\mathcal{R}(T)(n) \upharpoonright B \subseteq \mathcal{Y}$ or $(\mathcal{R}(T)(n) \upharpoonright B) \cap \mathcal{Y} = \emptyset$.

Claim 4 implies that B rejects s since $B \leq A$. Toward a contradiction suppose that $(\mathcal{R}(T)(n) \upharpoonright B) \cap \mathcal{Y} = \emptyset$. Then for all $p \in \mathcal{R}(T)(n) \upharpoonright B/s$, B accepts $s \cup p$. Assume that C is such that $C/s \subseteq B$ and $s \sqsubseteq C$. Since $C(|s|) \in \mathcal{R}(T)(n) \upharpoonright B/s$, B accepts $s \cup C(|s|)$. Since $s \cup C(|s|) = r_{|s|+1}(C) \sqsubseteq C$ and $C/r_{|s|+1}(C) \subseteq B$, we find that $C \in \mathcal{X}$. Therefore $\{C \in \mathcal{C} : s \sqsubseteq C \text{ \& } C/s \subseteq B\} \subseteq \mathcal{X}$ which is a contradiction to the fact that B rejects s . So it must be the case that $\mathcal{R}(T)(n) \upharpoonright B \subseteq \mathcal{Y}$.

Since $B \leq A$, Claim 4 implies that B decides all of its finite approximations. Thus, for all $p \in \mathcal{R}(T)(n) \upharpoonright B/s$, B rejects $s \cup p$, since $\mathcal{R}(T)(n) \upharpoonright B \subseteq \mathcal{Y}$ \square

Claim 7. *Suppose that A decides all of its finite subsets. If A rejects \emptyset , then there exists $B \leq A$ such that B rejects all of its finite subsets.*

Proof. Let $P(s, A)$ be the statement: A rejects s . By Claim 4 and Claim 5, $P(\cdot, \cdot)$ satisfies the hypotheses of Lemma 2.6.4. Since $P(A, \emptyset)$ holds, there exists $B \leq A$ such that for all $s \in \mathcal{AR}|B$, $P(s, B)$. \square

This concludes the series of Claims about the combinatorial forcing relative to the fixed set \mathcal{X} .

Claim 8. *Every open subset of $\mathcal{R}(T)$ is \mathcal{C} -Ramsey.*

Proof. Let \mathcal{X} be an open subset of $\mathcal{R}(T)$. In this proof we use the above forcing Lemmas relativized to the basic open set $[s, A]$ in place of $[0, T]$. Let $B \in [s, A]$ be a set that decides all approximations t such that $s \sqsubseteq t$. If B accepts s , then $\{C \in \mathcal{C} : s \sqsubseteq C \ \& \ C/s \subseteq B\} \subseteq \mathcal{X}$. In particular, $[s, B] \subseteq \mathcal{X}$. So, without loss of generality, we may assume that B rejects s .

By Claim 7, there exists $C \in [s, B]$ such that C rejects all of its finite approximations t such that $s \sqsubseteq t$. Toward a contradiction suppose that $[s, C] \cap \mathcal{X} \neq \emptyset$. Since $[s, C] \cap \mathcal{X}$ is an open subset of $\mathcal{R}(T)$, there exists t and D such that $[t, D]$ is nonempty and $[t, D] \subseteq [s, C] \cap \mathcal{X}$. However, this would imply that $D \leq C$, D accepts t and $s \sqsubseteq t$, a contradiction. Therefore, $[s, C] \cap \mathcal{X} = \emptyset$. \square

Claim 9. *Every meager subset of $\mathcal{R}(T)$ is \mathcal{C} -Ramsey null.*

Proof. By Corollary 1.2.5 it is enough to show that the result holds for nowhere dense subsets of $\mathcal{R}(T)$. Let \mathcal{M} be a nowhere dense subset of $\mathcal{R}(T)$. Applying the previous claim to the open set $\mathcal{R}(T) \setminus \bar{\mathcal{M}}$ (here $\bar{\mathcal{M}}$ denotes the topological closure in $\mathcal{R}(T)$) we find that for each nonempty basic open set $[s, A]$, there exist $B \in [s, A]$ such that either $[s, B] \subseteq \mathcal{R}(T) \setminus \bar{\mathcal{M}}$ or $[s, B] \cap (\mathcal{R}(T) \setminus \bar{\mathcal{M}}) = \emptyset$. The second alternative is impossible because it implies that $[s, B] \subseteq \bar{\mathcal{M}}$ and \mathcal{M} is nowhere dense. Hence, \mathcal{M} is \mathcal{C} -Ramsey null. \square

Suppose that \mathcal{X} has the Baire property. That is, there is an open set \mathcal{U} and meager set \mathcal{M} such that $\mathcal{X} = \mathcal{U} \Delta \mathcal{M}$. The previous two claims show that if $[s, A]$ is nonempty then there exists $B \in [s, A]$ such that $[s, B] \cap \mathcal{X} = \emptyset$, and either $[s, B] \cap \mathcal{U} = \emptyset$ or $[s, B] \subseteq \mathcal{U}$. Thus, either $[s, B] \subseteq \mathcal{U} \Delta \mathcal{M}$ or $[s, B] \cap (\mathcal{U} \Delta \mathcal{M}) = \emptyset$.

\square

Chapter 3

Selective but not Ramsey

The work in this chapter has been submitted for publication by the author in [48]. In this chapter, we give a partial answer to the following question of Dobrinen from Section 2.2:

Question 3.0.9. For a given topological Ramsey space \mathcal{R} , are the notions of selective for \mathcal{R} and Ramsey for \mathcal{R} equivalent?

Every topological Ramsey space \mathcal{R} has an associated notion of Ramsey ultrafilter for \mathcal{R} and selective ultrafilter for \mathcal{R} (see Definition 2.3.1 and [35]). If \mathcal{R} is taken to be the Ellentuck space then the two concepts reduce to the familiar notions of Ramsey and selective ultrafilters on ω ; so by a well-known result of Kunen in [5] the two are equivalent (see Example 2.2.2 and Example 2.3.2). In this chapter, we give the first example of an ultrafilter on a topological Ramsey space that is selective but not Ramsey for the space, and in fact a countable collection of such examples.

For each positive integer n , we show that for the topological Ramsey spaces \mathcal{R}_n defined by Dobrinen and Todorćević in [17] and [18] (see Example 2.2.5), the notions of selective for \mathcal{R}_n and Ramsey for \mathcal{R}_n are not equivalent. In particular, we

prove that forcing with a closely related space using almost-reduction, adjoins an ultrafilter that is selective but not Ramsey for \mathcal{R}_n . Furthermore, we show that forcing with closely related product spaces using almost-reduction, adjoins ultrafilters that are selective but not Ramsey for these product topological Ramsey spaces.

This chapter is concerned with giving examples of topological Ramsey spaces \mathcal{R} and ultrafilters that are selective for \mathcal{R} but not Ramsey for \mathcal{R} . The space \mathcal{R}_0 is taken to be the Ellentuck space; therefore, Ramsey for \mathcal{R}_0 is equivalent to selective for \mathcal{R}_0 (see Example 2.2.2 and Example 2.3.2). We show that for each positive integer n , there is a triple $(\mathcal{R}_n^*, \leq, r)$ such that forcing with the space using almost-reduction, adjoins an ultrafilter that is selective for \mathcal{R}_n but not Ramsey for \mathcal{R}_n .

Section 3.1 consists of the archetype example for the methods we apply in this chapter. In this section, we construct a closely associated tree T_1^* and prove that the associated triple $\mathcal{R}(T_1^*)$ forms a topological Ramsey space. Then we show that forcing with \mathcal{R}_1^* using almost-reduction, adjoins a selective but not Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$.

In Section 3.2, for each positive integer n , we construct a closely associated tree T_n^* and space $\mathcal{R}(T_n^*)$. We show that for each $n < \omega$, forcing with \mathcal{R}_n^* using almost-reduction, adjoins a selective but not Ramsey for \mathcal{R}_n ultrafilter on $[T_n]$.

In Section 3.3, we consider finite sequences $\langle S_i : i \leq n \rangle$ where each S_i is one of the trees T_j for some $j < \omega$. We introduce the product $\bigotimes_{i=0}^n \mathcal{R}(S_i)$ from [15]. Then we construct a closely associated tree $\bigotimes_{i=0}^n S_i^*$ and space $\bigotimes_{i=0}^n \mathcal{R}^*(S_i)$. We prove that forcing with $\bigotimes_{i=0}^n \mathcal{R}^*(S_i)$ using almost-reduction, adjoins a selective but not Ramsey for $\bigotimes_{i=0}^n \mathcal{R}(S_i)$ ultrafilter on $[\bigotimes_{i=0}^n S_i]$.

In section 3.4, we discuss why the methods used in this chapter fail for some topological Ramsey spaces defined from similar types of trees. We conclude with

some questions about the generalizations of Ramsey and selective ultrafilters to the spaces where our methods fail.

Throughout this chapter we use the methods of forcing but all of our constructions can be carried out using the continuum hypothesis or Martin's axiom. We work with σ -closed partial orders and all of the constructions only require 2^{\aleph_0} conditions to be met. For example, assuming the continuum hypothesis, we can guarantee the conditions hold at successor stages and use σ -closure at limit stages.

3.1 Selective but not Ramsey for \mathcal{R}_1

The purpose of this section is to introduce a closely related triple $(\mathcal{R}_1^*, \leq, r)$ and show that forcing with \mathcal{R}_1^* using almost-reduction, adjoins an ultrafilter that is selective but not Ramsey for \mathcal{R}_1 .

Next we define the topological Ramsey space \mathcal{R}_1^* . The space is constructed from a modified version of the tree T_1 from Chapter 2. The modified tree T_1^* has height 3 and for each $i < \omega$, the maximal nodes of $T_1^*(i)$ are in one-to-one correspondence with the maximal nodes of $T_1(i)$. These two properties of T_1^* are used below to show that forcing with \mathcal{R}_1^* using almost-reduction adjoins an ultrafilter that is selective but not Ramsey for \mathcal{R}_1 . The one-to-one correspondence is used to show that the adjoined ultrafilter is selective for \mathcal{R}_1 , and the extra level of T_1^* is used to show that the adjoined ultrafilter fails to be Ramsey for \mathcal{R}_1 .

Definition 3.1.1 ($(\mathcal{R}_1^*, \leq, r)$, T. [48]). For each $i < \omega$, let

$$T_1^*(i) = cl(\{\langle i, j, k \rangle : k \leq i \text{ \& } \langle j, k \rangle \in T_1\}). \quad (3.1.1)$$

Let $T_1^* = \bigcup_{i < \omega} T_1^*(i)$ and $(\mathcal{R}_1^*, \leq, r)$ denote the triple $(\mathcal{R}(T_1^*), \leq, r)$. Figure 3.1 is a graph of the tree T_1^* .

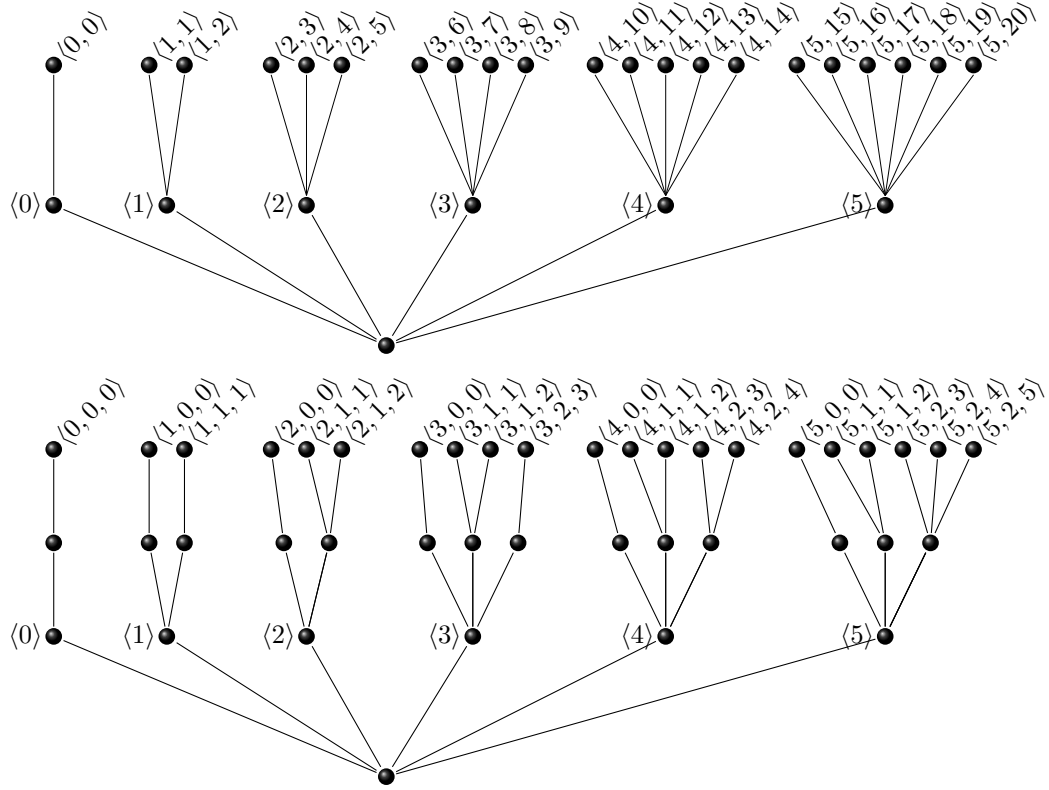


Figure 3.1: Graph of T_1 and T_1^*

The next two partition properties are needed to show that \mathcal{R}_1^* satisfies axiom A.4.

Lemma 3.1.2 (T. [48]). *For each pair of positive integers k and n with $k \leq n$, there exists $m < \omega$ such that*

$$T_1^*(m) \rightarrow (T_1^*(n))^{T_1^*(k)}. \quad (3.1.2)$$

Proof. Let k and n be positive integers. For each $i < \omega$, let i' be the smallest natural number such that $T_1^*(i)$ is isomorphic to a subtree of $cl(\{\langle i' \rangle \frown s : s \in r_{i'}(T_1)\})$.

For example, $1' = 2$ and $2' = 2$ since $T_1^*(1)$ and $T_1^*(2)$ are isomorphic to subtrees of $cl(\{\langle 2 \rangle^\frown s : s \in r_2(T_1)\})$ but not isomorphic to any subtree of $cl(\{\langle 1 \rangle^\frown s : s \in r_1(T_1)\})$. By Lemma 2.2.8 there exists $m < \omega$ such that $r_m(T_1) \rightarrow (r_{n'}(T_1))^{r_{k'}(T_1)}$. Suppose that $\{\Pi_0, \Pi_1\}$ is a partition of $\binom{T_1^*(m')}{T_1^*(k)}$. For each $j < 2$, let $\Pi'_j = \{S \in \binom{r_m(T_1)}{r_{k'}(T_1)} : cl(\{\langle m' \rangle^\frown s : s \in [\hat{S}]\}) \in \Pi_j\}$ where $[\hat{S}]$ consists of the lexicographically first k elements of $[S]$. $\{\Pi'_0, \Pi'_1\}$ forms a partition of $\binom{r_m(T_1)}{r_{k'}(T_1)}$. Hence, there exists $j < 2$ and $S \in \binom{r_m(T_1)}{r_{k'}(T_1)}$ such that $\binom{S}{r_{k'}(T_1)} \subseteq \Pi'_j$. If we let $S' := cl(\{\langle m' \rangle^\frown s : s \in [\hat{S}]\}) \in \Pi_j$ where $[\hat{S}]$ consists of the lexicographically first n elements of $[S]$, then $S' \in \binom{T_1^*(m')}{T_1^*(n)}$, and $\binom{S'}{T_1^*(k)} \subseteq \Pi_j$. Therefore the lemma holds. \square

Lemma 3.1.3 (T. [48]). *For each positive integer k ,*

$$T_1^* \rightarrow (T_1^*)^{T_1^*(k)}. \quad (3.1.3)$$

Proof. Let k be a positive integer. Lemma 3.1.2 shows that there exists a strictly increasing sequence $(m_i)_{i < \omega}$ such that for each $i < \omega$, $T_1^*(m_i) \rightarrow (T_1^*(i))^{T_1^*(k)}$. Let $\{\Pi_0, \Pi_1\}$ be a partition of $\binom{T_1^*}{T_1^*(k)}$ and (S_0, S_1, \dots) be a sequence of trees such that for each $i < \omega$, $S_i \in \binom{T_1^*}{T_1^*(i)}$ and $\binom{S_i}{T_1^*(k)}$ is contained in one piece of the partition $\{\Pi_0, \Pi_1\}$. By the pigeonhole principle there exists $j < 2$ and a strictly increasing sequence (i_0, i_1, \dots) such that for all $l < \omega$, $S_{i_l} \in \binom{T_1^*(m_{i_l})}{T_1^*(i)}$ and $\binom{S_{i_l}}{T_1^*(k)} \subseteq \Pi_j$. Let $S = \bigcup_{l < \omega} S_{i_l}$. If S' is any element of $\binom{S}{T_1^*}$ then $\binom{S'}{T_1^*(k)} \subseteq \Pi_j$. Therefore the lemma holds. \square

Theorem 3.1.4 (T. [48]). *$(\mathcal{R}_1^*, \leq, r)$ satisfies A.1-A.4 and forms a topological Ramsey space.*

Proof. By the abstract Ellentuck theorem it is enough to show that $(\mathcal{R}_1^*, \leq, r)$ satisfies A.1-A.4 and forms a closed subspace of $(\mathcal{AR}_1)^\omega$. The proof that \mathcal{R}_1^* is a closed

subspace of $(\mathcal{AR}_1)^\omega$ and satisfies axioms **A.1-A.3** follows by trivial modifications to the proofs of the same facts for the space \mathcal{R}_1 in [17]. For this reason, we omit the proof that \mathcal{R}_1^* forms a closed subspace of $(\mathcal{AR}_1)^\omega$ and satisfies axioms **A.1 – A.3**. By definition of \mathcal{R}_1^* , **A.4** is equivalent to Lemma 3.1.3. Hence **A.4** holds for \mathcal{R}_1^* . By the abstract Ellentuck theorem, \mathcal{R}_1^* forms a topological Ramsey space. \square

Lemma 3.1.5 (T. [48]). *For each sequence $S_0^* \geq S_1^* \geq S_2 \dots$ of elements of \mathcal{R}_1^* there exists $S \in \mathcal{R}_1^*$ such that for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$.*

Proof. Let $S_0^* \geq S_1^* \geq S_2 \dots$ be a decreasing sequence in \mathcal{R}_1^* . Hence, there is a strictly increasing sequence $(k_i)_{i < \omega}$ of natural numbers such that for all $i < \omega$ and all $j < i$, $S_{i+1} \setminus r_{k_i}(S_{i+1}) \subseteq S_j$. For each $i < \omega$, let $S(i)$ be an element of $\binom{S_{i+1} \setminus r_{k_i}(S_{i+1})}{T^*(i)}$. Let $S = \bigcup_{i < \omega} S(i)$. Then $S \in \mathcal{R}_1^*$ and for all $i < \omega$, $S \setminus r_i(S) \subseteq S_i$. \square

Next we define maps γ and Γ that will be used to transfer an ultrafilter on $[T_1^*]$ generated by a subset of \mathcal{R}_1^* to an ultrafilter on $[T_1]$ generated by a subset of \mathcal{R}_1 .

Definition 3.1.6 (T. [48]). Let $\{t_0, t_1, t_2, \dots\}$ and $\{s_0, s_1, s_2, \dots\}$ be the lexicographically increasing enumeration of $[T_1]$ and $[T_1^*]$, respectively. Let $\gamma : [T_1^*] \rightarrow [T_1]$ such that for all $i < \omega$,

$$\gamma(s_i) = t_i. \tag{3.1.4}$$

Let $\Gamma : \mathcal{R}_1^* \rightarrow \mathcal{R}_1$ be the map given by

$$\Gamma(S) = cl(\gamma''[S]). \tag{3.1.5}$$

Remark 3.1.7. γ is bijective and Γ is injective but not surjective.

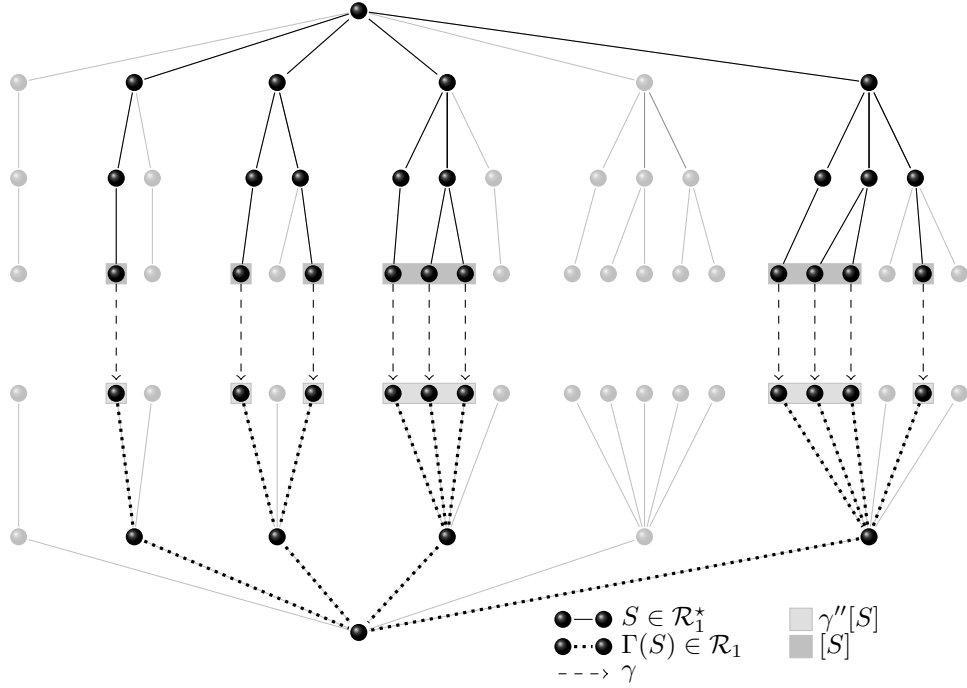


Figure 3.2: Graph of $S \in \mathcal{R}_1^*$ and $\Gamma(S) \in \mathcal{R}_1$.

Theorem 3.1.8 (T. [48]). $(\mathcal{R}_1^*, \leq^*)$ is σ -closed, and if \mathcal{G} is a generic filter for $(\mathcal{R}_1^*, \leq^*)$ over some ground model V , then $\Gamma''\mathcal{G}$ generates an ultrafilter on $[T_1]$ that is selective for \mathcal{R}_1 but not Ramsey for \mathcal{R}_1 in $V[\mathcal{G}]$.

Proof. By Lemma 3.1.5, $(\mathcal{R}_1^*, \leq^*)$ is a σ -closed partial order. Suppose that \mathcal{G} is a generic filter for $(\mathcal{R}_1^*, \leq^*)$ and $X \subseteq [T_1]$. By the previous remarks, X is in the ground model V . Since $[T_1]$ is in bijective correspondence with $\binom{T_1^*}{T_1^*(0)}$, Lemma 3.1.3 shows that for each $T \in \mathcal{R}_1^*$ there exists $S \in \binom{T}{T_1^*}$ such that either $\gamma''[S] \subseteq X$ or $\gamma''[S] \cap X = \emptyset$. Hence,

$$\Delta_X = \{S \in \mathcal{R}_1^* : [\Gamma(S)] \subseteq X \text{ or } [\Gamma(S)] \cap X = \emptyset\} \quad (3.1.6)$$

is dense in $(\mathcal{R}_1^*, \leq^*)$. Since \mathcal{G} is generic, for each $X \subseteq [T_1]$, $\mathcal{G} \cap \Delta_X \neq \emptyset$. In

particular, for each $X \subseteq [T_1]$ there exists $S \in \mathcal{G}$ such that $[\gamma(S)] \subseteq X$ or $[\gamma(S)] \cap X = \emptyset$. Therefore $\Gamma''\mathcal{G}$ generates an ultrafilter on $[T_1]$. Let \mathcal{V}_1 denote the ultrafilter on $[T_1]$ generated by $\Gamma''\mathcal{G}$.

Let (S_0, S_1, \dots) be a sequence of elements of \mathcal{R}_1 such that $\Gamma(S_0) \geq \Gamma(S_1) \geq \Gamma(S_2) \geq \dots$ is a decreasing sequence. Since (\mathcal{R}_1, \leq^*) is σ -closed,

$$\Delta_{(S_0, S_1, \dots)} = \bigcup_{i < \omega} \{S \in \mathcal{R}_1^* : S \cap S_i = \{\langle \rangle\}\} \cup \bigcap_{i < \omega} \{S \in \mathcal{R}_1^* : S \setminus r_i(S) \subseteq S_i\} \quad (3.1.7)$$

is in the ground model V .

Next we show that $\Delta_{(S_0, S_1, \dots)}$ is dense in $(\mathcal{R}_1^*, \leq^*)$. To this end, suppose that $T \in \mathcal{R}_1^*$. Since \mathcal{R}_1^* forms a topological Ramsey space, either there exists $S' \leq T$ and $i < \omega$ such that $S_i \cap S' = \{\langle \rangle\}$ or there exists a sequence (S'_0, S'_1, \dots) in $[\emptyset, T]$ such that for each $i < \omega$, $S'_i \leq S_i$. In the first case, $S' \leq^* T$ and $S' \in \Delta_{(S_0, S_1, \dots)}$. In the second case, Lemma 3.1.5 shows that there exists $S \leq T$ such that $S \setminus r_i(S) \subseteq S'_i \subseteq S_i$. In particular, $S \in \Delta_{(S_0, S_1, \dots)}$. In both cases, there exists $S \leq T$ such that $S \in \Delta_{(S_0, S_1, \dots)}$. Therefore $\Delta_{(S_0, S_1, \dots)}$ is dense in $(\mathcal{R}_1^*, \leq^*)$.

Suppose that the sequence $\Gamma(S_0) \geq \Gamma(S_1) \geq \Gamma(S_2) \dots$ consists of members $\Gamma''\mathcal{G}$. Then $\mathcal{G} \cap \Delta_{(S_0, S_1, \dots)} \neq \emptyset$ shows that there exists $S \in \mathcal{G}$ such that for all $i < \omega$, $\Gamma(S) \setminus r_i(\Gamma(S)) \subseteq \Gamma(S_i)$. Therefore \mathcal{V}_1 is a selective for \mathcal{R}_1 ultrafilter on $[T_1]$.

Next we construct a partition of $\binom{T_1}{r_2(T_1)}$ and show that the partition witnesses that \mathcal{V}_1 is not Ramsey for \mathcal{R}_1 . For each $s \in \binom{T_1}{r_2(T_1)}$, we let s' and s'' denote the two lexicographically smallest elements of $[s \setminus r_1(s)]$. Notice that for each $s \in \binom{T_1}{r_2(T_1)}$ the length of the longest common initial segment of $\gamma^{-1}(s')$ and $\gamma^{-1}(s'')$ is either 1 or 2. For each $j < 2$, let Π_j denote the set of all $s \in \binom{T_1}{r_2(T_1)}$ such that the length of the longest common initial segment of $\gamma^{-1}(s')$ and $\gamma^{-1}(s'')$ is $1 - j$. For each

$S \in \Gamma''\mathcal{G}$, $\binom{S}{r_2(T_1)}$ is neither a subset of Π_0 nor Π_1 . Therefore \mathcal{V}_1 is not a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$. \square

3.2 Selective but not Ramsey for \mathcal{R}_n

The purpose of this section is to introduce a closely related triple $(\mathcal{R}_n^*, \leq, r)$ and show that forcing with \mathcal{R}_n^* using almost-reduction, adjoins an ultrafilter that is selective but not Ramsey for \mathcal{R}_n .

Definition 3.2.1 ($(\mathcal{R}_n^*, \leq, r)$, T. [48]). Assume n is a positive integer and T_1^*, T_2^*, \dots and T_n^* have been defined. For each $i < \omega$, let

$$T_{n+1}^*(i) = \left\{ \langle \rangle, \langle i \rangle, \langle i \rangle \frown s : s \in T_n^*(j) \ \& \ \frac{i(i+1)}{2} \leq j < \frac{(i+1)(i+2)}{2} \right\}. \quad (3.2.1)$$

Let $T_{n+1}^* = \bigcup_{i < \omega} T_{n+1}^*(i)$ and $(\mathcal{R}_{n+1}^*, \leq, r)$ denote the triple $(\mathcal{R}(T_{n+1}^*), \leq, r)$.

The next lemma isolates the argument needed to show by induction that for each positive integer n , \mathcal{R}_n^* satisfies A.1-A.4 and forms a topological Ramsey space.

Lemma 3.2.2 (T. [48]). *For each positive integer l , if \mathcal{R}_l^* forms a topological Ramsey space then \mathcal{R}_{l+1}^* satisfies A.4.*

Proof. Suppose that l is a positive integer and \mathcal{R}_l^* forms a topological Ramsey space. By Theorem 3.5 of Mijares in [36] applied to \mathcal{R}_l^* , we find that for each pair of integers k and n with $k \leq n$ there exists $m < \omega$ such that

$$r_m(T_l^*) \rightarrow (r_{(n+1)(n+2)/2}(T_l^*))^{r_{(k+1)(k+2)/2}(T_l^*)}. \quad (3.2.2)$$

Next we prove a partition relation needed to show that **A.4** holds. Suppose that $\{\Pi_0, \Pi_1\}$ is a partition of $\binom{T_{l+1}^*(m)}{T_{l+1}^*(k)}$. Let m' be the unique integer such that $m'(m'+1) \leq 2m < (m'+1)(m'+2)$. Define a new partition $\{\bar{\Pi}_0, \bar{\Pi}_1\}$ of $\binom{r_m(T_l^*)}{r_{(k+1)(k+2)/2}(T_l^*)}$ by letting,

$$\bar{\Pi}_0 = \left\{ S \in \binom{r_m(T_l^*)}{r_{(k+1)(k+2)/2}(T_l^*)} : cl(\{\langle m' \rangle \wedge s : s \in S \setminus r_{k(k+1)/2}(S)\}) \in \Pi_0 \right\} \quad (3.2.3)$$

and $\bar{\Pi}_1$ be its complement. By equation (3.2.2) there exists $S \in \binom{r_m(T_l^*)}{r_{(n+1)(n+2)/2}(T_l^*)}$ and $j < 2$ such that $\binom{S}{r_{(k+1)(k+2)/2}(T_l^*)} \subseteq \bar{\Pi}_j$.

If we let $U = cl(\{\langle m' \rangle \wedge s : s \in S \setminus r_{n(n+1)/2}(S)\})$ then $U \in \binom{T_{l+1}^*(m)}{T_{l+1}^*(n)}$ and $\binom{U}{T_{l+1}^*(k)} \subseteq \Pi_j$. Therefore for each pair of positive integers k and n with $k \leq n$,

$$T_{l+1}^*(m) \rightarrow (T_{l+1}^*(n))^{T_{l+1}^*(k)}. \quad (3.2.4)$$

By the definition of \mathcal{R}_{l+1} , **A.4** for \mathcal{R}_{l+1} is equivalent to the following: for all $k < \omega$, $T_{l+1}^* \rightarrow (T_{l+1}^*)^{T_{l+1}^*(k)}$. To show that this partition relation holds, let k be a positive integer and $\{\Pi_0, \Pi_1\}$ be a partition of $\binom{T_{l+1}^*}{T_{l+1}^*(k)}$. Since n and k were arbitrary in the proof of (3.2.4), there exists a strictly increasing sequence $(m_n)_{n < \omega}$ such that for each $n < \omega$, $T_{l+1}^*(m_n) \rightarrow (T_{l+1}^*(n))^{T_{l+1}^*(k)}$. Let (S_0, S_1, \dots) be a sequence of trees such that for each $n < \omega$, $S_n \in \binom{T_{l+1}^*(m_n)}{T_{l+1}^*(i)}$ and $\binom{S_n}{T_{l+1}^*(k)}$ is contained in one piece of the partition $\{\Pi_0, \Pi_1\}$. By the pigeonhole principle, there exists $j < 2$ and a strictly increasing sequence (n_0, n_1, \dots) such that for all $i < \omega$, $S_{n_i} \in \binom{T_{l+1}^*(m_{n_i})}{T_{l+1}^*(n_i)}$ and $\binom{S_{n_i}}{T_{l+1}^*(k)} \subseteq \Pi_j$. If $S = \bigcup_{i < \omega} S_{n_i}$ and S' is any element of $\binom{S}{T_{l+1}^*}$ then $\binom{S'}{T_{l+1}^*(k)} \subseteq \Pi_j$. Therefore for each positive integer k , $T_{l+1}^* \rightarrow (T_{l+1}^*)^{T_{l+1}^*(k)}$ holds. Equivalently, **A.4** holds for \mathcal{R}_{l+1} . \square

Theorem 3.2.3 (T. [48]). *For each positive integer n , $(\mathcal{R}_n^*, \leq, r)$ satisfies A.1-A.4 and forms a topological Ramsey space.*

Proof by induction on n . The base case when $n = 1$ follows from Theorem 3.1.2. Suppose that \mathcal{R}_n^* satisfies A.1-A.4 and forms a topological Ramsey space. By the abstract Ellentuck theorem it is enough to show that $(\mathcal{R}_{n+1}^*, \leq, r)$ satisfies A.1-A.4 and forms a closed subspace of $(\mathcal{AR}_{n+1})^\omega$. The proof that \mathcal{R}_{n+1}^* is a closed subspace of $(\mathcal{AR}_{n+1})^\omega$ and satisfies axioms A.1-A.3 follows by trivial modifications to the proofs of the same facts for the space \mathcal{R}_{n+1} in [18]. For this reason, we omit the proof that \mathcal{R}_{n+1}^* forms a closed subspace of $(\mathcal{AR}_{n+1})^\omega$ and satisfies axioms A.1-A.3. The induction hypothesis and Lemma 3.2.2 show that A.4 holds for \mathcal{R}_{n+1}^* . By the abstract Ellentuck theorem, \mathcal{R}_{n+1}^* forms a topological Ramsey space. \square

Next, for each positive integer n , we define maps γ_n and Γ_n that will be used to transfer an ultrafilter on $[T_n^*]$ generated by a subset of \mathcal{R}_n^* to an ultrafilter on $[T_n]$ generated by a subset of \mathcal{R}_n .

Definition 3.2.4 (T. [48]). Let $\{t_0, t_1, t_2, \dots\}$ and $\{s_0, s_1, s_2, \dots\}$ be the lexicographically increasing enumeration of $[T_n]$ and $[T_n^*]$, respectively. Let $\gamma_n : [T_n^*] \rightarrow [T_n]$ such that for all $i < \omega$,

$$\gamma_n(s_i) = t_i. \tag{3.2.5}$$

Let $\Gamma_n : \mathcal{R}_n^* \rightarrow \mathcal{R}_n$ be the map given by

$$\Gamma_n(S) = cl(\gamma_n''[S]). \tag{3.2.6}$$

Remark 3.2.5. Let n be a positive integer. γ_n is bijective and Γ_n is injective but not surjective.

Theorem 3.2.6 (T. [48]). *Let n be a positive integer. $(\mathcal{R}_n^*, \leq^*)$ is σ -closed, and if \mathcal{G} is a generic filter for $(\mathcal{R}_n^*, \leq^*)$ over some ground model V , then $\Gamma_n''\mathcal{G}$ generates an ultrafilter on $[T_n]$ that is selective for \mathcal{R}_n but not Ramsey for \mathcal{R}_n in $V[\mathcal{G}]$.*

Proof. The proof that $(\mathcal{R}_n^*, \leq^*)$ is σ -closed, and if \mathcal{G} is a generic filter for $(\mathcal{R}_n^*, \leq^*)$ over some ground model V , then $\Gamma_n''\mathcal{G}$ generates an ultrafilter on $[T_n]$ that is selective for \mathcal{R}_n is completely analogous to the the proof for \mathcal{R}_1 given in the previous section. The only difference in the argument is an application of Theorem 3.2.3 instead of Theorem 3.1.4. So we omit the proof and let \mathcal{V}_n denote the selective for \mathcal{R}_n ultrafilter generated by $\Gamma_n''\mathcal{G}$.

Next we construct a partition of $\binom{T_n}{r_2(T_n)}$ and show that the partition witnesses that \mathcal{V}_n is not Ramsey for \mathcal{R}_n . For each $s \in \binom{T_n}{r_2(T_1)}$, we let s' and s'' denote the two lexicographically smallest elements of $[s \setminus r_1(s)]$. Notice that for each $s \in \binom{T_n}{r_2(T_n)}$ the length of the longest common initial segment of $\gamma^{-1}(s')$ and $\gamma^{-1}(s'')$ is either n or $n - 1$. For each $j < 2$, let Π_j denote the set of all $s \in \binom{T_n}{r_2(T_n)}$ such that length of the longest common initial segment of $\gamma^{-1}(s')$ and $\gamma^{-1}(s'')$ is $n - j$. For each $S \in \Gamma_n''\mathcal{G}$, $\binom{S}{r_2(T_n)}$ is neither a subset of Π_0 nor Π_1 . Since \mathcal{V}_n is generated by $\Gamma_n''\mathcal{G}$ it is not a Ramsey for \mathcal{R}_n ultrafilter on $[T_n]$. \square

3.3 Selective but not Ramsey for $\bigotimes_{i=0}^n \mathcal{R}(S_i)$

In Sections 3.1 and 3.2 we only considered trees on ω . In this section, in order to introduce the product of two spaces of the form $\mathcal{R}(T)$ and $\mathcal{R}(S)$ we must consider trees on ω^2 . Dobrinen, Mijares and Trujillo in [15] have introduced a notion of product among special types of topological Ramsey spaces. Included among these special spaces are $\mathcal{R}(T_i)$ and $\mathcal{R}^*(T_i)$ for $i < \omega$. In fact, for such spaces the product

of $\mathcal{R}(S)$ and $\mathcal{R}(T)$ is defined by introducing the tree $S \otimes T$ on ω^2 and letting $\mathcal{R}(S) \otimes \mathcal{R}(T)$ denote the triple $(\mathcal{R}(S \otimes T), \leq r)$.

The simplest possible non-trivial product space is $\mathcal{R}_1 \otimes \mathcal{R}_1$ which we denote by \mathcal{H}^2 (see Example 2.2.9). It was first considered by Dobrinen, Mijares and Trujillo in [15]. The construction in [15] was inspired by the work of Blass in [3] which uses forcing to adjoin a p-point ultrafilter having two Rudin-Keisler incomparable predecessors, and subsequent work of Dobrinen and Todorčević in [16] which shows that the same forcing adjoins a p-point ultrafilter with two Tukey-incomparable p-point Tukey-predecessors. In Chapter 5, we show that forcing with \mathcal{H}^2 using almost-reduction adjoins an ultrafilter whose Rudin-Keisler predecessors form a four element Boolean algebra. Before giving the general construction of the finite product we give the precise definition of the prototype example \mathcal{H}^2 .

Definition 3.3.1 ((\mathcal{H}^2, \leq, r) , Dobrinen, Mijares and T. [15]). Let

$$T_1 \otimes T_1 = \bigcup_{i < \omega} cl(\{(s_j, t_j)_{j < |s|} \in [\omega^2]^{< \omega} : s, t \in [T_1(i)]\}). \quad (3.3.1)$$

We let (\mathcal{H}^2, \leq, r) denote the space $(\mathcal{R}(T_1 \otimes T_1), \leq, r)$.

Remark 3.3.2. By the previous definition, for each $i < \omega$,

$$T_1 \otimes T_1(i) = cl(\{\langle (i, i), (j, k) \rangle : \frac{i(i+1)}{2} \leq j, k < \frac{(i+1)(i+2)}{2}\})$$

and $T_1 \otimes T_1 = \bigcup T_1 \otimes T_1(i)$. The elements of \mathcal{H}^2 are subtrees of $T_1 \otimes T_1$ that are isomorphic to $T_1 \otimes T_1$ (see Example 2.2.9).

The main theorem of this section implies that forcing with the similarly defined product $\mathcal{R}_1^* \otimes \mathcal{R}_1^*$ using almost-reduction adjoins an ultrafilter on $[T_1 \otimes T_1]$ that is

selective but not Ramsey for \mathcal{H}^2 . In order to define the general finite product, we introduce a notion of finite product among the trees $\{T_i, T_i^*, i < \omega\}$. If $S_0 \otimes \cdots \otimes S_{k-1}$ is a finite product of k such trees then $S_0 \otimes \cdots \otimes S_{k-1}$ forms a tree on ω^k . For example, $T_1 \otimes T_1$ is a tree on ω^2 .

Definition 3.3.3. Suppose k and k' are positive integers and s and t are finite sequences of k -tuples and k' -tuples, respectively, such that $|s| = |t|$. We let (s, t) denote the sequence on $\omega^{k+k'}$ given by $(s_i, t_i)_{i < k+k'}$.

For example if $s = \langle 1, 2 \rangle$ and $t = \langle 3, 4 \rangle$ then (s, t) denotes the sequence $\langle (1, 3), (2, 4) \rangle$ on ω^2 . Before giving the definition of the product of two triples we introduce the product of two trees.

Definition 3.3.4. Let S and T be trees on ω^k and $\omega^{k'}$, respectively. Assume that for all $s, t \in [S] \cup [T]$, $|s| = |t|$, $\pi_0''[S] = \{\langle (n, \dots, n) \rangle \in \omega^k : n < \omega\}$, and $\pi_0''[T] = \{\langle (n, \dots, n) \rangle \in \omega^{k'} : n < \omega\}$. We let

$$S \otimes T = \bigcup_{i < \omega} cl(\{(s, t) \in (\omega^{(k+k')})^{<\omega} : s \in [S(i)] \ \& \ t \in [T(i)]\}). \quad (3.3.2)$$

Remark 3.3.5. For positive integer n and each $m > n$, there exists $s, t \in [S] \cup [T]$ such that $|s| \neq |t|$. Therefore the product $T_n \otimes T_m$ is not well-defined. For each $m > n$, the elements of $[T_n]$ can be extended to length m sequences by repeating the last element of a given sequence for the final $(m - n)$ -elements of the extended sequence. (For example, the length 2 sequence $(2, 3)$ would extend to the length 4 sequence $(2, 3, 3, 3)$.) The space constructed using the extended tree is isomorphic to the original space. In this way the product $T_n \otimes T_m$ is well-defined.

Definition 3.3.6 (T. [48]). Suppose that $\langle S_i : i \leq n \rangle$ is a finite sequence of trees

where each S_i is one of the trees T_j for some $j < \omega$. Without loss of generality we may extend all of the trees so that for each $s, t \in \bigcup_{i \leq n} [S_i]$, $|s| = |t|$. If $n = 1$ then we let $\bigotimes_{i=0}^1 S_i = S_0 \otimes S_1$. If $n > 1$ then we recursively define the product by letting, $\bigotimes_{i=0}^n S_i = S_n \otimes \bigotimes_{i=0}^{n-1} S_i$. For each sequence let, $\bigotimes_{i=0}^n \mathcal{R}(S_i) = \mathcal{R}(\bigotimes_{i=0}^n S_i)$ and $\bigotimes_{i=0}^n \mathcal{R}^*(S_i) = \mathcal{R}(\bigotimes_{i=0}^n S_i^*)$.

The next theorem follows from a more general theorem in [15] about products of sequences of structures in a relational language.

Theorem 3.3.7 (Dobrinen, Mijares and T., [15]). *If $\langle S_i : i \leq n \rangle$ is a finite sequence of trees where each S_i is one of the trees T_j or T_j^* for some $j < \omega$, then $(\bigotimes_{i=0}^n \mathcal{R}(S_i), \leq, r)$ satisfies A.1-A.4 and forms a topological Ramsey space.*

In Chapter 5, we give a complete proof of the previous theorem for the space \mathcal{H}^2 and develop its canonical Ramsey theory.

Definition 3.3.8 (T. [48]). Suppose that $\langle S_i : i \leq n \rangle$ is a finite sequence of trees where each S_i is one of the trees T_j for some $j < \omega$. Let $\{t_0, t_1, t_2, \dots\}$ and $\{s_0, s_1, s_2, \dots\}$ be a lexicographically non-decreasing enumeration of $[\bigotimes_{i=0}^n S_i]$ and $[\bigotimes_{i=0}^n S_i^*]$, respectively. Let $\gamma : [\bigotimes_{i=0}^n S_i] \rightarrow [\bigotimes_{i=0}^n S_i^*]$ such that for all $i < \omega$,

$$\gamma(s_i) = t_i. \quad (3.3.3)$$

Let $\Gamma : \bigotimes_{i=0}^n \mathcal{R}^*(S_i) \rightarrow \bigotimes_{i=0}^n \mathcal{R}(S_i)$ be the map given by

$$\Gamma(S) = cl(\gamma''[S]). \quad (3.3.4)$$

Remark 3.3.9. For each $\langle S_i : i \leq n \rangle$ sequence, γ is bijective and Γ is injective but not surjective.

Theorem 3.3.10 (T. [48]). *Suppose that $\langle S_i : i \leq n \rangle$ is a finite sequence of trees where each S_i is one of the trees T_j for some $j < \omega$. $(\bigotimes_{i=0}^n \mathcal{R}^*(S_i), \leq^*)$ is σ -closed, and if \mathcal{G} is a generic filter for $(\bigotimes_{i=0}^n \mathcal{R}^*(S_i), \leq^*)$ over some ground model V , then $\Gamma''\mathcal{G}$ generates an ultrafilter on $[\bigotimes_{i=0}^n S_i]$ that is selective for $\bigotimes_{i=0}^n \mathcal{R}(S_i)$ but not Ramsey for $\bigotimes_{i=0}^n \mathcal{R}(S_i)$ in $V[\mathcal{G}]$.*

Proof. The proof that $(\bigotimes_{i=0}^n \mathcal{R}^*(S_i), \leq^*)$ is σ -closed, and if \mathcal{G} is a generic filter for $(\bigotimes_{i=0}^n \mathcal{R}^*(S_i), \leq^*)$ over some ground model V , then $\Gamma''\mathcal{G}$ generates an ultrafilter on $[\bigotimes_{i=0}^n S_i]$ that is selective for $\bigotimes_{i=0}^n \mathcal{R}(S_i)$ is completely analogous to the the proof for \mathcal{R}_1 given in the Section 3.3. The only difference in the argument is an application of Theorem 3.3.7 instead of Theorem 3.1.4. So we omit the proof and let \mathcal{V} denote the selective for $\bigotimes_{i=0}^n \mathcal{R}^*(S_i)$ ultrafilter generated by $\Gamma''\mathcal{G}$.

Let τ_0 be the map which takes a tree on ω^{n+1} to a tree on ω by sending each sequence of $(n + 1)$ -tuples to the sequence of first elements of the $(n + 1)$ -tuple. (For example, if $n = 2$ and $S = cl(\{\langle (1, 3), (2, 4) \rangle\})$ then $\tau_0(S) = cl(\{\langle 1, 2 \rangle\})$).

Next we construct a partition of $(\bigotimes_{i=0}^n S_i, r_2(\bigotimes_{i=0}^n S_i))$ and show that the partition witnesses that \mathcal{V} is not Ramsey for $\bigotimes_{i=0}^n \mathcal{R}^*(S_i)$. For each $s \in (\bigotimes_{i=0}^n S_i, r_2(\bigotimes_{i=0}^n S_i))$, we let s' and s'' denote any two elements of $[s \setminus r_1(s)]$ such that $\tau_0(s'), \tau_0(s'')$ are the two lexicographically smallest elements of $[\tau_0(s) \setminus r_1(\tau_0(s))]$. Notice that for each $s \in (\bigotimes_{i=0}^n S_i, r_2(\bigotimes_{i=0}^n S_i))$ the length of the longest common initial segment of $\gamma^{-1}(\tau_0(s'))$ and $\gamma^{-1}(\tau_0(s''))$ is either j or $j - 1$ where j is that natural number such that $S_0 = T_j$. For each $k < 2$, let Π_k denote the set of all $s \in (\bigotimes_{i=0}^n S_i, r_2(\bigotimes_{i=0}^n S_i))$ such that length of the longest common initial segment of $\gamma^{-1}(\tau_0(s'))$ and $\gamma^{-1}(\tau_0(s''))$ is $j - k$. For each $S \in \Gamma''\mathcal{G}$, $(\bigotimes_{i=0}^n S_i, r_2(\bigotimes_{i=0}^n S_i))$ is neither a subset of Π_0 nor Π_1 . Since \mathcal{V} is generated by $\Gamma''\mathcal{G}$, it is not a Ramsey for $\bigotimes_{i=0}^n \mathcal{R}(S_i)$ ultrafilter on $[\bigotimes_{i=0}^n S_i]$. \square

3.4 Open questions

For each positive integer k , we have presented countably many examples of trees on ω^k where forcing can be used to adjoin an ultrafilter on $[T]$ that is selective but not Ramsey for $\mathcal{R}(T)$. In each case, a new topological Ramsey space $\mathcal{R}^*(T)$, a map $\Gamma : \mathcal{R}^*(T) \rightarrow \mathcal{R}(T)$ and a partition $\{\Pi_0, \Pi_1\}$ of $\binom{T}{r_2(T)}$ were constructed in such a way that for all $S \in \Gamma''\mathcal{R}^*(T)$, neither $\binom{S}{r_2(T)} \subseteq \Pi_0$ nor $\binom{S}{r_2(T)} \subseteq \Pi_1$. The main results follow by showing that if \mathcal{G} is generic for $(\mathcal{R}^*(T), \leq^*)$ then $\Gamma''\mathcal{G}$ generates a selective but not Ramsey for $\mathcal{R}(T)$ ultrafilter on $[T]$. In each case, the partition $\{\Pi_0, \Pi_1\}$ witnesses that the generated ultrafilter cannot be Ramsey for $\mathcal{R}(T)$.

Dobrinen and Todorćević in [18] have also introduced the spaces \mathcal{R}_α where $\omega \leq \alpha < \omega_1$. These spaces are constructed from well-founded trees of unbounded height in a slightly different manner than those considered in this dissertation. In these cases it is possible to construct trees T_α and T_α^* , and a modified version of \mathcal{R}_α^* for $\omega \leq \alpha < \omega_1$. However, the partition given in the finite case can not be extended to the case for $\omega \leq \alpha < \omega_1$ since the trees being used have unbounded height. In particular the next question remains open.

Question 3.4.1. For α between ω and ω_1 , are the notions of selective for \mathcal{R}_α and Ramsey for \mathcal{R}_α equivalent?

For each positive integer n , let \mathcal{H}^n denote the space $\otimes_{i=1}^n \mathcal{R}_1$. By Theorem 3.3.10 forcing with $\otimes_{i=1}^n \mathcal{R}_1^*$ using almost-reduction adjoins an ultrafilter that is selective but not Ramsey for \mathcal{H}^n . Dobrinen, Mijares and Trujillo in [14] have also defined the topological Ramsey spaces \mathcal{H}^α for $\omega \leq \alpha < \omega_1$. For similar reasons to the \mathcal{R}_α , case our methods fail to produce an ultrafilter that is selective but not Ramsey for \mathcal{H}^α . Hence, our next question also remains open.

Question 3.4.2. For α between ω and ω_1 , are the notions of selective for \mathcal{H}^α and Ramsey for \mathcal{H}^α equivalent?

All the spaces studied in this dissertation except the Ellentuck space either support ultrafilters which are selective but not Ramsey for the space, or it is unknown if the notions of selective and Ramsey for the space are equivalent. In fact, the following question remains open.

Question 3.4.3. Is the Ellentuck space the only topological Ramsey space for which the notions of selective and Ramsey for the space are equivalent?

Chapter 4

Ramsey for \mathcal{R}_1 ultrafilters & their Dedekind cuts

The work in this chapter has been submitted for publication by the author in [47]. Associated to each ultrafilter \mathcal{U} on ω and each map $p : \omega \rightarrow \omega$ is a Dedekind cut in the ultrapower $\omega^\omega/p(\mathcal{U})$. Blass has characterized, under the continuum hypothesis, the cuts obtainable when \mathcal{U} is taken to be either a p-point ultrafilter, a weakly-Ramsey ultrafilter, or a Ramsey ultrafilter.

Dobrinen and Todorćević have introduced the topological Ramsey space \mathcal{R}_1 . Associated to the space \mathcal{R}_1 is a notion of Ramsey ultrafilter for \mathcal{R}_1 generalizing the familiar notion of Ramsey ultrafilter on ω (see Definition 2.3.1). We characterize, under the continuum hypothesis, the cuts obtainable when \mathcal{U} is taken to be a Ramsey for \mathcal{R}_1 ultrafilter and p is taken to be any map. In particular, we show that the only cut obtainable is the standard cut, whose lower half consists of the collection of equivalence classes of constant maps.

Forcing with \mathcal{R}_1 using almost-reduction adjoins an ultrafilter which is Ramsey

for \mathcal{R}_1 (see Theorem 2.4.5). For such ultrafilters \mathcal{U}_1 , Dobrinen and Todorćević motivated by the work of Laflamme in [31], have shown that the Rudin-Keisler types of the p-points within the Tukey type of \mathcal{U}_1 consists of a strictly increasing chain of rapid p-points of order type ω . We show that for any Rudin-Keisler mapping between any two p-points within the Tukey type of \mathcal{U}_1 , the only cut obtainable is the standard cut. These results imply existence theorems for special kinds of ultrafilters.

4.1 Ultrafilter mappings and their Dedekind cuts

In this section, we define the notion of a Rudin-Keisler mapping and associate to each mapping a Dedekind cut. Then we state some results of Blass in [4] characterizing, under the continuum hypothesis, the types of cuts obtainable for Rudin-Keisler mappings from a p-point or a weakly-Ramsey ultrafilter on ω . In last part of this section, we provide an outline of the rest of the chapter and highlight its main results.

We remind the reader of the Rudin-Keisler reducibility relation. If \mathcal{U} is an ultrafilter on the base set X and \mathcal{V} is an ultrafilter on the base set Y , then we say that \mathcal{V} is *Rudin-Keisler reducible to \mathcal{U}* and write $\mathcal{V} \leq_{RK} \mathcal{U}$ if there there exists a function $f : X \rightarrow Y$ such that $\mathcal{V} = f(\mathcal{U})$, where

$$f(\mathcal{U}) = \langle \{f(Z) : Z \in \mathcal{U}\} \rangle. \quad (4.1.1)$$

A *Rudin-Keisler mapping from \mathcal{U} to \mathcal{V}* is a function $f : X \rightarrow Y$ such that $\mathcal{V} = f(\mathcal{U})$.

Associated to each ultrafilter \mathcal{U} on X is an equivalence relation on ω^X . If f and g are two functions from X to ω then we say that f and g are *equivalent mod \mathcal{U}* if there exists $Z \in \mathcal{U}$ such that $f \upharpoonright Z = g \upharpoonright Z$. The *ultrapower ω^X/\mathcal{U}* is the collection of

all equivalence classes with respect to this equivalence. All operations and relations defined on ω have natural extensions making the ultrapower an elementary extension of the standard model of ω . In particular, ω^X/\mathcal{U} forms a linearly ordered set. (In this case, $[f] \leq [g]$ if and only if $\{x \in X : f(x) \leq g(x)\} \in \mathcal{U}$.)

Recall that, a *Dedekind cut of a linearly ordered set* is a partition (S, L) of the linear order such that no element of L precedes any element of S . We follow the work of Blass in [4] and associate to each Rudin-Keisler mapping from \mathcal{U} on X to \mathcal{V} on Y a Dedekind cut in the ultrapower ω^Y/\mathcal{V} . A cut (S, L) in the ultrapower is said to be *proper* if L is nonempty and S contains the equivalence class of each constant map. The cut given by taking S to be the set of equivalence classes of constant maps is called *the standard cut*.

Definition 4.1.1 (Blass, [4]). Let \mathcal{U} be an ultrafilter on the base set X and $p : X \rightarrow Y$. For any $A \subseteq X$, we define *the cardinality function of A relative to p* by

$$C_A(y) = |A \cap p^{-1}\{y\}| \text{ for } y \in Y. \quad (4.1.2)$$

The set of all equivalence classes of cardinality functions of sets in \mathcal{U} , and all larger elements of $\omega^X/p(\mathcal{U})$, constitute the upper part L of a cut (S, L) of $\omega^X/p(\mathcal{U})$, which we call *the cut associated to p and \mathcal{U}* . (If $C_A(n)$ is infinite for some y then $C_A \notin \omega^X$ and has no equivalence class, so it makes no contribution to L ; it is entirely possible for L to be empty.)

The cut associated to p and \mathcal{U} is proper if and only if p is finite-to-one but not one-to-one on any set in \mathcal{U} . Additionally, the existence of a proper cut in ω^X/\mathcal{U} implies that \mathcal{U} is nonprincipal. The next three theorems are due to Blass and appear as Theorems 1, 2 and 4, in [4]. The first theorem shows that certain Dedekind cuts

are not obtainable from Rudin-Keisler mappings. In the remaining theorems of this section, (S, L) is assumed to be a proper cut.

Theorem 4.1.2 (Blass, [4]). *(S, L) is the cut associated to some map of some ultrafilter to \mathcal{U} if and only if \mathcal{U} is an ultrafilter and S is closed under addition.*

Before stating the next two theorems we remind the reader of the definitions of some special types of ultrafilters on ω .

Definition 4.1.3. Let \mathcal{U} be an ultrafilter on ω .

1. \mathcal{U} is a *p-point ultrafilter*, if for each sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ of members of \mathcal{U} there exists $A \in \mathcal{U}$ such that for each $i < \omega$, $A \subseteq^* A_i$. (Here \subseteq^* denotes the almost-inclusion relation.)
2. \mathcal{U} is a *weakly-Ramsey ultrafilter*, if for each partition of the two-element subsets of ω into three parts there exists an element of \mathcal{U} all of whose two-element subsets lie in two parts of the partition.
3. \mathcal{U} is a *Ramsey ultrafilter*, if for each partition of the two-element subsets of ω into two parts there exists an element of \mathcal{U} all of whose two-element subsets lie in exactly one part of the partition.

Remark 4.1.4. Weakly-Ramsey ultrafilters were introduced by Blass in [4]. It is clear that every Ramsey ultrafilter is weakly-Ramsey. In Theorem 5 of [4], Blass has shown that every weakly-Ramsey ultrafilter is a p-point.

Theorem 4.1.5 (Blass, [4]). *Assume the continuum hypothesis. (S, L) is the cut associated to some map of some p-point ultrafilter to \mathcal{U} if and only if \mathcal{U} is a p-point, S is closed under addition, and every countable subset of L has a lower bound in L .*

Theorem 4.1.6 (Blass, [4]). *Assume the continuum hypothesis. (S, L) is the cut associated to some map of some weakly Ramsey ultrafilter to \mathcal{U} if and only if \mathcal{U} is Ramsey, S is closed under exponentiation, and every countable subset of L has a lower bound in L .*

Blass has remarked in [4] that many of the ultrafilter-theoretic concepts involved in the previous theorems have natural model-theoretic interpretations in terms of ultrapowers. Following Blass, we consider models that are elementary extensions of the standard model whose universe is ω and whose relations and functions are all the relations and functions on ω . Suppose that N is such a model. An element $x \in N$ is said to *generate* N over the submodel M if and only if no proper submodel of N includes $M \cup \{x\}$.

In [4], Blass notes that, if $f : X \rightarrow Y$ and \mathcal{U} is an ultrafilter on X , then

$$f^* : \omega^Y / f(\mathcal{U}) \rightarrow \omega^X / \mathcal{U} \tag{4.1.3}$$

is an elementary embedding. Furthermore, f is an isomorphism of ultrafilters if and only if f^* is an isomorphism of models. The image of f^* is cofinal in ω^Y / \mathcal{U} if and only if f is finite-to-one on some set of \mathcal{U} . Hence, \mathcal{U} is a p-point (Ramsey ultrafilter) if and only if every nonstandard submodel of ω^X / \mathcal{V} is cofinal in (equal to) ω^X / \mathcal{U} . An element $x \in \omega^Y / f(\mathcal{V})$ is in the upper half of the cut associated to f and \mathcal{U} if and only if $f^*(x)$ is greater than some generator of ω^X / \mathcal{U} over $f^*(\omega^Y / f(\mathcal{U}))$.

Remark 4.1.7. Let \mathcal{U} be an ultrafilter on the base set X and $p : X \rightarrow Y$. Suppose that $g : W \rightarrow X$ is a bijection and \mathcal{W} is an ultrafilter on W such that $g(\mathcal{W}) = \mathcal{U}$. Then the cut associated to $p \circ g$ and \mathcal{W} is exactly the cut associated to p and \mathcal{U} . Additionally, suppose that $h : Y \rightarrow Z$ is a bijection. Since h^* is an isomorphism of

models, it follows that if (S, L) is the cut associated to $h \circ p$ and \mathcal{U} then $(h^{**}S, h^{**}L)$ is the cut associated to p and \mathcal{U} . In particular, if (S, L) is the standard cut in ω^ω then $(h^{**}S, h^{**}L)$ is the standard cut in $\omega^\omega/p(\mathcal{U})$.

The purpose of this chapter is to prove analogous results for ultrafilters satisfying similar properties. In Section 4.2, we remind the reader of the definition of the space used for the main results of this chapter, namely the topological Ramsey space \mathcal{R}_1 . In Section 4.3, we characterize the cuts obtainable, under the continuum hypothesis, from Ramsey for \mathcal{R}_1 ultrafilters. The next theorem which we prove in Section 4.4 is one of the two main results of this chapter.

Theorem 4.1.8 (T. [47]). *Assume the continuum hypothesis. (S, L) is the cut associated to some map of some Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ to \mathcal{V} if and only if \mathcal{V} is selective and (S, L) is the standard cut in $\omega^\omega/\mathcal{V}$.*

In Section 2.5, we introduced the basic definitions associated with the Tukey theory of ultrafilters. Applying theorems of Dobrinen and Todorćević in [17] about ultrafilters generated from generic subsets of (\mathcal{R}_1, \leq^*) , we prove the second main result of this chapter.

Theorem 4.1.9 (T. [47]). *Suppose \mathcal{U}_1 is a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ generated by a generic subset of (\mathcal{R}_1, \leq^*) . If (S, L) is the cut associated to some map from some p -point ultrafilter in the Tukey type of \mathcal{U}_1 to some ultrafilter \mathcal{V} then \mathcal{V} is a p -point ultrafilter and (S, L) is the standard cut.*

In Section 4.5, we show that the main results imply the existence of special ultrafilters. We then conclude with some questions about the types of cuts obtainable from ultrafilters defined from other similar topological Ramsey spaces.

4.2 The topological Ramsey space \mathcal{R}_1

In this section we collect together a theorem, a lemma, and a definition concerning the topological Ramsey space \mathcal{R}_1 .

Theorem 4.2.1 (The abstract Ellentuck theorem for \mathcal{R}_1 , [17]). *The triple (\mathcal{R}_1, \leq, r) forms a topological Ramsey space. That is, every subset of \mathcal{R}_1 with the Baire property is Ramsey and every meager subset of \mathcal{R}_1 is Ramsey null.*

The next theorem, which we use in Section 4.5, is equivalent to the finite version of the Ramsey theorem.

Theorem 4.2.2 (Ramsey theorem in terms of \mathcal{R}_1 , [17]). *Let $k, n < \omega$ with $k \leq n$ be given. Then, there exists $m < \omega$ such that for each $p \in \mathcal{R}_1(m)$ and each partition of $\mathcal{R}_1(k) \upharpoonright p$ into two parts there exists $q \in \mathcal{R}_1(n) \upharpoonright p$ such that $\mathcal{R}_1(k) \upharpoonright q$ lies in exactly one one part of the partition. In other words, for each $k, n < \omega$ with $k \leq n$, there exists $m < \omega$ such that*

$$T_1(m) \rightarrow (T_1(n))^{T_1(k)}. \quad (4.2.1)$$

Next we define a collection of natural projection maps related to the space \mathcal{R}_1 . Dobrinen and Todorćević in [17] have used these projections to completely characterize the Tukey ordering below weakly-Ramsey ultrafilters obtained from forcing with \mathcal{R}_1 using almost-reduction.

Definition 4.2.3. Each $x \in [T_1]$ is a sequence of natural numbers of length two, which we denote by $\langle x_0, x_1 \rangle$. Just as in Chapter 2, we let $\pi : [T_1] \rightarrow \omega$ be the map which sends $x \in [T_1]$ to x_0 . For each $i < j < \omega$, define $\pi_{T(i)} : \mathcal{R}_1(j) \rightarrow \mathcal{R}_1(i)$ to be the map that removes the right-most $j - i$ branches of a given element of $\mathcal{R}_1(j)$.

4.3 Ramsey for \mathcal{R}_1 ultrafilters and their Dedekind cuts

In this section, assuming the continuum hypothesis, we characterize the types of proper Dedekind cuts that can be obtained from a map $p : [T_1] \rightarrow \omega$ and a Ramsey for \mathcal{R}_1 ultrafilter. In the following theorems, all cuts are assumed to be proper.

Lemma 4.3.1 (T. [47]). *Let \mathcal{U} be a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ generated by $\mathcal{C} \subseteq \mathcal{R}_1$ and p be a map from $[T_1]$ to ω . The cut associated to p and \mathcal{U} is the standard cut in $\omega^\omega/p(\mathcal{U})$.*

Proof. Suppose that (S, L) is the cut associated to p and \mathcal{U} . Let $f : \omega \rightarrow \omega$ be given and suppose that f is not constant mod $p(\mathcal{U})$. For each $s \in \mathcal{AR}_2$ let $\{s_0, s_1, s_2\}$ be the lexicographically increasing enumeration of $[s]$. Let $\{\Pi_0, \Pi_1, \Pi_2\}$ be the partition of \mathcal{AR}_2 given by letting

$$\Pi_0 = \{s \in \mathcal{AR}_2 : p(s_0) < p(s_1) \ \& \ p(s_1) = p(s_2)\}, \quad (4.3.1)$$

$$\Pi_1 = \{s \in \mathcal{AR}_2 : p(s_0) < p(s_1) \ \& \ p(s_1) \neq p(s_2)\} \text{ and} \quad (4.3.2)$$

$$\Pi_2 = \{s \in \mathcal{AR}_2 : p(s_0) \geq p(s_1)\}. \quad (4.3.3)$$

Since \mathcal{R}_1 is Ramsey for \mathcal{R}_1 there exists $S' \in \mathcal{C}$ and $j < 3$ such that $\mathcal{AR}_2|S' \subseteq \Pi_j$. If $j = 2$ then p is bounded by $p(x) \bmod \mathcal{U}$ where x is the lexicographically least element of $[S']$. Thus, if $j = 2$ then p is constant mod \mathcal{U} and (S, L) is not a proper cut. If $j = 1$ then p is one-to-one mod \mathcal{U} , so (S, L) is not a proper cut. Hence, if (S, L) is a proper cut then $\mathcal{AR}_2|S' \subseteq \Pi_0$. In particular, if $T \leq S'$ and $(k_i)_{i < \omega}$ is the increasing enumeration of $\pi''[T]$, then for each $i < \omega$, $C_{[T]}(k_i) = |[T] \cap p^{-1}\{k_i\}| = i$.

For each $n < \omega$, let

$$X_n = \{x \in [S'] : f(p(x)) \geq n\}. \quad (4.3.4)$$

Since \mathcal{U} is an ultrafilter on $[T_1]$ we find that for each $n < \omega$, either $X_n \in \mathcal{U}$ or $[T_1] \setminus X_n \in \mathcal{U}$. If there exists $n < \omega$ such that $[T_1] \setminus X_n \in \mathcal{U}$ then f would be bounded by $n \bmod p(\mathcal{U})$. However, this can not happen since we assumed that f is not constant mod $p(\mathcal{U})$. Hence for each $n < \omega$, $X_n \in \mathcal{U}$.

Note that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ is a decreasing sequence of members of \mathcal{U} . Since \mathcal{U} is selective for \mathcal{R}_1 there exists $S'' \in \mathcal{C}$ such that for each $n < \omega$, $[S'' \setminus r_n(S'')] \subseteq X_n$. Let $\{k_n : n < \omega\}$ be the increasing enumeration of $p''[S'']$. Since $\mathcal{AR}_2|S'' \subseteq \Pi_0$, we find that for each $n < \omega$ and each $[S''] \cap p^{-1}\{k_n\} = [S''(n)]$. So for each $n < \omega$, $[S''] \cap p^{-1}\{k_n\} \subseteq [S'' \setminus r_n(S'')] \subseteq X_n$. Hence, for each $n < \omega$ and each $x \in [S''] \cap p^{-1}\{k_n\}$,

$$f(k_n) = f(p(x)) \geq n = |[S''] \cap p^{-1}\{k_n\}| = C_{[S'']}(k_n). \quad (4.3.5)$$

Since $\{k_n : n < \omega\} = p''[S''] \in p(\mathcal{U})$, we find that $[f] \geq [C_{[S'']}]$. Thus, $[f] \in L$ as $S'' \in \mathcal{C}$. Additionally, note that the cardinality function of any member of \mathcal{C} is not constant mod $p(\mathcal{U})$. Therefore the cut (S, L) is the standard cut in $\omega^\omega/p(\mathcal{U})$. \square

In the next Lemma and Theorem, \mathcal{V} is an ultrafilter on ω , and (S, L) is a proper cut in $\omega^\omega/\mathcal{V}$.

Lemma 4.3.2 (T. [47]). *Assume the continuum hypothesis. If \mathcal{V} is selective and (S, L) is the standard cut in $\omega^\omega/\mathcal{V}$ then there exists a Ramsey for \mathcal{R}_1 ultrafilter \mathcal{U} such that (S, L) is the cut associated to \mathcal{U} and π .*

Proof. Let \mathcal{V} be selective and (S, L) be the standard cut in $\omega^\omega/\mathcal{V}$. We will construct a \leq^* -decreasing sequence $\{A_\alpha \in \mathcal{R}_1 : \alpha < \omega_1\}$ that generates a Ramsey for \mathcal{R}_1 ultrafilter \mathcal{U} on $[T_1]$ such that $\pi(\mathcal{U}) = \mathcal{V}$. Let $\{Z_\alpha : \alpha < \omega_1\}$ be an enumeration of the elements of $\{Z : (\exists k)Z \subseteq \mathcal{R}_1(k)\}$. We impose the following requirements on the sequence, for each $\alpha < \omega_1$:

$$\text{Either } Z_\alpha \text{ or } \mathcal{R}_1(k) \setminus Z_\alpha \text{ includes } \mathcal{R}_1(k) \upharpoonright A_{\alpha+1} \quad (\alpha)$$

where k is the natural number such that $Z_\alpha \subseteq \mathcal{R}_1(k)$. Since $\mathcal{R}_1(0)$ is in bijective correspondence with $[T_1]$ and $\{A_\alpha \in \mathcal{R}_1 : \alpha < \omega_1\}$ is an almost-decreasing sequence, the sequence will generate a p-point ultrafilter on $[T_1]$. Each A_α we be *large* in the sense that $\pi''[A_\alpha] \in \mathcal{V}$; this suffices to guarantee that $\pi(\mathcal{U}) = \mathcal{V}$. By Theorem 2.5.1 this is enough to show that \mathcal{U} is selective for \mathcal{R}_1 . The conditions (α) for $\alpha < \omega_1$, guarantee that for each $k < \omega$, $\mathcal{U} \upharpoonright \mathcal{R}_1(k)$ is an ultrafilter on $\mathcal{R}_1(k)$. By Theorem 2.6.2 this is enough to show that \mathcal{U} is a Ramsey for \mathcal{R}_1 ultrafilter. By the previous lemma this guarantees that the cut associated to \mathcal{U} and p in $\omega^\omega/\mathcal{V}$ is standard.

First let $A_0 = T_1$. Suppose that $\beta < \omega_1$ and $\{A_\alpha : \alpha < \beta\}$ have been defined so that $A_{\alpha+1}$ satisfies the α^{th} condition. Assume that for each $\alpha < \beta$, A_α is large and these A_α 's form a \leq^* -decreasing chain.

Consider the case when β is a successor ordinal. Let α be the ordinal such that $\beta = \alpha + 1$. Let k be the natural number such that $Z_\alpha \subseteq \mathcal{R}_1(k)$. Let $(k_m)_{m < \omega}$ be the increasing enumeration of $\pi([A_\alpha]) \in \mathcal{V}$. By Theorem 4.2.2, there exists a subsequence $(k'_m)_{m < \omega}$ such that for each $m < \omega$, there exists a set $A'_m \in \mathcal{R}_1(m) \upharpoonright A_\alpha(k'_m)$ such that either $(\dagger) \mathcal{R}_1(k) \upharpoonright A'_m \subseteq Z$ or $(\ddagger) \mathcal{R}_1(k) \upharpoonright A'_m \subseteq \mathcal{R}_1(k) \setminus Z$.

Since \mathcal{V} is an ultrafilter, there exists a strictly increasing sequence $(m_i)_{i < \omega}$ such

that $\{k'_{m_i} : i < \omega\} \in \mathcal{V}$ and either for all $i < \omega$, $\mathcal{R}_1(k) \upharpoonright A'_{m_i} \subseteq Z$ or for all $i < \omega$, $\mathcal{R}_1(k) \upharpoonright A'_{m_i} \subseteq \mathcal{R}_1(k) \setminus Z$. Let $B = \bigcup_{i < \omega} A'_{m_i}$. So for each $i < \omega$,

$$\begin{aligned} C_{[B]}(k'_{m_i}) &= |[B] \cap \pi^{-1}(k'_{m_i})|, \\ &= |[A'_{m_i}]|, \\ &\geq i. \end{aligned}$$

Consequently, the following construction of $C \in \mathcal{R}_1$ is well-defined:

$$C = \bigcup_{i < \omega} \pi_{T(i)}(A'_{m_i}).$$

By construction, $C \leq A_\alpha$ and either $\mathcal{R}_1(k) \upharpoonright C \subseteq Z$ or $\mathcal{R}_1(k) \upharpoonright C \subseteq \mathcal{R}_1(k) \setminus Z$. Note that C is large since $\pi''[C] = \{k'_{m_i} : i < \omega\} \in \mathcal{V}$. Let $A_\beta = C$, then A_β is large and satisfies the condition (α) .

Next consider the case when β is a limit ordinal. Since the continuum hypothesis holds the cofinality of β is ω . Let $\{B_n : n < \omega\}$ be a \leq^* -cofinal sequence in $\{A_\alpha : \alpha < \beta\}$. Thus, for each $i < \omega$, there exists $H_i \in \mathcal{V}$ such that for all $n \in H_i$, $i \leq C_{B_i}(n)$. Without loss of generality, we may assume that $H_0 \supseteq H_1 \supseteq H_2 \supseteq \dots$. Since \mathcal{V} is selective, there exist $H \in \mathcal{V}$, $H = \{h_0, h_1, \dots\}$ and for all $i < \omega$, $h_{i+1} \in H_{h_i}$. Therefore for all $i < \omega$, $h_{i+1} \leq C_{B_{h_i}}(h_{i+1})$. Hence for all $i < \omega$, $i \leq C_{B_{h_i}}(h_{i+1})$. For each $n < \omega$, let $A'(n) = \pi_{T(n)}(B_{h_n}(h_{n+1}))$. Let $A_\beta = \bigcup_{n < \omega} A'(n)$, then $A_\beta \in \mathcal{R}_1$ is large because $\pi''[A_\beta] = \{h_1, h_2, h_3, \dots\} \in \mathcal{V}$. Moreover, for each $n < \omega$, $A_\beta \leq^* B_{h_n}$. So $\{A_\alpha : \alpha \leq \beta\}$ is a \leq^* -decreasing sequence of large sets.

This completes the inductive construction of $\{A_\beta : \beta < \aleph_1\}$ and the proof of the lemma. \square

We can now prove the main result of this section.

Proof of Theorem 4.1.8. Lemma 4.3.2 shows that necessity holds and Lemma 4.3.1 shows that sufficiency holds. \square

4.4 The cuts obtained from $\pi_{T(i)}$ and Ramsey for \mathcal{R}_1 ultrafilters

If \mathcal{U}_1 is a Ramsey for \mathcal{R}_1 ultrafilter generated by a generic subset of (\mathcal{R}_1, \leq^*) and $i < \omega$, then we let \mathcal{Y}_{i+1} denote the ultrafilter $\mathcal{U}_1|_{\mathcal{R}_1(i)}$. In this section, we show that for any Rudin-Keisler mapping between any two p-points within the Tukey type of \mathcal{U}_1 , the only cut obtainable is the standard cut. Notice that for each $i < j < \omega$ and each Ramsey for \mathcal{R}_1 ultrafilter \mathcal{U}_1 ,

$$\pi_{T(i)}(\mathcal{U}_1|_{\mathcal{R}_1(j)}) = \mathcal{U}_1|_{\mathcal{R}_1(i)}. \quad (4.4.1)$$

Additionally, notice that if $Z \in \mathcal{U}_1|_{\mathcal{R}_1(j)}$ then the cardinality function of Z with respect to $\pi_{T(i)}$ is defined as

$$C_Z(p) = |Z \cap \pi_{T(i)}^{-1}(p)|, \text{ for } p \in \mathcal{R}_1(i). \quad (4.4.2)$$

A simple counting argument shows that for each $S \in \mathcal{R}_1$, each $n < \omega$, and each $p \in \mathcal{R}_1(i) \upharpoonright S(n)$,

$$C_{\mathcal{R}_1(j) \upharpoonright S}(p) = |\mathcal{R}_1(j) \upharpoonright S \cap \pi_{T(i)}^{-1}(p)| \leq \binom{n-i}{j-i}. \quad (4.4.3)$$

Next we remind the reader of the basic definitions of the Tukey theory of partial orders restricted to ultrafilters ordered under reverse inclusion.

Definition 4.4.1. (Tukey, [49]) Suppose that \mathcal{U} and \mathcal{V} are ultrafilters. A function f from \mathcal{U} to \mathcal{V} is *cofinal* if every cofinal subset of (\mathcal{U}, \supseteq) is mapped by f to a cofinal subset of (\mathcal{V}, \supseteq) . We say that \mathcal{V} is *Tukey reducible to \mathcal{U}* and write $\mathcal{V} \leq_T \mathcal{U}$ if there exists a cofinal map $f : \mathcal{U} \rightarrow \mathcal{V}$. If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ then we write $\mathcal{V} \equiv_T \mathcal{U}$ and say that \mathcal{U} and \mathcal{V} are *Tukey equivalent*. The relation \equiv_T is an equivalence relation and \leq_T is a partial order on its equivalence classes. The equivalence class are also called *Tukey types* or *Tukey degrees*.

The Tukey reducibility relation is a generalization of the Rudin-Keisler reducibility relation. If $h(\mathcal{U}) = \mathcal{V}$ then the map sending $X \in \mathcal{U}$ to $h''X \in \mathcal{V}$ is Tukey. So if $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{V} \leq_T \mathcal{U}$. This leads to the following question of Dobrinen: For a given ultrafilter \mathcal{U} , what is the structure of the Rudin-Keisler ordering within the Tukey type of \mathcal{U} ?

Dobrinen and Todorćević in [17], have given an answer to this question if \mathcal{U} is a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ generated by a generic subset of (\mathcal{R}_1, \leq^*) . Furthermore, Dobrinen and Todorćević describe the Rudin-Keisler structure of the p-points within the Tukey type of a Ramsey for \mathcal{R}_1 ultrafilter generated by a generic subset of (\mathcal{R}_1, \leq^*) . In particular, the Tukey type of such an ultrafilter \mathcal{U}_1 consists of a strictly increasing chain of rapid p-points of order type ω : $\mathcal{Y}_1 <_{RK} \mathcal{Y}_2 <_{RK} \dots$ where $\mathcal{Y}_{i+1} = \mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$ for $i < \omega$. The next proof is the main result of this section.

Proof of Theorem 4.1.9. Suppose that \mathcal{U}_1 is a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ generated by a generic subset of (\mathcal{R}_1, \leq^*) . Let \mathcal{Y} be a p-point ultrafilter in the Tukey-type of \mathcal{U}_1 and g be a map from the base of \mathcal{Y} to ω . Let (S, L) be the cut associated to

\mathcal{Y} and g . By Theorem 5.10 and Example 5.17 from [17] there exists $i, j < \omega$ such that $i \leq j$, $\mathcal{Y} \cong \mathcal{Y}_{j+1}$ and $g(\mathcal{Y}) \cong \mathcal{Y}_{i+1}$. Notice that since (S, L) is not proper we have $i < j$. By Remark 4.1.7, it is enough to prove the this theorem in the case when $\mathcal{Y} = \mathcal{Y}_{j+1}$ and $g(\mathcal{Y}) = \mathcal{Y}_{i+1}$. In particular, we may assume without loss of generality that $g : \mathcal{R}_1(j) \rightarrow \mathcal{R}_1(i)$.

Since \mathcal{U}_1 is generated by a generic subset of (\mathcal{R}_1, \leq^*) , we find that Theorem 4.25 and Proposition 5.8 in [17] imply that there exists $T \in \mathcal{C}$ and $i' \leq j$ such that for all $p, q \in \mathcal{R}_1(j) \upharpoonright T$, $g(p) = g(q)$ if and only if $\pi_{T(i')}(p) = \pi_{T(i')}(q)$. So if $T' \in \mathcal{C}$, then for each $n < \omega$ and each $p \in \mathcal{R}_1(i) \upharpoonright T(n)$,

$$\begin{aligned} C_{\mathcal{R}_1(j) \upharpoonright (T' \cap T)}(p) &= |\mathcal{R}_1(j) \upharpoonright (T' \cap T) \cap g^{-1}(p)| \\ &= |\mathcal{R}_1(j) \upharpoonright (T' \cap T) \cap \pi_{T(i')}^{-1}(p)| \leq \binom{n - i'}{j - i'}. \end{aligned} \quad (4.4.4)$$

Let $f : \mathcal{R}_1(i) \rightarrow \omega$ be given and suppose that f is not constant mod $\mathcal{U} \upharpoonright \mathcal{R}_1(i)$.

For each $n < \omega$, let

$$X_n = \{p \in \mathcal{R}_1(i) : f(p) \geq \binom{n - i'}{j - i'}\}. \quad (4.4.5)$$

Since $\mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$ is an ultrafilter on $\mathcal{R}_1(i)$ we find that for each $n < \omega$, either $X_n \in \mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$ or $\mathcal{R}_1(i) \setminus X_n \in \mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$. If there exists $n < \omega$ such that $\mathcal{R}_1(i) \setminus X_n \in \mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$ then f would be bounded by n mod $\mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$. However, this can not happen since we assumed that f is not constant mod $\mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$. Hence for each $n < \omega$, $X_n \in \mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$.

Note that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ is a decreasing sequence of members of $\mathcal{U}_1 \upharpoonright \mathcal{R}_1(i)$. Since \mathcal{U}_1 is selective for \mathcal{R}_1 there exists $S' \in \mathcal{C}$ such that for each $n < \omega$,

$\mathcal{R}_1(i) \upharpoonright (S' \setminus r_n(S')) \subseteq X_n$. For each $n < \omega$ and each $p \in \mathcal{R}_1(i) \upharpoonright S'(n)$, $p \in \mathcal{R}_1(i) \upharpoonright (S' \setminus r_n(S')) \subseteq X_n$. So (4.4.4) implies that for each $n < \omega$ and each $p \in \mathcal{R}_1(i) \upharpoonright S'(n)$,

$$f(p) \geq \binom{n - i'}{j - i'} \geq C_{\mathcal{R}_1(j) \upharpoonright (T \cap S')}(p). \quad (4.4.6)$$

Thus, $\mathcal{R}_1(i) \upharpoonright (T \cap S') \in \mathcal{U}_1 | \mathcal{R}_1(i)$, $[f] \geq [C_{\mathcal{R}_1(j) \upharpoonright (T \cap S')}]$, and $[f] \in L$ as $\mathcal{R}_1(j) \upharpoonright (T \cap S') \in \mathcal{U}_1 | \mathcal{R}_1(i)$. Additionally, note that the cardinality function of any member of \mathcal{Y}_{j+1} is not constant mod $g(\mathcal{Y}_{j+1}) = \mathcal{U}_1 | \mathcal{R}_1(i)$. Therefore the cut (S, L) is the standard cut in $\omega^{\mathcal{R}_1(i)} / \mathcal{U}_1 | \mathcal{R}_1(i)$. \square

4.5 Applications and open questions

In this section we use the two main results to prove that, under the continuum hypothesis, certain special ultrafilters exist. Additionally, we ask some questions about the types of cuts that can be obtained from similarly defined topological Ramsey spaces. The two main results of this chapter only associate the standard cut to a given ultrafilter mapping. However, the results of Blass from [4] only require the lower half of the cuts to be closed under certain operations. These differences allow us to show that, under the continuum hypothesis, certain special ultrafilters exist which are not Ramsey for \mathcal{R}_1 . The next theorem corollary can also be proved using the results of Blass characterizing the Dedekind cuts associated to weakly-Ramsey ultrafilters and p -points.

Corollary 4.5.1 (Blass, [4]). *Assume the continuum hypothesis. There exists a p -point ultrafilter on $[T_1]$ which is not weakly Ramsey.*

Proof. Recall that under the continuum hypothesis, p-point ultrafilters exist. Let \mathcal{U} be a p-point ultrafilter on ω , $[f]$ be a nonstandard element of $\omega^\omega/\mathcal{U}$, and

$$S = \bigcup_{n < \omega} \{[g] \in \omega^\omega/\mathcal{U} : [g] \leq n \cdot [f]\}. \quad (4.5.1)$$

Notice that S is closed under addition and contains the standard part of $\omega^\omega/\mathcal{U}$. Let L be the complement of S . The condition that L is nonempty and every countable subset of L has a lower bound in L , will be automatically satisfied if S has a countable cofinal subset, because $\omega^\omega/\mathcal{U}$ is countably saturated (see [29]). Thus, (S, L) is a proper cut such that S is closed under addition and every countable subset of L has a lower bound in L . By Theorem 4.1.5, there is a p-point ultrafilter \mathcal{V} and a Rudin-Keisler map p such that $p(\mathcal{V}) = \mathcal{U}$ and (S, L) is the cut associated to p and \mathcal{V} . Since (S, L) is not the standard cut, Theorem 4.1.8 implies that \mathcal{V} is not Ramsey for \mathcal{R}_1 . Since (S, L) is not closed under exponentiation, Theorem 4.1.6 implies that \mathcal{V} is not weakly-Ramsey. \square

We remind the reader of the recursive definition of *iterated exponentials*. For each natural number n and i we let,

$$\begin{cases} {}^0n = 1, \\ {}^{i+1}n = n^{i n}. \end{cases} \quad (4.5.2)$$

For example, ${}^32 = 2^{2^2}$ and ${}^26 = 6^6$. Note that if k, n and m are natural numbers, then $({}^n k)^{m k} \leq {}^{n+m} k$.

Corollary 4.5.2 (T. [47]). *Assume the continuum hypothesis. There exists a weakly Ramsey ultrafilter on $[T_1]$ which is not Ramsey for \mathcal{R}_1 .*

Proof. Recall that under the continuum hypothesis, selective ultrafilters exist. Let \mathcal{U} be a selective ultrafilter on ω , $[f]$ be a nonstandard element of $\omega^\omega/\mathcal{U}$, and

$$S = \bigcup_{n < \omega} \{[g] \in \omega^\omega/\mathcal{U} : [g] \leq^n [f]\}. \quad (4.5.3)$$

If $[g]$ and $[h]$ are in S then there exists natural numbers n and m such that $[g] \leq^n [f]$ and $[h] \leq^m [f]$. So $[g]^{[h]} \leq^{n+m} [f]$. Therefore, S is closed under exponentiation. Additionally, S contains the standard part of $\omega^\omega/\mathcal{U}$. Let L be the complement of S . The condition that L is nonempty and every countable subset of L has a lower bound in L , will be automatically satisfied if S has a countable cofinal subset, because $\omega^\omega/\mathcal{U}$ is countably saturated (see [29]). Thus, (S, L) is a proper cut such that S is closed under exponentiation and every countable subset of L has a lower bound in L . By Theorem 4.1.6, there is a weakly Ramsey ultrafilter \mathcal{V} and a Rudin-Keisler map p such that $p(\mathcal{V}) = \mathcal{U}$ and (S, L) is the cut associated to p and \mathcal{V} . Since (S, L) is not the standard cut, Theorem 4.1.8 implies that \mathcal{V} is not Ramsey for \mathcal{R}_1 . \square

In Chapter 3, we showed that, assuming the continuum hypothesis, there exists a selective for \mathcal{R}_1 ultrafilter which is not Ramsey for \mathcal{R}_1 . Using a similar proof to that in Theorem 4.1.8, it is possible to characterize the cuts obtainable from a selective for \mathcal{R}_1 ultrafilter and the map π ; they are exactly the standard cuts. However, it is unclear if there is a selective for \mathcal{R}_1 ultrafilter \mathcal{U} and a Rudin-Keisler map p such that the cut associated to p and \mathcal{U} is not standard. This motivates the following question:

Question 4.5.3. Can the notions of selective for \mathcal{R}_1 and Ramsey for \mathcal{R}_1 be distinguished by characterizing the Dedekind cuts obtainable from selective for \mathcal{R}_1 ultrafilters on $[T_1]$?

If it is shown that the only cuts obtainable are standard, then this method will not distinguish between the two notions. However, if there exists a selective for \mathcal{R}_1 ultrafilter and a Rudin-Keisler map p such that (S, L) is not standard, then \mathcal{U} will not be Ramsey for \mathcal{R}_1 and the two notions will be distinguished.

Dobrinen and Todorćević in [18] have defined generalizations of the space \mathcal{R}_1 for $1 < \alpha < \omega_1$. The spaces are built from trees T_α in much the same way that \mathcal{R}_1 is built from T_1 . In Chapter 3, we showed that for each positive integer n , under the continuum hypothesis, there are selective for \mathcal{R}_n ultrafilters on $[T_n]$ which are not Ramsey for \mathcal{R}_n . However, it is still unknown if, under the continuum hypothesis, there are selective for \mathcal{R}_α ultrafilters on $[T_\alpha]$ which are not Ramsey for \mathcal{R}_α , for $\omega \leq \alpha < \omega_1$. The methods used in Chapter 3 fail for infinite α as the well-founded trees T_α have unbounded height.

Question 4.5.4. Assume that $1 < \alpha < \omega_1$. Can the notions of selective for \mathcal{R}_α and Ramsey for \mathcal{R}_α be distinguished by characterizing the Dedekind cuts obtainable from selective and Ramsey for \mathcal{R}_α ultrafilters on $[T_\alpha]$?

We may also prove similar existence results using the second main result, Theorem 4.1.9.

Corollary 4.5.5 (T. [47]). *Assume the continuum hypothesis holds and \mathcal{U}_1 is a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ generated by a generic subset of (\mathcal{R}_1, \leq^*) . There exists a weakly Ramsey ultrafilter \mathcal{V} on $[T_1]$ such that $\mathcal{V} \not\leq_T \mathcal{U}_1$.*

Proof. Notice that $\pi(\mathcal{U}_1)$ is a selective ultrafilter by Theorem 2.5.1. Let $[f]$ be a nonstandard element of $\omega^\omega / \pi(\mathcal{U}_1)$ and

$$S = \bigcup_{n < \omega} \{[g] \in \omega^\omega / \pi(\mathcal{U}_1) : [g] \leq^n [f]\}. \quad (4.5.4)$$

S is closed under exponentiation. Additionally, S contains the standard part of $\omega^\omega/\pi(\mathcal{U}_1)$. Let L be the complement of S . The condition that L is nonempty and every countable subset of L has a lower bound in L , is automatically satisfied if S has a countable cofinal subset, because $\omega^\omega/\pi(\mathcal{U}_1)$ is countably saturated (see [29]). So (S, L) is a proper cut such that S is closed under exponentiation and every countable subset of L has a lower bound in L . By Theorem 4.1.6, there is a weakly Ramsey ultrafilter \mathcal{V} and a Rudin-Keisler map p such that $p(\mathcal{V}) = \pi(\mathcal{U}_1)$ and (S, L) is the cut associated to p and \mathcal{V} . Since (S, L) is not the standard cut Theorem 4.1.9 implies that \mathcal{V} is not Tukey-reducible to \mathcal{U}_1 . \square

Corollary 4.5.6 (T. [47]). *Assume the continuum hypothesis holds and \mathcal{U}_1 is a Ramsey for \mathcal{R}_1 ultrafilter on $[T_1]$ generated by a generic subset of (\mathcal{R}_1, \leq^*) . There exists a p -point ultrafilter \mathcal{W} on $[T_1]$ which is not weakly Ramsey such that $\mathcal{W} >_T \mathcal{U}_1$.*

Proof. Notice that $\pi_{T(1)}(\mathcal{U}_1)$ is a p -point ultrafilter by Theorem 2.5.1. Let $[f]$ be a nonstandard element of $\omega^\omega/\pi_{T(1)}(\mathcal{U}_1)$ and

$$S = \bigcup_{n < \omega} \{[g] \in \omega^\omega/\pi_{T(1)}(\mathcal{U}_1) : [g] \leq n \cdot [f]\} \text{ and} \quad (4.5.5)$$

S is under addition. Additionally, S contains the standard part of $\omega^\omega/\pi_{T(1)}(\mathcal{U})$. Let L be the complement of S . The condition that L is nonempty and every countable subset of L has a lower bound in L , is automatically satisfied if S has a countable cofinal subset, because $\omega^\omega/\pi_{T(1)}(\mathcal{U}_1)$ is countably saturated (see [29]). Thus, (S, L) is a proper cut such that S is closed under addition and every countable subset of L has a lower bound in L . By Theorem 4.1.5, there is a p -point ultrafilter \mathcal{W} and a Rudin-Keisler map p such that $p(\mathcal{W}) = \pi_{T(1)}(\mathcal{U}_1)$ and (S, L) is the cut associated to p and \mathcal{W} . Since (S, L) is not the standard cut, Theorem 4.1.9 implies that \mathcal{W} is

not Tukey-reducible to \mathcal{U}_1 . On the other hand, $\pi_{T(1)}(\mathcal{U}_1)$ is Tukey equivalent to \mathcal{U}_1 . Therefore $\mathcal{W} >_T \mathcal{U}_1$. \square

Remark 4.5.7. Notice that the previous corollary shows that the converse of Theorem 4.1.9, under the continuum hypothesis, does not hold. In particular, there is a p-point \mathcal{W} such that there is no p-point in the Tukey-type of \mathcal{U}_1 which gives the standard cut in $\omega^\omega/\mathcal{W}$, since otherwise \mathcal{W} would be Tukey reducible to \mathcal{U}_1 .

This leads naturally to the next question.

Question 4.5.8. Is it possible to strengthen the conclusion of Theorem 4.1.9 so that, under the continuum hypothesis, a converse to the modified theorem holds?

The previous remark shows that at the very least one needs to assume that the not only is \mathcal{V} a p-point but it is also Tukey-reducible to \mathcal{U}_1 .

Chapter 5

Canonical Ramsey theory for \mathcal{H}^2 & an application to the Tukey theory of ultrafilters

The work in this chapter has been submitted for publication with the more general results of Dobrinen, Mijares, and Trujillo in [15]. Although the work in this chapter follows from [15], it also preceded and influenced the work in [15]. In this chapter, we explore the canonical Ramsey theory of the space \mathcal{H}^2 (see Example 2.2.9 for the definition of \mathcal{H}^2) and apply it to the Tukey theory of ultrafilters (see Section 1.4 for an introduction to the basic concepts in the Tukey theory of ultrafilters). As an application, we prove that it is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure. Recall that, an initial Tukey structure is a family of Tukey types of nonprincipal ultrafilters which is closed under Tukey reducibility. In [15], the authors show that it is consistent with ZFC that, along with other initial Tukey structures, all finite Boolean algebras appear as initial Tukey

structures. The methods and proofs we use closely follow the work of Dobrinen and Todorčević in [17] and [18] where a new canonical Ramsey theory is developed and then used to show that it is consistent with ZFC that for all $\alpha < \omega_1$, the decreasing chain of order type $\alpha + 1$ appears as an initial Tukey structure.

In Section 5.1, we give a proof that the space \mathcal{H}^2 is a topological Ramsey space. Dobrinen, Mijares and Trujillo in [15] use a theorem of Sokić in [44] to verify that the more general triples under consideration in [15], including the space \mathcal{H}^2 , form topological Ramsey spaces. Theorem 5.1.2 below is equivalent to the theorem of Sokić from [44] used in [15] when restricted to the family of finite trees $\{S(n) : S \in \mathcal{H}^2 \text{ \& } n < \omega\}$. The proof of Theorem 5.1.2 and the definition of \mathcal{H}^2 in this dissertation are both due to Dobrinen. We include the proof of Theorem 5.1.2 since it was carried out independently of the more general Sokić result from [44] used in [15] for products. In the final part of Section 5.1, we use this result to verify that \mathcal{H}^2 forms a topological Ramsey space.

In Section 5.2, we show that an extension of the Erdős-Rado Theorem, Theorem 1.3.3, holds for $\mathcal{H}^2(n) = \{S(n) : S \in \mathcal{H}^2\}$, $n < \omega$. In Section 5.3, we apply this extension in conjunction with the theory of mixing and separating to derive a canonical Ramsey theory for the space \mathcal{H}^2 . We conclude with Section 5.4 where we prove, using the canonical Ramsey theory for \mathcal{H}^2 , that the Tukey-types of nonprincipal ultrafilters Tukey which are reducible to a Ramsey for \mathcal{H}^2 ultrafilter is isomorphic to the four-element Boolean algebra. Thus, forcing with \mathcal{H}^2 using almost-reduction proves that it is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure.

5.1 The Ellentuck theorem for \mathcal{H}^2

In this section, we give a complete proof that the space \mathcal{H}^2 , generated by the tree $T_1 \otimes T_1$, forms a topological Ramsey space. We begin by proving an extension of the finite Ramsey theorem to finite products of sets. This extension will be used to verify that \mathcal{H}^2 satisfies axiom **A.4** from Section 1.2.

For each pair of natural numbers k and l , define an l -coloring of $[\omega]^k$ to be a map from $[\omega]^k$ to l . If f is an l -coloring $[\omega]^k$ and X is an infinite subset of ω , then we say that f is *monochromatic on X* , if f is constant on $[X]^k$. The finite Ramsey theorem can be restated in terms of colorings : *For all positive integers k, l , and n with $k \leq n$ there is a positive integer m such that for every m -element set M and every coloring $f : [M]^k \rightarrow l$, there is a $N \subseteq M$ of cardinality n such that $[N]^k$ is monochromatic, that is f is constant on $[N]^k$.*

The *Ramsey number* for n, k and l denoted by $R(n, k, l)$ is the smallest ordinal m satisfying the conclusion of the finite Ramsey theorem with the parameters n, k and l . Additionally, for each $i < \omega$ define $R^i(n, k, l)$ recursively by

$$R^i(n, k, l) = \begin{cases} R(R^{i-1}(n, k, l), k, l) & \text{if } i > 0 \\ n & \text{otherwise.} \end{cases}$$

Notice that by construction, for each $i < \omega$ and each $f : [R^{i+1}(n, k, l)]^k \rightarrow l$ there is a $N \in [R^{i+1}(n, k, l)]^{R^i(n, k, l)}$ such that $[N]^k$ is monochromatic.

Lemma 5.1.1. *For all positive integers k, l, n , and i with $k \leq n$ there is a positive integer m such that for every m -element set M and every sequence $\langle f_0, f_1, \dots, f_{i-1} \rangle$ of maps from $[M]^k$ to l , there is $N \subseteq M$ of cardinality n such that $[N]^k$ is monochro-*

matic for each of the colorings in the sequence.

Proof. Fix positive integers k, l, n , and i with $k \leq n$. Let $m = R^i(n, k, l)$, M be an m -element subset of ω and $\langle f_0, f_1, \dots, f_{i-1} \rangle$ be a sequence of maps from $[M]^k$ to l . By definition of $R^i(m, k, l)$ there is an $R^{i-1}(m, k, l)$ -element subset, A_0 , of M such that $[A_0]^k$ is f_0 -monochromatic. Again by definition, there is an $R^{i-2}(m, k, l)$ -element subset, A_1 , of A_0 such that $[A_1]^k$ is f_1 -monochromatic. Continuing this procedure i times we obtain a sequence $\langle A_0, \dots, A_{i-1} \rangle$ of subsets of M such that:

1. $A_0 \supseteq A_1 \supseteq \dots \supseteq A_{i-1}$.
2. For each $j < i$, $[A_j]^k$ is f_j -monochromatic.
3. A_{i-1} is an n -element subset of M .

Let $N = A_{i-1}$. (1) and (2) imply that $[M]^k$ is monochromatic for each of the colorings in the sequence. (3) implies that $N \subseteq M$ and has cardinality n . \square

A subset A of $\omega \times \omega$ is an n -square if there exists two n -element sets P and Q such that $A = P \times Q$. For each $B \subseteq \omega \times \omega$, let $\mathcal{S}_n(B)$ denote the set of n -squares that are also subsets of B . An l -coloring of the k -squares in B is a map $f : \mathcal{S}_k(B) \rightarrow l$.

Theorem 5.1.2. *For all positive integers k, l , and n with $k \leq n$ there is a positive integer m such that for every m -square M and every coloring $F : \mathcal{S}_k(M) \rightarrow l$, there is an n -square $N \subseteq M$ such that $\mathcal{S}_k(N)$ is monochromatic.*

Proof. Choose m greater than the maximum of m' and $R(n, k, l)$ where m' is the smallest ordinal satisfying the conclusion of Lemma 5.1.1 for k, l, n and

$$i = |[R(n, k, l)]^k|.$$

Let M be an m -square, $F : \mathcal{S}_k(M) \rightarrow l$, and X be the first $R(n, k, l)$ elements of $\pi_0''M$. For each $s \in [X]^k$, define the coloring $f_s : [\pi_1''M]^k \rightarrow l$ by $f_s(t) = F(s \times t)$.

Since $m > m'$, there exists $N_1 \subseteq \pi_1''M$ of cardinality n such that for all $s \in [X]^2$ N_1 is f_s -monochromatic. Now define $g : [X]^k \rightarrow l$ by letting $g(s)$ be the unique value of f_s on $[N_1]^k$. Since $|X| = R(n, k, l)$, there exists $N_0 \subseteq \pi_0''M$ of cardinality n and $c < l$ such that for all $s \in [N_0]^k$, $g(s) = c$. Let $N = N_0 \times N_1$. For all $X \in \mathcal{S}_k(N)$, $F(X) = f_{\pi_0''X}(\pi_1''X) = g(\pi_0''X) = c$. So $\mathcal{S}_k(N)$ is F -monochromatic. \square

Recall that the space \mathcal{H}^2 from Example 2.2.9 was taken to be the collection of subtrees of

$$T_1 \otimes T_1 = cl\left(\bigcup_{n < \omega} \left\{ \langle (n, n), (i, j) \rangle : \frac{n(n+1)}{2} \leq i, j < \frac{(n+1)(n+2)}{2} \right\}\right)$$

isomorphic to $T_1 \otimes T_1$. In order to simplify notation, for each $n < \omega$, we let (n, n) be denoted by simply n . Thus, we let

$$T_1 \otimes T_1 = \bigcup_{n < \omega} \left\{ \langle \rangle, \langle n \rangle, \langle n, (i, j) \rangle : \frac{n(n+1)}{2} \leq i, j < \frac{(n+1)(n+2)}{2} \right\}.$$

See Figure 2.6 for a graph of the tree $T_1 \otimes T_1$.

In the statement of the axioms **A.1** – **A.4** used in applying the abstract Ellentuck theorem, Theorem 1.2, it is necessary to define a quasi-order \leq_{fin} on the set of finite approximations, in this case $\mathcal{A}\mathcal{H}^2$. From the definition of subtree in Section 2.2, it is clear that for $X, Y \in \mathcal{H}^2$, $X \leq Y$ if and only if there is a strictly increasing sequence $(k_n)_{n < \omega}$ such that for each n , $Y(n)$ is a subtree of $X(k_n)$. We define the quasi-ordering \leq_{fin} on $\mathcal{A}\mathcal{H}^2$ in similar way. For $a, b \in \mathcal{A}\mathcal{H}^2$ we let, $b \leq_{fin} a$ if and only if there are $n \leq m$ and a strictly increasing sequence $(k_i)_{i < n}$ with $k_{n-1} < m$

such that $a \in \mathcal{AH}_m^2$, $b \in \mathcal{AH}_n^2$, and for each $i < n$, $b(i)$ is a subtree of $a(k_i)$. We write $a \leq_{\text{fin}} B$ if and only if there is an n such that $a \leq_{\text{fin}} r_n(B)$. Note that from these definitions one can prove that, $a \leq_{\text{fin}} b$ if and only if a is a subtree of b , we opt for the above definition as it makes the proof of Theorem 5.1.4 simpler.

Recall that the basic open sets of the Ellentuck topology for \mathcal{H}^2 are given by $[a, B] = \{X \in \mathcal{H}^2 : a \sqsubseteq X \ \& \ X \leq B\}$. The following notation will be useful in the next proof and in the next section.

Notation 5.1.3. A/b denotes the set $A \setminus r_n(A)$, where n is the least ordinal such that $\text{depth}_{T_1 \otimes T_1}(r_n(A)) \geq \text{depth}_{T_1 \otimes T_1}(b)$. $\mathcal{H}^2(k) = \{X(k) : X \in \mathcal{H}^2\}$, $\mathcal{H}^2(k)|A = \{X(k) : X \in \mathcal{H}^2 \ \& \ X(k) \subseteq A\}$, and $\mathcal{H}^2(k)|A/b = \{X(k) : X \in \mathcal{H}^2 \ \& \ X(k) \subseteq A/b\}$.

Theorem 5.1.4 (The Ellentuck Theorem for \mathcal{H}^2). (\mathcal{H}^2, \leq, r) is a topological Ramsey space.

Proof. By the abstract Ellentuck Theorem, Theorem 1.2.2, it is enough to show that (\mathcal{H}^2, \leq, r) satisfies **A.1 - A.4** and \mathcal{H}^2 is a closed subspace of the Tychonov power $(\mathcal{AH}^2)^\omega$ of \mathcal{AH}^2 with its discrete topology. A metric compatible with this topology is given by

$$d((a_i)_{i < \omega}, (b_j)_{j < \omega}) = \begin{cases} 2^{-m} & \text{if } m = \min\{n : a_n \neq b_n\} \text{ exists;} \\ 0 & \text{otherwise.} \end{cases} \quad (5.1.1)$$

\mathcal{H}^2 is identified with the subspace of $(\mathcal{AH}^2)^\omega$ of all sequences $(a_i)_{i < \omega}$ such that there is an $A \in \mathcal{H}^2$ such that for each $n < \omega$, $a_n = r_n(A)$. Let $(A_i)_{i < \omega}$ be a sequence of elements in \mathcal{H}^2 that converges to $A = (a_j)_{j < \omega}$ in $(\mathcal{AH}^2)^\omega$. Then there is a strictly increasing sequence $(i_n)_{n < \omega}$ such that for all $n < \omega$, $d(A_{i_n}, A) < 2^{-n-1}$.

By definition of d , we find that for all $k < \omega$, if $n \leq k$ then $a_n = r_n(A_{i_k})$. So for all $k < \omega$, $a_k = r_k(A_{i_k}) \in \mathcal{AH}_k^2$. Furthermore, for all $n < k$, $r_n(a_k) = r_n(r_k(A_{i_k})) = r_n(A_{i_k}) = a_n$. The previous two facts imply that $\bigcup_{i < \omega} a_i \in \mathcal{H}^2$. Hence $(a_i)_{i < \omega}$ is in the subspace of $(\mathcal{AH}^2)^\omega$ that is identified with \mathcal{H}^2 . Since the sequence was an arbitrary convergent sequence, we find that \mathcal{H}^2 is a closed subspace of the Tychonov power $(\mathcal{AH}^2)^\omega$.

A.1 By definition of r_n we have: $\forall A \in \mathcal{H}^2$, $r_0(A) = \emptyset$. Hence, (a) holds. If for all $n < \omega$, $r_n(A) = r_n(B)$ then $A = \bigcup_{n < \omega} r_n(A) = \bigcup_{n < \omega} r_n(B) = B$. Taking the contrapositive of the previous statement we find that (b) holds. Next suppose that $r_n(A) = r_m(B)$. Then by definition of \mathcal{H}^2 it must be the case that the last subtrees of $r_n(A)$ and $r_m(B)$ coincide, i.e., $A(n-1) = B(m-1)$. Thus, $(n-1)^2 + 2 = |A(n-1)| = |B(m-1)| = (m-1)^2 + 2$ and $n = m$. Since $n = m$ and $r_n(A) = r_m(B)$ we find that, by definition of \mathcal{H}^2 , for all $k < n$, $r_k(A) = r_k(B)$. Thus, (c) holds for \mathcal{H}^2 .

A.2 For each $t \in \mathcal{AH}^2$, there is a unique $n < \omega$ such that $t \in \mathcal{AH}_n^2$. Thus,

$$\{s \in \mathcal{AH}^2 : s \leq_{\text{fin}} t\} = \bigcup_{k \leq n} \{s \in \mathcal{AH}_k^2 : \forall j \leq k, \exists m_j \leq n (s(j) \subseteq t(m_j))\}. \quad (5.1.2)$$

Since each set participating in the union on the right hand side of (5.1.2) is finite we find (a) holds for \mathcal{H}^2 .

If $S \leq T$ then there is a strictly increasing sequence $(k_i)_{i < \omega}$ such that for each $n < \omega$, $r_n(S) \leq_{\text{fin}} r_{k_n}(T)$ as witnessed by the finite sequence $(k_i)_{i < n}$. If we let $m = k_n$, then $r_n(S) \leq_{\text{fin}} r_m(T)$. Thus, for all $n < \omega$ there is an $m < \omega$ such that $r_n(S) \leq_{\text{fin}} r_m(T)$. On the other hand, if $S \not\leq T$ then for all strictly increasing sequences $(k_i)_{i < \omega}$, there is an $n < \omega$ such that $r_n(S) \not\leq_{\text{fin}} r_{k_n}(T)$. In particular,

there exists an $n < \omega$ such that for all $m < \omega$, $r_n(S) \not\leq_{\text{fin}} r_m(T)$. Hence (b) holds for \mathcal{H}^2 . Additionally, for each $a, b \in \mathcal{AH}^2$, if $a \sqsubseteq b$ and $b \leq_{\text{fin}} c$ then $a \leq_{\text{fin}} c$. Thus, (3) holds for \mathcal{H}^2 .

A.3 If $\text{depth}_B(a) = n < \infty$ then $a \leq_{\text{fin}} r_n(B)$. If $A \in [\text{depth}_B(a), B]$ then $r_n(A) = r_n(B)$ and there is a strictly increasing sequence $(k_i)_{i < \omega}$ such that for all $i < \omega$, $A(i) \subseteq B(k_i)$. For each $i > |a|$, let $a'(i)$ be any tree in $\mathcal{H}^2(i)|A(n+i)$. Let $A' = a \cup \bigcup\{a'(i) : |a| < i < \omega\}$. Then $A' \in [a, A]$, so $[a, A] \neq \emptyset$. Thus, (a) holds for \mathcal{H}^2 .

Assume that $A \leq B$ and $[a, A] \neq \emptyset$. It follows that $\text{depth}_B(a) = n < \infty$. Let $A' = r_n(B) \cup \bigcup\{A(n+i) : i < \omega\}$. Thus, $A' \in [\text{depth}_B(a), B]$ and (a) implies that $[a, A'] \neq \emptyset$. By construction of A' , it is clear that $[a, A'] \subseteq [a, A]$. Therefore (b) holds for \mathcal{H}^2 .

A.4 Suppose that $\text{depth}_B(a) = n < \infty$ and $\mathcal{O} \subseteq \mathcal{AH}^2_{|a|+1}$. Let F be a map from $\mathcal{H}^2(|a|)|B/r_n(B)$ to 2 be given by

$$F(u) = \begin{cases} 0 & \text{if } a \cup u \in \mathcal{O}, \\ 1 & \text{if } a \cup u \notin \mathcal{O}. \end{cases} \quad (5.1.3)$$

Identifying each element of the domain of F with its leaves, *i.e.* maximal nodes, the finite square Ramsey Theorem may be applied. By Theorem 5.1.2, taking N_0 large enough, there is a subtree $w(n) \subseteq B(N_0)$ which has the same structure as $T_1 \otimes T_1(n)$ such that the collection of all subtrees of $w(n)$ which have the same structure as $T_1 \otimes T_1(|a|)$ is monochromatic. Take $N_1 > N_0$ large enough that there is a subtree $w(n+1) \subseteq B(N_1)$ that has the same structure as $T_1 \otimes T_1(n+1)$ such that the collection of subtrees of $w(n+1)$ which have the same structure as $T_1 \otimes T_1(|a|)$ is monochromatic. In general, given N_i and $w(n+i)$, take $N_{i+1} > N_i$, large enough

that there is a subtree $w(n + i + 1) \subseteq B(N_{i+1})$ that has the same structure as $T_1 \otimes T_1(n + i + 1)$ such that the collection of all subtrees of $w(n + i + 1)$, which have the same structure as $T_1 \otimes T_1(|a|)$, is monochromatic. Take a subsequence $(k_i)_{i < \omega}$ of $(n + i)_{i < \omega}$ such that all subtrees of $w(k_i)$ that have the same structure as $T_1 \otimes T_1(|a|)$ and have the same color for all $i < \omega$. Now take any subtree $u(n + i) \subseteq w(k_i)$ that has the same structure as $T_1 \otimes T_1(n + i)$, for each $i < \omega$. Let $A = r_n(B) \cup \bigcup_{i < \omega} u(n + i)$. So $A \in [\text{depth}_B(a), B]$. If the color omitted by u is 1 then, by construction of A , $r_{|a|+1}[a, A] \subseteq \mathcal{O}$; otherwise, u omits 0 and $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$. \square

5.2 The abstract Erdős-Rado theorem for \mathcal{H}^2

In this section, we establish a version of the Erdős-Rado Theorem, Theorem 1.3.3, for equivalence relations on barriers of the form $\mathcal{A}\mathcal{H}_n^2$ for $n < \omega$. The result follows by a more general proof of Dobrinen in [15]; however, the proof we present for \mathcal{H}^2 uses a different method from the proof of Dobrinen in [15] and was done prior to [15]. We begin by introducing families of subtrees of $T_1 \otimes T_1$ given by projections maps on $T_1 \otimes T_1(n)$, for $n < \omega$.

Definition 5.2.1. For each $n < \omega$, let $\tilde{S}(n)$ denote the tree $\{\langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in (n+1) \times (n+1)\}$. Let $T_{\langle \rangle} = \{\langle \rangle\}$ and let $T_{\langle 0 \rangle} = \{\langle \rangle, \langle 0 \rangle\}$. For $I, J \subseteq n+1$ with either I or J nonempty, let

$$T(I, J) = \begin{cases} \{\langle \rangle, \langle 0 \rangle, \langle 0, (i, \cdot) \rangle : i \in I\} & \text{if } I \neq \emptyset \text{ \& } J = \emptyset, \\ \{\langle \rangle, \langle 0 \rangle, \langle 0, (\cdot, j) \rangle : j \in J\} & \text{if } I = \emptyset \text{ \& } J \neq \emptyset, \\ \{\langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in I \times J\} & \text{if } I \neq \emptyset \text{ \& } J \neq \emptyset. \end{cases} \quad (5.2.1)$$

Let $\mathcal{T}(n)$ denote the collection of trees of the form $T_{\langle \rangle}, T_{\langle 0 \rangle}$ and $T(I, J)$ with $I, J \subseteq n + 1$ and either I or J nonempty.

Let $T \in \mathcal{T}(n)$, $X \in \mathcal{H}^2$, and suppose that $X(n) = \{\langle \rangle, \langle m \rangle, \langle m, (l, k) \rangle : (l, k) \in L \times K\}$ with $L = \{l_0, \dots, l_n\}$ and $K = \{k_0, \dots, k_n\}$. The T -projection of $X(n)$ denoted by $\pi_T(X(n))$ is given by

$$\pi_T(X(n)) = \begin{cases} \{\langle \rangle\} & \text{if } T = T_{\langle \rangle}, \\ \{\langle \rangle, \langle m \rangle\} & \text{if } T = T_{\langle 0 \rangle}, \\ \{\langle \rangle, \langle m \rangle, \langle m, k_j \rangle : j \in J\} & \text{if } T = T(\emptyset, J) \text{ \& } J \neq \emptyset, \\ \{\langle \rangle, \langle m \rangle, \langle m, l_i \rangle : i \in I\} & \text{if } T = T(I, \emptyset) \text{ \& } I \neq \emptyset, \\ \{\langle \rangle, \langle m \rangle, \langle m, (l_i, k_j) \rangle : (i, j) \in I \times J\} & \text{otherwise.} \end{cases} \quad (5.2.2)$$

The graphs in Figure 5.1 give examples of the image of a typical element of $\mathcal{H}^2(2)$ under various projection maps. The trees in Figure 5.1 with nodes \bullet form subtrees of T_1 ; hence, they are trees on ω . The trees in Figure 5.1 with nodes \bullet form subtrees of $T_1 \otimes T_1$; hence, they are trees on ω^2 . Each $T \in \mathcal{T}(n)$ induces an equivalence relation E_T , on $\mathcal{H}^2(n)$ by

$$X(n)E_T Y(n) \Leftrightarrow \pi_T(X(n)) = \pi_T(Y(n)). \quad (5.2.3)$$

Let $\mathcal{E}(n)$ denote the collection of equivalence relations E_T , for $T \in \mathcal{T}(n)$. The next result is an extension of the Erdős-Rado Theorem to $\mathcal{H}^2(n)$, $n < \omega$, and gives a canonization of the equivalence relations on $\mathcal{H}^2(n)$ in terms of $\mathcal{E}(n)$. Figure 5.2 gives the graph of the equivalence relations on $\mathcal{H}^2(0)$ associated to the five trees on

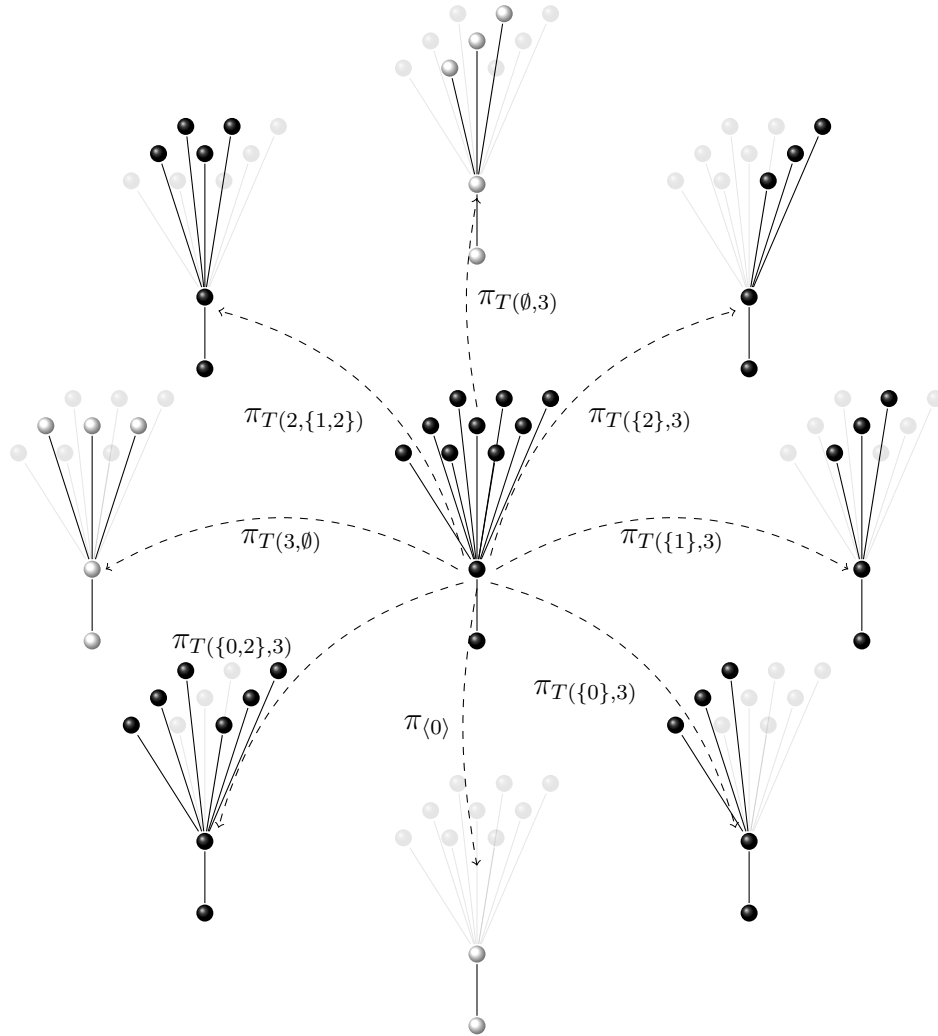


Figure 5.1: Graphs of various projections of an element of $\mathcal{H}^2(2)$

$\mathcal{T}(0)$. In Figure 5.2, two elements of $\mathcal{H}^2(0)$ are equivalent if they have the same shape (note that the elements of $\mathcal{H}^2(0)$ are in one-to-one correspondence with $[T_1 \otimes T_1]$).

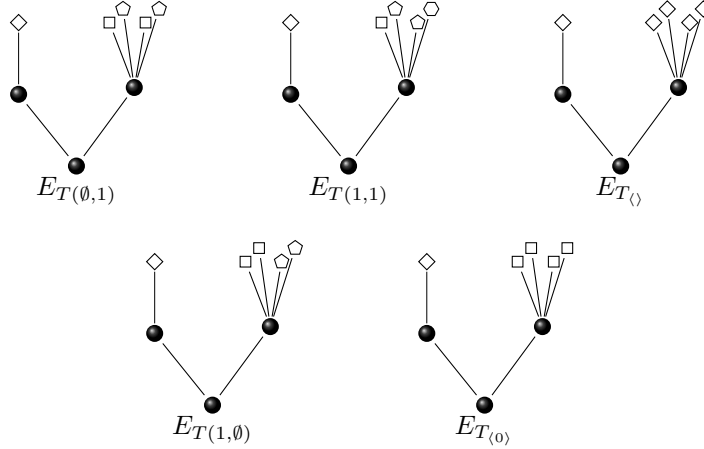


Figure 5.2: Graphs of the equivalence classes of the five canonical relations on $\mathcal{H}^2(0)$ restricted to an arbitrary element of \mathcal{AH}_2^2 .

Theorem 5.2.2 (Erdős-Rado Theorem for \mathcal{H}^2). *For each $A \in \mathcal{H}^2$ and equivalence relation E on $\mathcal{H}^2(n)|A$, there exists $B \leq A$ and $E_T \in \mathcal{E}(n)$ such that $E_T \upharpoonright (\mathcal{H}^2(n)|B) = E \upharpoonright (\mathcal{H}^2(n)|B)$.*

Proof. Let $A \in \mathcal{H}^2$ and E be an equivalence relation on $\mathcal{H}^2(n)|A$. Recall that $\tilde{S}(n) = \{\langle \rangle, \langle 0 \rangle, \langle 0, (i, j) \rangle : (i, j) \in (n+1) \times (n+1)\}$. Consider the set

$$\mathcal{X} = \{X \in \mathcal{H}^2 : X(n)E\pi_{\tilde{S}(n)}(X(n+1))\}. \quad (5.2.4)$$

If $X \in \mathcal{X}$ and $Y \in [r_{n+2}(X), X]$ then $Y \in \mathcal{X}$. So $\mathcal{X} = \bigcup_{X \in \mathcal{X}} [r_{n+2}(X), X]$, i.e., \mathcal{X} is open. Since \mathcal{H}^2 is a topological Ramsey space we find that \mathcal{X} is Ramsey. Thus, there exists an $X \leq A$ such that either $[\emptyset, X] \subseteq \mathcal{X}$ or $[\emptyset, X] \cap \mathcal{X} = \emptyset$. Let $Y \leq X$

be obtained by removing the final row and final column of the leaves of each subtree of X , i.e., $Y = \bigcup_{n < \omega} \pi_{\tilde{S}(n)}(X(n+1))$. If $s \in \mathcal{H}^2(n)|X \setminus \{X(n)\}$ then there is a $w \in \mathcal{H}^2(n+1)|X$ such that $\pi_{\tilde{S}(n)}(w) = s$. (Such a w can be obtained by adding elements to s from the final rows and columns of the leaves of $X(l)$ where l is such that $s \subseteq X(l)$.) Now define $Y' \leq X$ by

$$Y'(i) = \begin{cases} X(i) & \text{if } i \leq n, \\ w & \text{if } i = n+1, \\ \pi_{\tilde{S}(i)}(X(i+l)) & \text{if } i > n+1. \end{cases} \quad (5.2.5)$$

If $[\emptyset, X] \subseteq \mathcal{X}$ holds then $X(n) = Y'(n)E\pi_{\tilde{S}(n)}(Y'(n+1)) = \pi_{\tilde{S}(n)}(w)$, i.e., $X(n)Es$. Thus, all $s \in \mathcal{H}^2(n)|Y$ are E -related to $X(n)$. By transitivity, any two elements of $\mathcal{H}^2(n)|Y$ are E -related. Hence, $E \upharpoonright (\mathcal{H}^2(n)|Y) = E_{T_\emptyset} \upharpoonright (\mathcal{H}^2(n)|Y)$. In this case, if we let $B = Y$ then the Lemma holds. So, without loss of generality, we may assume that $[\emptyset, X] \cap \mathcal{X} = \emptyset$.

Claim 10. *There exists a $Y \leq X$ such that for all $i < j < \omega$, if $s \in \mathcal{H}^2(n)|Y(i)$ and $t \in \mathcal{H}^2(n)|Y(j)$ then $s \not E t$.*

Proof. Let $Y \leq X$ obtained by removing the final row and final column of the leaves of each subtree of X , i.e., $Y = \bigcup_{n < \omega} \pi_{\tilde{S}(n)}(X(n+1))$. Suppose that $s, t \in \mathcal{H}^2(n)|Y$ come from the subtrees $Y(i)$ and $Y(j)$, respectively, with $i < j$. There exists a $w \in \mathcal{H}^2(n+1)|Y$ such that $\pi_{\tilde{S}(n)}(w) = t$. (Such a w can be obtained by adding elements to t from the final rows and columns of the leaves of $X(l)$ where l is such

that $t \subseteq X(l)$.) Now define $Y' \leq X$ by

$$Y'(i) = \begin{cases} X(i) & \text{if } i < n, \\ s & \text{if } i = n, \\ w & \text{if } i = n + 1, \\ \pi_{\tilde{S}(i)}(X(i + l)) & \text{if } i > n + 1. \end{cases} \quad (5.2.6)$$

Since $[\emptyset, X] \cap \mathcal{X} = \emptyset$ and $Y' \leq X$, we find that $s = Y'(n) \not\equiv \pi_{\tilde{S}(n)}(Y'(n)) = \pi_{\tilde{S}(n)}(w)$, i.e., $s \not\equiv t$. \square

For each $n < \omega$, $n < M < \omega$, $w \in \mathcal{H}^2(M)|Y$, and $s \in \mathcal{H}^2(n)|w$ there are p, K, K', L' and L such that $s = \{\langle \rangle, \langle p \rangle, \langle p, (l, k) \rangle : (l, k) \in L \times K\}$, $|L| = |K| = n+1$ and $w = \{\langle \rangle, \langle p \rangle, \langle p, (l, k) \rangle : (l, k) \in L' \times K'\}$ with $|L'| = |K'| = M > n+1$. For each $P \in [L' \cup K']^{k+1}$, let

$$P_s = \{\langle \rangle, \langle p \rangle, \langle p, (l, k) \rangle : (l, k) \in P \times K\} \quad (5.2.7)$$

$$P^s = \{\langle \rangle, \langle p \rangle, \langle p, (l, k) \rangle : (l, k) \in L \times P\}. \quad (5.2.8)$$

Figure 5.3 includes a graphical depiction of P_s and P^s for some $P \in [L' \cup K']^{k+1}$. For each such s and w define equivalence relations $E_{s,w}$ and $E^{s,w}$ on $[L]^{n+1}$ and $[K']^{n+1}$ by letting

$$PE_{s,w}Q \Leftrightarrow P_sEQ_s, \quad (5.2.9)$$

$$PE^{s,w}Q \Leftrightarrow P^sEQ^s. \quad (5.2.10)$$

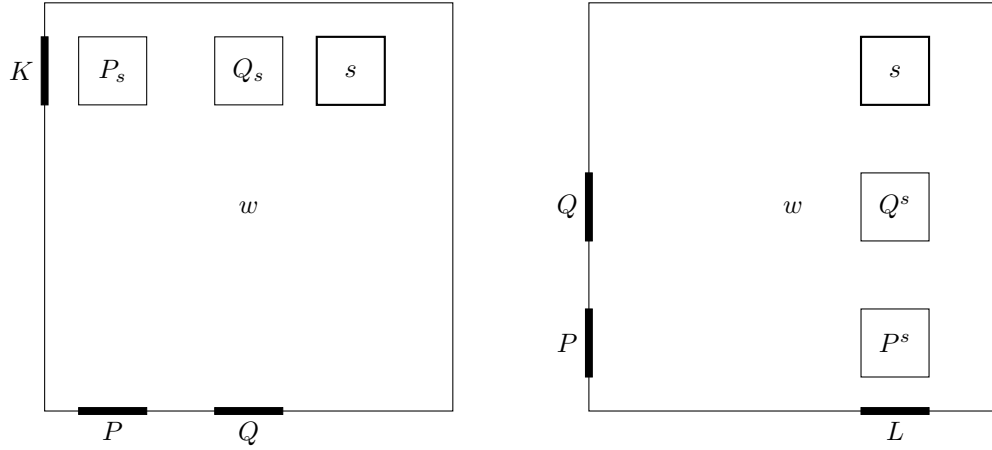


Figure 5.3: Graphs of s , w , P , Q , L , K , P_s , P^s , Q_s , and Q^s used in definition of $E_{s,w}$ and $E^{s,w}$.

In the next two claims, we use the word canonical in the sense of Theorem 1.3.3.

Claim 11. *For each $X \in \mathcal{H}^2$, $n < \omega$, and $n < M < \omega$, there exists $w \in \mathcal{H}^2(M)|X$ such that for all $s \in \mathcal{H}^2(n)|w$, $E_{s,w}$ is canonical.*

Proof. Let $X \in \mathcal{H}^2$, $n < \omega$ and $n < M < M' < \omega$. Suppose that $X(M') = \{\langle \rangle, \langle p \rangle, \langle p, (l, k) \rangle : (l, k) \in L \times K\}$ with $|L| = |K| = M' + 1$. Let $K' \in [K]^{M+1}$. For each $P \in [K']^{n+1}$ there exists $s_P \in \mathcal{H}^2(n)|X(M')$ such that

$$\pi_{T(\emptyset, n+1)}(s_P) = \{\langle \rangle, \langle p \rangle, \langle p, l \rangle : l \in P\}. \quad (5.2.11)$$

By applying the finite version of Theorem 1.3.3 successively, once for each $P \in [K']^{n+1}$, we can pick M' large enough that there exists $L' \in [L]^{M+1}$ such that for all $P \in [K']^{n+1}$, $E_{s_P, X(M')} \upharpoonright [L']^{n+1}$ is canonical. Now define $w \in \mathcal{H}^2(M)|X$ by letting

$$w = \{\langle \rangle, \langle p \rangle, \langle p, (l, k) \rangle : (l, k) \in L' \times K'\}. \quad (5.2.12)$$

Then for each $s \in \mathcal{H}^2(n)|w$ there exists $P \in [K']^{n+1}$ such that $\pi_{T(\emptyset, n+1)}(s_P) = \pi_{T(\emptyset, n+1)}(s)$. Therefore $E_{s,w} = E_{s_P, X(M')} \upharpoonright [L']^{n+1}$ is canonical. \square

By an identical argument we also have the following claim.

Claim 12. *For each $X \in \mathcal{H}^2$, $n < \omega$, and $n < M < \omega$, there exists $w \in \mathcal{H}^2(M)|X$ such that for all $s \in \mathcal{H}^2(n)|w$, $E^{s,w}$ is canonical.*

Now using Claim 11 we can build a sequence $\langle w_i \in \mathcal{H}^2(i)|Y : i < \omega \rangle$ such that:

- (1) If $i < j < \omega$ then there exists $i' < j' < \omega$ such that $w_i \subseteq Y(i')$ and $w_j \subseteq Y(j')$.
- (2) For all $i < \omega$ and each $s \in \mathcal{H}^2(n)|w_i$, E_{s,w_i} is canonical.

Let $Z = \bigcup_{i < \omega} w_i$. Then $Z \leq Y$ and for all $i < \omega$ and $s \in \mathcal{H}^2(n)|Z(i)$, $E_{s,Z(i)}$ is canonical. By an identical argument using Claim 12 in place of Claim 11, there is a $Z' \leq Z$ such that for all $j < \omega$ and $s \in \mathcal{H}^2(n)|Z'(j)$, $E^{s,Z'(j)}$ is canonical. In particular, for all $i < \omega$ and $s \in \mathcal{H}^2(n)|Z'(i)$ both $E^{s,Z'(i)}$ and $E_{s,Z'(i)}$ are canonical. So for each $s \in \mathcal{H}^2(n)|Z'$ there exists $I_s, J_s \subseteq n+1$ such that $E_{s,Z'(i)} = E_{I_s}$ and $E^{s,Z'(i)} = E_{J_s}$.

For each $I, J \subseteq n+1$, consider the following sets:

$$\mathcal{X}_I = \{W \leq Z' : I_{W(n)} = I\}, \quad (5.2.13)$$

$$\mathcal{Y}_J = \{W \leq Z' : I_{W(n)} = J\}. \quad (5.2.14)$$

Notice that $\mathcal{X}_I = \bigcup_{W \in \mathcal{X}_I} [r_{n+1}(W), W]$ and $\mathcal{Y}_J = \bigcup_{W \in \mathcal{Y}_J} [r_{n+1}(W), W]$. It follows that for each $I, J \subseteq n+1$, \mathcal{X}_I and \mathcal{Y}_J are open and Ramsey since \mathcal{H}^2 is a topological Ramsey space. Applying the Ramsey property finitely many times, we find a $W \leq$

Z' such that for each pair I, J of subsets of $n+1$, either $[\emptyset, W] \subseteq \mathcal{X}_I$ or $[\emptyset, W] \cap \mathcal{X}_I = \emptyset$, and either $[\emptyset, W] \subseteq \mathcal{Y}_J$ or $[\emptyset, W] \cap \mathcal{Y}_J = \emptyset$.

Toward a contradiction suppose that for all $I \subseteq n+1$, $[\emptyset, W] \cap \mathcal{X}_I = \emptyset$. Then for all $I \subseteq n+1$, $I_{W(n)} \neq I$, contradiction. So there exists $I \subseteq n+1$ such that $[\emptyset, W] \subseteq \mathcal{X}_I$. Likewise there exists $J \subseteq n+1$ such that $[\emptyset, W] \subseteq \mathcal{Y}_J$. If I' is another subset of $n+1$ such that $[\emptyset, W] \subseteq \mathcal{X}_{I'}$ then $I = I_{W(n)} = I'$. So there is a unique $I \subseteq n+1$ such that $[\emptyset, W] \subseteq \mathcal{X}_I$. Similarly, there exists a unique $J \subseteq n+1$ such that $[\emptyset, W] \subseteq \mathcal{Y}_J$. For all $s \in \mathcal{H}^2(n)|W$ there exists $W' \leq W$ such that $W'(n) = s$. Since $[\emptyset, W] \subseteq \mathcal{X}_I \cap \mathcal{Y}_J$, we find that for all $s \in \mathcal{H}^2(n)|W$, $I_{W'(n)} = I_s = I$ and $J_{W'(n)} = J_s = J$.

Claim 13. $E_{T(I,J)} \upharpoonright (\mathcal{H}^2(n)|W) \subseteq E \upharpoonright (\mathcal{H}^2(n)|W)$. Furthermore, if $I = J = \emptyset$ then $E_{T(\emptyset)} \upharpoonright (\mathcal{H}^2(n)|W) = E \upharpoonright (\mathcal{H}^2(n)|W)$.

Proof. Suppose that $s, t \in \mathcal{H}^2(n)|W$ and $sE_{T(I,J)}t$. (If I and J are both empty let $T(I, J)$ denote $T_{(\emptyset)}$.) By definition, there exists m, K, L, K' , and L' such that $s = \{\langle \rangle, \langle m \rangle, \langle m, (l, k) \rangle : (l, k) \in L \times K\}$, $t = \{\langle \rangle, \langle m \rangle, \langle m, (l, k) \rangle : (l, k) \in L' \times K'\}$ and $|L| = |K| = |L'| = |K'| = n+1$. Since $sE_{T(I,J)}t$ we find that $L'E_I L$ and $K'E_J K$. Next define $u \in \mathcal{H}^2(n)|W$ by letting $u = \{\langle \rangle, \langle m \rangle, \langle m, (l, k) \rangle : (l, k) \in L' \times K'\}$. Since $\pi_{T(\emptyset, n+1)}(s) = \pi_{T(\emptyset, n+1)}(u) = \{\langle \rangle, \langle m \rangle, \langle m, k \rangle : k \in K\}$ and $L'E_I L$ we find that sEu . Similarly, since $\pi_{T(n+1, \emptyset)}(t) = \pi_{T(n+1, \emptyset)}(u) = \{\langle \rangle, \langle m \rangle, \langle m, l \rangle : l \in L'\}$ and $K'E_J K$ we find that uEt . By transitivity of E we find that sEt .

If $I = J = \emptyset$ then for all $l < \omega$ and for all $s, t \in \mathcal{H}^2(n)|W(l)$, sEt . This along with Claim 10 implies that $E_{T(\emptyset)} \upharpoonright (\mathcal{H}^2(n)|W) = E \upharpoonright (\mathcal{H}^2(n)|W)$. \square

If $I = J = \emptyset$ then we can take $B = W$ and the Lemma holds. Thus, without loss

of generality, we may assume that $I \neq \emptyset$ or $J \neq \emptyset$. For each $P, Q \in \{T(L, K) \in \mathcal{T}(2n+2) : |L| = |K| = n+1\}$ consider the sets:

$$\mathcal{F}_0(P, Q) = \{w \in \mathcal{H}^2(2n+2) | W : \pi_P(w) E \pi_Q(w)\}, \quad (5.2.15)$$

$$\mathcal{F}_1(P, Q) = \{w \in \mathcal{H}^2(2n+2) | W : \pi_P(w) \not E \pi_Q(w)\}. \quad (5.2.16)$$

Notice that $\mathcal{F}_0(P, Q) \cup \mathcal{F}_1(P, Q) = \mathcal{H}^2(2n+2) | W$ is Nash-Williams. Applying Theorem 1.2.7 once for each pair (P, Q) gives a $V \leq W$ such that for each such pair (P, Q) , either $\mathcal{F}_0(P, Q) | V = \emptyset$ or $\mathcal{F}_1(P, Q) | V = \emptyset$. Let $V' \leq V$ be obtained from V by letting $V' = \bigcup_{i < \omega} V'(i)$ where $V'(i) = \pi_{T(H_i, H_i)}(V(2i+1))$ and $H_i = \{2j : j < i\}$. V' then has the property that between any two rows or columns of leaves in a subtree of V' there is one row and one column of the leaves of V , respectively. Next let

$$\Sigma = \{(P, Q) : \mathcal{F}_1(P, Q) | V = \emptyset\}. \quad (5.2.17)$$

Claim 14. For each $P, Q \in \{T(L, K) \in \mathcal{T}(2n+2) : |L| = |K| = n+1\}$ and each $i, j < n+1$ we have,

$$i \in I \ \& \ P \not E_{T(\{i\}, \emptyset)} Q \Rightarrow (P, Q) \notin \Sigma, \quad (5.2.18)$$

$$j \in J \ \& \ P \not E_{T(\emptyset, \{j\})} Q \Rightarrow (P, Q) \notin \Sigma. \quad (5.2.19)$$

Proof. Suppose that $i \in I$, and $P \not E_{T(\{i\}, \emptyset)} Q$. Fix $w \in \mathcal{H}^2(2n+2) | V'$ and let $s = \pi_P(w)$ and $t = \pi_Q(w)$. Note that by definition there are i', i'', M and L such

that

$$\pi_{T(\{i\},\emptyset)}(s) = \pi_{T(\{i'\},\emptyset)}(w), \quad (5.2.20)$$

$$\pi_{T(\{i'\},2n+2)}(w) = \pi_{T(\{i''\},L)}(V(M)), \quad (5.2.21)$$

and $|L| = 2n + 2$. In other words, i' and i'' give the position of columns in w and $V(M)$ that contain $\pi_{T(\{i\},n+1)}(s)$. Now construct $w' \in \mathcal{H}^2(2n + 2)|V$ by letting

$$w' = (w \setminus \pi_{T(\{i'\},2n+2)}(w)) \cup \pi_{T(\{i''+1\},L)}(V(M)). \quad (5.2.22)$$

That is, w' is obtained by taking the column containing $\pi_{T(\{i\},n+1)}(s)$ in w and moving it one column to the right in V . If we let $s' = \pi_P(w')$ then $\pi_{T(\{i\},\emptyset)}(s) \neq \pi_{T(\{i\},\emptyset)}(s')$ and $\pi_{T(\emptyset,n+1)}(s) = \pi_{T(\emptyset,n+1)}(s')$. Hence, $s \not\mathcal{E} s'$ since $E_{s,V(M)} = E_I$ and $i \in I$.

Now let $t' = \pi_Q(w')$ and assume tEt' . By the contrapositive of transitivity, either $s \mathcal{E} t$ or $t \mathcal{E} s'$. Thus, either $\pi_P(w) \mathcal{E} \pi_Q(w)$ or $\pi_P(w') \mathcal{E} \pi_Q(w')$. Since $w, w' \in \mathcal{H}^2(2n + 2)|V$ we find that either w or w' is in $\mathcal{F}_1|V$, i.e., $(P, Q) \notin \Sigma$.

Otherwise $t' \mathcal{E} t$. Since $\pi_{T(\emptyset,n+1)}(t) = \pi_{T(\emptyset,n+1)}(t')$, there exists $i_1 \in I \setminus \{i\}$ such that:

$$\pi_{T(\{i_1\},\emptyset)}(t) = \pi_{T(\{i\},\emptyset)}(s), \quad (5.2.23)$$

$$P \mathcal{E}_{T(\{i_1\},\emptyset)} Q. \quad (5.2.24)$$

Now suppose that $i_1 > i$. Applying the same construction above to s and t but with

i_1 in place of i we find that either $(P, Q) \notin \Sigma$ or there exists $i_2 \in I \setminus \{i_1\}$ such that:

$$\pi_{T(\{i_2\}, \emptyset)}(t) = \pi_{T(\{i_1\}, \emptyset)}(s), \quad (5.2.25)$$

$$P \not\leq_{T(\{i_2\}, \emptyset)} Q. \quad (5.2.26)$$

Notice that $\pi_{T(\{i_1\}, n+1)}(s)$ is to the right of $\pi_{T(\{i\}, n+1)}(s)$; so, it must also be the case that $\pi_{T(\{i_2\}, n+1)}(t)$ is to the right of $\pi_{T(\{i_1\}, n+1)}(t)$ because $\pi_{T(\{i_1\}, \emptyset)}(t) = \pi_{T(\{i\}, \emptyset)}(s)$ and $\pi_{T(\{i_2\}, \emptyset)}(t) = \pi_{T(\{i_1\}, \emptyset)}(s)$. Therefore $i_2 > i_1$.

It follows that if $(P, Q) \in \Sigma$ then we obtain an infinite sequence $i < i_1 < i_2 < i_3 \dots$, such that for all $l < \omega$,

$$\pi_{T(\{i_{l+1}\}, \emptyset)}(t) = \pi_{T(\{i_l\}, \emptyset)}(s) \text{ and} \quad (5.2.27)$$

$$P \not\leq_{T(\{i_l\}, \emptyset)} Q, \quad (5.2.28)$$

which is a contradiction since P and Q are finite. So in the case that $i_1 > i$, we find that $(P, Q) \notin \Sigma$. By a similar argument, we also find that if $i_1 < i$ then $(P, Q) \notin \Sigma$. Since $i_1 \neq i$ we find that it must be the case that $(P, Q) \notin \Sigma$.

By an similar argument, moving rows instead of columns, we find that if $j \in J$ and $P \not\leq_{T(\emptyset, \{j\})} Q$ then $(P, Q) \notin \Sigma$. \square

Let $B = V$ and suppose that $s, t \in \mathcal{H}^2(n)|B(l)$ for some $l < \omega$. Since $|s \cup t| < |w|$ for each $w \in \mathcal{H}^2(2n+2)$, there are $P, Q \in \{T(L, K) \in \mathcal{T}(2n+2) : |L| = |K| = n+1\}$ and $w \in \mathcal{H}^2(2n+2)$ such that $s = \pi_P(w)$ and $t = \pi_Q(w)$. Recall

that either $I \neq \emptyset$ or $J \neq \emptyset$. By the previous claim,

$$\begin{aligned}
s \not E_{T(I,J)} t &\Rightarrow \exists i < n + 1 (i \in I \ \& \ P \not E_{T(\{i\}, \emptyset)} Q) \\
&\text{or } \exists j < n + 1 (j \in J \ \& \ P \not E_{T(\emptyset, \{j\})} Q), \\
&\Rightarrow (P, Q) \notin \Sigma, \\
&\Rightarrow \pi_P(w) \not E \pi_Q(w), \\
&\Rightarrow s \not E t.
\end{aligned}$$

Therefore $E \upharpoonright (\mathcal{H}^2(n)|B) \subseteq E_{T(I,J)} \upharpoonright (\mathcal{H}^2(n)|B)$. Together with Claim 13 we have $E \upharpoonright (\mathcal{H}^2(n)|B) = E_{T(I,J)} \upharpoonright (\mathcal{H}^2(n)|B)$. \square

5.3 Canonical Ramsey theory for \mathcal{H}^2

In this section, we prove a version of the Pudlak-Rödl Theorem, Theorem 1.3.6, for equivalence relations on fronts on the topological Ramsey space \mathcal{H}^2 . We follow Dobrinen and Todorćević in [17] and apply the method of mixing and separating introduced by Pröml and Voigt in [39] to canonize Borel mappings from $[\omega]^\omega$ into the real numbers. In the next Lemma $X/(s, t)$ denotes the set $X/s \cap X/t$.

Lemma 5.3.1. (1) *Suppose $P(\cdot, \cdot)$ is a property such that for each $s \in \mathcal{A}\mathcal{H}^2$ and each $X \in \mathcal{H}^2$, there is a $Z \leq X$ such that $P(s, Z)$ holds. Then for each $X \in \mathcal{H}^2$ there is a $Y \leq X$ such that for each $s \in \mathcal{A}\mathcal{H}^2|Y$ and each $Z \leq Y$, $P(s, Z/s)$ holds.*

(2) *Suppose $P(\cdot, \cdot, \cdot)$ is a property such that for each $s, t \in \mathcal{A}\mathcal{H}^2$ and each $X \in \mathcal{H}^2$, there is a $Z \leq X$ such that $P(s, t, Z)$ holds. Then for each $X \in \mathcal{H}^2$ there is a $Y \leq X$ such that for each $s, t \in \mathcal{A}\mathcal{H}^2|Y$ and each $Z \leq Y$, $P(s, t, Z/(s, t))$ holds.*

Proof. The proofs are by fusion arguments. Let X be given. By the hypothesis, there is an $X_1 \leq X$ for which $P(\emptyset, X_1)$ holds. Fix $y_1 = r_1(X_1)$. For $n \geq 1$, given X_n and y_n , enumerate $\mathcal{AH}^2|y_n$ as $s_i, i < |\mathcal{AH}^2|y_n|$. Applying the hypothesis finitely many times, we obtain an $X_{n+1} \leq X_n$ such that $P(s_i, X_{n+1}/s_i)$ holds for all $i < |\mathcal{AH}^2|y_n|$. Let $y_{n+1} = y_n \cup X_{n+1}(n)$. Continuing in this manner, we obtain $Y = \bigcup_{n \geq 1} y_n$ which satisfies (1).

Let X be given. Fix $s = r_0(X) = \emptyset$ and $t = r_1(X)$, and let $y_1 = r_1(X)$. By the hypothesis, there is an $X_2 \leq X$ such that $P(s, t, X_2)$. Let $y_2 = y_1 \cup X_2(1)$. Let $n \geq 2$ be given, and suppose X_n and y_n have been constructed. Enumerate the pairs of distinct elements $s, t \in \mathcal{AH}^2|y_n$ as (s_i, t_i) , for all $i < |[\mathcal{AH}^2|y_n|^2]$. By finitely many applications of the hypothesis, we obtain an $X_{n+1} \leq X_n$ such that for each $i < |[\mathcal{AH}^2|y_n|^2]$, $P(s_i, t_i, X_n)$ holds. Let $y_{n+1} = y_n \cup X_{n+1}(n)$. In this way we obtain $Y = \bigcup_{n \geq 1} y_n$ which satisfies (2). \square

Given a front \mathcal{F} on $[\emptyset, A]$ for some $A \in \mathcal{H}^2$ and $f : \mathcal{F} \rightarrow \omega$, we adhere to the following convention from [17]: If we write $f(a)$ or $f(s \cup u)$, it is assumed that a and $a \cup u$ are in \mathcal{F} . We also use the following notation from [17]. Define

$$\hat{\mathcal{F}} = \{r_m(b) : b \in \mathcal{F}, m \leq n < \omega, \text{ where } b \in \mathcal{AH}_n^2\}. \quad (5.3.1)$$

Note that $\emptyset \in \hat{\mathcal{F}}$, since $\emptyset = r_0(b)$ for any $b \in \mathcal{F}$. For any $X \leq A$, define

$$\text{Ext}(X) = \{s \setminus r_m(s) : m < \omega, \exists n \geq m (s \in \mathcal{AH}_n^2, \& s \setminus r_m(s) \subseteq X)\}. \quad (5.3.2)$$

As in [17], $\text{Ext}(X)$ is the collection of possible legal extensions into X . For $s \in \mathcal{AH}^2$, let $\text{Ext}(X/s)$ denote the set of $y \in \text{Ext}(X)$ such that $y \subseteq X/s$. For $u \in \mathcal{AH}^2$,

we write $v \in \text{Ext}(u)$ to mean that $v \in \text{Ext}(T_1 \otimes T_1)$ and $v \subseteq u$.

The next notions of separating and mixing is essentially identical to the corresponding notions in [17]. The definitions of mixing and separating in [17] are based on definitions from the paper [39], where Pröml and Voigt canonized Borel mappings from $[\omega]^\omega$ into the real numbers. Next we reproduce the definitions and proofs about mixing and separating from the paper [17] in the context of the topological Ramsey space \mathcal{H}^2 .

Definition 5.3.2. Fix $s, t \in \hat{\mathcal{F}}$ and $X \in \mathcal{H}^2$. We say that X *separates* s and t if and only for all $x \in \text{Ext}(X/s)$ and $y \in \text{Ext}(X/t)$ such that $s \cup x$ and $t \cup y$ are in \mathcal{F} , $f(s \cup x) \neq f(t \cup y)$. We say that X *mixes* s and t if and only if there is no $Y \leq X$ which separates s and t . X *decides for* s and t if and only if either X separates s and t or else X mixes s and t .

Definition 5.3.3. Fix $s, t \in \hat{\mathcal{F}}$ and $X \in \mathcal{H}^2$. In the following we let $\text{Ext}(X/(s, t))$ denote $\text{Ext}(X/s) \cap \text{Ext}(X/t)$. We say that $X/(s, t)$ *separates* s and t if and only if for all $x, y \in \text{Ext}(X/(s, t))$ such that $s \cup x$ and $t \cup y$ are in \mathcal{F} , $f(s \cup x) \neq f(t \cup y)$. We say that $X/(s, t)$ *mixes* s and t if and only if there is no $Y \leq X$ which separates s and t . $X/(s, t)$ *decides for* s and t if and only if either $X/(s, t)$ separates s and t or else $X/(s, t)$ mixes s and t .

Remark 5.3.4. The relation of mixing is both reflexive and symmetric. That is, for all $s, t \in \hat{\mathcal{F}}$ and $X \in \mathcal{H}^2$, if X mixes s and t then X mixes t and s , and X mixes s and s . For $s, t \in \hat{\mathcal{F}}$ and $X \in \mathcal{H}^2$, $X/(s, t)$ decides for s and t if and only if for all $x, y \in \text{Ext}(X/(s, t))$, $f(s \cup x) \neq f(t \cup y)$, or else there is no $Y \leq X$ which has this property.

The next lemma is a straightforward generalization of the same result for the

space \mathcal{R}_1 in [17].

Lemma 5.3.5 (Transitivity of Mixing). *For any $X \in \mathcal{H}^2$ and any $s, t, u \in \hat{\mathcal{F}}$, if X mixes s and t and X mixes t and u , then X mixes s and u .*

Proof by Contrapositive. Suppose that X does not mix s and u . Then there is a $Y \leq X$ that separates s and u . Let $k = |s|, l = |t|$, and $m = |u|$. Shrinking Y if necessary, we may assume that $\text{depth}_{T_1 \otimes T_1}(Y(1)) > \max(\text{depth}_{T_1 \otimes T_1}(s), \text{depth}_{T_1 \otimes T_1}(t))$. Let $Y_s = s \cup (Y \setminus r_k(Y))$ and $Y_t = t \cup (Y \setminus r_l(Y))$. Then Y_s and Y_t are both members of \mathcal{H}^2 . Let $\mathcal{F}_t = \{v \in \mathcal{F} : t \sqsubseteq v\}$ and

$$\mathcal{G} = \{v \in \mathcal{F}_t | Y_t : \exists w \in \mathcal{F}_s | Y_s (f(v) = f(w))\}. \quad (5.3.3)$$

Note that $\mathcal{F}_t | Y_t$ is a Nash-Williams family because otherwise \mathcal{F} would not be a Nash-Williams family. Furthermore, if $\mathcal{G}' = (\mathcal{F}_t | Y_t) \setminus \mathcal{G}$ then $\mathcal{G} \cup \mathcal{G}' = \mathcal{F}_t | Y_t$. By Theorem 1.2.7 relativized to \mathcal{F}_t , there is a $Z \in [t, Y_t]$ such that either $\mathcal{G} | Z = \emptyset$ or else $\mathcal{G}' | Z = \emptyset$.

If $\mathcal{G}' | Z = \emptyset$ then for each $v \in \mathcal{F}_t | Z$, there is a $w \in \mathcal{F}_s | Y_s$ such that $f(v) = f(w)$. Since Y separates s and u , we find that for each $y \in \text{Ext}(Z/u)$ such that $u \cup y \in \mathcal{F}$ and each $w \in \mathcal{F}_s | Y_s$, we have that $f(w) \neq f(u \cup y)$. Therefore, for each $y \in \text{Ext}(Z/u)$ such that $u \cup y \in \mathcal{F}$ and each $v \in \mathcal{F}_t | Z$, $f(u \cup y) \neq f(v)$. If $x \in \text{Ext}(Z/t)$ such that $t \cup x \in \mathcal{F}$ then $t \cup x \in \mathcal{F}_t | Z$. Hence, for all $x \in \text{Ext}(Z/t)$ and $y \in \text{Ext}(Z/u)$ such that $u \cup y$ and $t \cup x$ are in \mathcal{F} , $f(u \cup y) \neq f(t \cup x)$. Thus, Z does not mix t and u .

If $\mathcal{G} | Z = \emptyset$ then for each $v \in \mathcal{F}_t | Z$, for each $w \in \mathcal{F}_s | Y_s$, $f(v) \neq f(w)$. Note that for all $x \in \text{Ext}(Z/t)$ and $y \in \text{Ext}(Z/s)$ with $t \cup x$ and $s \cup y$ in \mathcal{F} , $t \cup x \in \mathcal{F}_t | Z$ and $s \cup y \in \mathcal{F}_s | Y_s$. Thus, Z does not mix t and s .

Note that if X mixes v and w then for all $Z \leq X$, Z mixes v and w . Thus, if X does not mix s and u then either X does not mix t and u or X does not mix t and s . \square

Remark 5.3.6. The previous Lemma shows that the mixing relation is an equivalence relation since it also reflexive and symmetric.

Lemma 5.3.7. *For each $X \in \mathcal{H}^2$, there is a $Y \leq X$ such that for each $s, t \leq_{\text{fin}} Y$ in $\hat{\mathcal{F}}$, $Y/(s, t)$ decides s and t .*

Proof. For $s, t \in \mathcal{AH}^2$ and $Y \in \mathcal{H}^2$, let $P(s, t, Y)$ be the following property: If $s, t \in \hat{\mathcal{F}}$, then $Y/(s, t)$ decides s and t . We will show that for each $s, t \in \hat{\mathcal{F}}$ and each $X \in \mathcal{H}^2$, there is a $Y \leq X$ which decides for s and t . The result will then follow from Lemma 5.3.1 (2).

Fix $X \in \mathcal{H}^2$ and $s, t \in \hat{\mathcal{F}}$. Let

$$\mathcal{X}_{s,t} = \{Y \leq X : \exists v, w \in \text{Ext}(Y)(f(s \cup v) = f(t \cup w))\}. \quad (5.3.4)$$

For each $Y \in \mathcal{X}_{s,t}$, let n be the smallest ordinal such that there exists v and w in $\text{Ext}(Y)$ with $v, w \subseteq r_n(Y)$. For each $Y \in \mathcal{X}$, let $\mathcal{X}_Y = [r_n(Y), Y]$. Notice that for each $Y \in \mathcal{X}$, $\mathcal{X}_Y \subseteq \mathcal{X}_{s,t}$. Since each \mathcal{X}_Y is a basic open set and $\mathcal{X}_{s,t} = \bigcup\{\mathcal{X}_Y : Y \in \mathcal{X}_{s,t}\}$ we find that $\mathcal{X}_{s,t}$ is open. Since \mathcal{H}^2 is a topological Ramsey space, there is a $Y \leq X$ such that either $[\emptyset, Y] \subseteq \mathcal{X}_{s,t}$ or $[\emptyset, Y] \cap \mathcal{X}_{s,t} = \emptyset$. If $[\emptyset, Y] \subseteq \mathcal{X}_{s,t}$ then for all $Y' \leq Y$ there are $v, w \in \text{Ext}(Y')$ such that $s \cup v, t \cup w \in \mathcal{F}$ and $f(s \cup v) = f(t \cup w)$. Hence, Y mixes s and t . On the other hand, if $[\emptyset, Y] \cap \mathcal{X}_{s,t} = \emptyset$ then for each $v, w \in \text{Ext}(Y)$ such that $s \cup v, t \cup w \in \mathcal{F}$, $f(s \cup v) \neq f(t \cup w)$. Thus, Y separates s and t . In both cases, Y decides for s and t . \square

Notation 5.3.8. Suppose that $\mathcal{F} \subseteq \mathcal{AH}^2$ and φ is a function on \mathcal{F} such that for all $a \in \mathcal{F}$, $\varphi(a)$ is a subtree of a . For $a, b \in \mathcal{F}$ we say that we write $\varphi(a) \sqsubseteq \varphi(b)$ if $\pi_0''[\varphi(a)] \subseteq \pi_0''[\varphi(b)]$ and for each $\langle k \rangle \in \pi_0''[\varphi(a)]$, $cl(\{s \in [\varphi(a)] : \pi_0 = \langle k \rangle\}) = cl(\{s \in [\varphi(b)] : \pi_0 = \langle k \rangle\})$.

Definition 5.3.9. Let \mathcal{F} be a front on $[\emptyset, X]$ for some $X \in \mathcal{H}^2$, and let φ be a function on \mathcal{F} .

- (1) φ is *inner* if for each $a \in \mathcal{F}$ there exists a family of trees $\{T_i : i < |a|\}$ such that for each $i < \omega$ $T(i) \in \mathcal{T}(i)$ and $\varphi(a) = \bigcup_{i < |a|} \pi_{T_i}(a(i))$.
- (2) φ is *Nash-Williams* if $\varphi(a) \not\sqsupseteq \varphi(b)$, for all $a \neq b \in \mathcal{F}$.
- (3) φ is *Sperner* if $\varphi(a) \not\subseteq \varphi(b)$, for all $a \neq b \in \mathcal{F}$.

Definition 5.3.10 ([17]). Let \mathcal{F} be a front on $[\emptyset, X]$, and R be an equivalence relation on \mathcal{F} . We say that R is *canonical* if and only if there is an inner Nash-Williams function φ on \mathcal{F} such that

- (1) for all $a, b \in \mathcal{F}$, aRb if and only if $\varphi(a) = \varphi(b)$; and
- (2) φ is maximal among all inner Nash-Williams functions satisfying (1). That is, for any other inner Nash-Williams function φ' on \mathcal{F} satisfying (1), there is a $Y \leq X$ such that $\varphi'(a) \subseteq \varphi(a)$, for all $a \in \mathcal{F}|Y$.

The following theorem is the main canonization theorem of this section. The proof of the theorem is nearly identical to the proof of Theorem 4.12 in [17]. The main difference is an application of Lemma 5.2.2 in Claim 15. Additionally, a simple modification of the proof of the subclaim within the proof of Claim 17 is also needed to complete the proof.

Theorem 5.3.11. *Suppose $A \in \mathcal{H}^2$, \mathcal{F} is a front on $[\emptyset, A]$ and R is an equivalence relation on \mathcal{F} . Then there is a $C \leq A$ such that R is canonical on $\mathcal{F}|C$.*

Proof. Let $A \in \mathcal{H}^2$, let \mathcal{F} be a given front on $[\emptyset, A]$, and R be an equivalence relation on \mathcal{F} . Let $f : \mathcal{F} \rightarrow \omega$ be any mapping which induces R . By thinning if necessary, we may assume A satisfies Lemma 5.3.7. Let $(\hat{\mathcal{F}} \setminus \mathcal{F})|X$ denote the collection of those $t \in \hat{\mathcal{F}} \setminus \mathcal{F}$ such that $t \leq_{\text{fin}} X$.

Claim 15. *There is a $B \leq A$ such that for all $s \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$, letting n denote $|s|$, there is an equivalence relation $E_s \in \mathcal{E}(n)$ such that, for all $u, v \in \mathcal{H}^2(n)|B/s$, B mixes $s \cup u$ and $s \cup v$ if and only if $uE_s v$.*

Proof. For any $X \leq A$ and $s \in \mathcal{AH}^2|A$, let $P(s, X)$ denote the following statement: "If $s \in \hat{\mathcal{F}} \setminus \mathcal{F}$, then there is an equivalence relation $E_s \in \mathcal{E}(|s|)$ such that for all $u, v \in \mathcal{H}^2(|s|)|X/s$, mixes $s \cup u$ and $s \cup v$ if and only if $uE_s v$." We shall show that for each $X \leq A$ and $s \in \mathcal{AH}^2|A$, there is a $Z \leq X$ for which $P(s, Z)$ holds. The claim will then follow from Lemma 5.3.1.

Let $X \leq A$ and $s \in \hat{\mathcal{F}} \setminus \mathcal{F}$ be given, and let $n = |s|$. Let R denote the following equivalence relation on $\mathcal{H}^2(n)|X/s$:

$$uRv \text{ if and only if } A \text{ mixes } s \cup u \text{ and } s \cup v. \quad (5.3.5)$$

By Lemma 5.2.2 there exists $Z \leq X$ and $T_s \in \mathcal{E}(n)$ such that $E_{T_s} \upharpoonright (\mathcal{H}^2(n)|Z/s) = R \upharpoonright (\mathcal{H}^2(n)|Z/s)$. If we let $E_s = E_{T_s}$ then for all $u, v \in \mathcal{H}^2(n)|Z/s$, Z mixes $s \cup u$ and $s \cup v$ if and only if $uE_s v$. Therefore there exists $Z \leq X$ such that $P(Z, s)$ holds. \square

Fix B be as in Claim 15. For $s \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$ and $n = |s|$, let E_s be the equivalence

relation for s from Claim 15. We say that s is E_s -mixed by B , meaning that for all $u, v \in \mathcal{H}^2(n)|B/s$, B mixes $s \cup u$ and $s \cup v$ if and only if $uE_s v$. Let T_s denote the subtree of $\tilde{S}(n)$ such that $E_s = E_{T_s}$.

Definition 5.3.12. For $s \in \hat{\mathcal{F}}|B$, $n = |s|$, and $i < n$, define

$$\varphi_{r_i(s)}(s(i)) = \pi_{T_{r_i(s)}}(s(i)). \quad (5.3.6)$$

For $s \in \mathcal{F}|B$, define

$$\varphi(s) = \bigcup_{i < |s|} \varphi_{r_i(s)}(s(i)). \quad (5.3.7)$$

Claim 16. *The following are true for all $X \leq B$ and all $s, t \in \hat{\mathcal{F}}|B$.*

- (A1) *Suppose $s \notin \mathcal{F}$ and $n = |s|$. Then X mixes $s \cup u$ and t for at most one E_s equivalence class of u 's in $\mathcal{H}^2(n)|B/s$.*
- (A2) *If $X/(s, t)$ separates s and t , then $X/(s, t)$ separates $s \cup x$ and $t \cup y$ for all $x, y \in \text{Ext}(X/(s, t))$ such that $s \cup x, t \cup y \in \hat{\mathcal{F}}$.*
- (A3) *Suppose $s \notin \mathcal{F}$ and $n = |s|$. Then $T_s = T_\emptyset$ if and only if X mixes s and $s \cup u$ for all $u \in \mathcal{H}^2(n)|B/s$.*
- (A4) *If $s \sqsubset t$ and $\varphi(s) = \varphi(t)$, then X mixes s and t .*

Proof. (A1) Toward a contradiction suppose that there are $u, v \in \mathcal{H}^2(n)|B/s$ such that $s \cup u, s \cup v \in \mathcal{F}$, $u \not E_s v$, X mixes $s \cup u$ and t , and X mixes t and $s \cup v$. By transitivity of mixing, X mixes $s \cup u$ and $s \cup v$. However, this contradicts the fact that X E_s -mixes s .

(A2) Suppose that $X/(s, t)$ separates s and t . Let $x, y \in \text{Ext}(X/(s, t))$ be such that $s \cup x, t \cup y \in \hat{\mathcal{F}}$. Then for any $x', y' \in \text{Ext}(X/(s, t))$ such that $s \cup x \cup x', t \cup y \cup y' \in$

\mathcal{F} , it must be the case that $f(s \cup x \cup x') = f(t \cup y \cup y')$. Hence, $X/(s, t)$ separates $s \cup x$ and $t \cup y$ for all $x, y \in \text{Ext}(X/(s, t))$ such that $s \cup x, t \cup y \in \hat{\mathcal{F}}$.

(A3) Let $n = |s|$ and suppose that X mixes s and $s \cup u$ for all $u \in \mathcal{H}^2(n)|B/s$. By the transitivity of mixing, we find that for all $u, v \in \mathcal{H}^2(n)|B/s$, X mixes $s \cup u$ and $s \cup v$. Thus, for all $u, v \in \mathcal{H}^2(n)|B/s$, $uE_s v$, i.e. $E_s = E_{T_\emptyset}$. Hence, $T_s = T_\emptyset$.

We prove the converse by contrapositive. Suppose that $X/(s \cup u)$ separates s and $s \cup u$ for some $u \in \mathcal{H}^2(n)|B/s$. By (A2), $X/(s \cup u)$ separates $s \cup v$ and $s \cup u \cup u'$, for all $v, u' \in \text{Ext}(X/(s \cup u))$ such that $s \cup v, s \cup u \cup u' \in \hat{\mathcal{F}}$. If we let $u' = \emptyset$ and $v \in \mathcal{H}^2(n)|X/(s \cup u)$ then $X/(s \cup u)$ separates $s \cup u$ and $s \cup v$. By Claim 15 we find that $u \not E_s v$. Hence $T_s \neq T_\emptyset$.

(A4) We proceed by induction on $|t| - |s|$. If there exists $|s| \leq i < |t|$ such that $T_{r_i(t)} \neq T_\emptyset$ then by definition of φ , $\varphi(s) \neq \varphi(t)$ since $s \sqsubset t$. Hence, for all $|s| \leq i < |t|$, $T_{r_i(t)} = T_\emptyset$. If $|t| - |s| = 1$ then (A3) implies that X mixes s and $s \cup t(n)$ since $t(n) \in \mathcal{H}^2(n)|B/s$. Notice that $t = s \cup t(n)$ because $|t| - |s| = 1$. Therefore X mixes s and t .

Next assume that the (A4) holds when $|t| - |s| = k$ and suppose that s, t are given with $|t| - |s| = k + 1$. Let $i = |t| - 1$. By the induction hypothesis, X mixes s and $r_i(t)$ since $|r_i(t)| - |s| = k$ and $s \sqsubset r_i(t)$. Note that since $|t| - |r_i(t)| = 1$ and $r_i(t) \sqsubset t$ the base case argument implies that X mixes $r_i(t)$ and t . By the transitivity of mixing, we find that X mixes s and t . \square

Claim 17. *If $s, t \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$ are mixed by $B/(s, t)$, then T_s and T_t are isomorphic. Moreover, there is a $C \leq B$ such that for all $s, t \in (\hat{\mathcal{F}} \setminus \mathcal{F})|C$, for all $u \in \mathcal{H}^2(|s|)|C/(s, t)$ and $v \in \mathcal{H}^2(|t|)|C/(s, t)$, C mixes $s \cup u$ and $t \cup v$ if and only if $\varphi_s(u) = \varphi_t(v)$.*

Proof. Suppose $s, t \in (\hat{\mathcal{F}} \setminus \mathcal{F})|B$ are mixed by $B/(s, t)$, and let $X \leq B$. Let $i = |s|$ and $j = |t|$.

Assume that $T_s = T_\emptyset$. Let

$$\mathcal{A}_0 = \{v \in \mathcal{H}^2(j)|X/t : X \text{ mixes } s \text{ and } t \cup v\} \text{ and} \quad (5.3.8)$$

$$\mathcal{A}_1 = \{v \in \mathcal{H}^2(j)|X/t : X \text{ separates } s \text{ and } t \cup v\}. \quad (5.3.9)$$

Since each element $\mathcal{A}_0 \cup \mathcal{A}_1$ has length j we find that $\mathcal{A}_0 \cup \mathcal{A}_1$ is a Nash-Williams family. By Theorem 1.2.7, there exists $Y \leq X$ and $i \in \{0, 1\}$ such that $\mathcal{A}_i|Y = \emptyset$.

Toward a contradiction, suppose that $\mathcal{A}_0|Y = \emptyset$. Then for each $v \in \mathcal{H}^2(j)|Y/t$, Y separates s and $t \cup v$. Note that since $T_s = T_\emptyset$, it follows that for all $u \in \mathcal{H}^2(i)|Y/s$

$$\begin{aligned} \varphi(s) &= \bigcup_{i < |s|} \varphi_{s|i}(s(i)), \\ &= \left(\bigcup_{i < |s|} \varphi_{s|i}(s(i)) \right) \cup \{\langle \rangle\}, \\ &= \left(\bigcup_{i < |s|} \varphi_{s|i}(s(i)) \right) \cup \pi_{T_s}(u), \\ &= \bigcup_{i < |s \cup u|} \varphi_{(s \cup u)|i}((s \cup u)(i)), \\ &= \varphi(s \cup u). \end{aligned} \quad (5.3.10)$$

Therefore (A4) implies that for all $u \in \mathcal{H}^2(i)|Y/s$, Y mixes s and $s \cup u$. If there are $u \in \mathcal{H}^2(i)|Y/s$ and $v \in \mathcal{H}^2(j)|Y/t$ such that Y mixes $s \cup u$ and $t \cup v$, then Y mixes s and $t \cup v$, by transitivity of mixing. This contradicts that for each $v \in \mathcal{H}^2(j)|Y/t$, Y separates s and $t \cup v$. Therefore, all extensions of s and t are separated. However, this implies that Y separates s and t which is a contradiction. Hence, it must be the

case the $\mathcal{A}_1|Y = \emptyset$.

If $\mathcal{A}_1|Y = \emptyset$ then for all $v \in \mathcal{H}^2(j)|Y/t$, Y mixes s and $t \cup v$. By (A1), Y mixes s and $t \cup v$ for at most one equivalence class of v 's in $\mathcal{H}^2(j)|Y/t$. By transitivity of mixing, we find that for all $u, v \in \mathcal{H}^2(j)|Y/t$, uE_tv . Hence, $T_t = T_\emptyset$.

By the same argument with s and t reversed we may conclude that, $T_s = T_\emptyset$ if and only if $T_t = T_\emptyset$. In this case, for all $u \in \mathcal{H}^2(i)|B$ and $v \in \mathcal{H}^2(j)|B$

$$\begin{aligned}\varphi_s(u) &= \pi_{T_s}(u), \\ &= \{\langle \rangle\}, \\ &= \pi_{T_t}(v), \\ &= \varphi_t(v).\end{aligned}\tag{5.3.11}$$

Thus, if we let $C = B$ then for all $s, t \in (\hat{\mathcal{F}} \setminus \mathcal{F})|C$, for all $u \in \mathcal{H}^2(|s|)|C/(s, t)$ and $v \in \mathcal{H}^2(|t|)|C/(s, t)$, C mixes $s \cup u$, and $t \cup v$ if and only if $\varphi_s(u) = \varphi_t(v)$.

Suppose now that both T_s and T_t are not T_\emptyset . Let $X \leq B$, $m = \max(i, j) + 1$ and $k = m^m$. Let

$$\mathcal{Z}_< = \{Y \leq X : B \text{ separates } s \cup Y(i) \text{ and } t \cup \pi_{\tilde{S}(j)}(Y(k))\} \text{ and}\tag{5.3.12}$$

$$\mathcal{Z}_> = \{Y \leq X : B \text{ separates } s \cup \pi_{\tilde{S}(i)}(Y(k)) \text{ and } t \cup Y(j)\}.\tag{5.3.13}$$

Since $i, j < k$ we have $\mathcal{Z}_< = \bigcup_{Y \in \mathcal{Z}_<} [r_{k+1}(Y), Y]$ and $\mathcal{Z}_> = \bigcup_{Y \in \mathcal{Z}_>} [r_{k+1}(Y), Y]$. So $\mathcal{Z}_<$ and $\mathcal{Z}_>$ are open and Ramsey since \mathcal{H}^2 is a topological Ramsey space. Hence there exists $Y \leq X$ such that either $\mathcal{Z}_< \cap [\emptyset, Y] = \emptyset$ or $[\emptyset, Y] \subseteq \mathcal{Z}_<$. Furthermore, either $\mathcal{Z}_> \cap [\emptyset, Y] = \emptyset$ or $[\emptyset, Y] \subseteq \mathcal{Z}_>$.

Toward a contradiction, suppose that $\mathcal{Z}_< \cap [\emptyset, Y] = \emptyset$. Recall that $\tilde{S}(n)$ denotes

the tree $\{\langle \rangle, \langle 0 \rangle, \langle 0, (p, q) \rangle : (p, q) \in (n+1) \times (n+1)\}$. Let $Y' \leq Y$ be given by removing from each subtree of Y the final k rows and columns of the leaves of each subtree, that is $Y' = \bigcup_{n < \omega} \pi_{\tilde{S}(n)}(Y(k+n))$. If u and v are elements of $\mathcal{H}^2(j)|Y'/t$ then we can construct $U, V \leq Y$ by letting

$$U(n) = \begin{cases} Y(n) & \text{if } n < k, \\ \hat{u} & \text{if } n = k, \\ \pi_{\tilde{S}(n)}(Y(n + \text{depth}_Y(u))) & \text{if } n > k, \end{cases} \quad (5.3.14)$$

and

$$V(n) = \begin{cases} Y(n) & \text{if } n < k, \\ \hat{v} & \text{if } n = k, \\ \pi_{\tilde{S}(n)}(Y(n + \text{depth}_Y(v))) & \text{if } n > k, \end{cases} \quad (5.3.15)$$

where \hat{u} and \hat{v} are elements of $\mathcal{H}^2(k)|Y/t$ such that $\pi_{\tilde{S}(j)}(\hat{u}) = u$ and $\pi_{\tilde{S}(j)}(\hat{v}) = v$. Such a \hat{u} and \hat{v} exists since Y' was obtained by removing the final k rows and columns of the leaves of each subtree of Y . Since $\mathcal{Z}_< \cap [\emptyset, Y] = \emptyset$ we find that, B mixes $s \cup U(i)$ and $t \cup \pi_{\tilde{S}(j)}(U(k))$ and B mixes $s \cup V(i)$ and $t \cup \pi_{\tilde{S}(j)}(V(k))$. Since $U(i) = V(i) = Y(i)$ we find, by transitivity of mixing, that B mixes $s \cup u$ and $t \cup v$. Since u and v where arbitrary we have, $T_t = T_\emptyset$ which is a contradiction. Thus, it must be the case that $[\emptyset, Y] \subseteq \mathcal{Z}_<$. A similar argument with s and t reversed shows that it must also be the case that $[\emptyset, Y] \subseteq \mathcal{Z}_>$.

Next let $X' \leq Y'$ be obtained from Y' such that between any two subtrees of X' there are at least k subtrees of Y' . Suppose that $u \in \mathcal{H}^2(i)|X'/s$ and $v \in \mathcal{H}^2(j)|X'/t$. Thus, there exists $l, l' < \omega$ such that $u \subseteq X'(l)$ and $v \subseteq X'(l')$.

First consider the case when $l < l'$. Define $Z \leq Y$ by letting

$$Z(n) = \begin{cases} Y'(n) & \text{if } n < i, \\ u & \text{if } n = i, \\ \hat{Y}_n & \text{if } i < n < k, \\ \hat{v} & \text{if } n = k, \\ \pi_{\tilde{S}(n)}(Y(n+l)) & \text{if } n > k, \end{cases} \quad (5.3.16)$$

where $\langle \hat{Y}_n \in \mathcal{H}^2(n)|Y/u : i < n < k \rangle$ is a sequence of $k-i$ subtrees coming from distinct subtrees of Y falling between $X'(l)$ and $X'(l')$ and $\hat{v} \in \mathcal{H}^2(k)|Y/t$ such that $\pi_{\tilde{S}(j)}(\hat{v}) = v$. Such a sequence exists since between any two subtrees of X' there are at least k subtrees of Y . Additionally, such a \hat{v} exists since Y' was obtained by removing the final k rows and columns of the leaves of each subtree of Y .

Since $[\emptyset, Y] \subseteq \mathcal{Z}_<$ and $Z \leq Y$ we find that B separates $s \cup Z(i)$ and $t \cup \pi_{\tilde{S}(j)}(Z(k))$. In other words, B separates $s \cup u$ and $t \cup v$. If $l > l'$ then by a similar construction and the fact that $[\emptyset, Y] \subseteq \mathcal{Z}_>$ we may conclude that B separates $s \cup u$ and $t \cup v$. Therefore for all $u \in \mathcal{H}^2(i)|X'/s$ and $v \in \mathcal{H}^2(j)|X'/t$, $s \cup u$ and $t \cup v$ are mixed by B if and only if u and v are subtrees of the same $X'(l)$ for some l , i.e. $l = l'$.

For each pair of trees $S, T \in \mathcal{T}(k)$ such that $\pi_S(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$ and $\pi_T(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$, let

$$\mathcal{X}_{S,T} = \{W \leq X' : B \text{ mixes } s \cup \pi_S(W(k)) \text{ and } t \cup \pi_T(W(k))\}. \quad (5.3.17)$$

$\mathcal{X}_{S,T}$ is an open set since $\mathcal{X}_{S,T} = \bigcup_{W \in \mathcal{X}_{S,T}} [r_{k+1}(W), W]$. Since \mathcal{H}^2 is a topological

Ramsey space we find that each $\mathcal{X}_{S,T}$ is Ramsey. The collection of all such pairs $S, T \in \mathcal{T}(k)$ is finite; so, we can successively apply the Ramsey property to obtain a $W \leq X'$ such that for each such pair S, T either $[\emptyset, W] \cap \mathcal{X}_{S,T} = \emptyset$ or $[\emptyset, W] \subseteq \mathcal{X}_{S,T}$.

Subclaim 1. *There is a $W' \leq W$ such that for each pair $S, T \in \mathcal{T}(k)$ such that $\pi_S(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$ and $\pi_T(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$, and each $Z' \leq W'$, if $\varphi_s(\pi_S(Z'(k))) \neq \varphi_t(\pi_T(Z'(k)))$, then $[\emptyset, Z'] \cap \mathcal{X}_{S,T} = \emptyset$.*

Proof. For each pair $S, T \in \mathcal{T}(k)$ such that $\pi_S(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$ and $\pi_T(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$, let

$$\mathcal{W}_{S,T} = \{W' \leq W : \pi_{T_s} \circ \pi_S(W'(k)) \neq \pi_{T_t} \circ \pi_T(W'(k))\}. \quad (5.3.18)$$

$\mathcal{W}_{S,T}$ is open since $\mathcal{W}_{S,T} = \cup_{W' \in \mathcal{W}_{S,T}} [r_{k+1}(W'), W']$. $\mathcal{W}_{S,T}$ is Ramsey because \mathcal{H}^2 is a topological Ramsey space. Hence, there exists $W' \leq W$ such that either $[\emptyset, W'] \subseteq \mathcal{W}_{S,T}$ or $[\emptyset, W'] \cap \mathcal{W}_{S,T} = \emptyset$.

Recall that $\varphi_s(\pi_S(Z'(k))) = \pi_{T_s} \circ \pi_S(Z'(k))$ and $\varphi_t(\pi_T(Z'(k))) = \pi_{T_t} \circ \pi_T(Z'(k))$. Suppose that $[\emptyset, W'] \cap \mathcal{W}_{S,T} = \emptyset$. It follows that for each $Z' \leq W'$, $\varphi_s(\pi_S(Z'(k))) = \varphi_t(\pi_T(Z'(k)))$. Therefore, vacuously, we have for each $Z' \leq W'$, if $\varphi_s(\pi_S(Z'(k))) \neq \varphi_t(\pi_T(Z'(k)))$, then $[\emptyset, Z'] \cap \mathcal{X}_{S,T} = \emptyset$.

Next suppose that $[\emptyset, W'] \subseteq \mathcal{W}_{S,T}$. Then for each $Z' \leq W'$, $\pi_{T_s} \circ \pi_S(Z'(k)) \neq \pi_{T_t} \circ \pi_T(Z'(k))$. So for each $Z' \leq W'$, either $\pi_{T_s} \circ \pi_S(Z'(k)) \setminus \pi_{T_t} \circ \pi_T(Z'(k))$ or $\pi_{T_t} \circ \pi_T(Z'(k)) \setminus \pi_{T_s} \circ \pi_S(Z'(k))$ is nonempty.

Suppose that $q \in \pi_{T_s} \circ \pi_S(\tilde{\mathcal{S}}(k)) \setminus \pi_{T_t} \circ \pi_T(\tilde{\mathcal{S}}(k))$. Take $w, w' \in \mathcal{H}^2(k)|Z'(l)$ for some l such that w and w' differ exactly on their elements in the place q and any extensions of q . (That is, for each $T \in \tilde{\mathcal{S}}(k)$, $\pi_{\{q\}}(w) \neq \pi_T(w')$ if and only if $T \supseteq q$.)

Toward a contradiction, assume that $[\emptyset, Z'] \subseteq \mathcal{X}_{S,T}$. Let $u = \pi_S(w)$, $u' = \pi_S(w')$, $v = \pi_T(w)$ and $v' = \pi_T(w')$. By construction, we have $uE'_s u'$ and $vE_t v'$. Since $[\emptyset, Z'] \subseteq \mathcal{X}_{S,T}$, B mixes $s \cup u$ and $t \cup v$, and B mixes $s \cup u'$ and $t \cup v'$. B mixes $t \cup v$ and $t \cup v'$ since $vE_t v'$. Hence, by transitivity of mixing, B mixes $s \cup u$ and $s \cup u'$, contradicting that $uE'_s u'$. Therefore $[\emptyset, Z'] \cap \mathcal{X}_{S,T} = \emptyset$.

By the same argument with s and t reversed we obtain a contradiction if $\pi_{T_t} \circ \pi_T(Z'(k)) \setminus \pi_{T_s} \circ \pi_S(Z'(k))$ and $[\emptyset, W'] \subseteq \mathcal{X}_{S,T}$. Thus, in both cases, for each $Z' \leq W'$, if $\varphi_s(\pi_S(Z'(k))) \neq \varphi_t(\pi_T(Z'(k)))$, then $[\emptyset, Z'] \cap \mathcal{X}_{S,T} = \emptyset$. \square

Let $W' \leq W$ satisfying the subclaim and $W'' \leq W'/(s, t)$ such that $W'' \subseteq W'/r_k(W')$. Let $u \in \mathcal{H}^2(i)|W''$ and $v \in \mathcal{H}^2(j)|W''$ such that B mixes $s \cup u$ and $t \cup v$. Since $W'' \leq X'$ it must be the case that u and v come from the subtree $W''(l)$ for some l . Since $k = m^m$ and $i + j < k$ we find that there exists $w \in \mathcal{H}^2(k)|W''(l)$ and $S, T \in \mathcal{T}(k)$ such that $\pi_S(w) = u$ and $\pi_T(w) = v$. Now define $Z' \leq W'$ by letting

$$Z'(n) = \begin{cases} W'(n) & \text{if } n < k, \\ w & \text{if } n = k, \\ \pi_{\tilde{S}(n)}(W''(n+l)) & \text{if } n > k. \end{cases} \quad (5.3.19)$$

B mixes $s \cup \pi_S(Z'(k))$ and $t \cup \pi_T(Z'(k))$ since $u = \pi_S(Z'(k))$ and $v = \pi_T(Z'(k))$. Therefore $[\emptyset, Z'] \cap \mathcal{X}_{S,T} \neq \emptyset$. By the contrapositive of the implication in the previous subclaim, we find that $\varphi_s(u) = \varphi_s(\pi_S(Z'(k))) = \varphi_t(\pi_T(Z'(k))) = \varphi_t(v)$. Hence, for all $u \in \mathcal{H}^2(i)|W''$ and $v \in \mathcal{H}^2(j)|W''$, if $s \cup u$ and $t \cup v$ are mixed by B , then $\varphi_s(u) = \varphi_t(v)$. T_s and T_t are isomorphic since otherwise there would be a u and v such that $\varphi_s(u) = \pi_{T_s}(u) \neq \pi_{T_t}(v) = \varphi_t(v)$ which is not the case.

Thus, we have shown that there is a $W'' \leq X$ such that for all $u \in \mathcal{H}^2(i)|W''$

and $v \in \mathcal{H}^2(j)|W''$, if W'' mixes $s \cup u$ and $t \cup v$, then $\varphi_s(u) = \varphi_t(v)$. It remains to show that there is a $C \leq W''$ such that for all $u \in \mathcal{H}^2(i)|W''$ and $v \in \mathcal{H}^2(j)|W''$, if $\varphi_s(u) = \varphi_t(v)$, then W'' mixes $s \cup u$ and $t \cup v$.

Suppose $S, T \in \mathcal{T}(k)$ is a pair such that $\pi_S(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$ and $\pi_T(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$, and for all $w \in \mathcal{H}^2(k)|W''$, $\varphi_s(\pi_S(w)) = \varphi_t(\pi_T(w))$. Assume toward a contradiction that $[\emptyset, W''] \cap \mathcal{X}_{S,T} = \emptyset$. Then for all $w \in \mathcal{H}^2(k)|W''$, W'' separates $s \cup \pi_S(w)$ and $t \cup \pi_T(w)$.

Let S' and T' be any pair in $\mathcal{T}(k)$ such that $\pi_{S'}(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$, $\pi_{T'}(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$, and $\varphi_s(\pi_{S'}(x)) = \varphi_t(\pi_{T'}(x))$, for all $x \in \mathcal{H}^2(k)|W''$. Let $x, y \in \mathcal{H}^2(k)|W''$ such that $\pi_S(x)E_s\pi_{S'}(y)$. Since $\varphi_s(\pi_{S'}(y)) = \varphi_t(\pi_{T'}(y))$ and $\varphi_s(\pi_S(x)) = \varphi_t(\pi_T(x))$ we also have $\pi_T(x)E_t\pi_{T'}(y)$. By Claim 15, W'' mixes $s \cup \pi_S(x)$ and $s \cup \pi_{S'}(y)$, and W'' mixes $t \cup \pi_T(x)$ and $t \cup \pi_{T'}(y)$. If W'' mixes $s \cup \pi_{S'}(y)$ and $t \cup \pi_{T'}(y)$ then, by transitivity of mixing, W'' mixes $s \cup \pi_T(x)$ and $t \cup \pi_S(x)$ which is not true. So W'' separates $s \cup \pi_{S'}(y)$ and $t \cup \pi_{T'}(y)$. It follows that $[\emptyset, W''] \cap \mathcal{X}_{S',T'} = \emptyset$, i.e. for all $w \in \mathcal{H}^2(k)|W''$, W'' separates $s \cup \pi_{S'}(w)$ and $t \cup \pi_{T'}(w)$.

Given any S', T' for which $\varphi_s(\pi_{S'}(x)) \neq \varphi_t(\pi_{T'}(x))$, W'' separates $s \cup \pi_{S'}(x)$ and $t \cup \pi_{T'}(x)$. Note that for each pair S, T such that $\pi_S(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$ and $\pi_T(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$, we either have $[\emptyset, W''] \subseteq \mathcal{W}_{S,T}$ or $[\emptyset, W''] \cap \mathcal{W}_{S,T} = \emptyset$. If $W'' \subseteq \mathcal{W}_{S,T}$ then we find that for all $w \in \mathcal{H}^2(k)|W''$, W'' separates $s \cup \pi_S(w)$ and $t \cup \pi_T(w)$. On the other hand, if $W'' \cap \mathcal{W}_{S,T} = \emptyset$ then the previous paragraph shows that for all $w \in \mathcal{H}^2(k)|W''$, W'' separates $s \cup \pi_S(w)$ and $t \cup \pi_T(w)$. We have shown that for any pair $S, T \in \mathcal{T}(k)$ such that $\pi_S(T_1 \otimes T_1(k)) \in \mathcal{H}^2(i)$, $\pi_T(T_1 \otimes T_1(k)) \in \mathcal{H}^2(j)$ and any $w \in \mathcal{H}^2(k)|W''$, W'' separates $s \cup \pi_S(w)$ and $t \cup \pi_T(w)$.

Let $Z'' \leq W''$ such that $Z'' \subseteq W''/r_k(W'')$. Now let $u \in \mathcal{H}^2(i)|Z''/s$ and

$v \in \mathcal{H}^2(j)|Z''/t$. If u and v come from different subtrees of Z'' then Z'' separates $s \cup u$ and $t \cup v$. Otherwise, there exists $S, T \in \mathcal{T}(k)$ and $x \in \mathcal{H}^2(k)|Z''$ such that $u = \pi_S(x)$ and $v = \pi_T(x)$. Hence, in the second case, Z'' separates $s \cup u$ and $t \cup v$. Thusm all extensions of s and t into Z'' are separated. But then s and t are separated by $Z'' \leq B$, contradiction. Therefore, $[\emptyset, W''] \subseteq \mathcal{X}_{S,T}$, and thus W'' mixes $s \cup \pi_S(Z''(k))$ and $t \cup \pi_T(Z''(k))$ for all $Z'' \leq W''$.

Hence, for all pairs (S, T) , we have that, if for all $w \in \mathcal{H}^2(k)|W''$, $\varphi_s(\pi_S(w)) = \varphi_t(\pi_T(w))$ then $[\emptyset, W''] \subseteq \mathcal{X}_{S,T}$. Let $Z'' \leq W''$ such that $Z'' \subseteq W''/r_k(W'')$. Now suppose that $u \in \mathcal{H}^2(i)|Z''$, $v \in \mathcal{H}^2(j)|Z''$ and $\varphi_s(u) = \varphi_t(v)$. By the definition of φ , it must be the case that u and v come from the same subtree of Z'' . Thus, there exists $w \in \mathcal{H}^2(k)|W''$ and S, T such that $\pi_S(w) = u$ and $\pi_T(w) = v$. Now let $V \leq W''$ be given by

$$V(n) = \begin{cases} W''(n) & \text{if } n < k, \\ w & \text{if } n = k, \\ \pi_{\hat{S}(n)}(W''(n + \text{depth}_{W''}(w))) & \text{if } n > k. \end{cases} \quad (5.3.20)$$

Hence, by assumption, $\varphi_s(\pi_S(V(k))) = \varphi_s(u) = \varphi_t(v) = \varphi_t(\pi_T(V(k)))$. Therefore, W'' mixes $s \cup \pi_S(V(k)) = s \cup u$ and $t \cup \pi_T(V(k)) = t \cup v$.

We have shown that for all $u \in \mathcal{H}^2(i)|Z''$ and $v \in \mathcal{H}^2(j)|Z''$, $\varphi_s(u) = \varphi_t(v)$ if and only if Z'' mixes $s \cup u$ and $t \cup v$. Thus, by Lemma 5.3.1, there is a $C \leq B$ such that for all $s, t \in (\hat{\mathcal{F}} \setminus \mathcal{F})|C$ with s and t mixed by B , for all $u \in \mathcal{H}^2(|S|)|C/(s, t)$ and $v \in \mathcal{H}^2(|t|)|C/(s, t)$, C mixes $s \cup u$ and $t \cup v$ if and only if $\varphi_s(u) = \varphi_t(v)$. \square

Claim 18. *For all $s, t \in \hat{\mathcal{F}}|C$, if $\varphi(s) = \varphi(t)$, then s and t are mixed by C . Hence, for all $s, t \in \mathcal{F}|C$, if $\varphi(s) = \varphi(t)$, then $f(s) = f(t)$.*

Proof. Let $s, t \in \hat{\mathcal{F}}|C$, and suppose $\varphi(s) = \varphi(t)$. It follows that for each l , $\varphi(s \cap r_l(C)) = \varphi(t \cap r_l(C))$.

The proof is by induction on $l \leq \max(\text{depth}_C(s), \text{depth}_C(t))$. For $l = 0$, $s \cap r_0(C) = t \cap r_0(C) = \emptyset$, so C mixes $s \cap r_0(C)$ and $t \cap r_0(C)$. Suppose that C mixes $s \cap r_l(C)$ and $t \cap r_l(C)$. If $s \cap r_{l+1}(C) = t \cap r_{l+1}(C) = \emptyset$, then $s \cap r_{l+1}(C) = s \cap r_l(C)$ and $t \cap r_{l+1}(C) = t \cap r_l(C)$; hence, $s \cap r_{l+1}(C)$ and $t \cap r_{l+1}(C)$ are mixed by C . If $s \cap r_{l+1}(C) \neq \emptyset$ and $t \cap r_{l+1}(C) = \emptyset$, then $\varphi(s \cap r_{l+1}(C)) = \varphi(t \cap r_{l+1}(C))$ implies that $T_{r_i(s)} = T_{\emptyset}$ where i is such that $s(i) \subseteq C(l)$. By (A4), $r_i(s) = s \cap r_l(C)$ and $r_{i+1}(s) = s \cap r_{l+1}(C)$ are mixed by C . Thus, $s \cap r_{l+1}(C)$ and $t \cap r_{l+1}(C) = t \cap r_l(C)$ are mixed by C . Similarly, if $s \cap r_{l+1}(C) = \emptyset$ and $t \cap r_{l+1}(C) \neq \emptyset$, mixing of $s \cap r_{l+1}(C)$ and $t \cap r_{l+1}(C)$ again follows from (A4). If both $s \cap C(l) \neq \emptyset$ and $t \cap C(l) \neq \emptyset$, then $s(i) \in \mathcal{H}^2(i)|C/(r_i(s), r_j(t))$ and $t(j) \in \mathcal{H}^2(j)|C/(r_i(s), r_j(t))$ where i, j are such that $s(i) \subseteq C(l)$ and $t(j) \subseteq C(l)$. By the inductive hypothesis, $r_i(s) = s \cap r_l(C)$ and $r_j(t) = t \cap r_l(C)$ are mixed by C . Since $\varphi_{r_i(s)}(s(i)) = \varphi_{r_j(t)}(t(j))$, Claim 17 implies that $s \cap r_{l+1}(C) = r_i(s) \cup s(i)$ and $t \cap r_{l+1}(C) = r_j(t) \cup t(j)$ are mixed by C .

By induction, s and t are mixed by C . In particular, if $s, t \in \mathcal{F}|C$, then $f(s) = f(t)$. □

Claim 19. For all $s, t \in \mathcal{F}|C$, $\varphi(s) \not\sqsubseteq \varphi(t)$.

Proof. Toward a contradiction, suppose $\varphi(s) \sqsubset \varphi(t)$. Let j be maximal such that $\varphi(s) = \varphi(r_j(t))$. Then $T_{r_j(t)} \neq T_{\emptyset}$. Let l be such that $t(j) \subseteq C(l)$. Then $r_j(t) = t \cap r_l(C)$, and $\varphi(s \cap r_l(C)) = \varphi(s) = \varphi(r_j(t)) = \varphi(t \cap r_l(C))$. C mixes $s \cap r_l(C)$

and $t \cap r_l(C)$, by Claim 18. Let

$$\mathcal{Q} = \{X \leq C : C \text{ mixes } s \cap r_l(C) \text{ and } (t \cap r_l(C)) \cup X(j)\}. \quad (5.3.21)$$

Note that $\mathcal{Q} = \bigcup_{X \in \mathcal{Q}} [r_{j+1}(X), X]$. So \mathcal{Q} is open and Ramsey since \mathcal{H}^2 is a topological Ramsey space. Therefore there exists $X \leq C$ such that either $[\emptyset, X] \subseteq \mathcal{Q}$ or $[\emptyset, X] \cap \mathcal{Q} = \emptyset$. Next we consider two cases.

Case 1, assume that $[\emptyset, X] \subseteq \mathcal{Q}$ and let $v \in \mathcal{H}^2(j)|X/r_j(t)$. Let $W \leq X$ be given by

$$W(n) = \begin{cases} X(n) & \text{if } n < j, \\ v & \text{if } n = j, \\ \pi_{\bar{s}(n)}(X(n+l)) & \text{if } n > j, \end{cases} \quad (5.3.22)$$

where l is such that $v \subseteq X(l)$. Since $[\emptyset, X] \subseteq \mathcal{Q}$ and $W \leq X$ we find that C mixes $s \cap r_l(C)$ and $(t \cap r_l(C)) \cup W(j) = (t \cap r_l(C)) \cup v$. By (A1) for all $u, v \in \mathcal{H}^2(j)|X/r_j(t)$, $uE_{r_j(t)}v$. So $T_{r_j(t)} = T_{\langle \rangle}$ which is a contradiction.

Case 2, assume that $[\emptyset, X] \cap \mathcal{Q} = \emptyset$ and let $v \in \mathcal{H}^2(j)|X/r_j(t)$. Let $W \leq X$ be defined as in equation 5.3.22. Then C separates $s \cap r_l(C)$ and $(t \cap r_l(C)) \cup W(j) = (t \cap r_l(C)) \cup v$. Since v was an arbitrary element of $\mathcal{H}^2(j)|X/r_j(t)$, X separates $s \cap r_l(C)$ and $t \cap r_l(C)$, contradicting that $s \cap r_l(C)$ and $t \cap r_l(C)$ are mixed by C . In both cases we obtain a contradiction; therefore, $\varphi(s) \not\sqsubseteq \varphi(t)$. \square

Claim 20. For all $s, t \in \mathcal{F}|C$, if $f(s) = f(t)$, then $\varphi(s) = \varphi(t)$.

Proof. Let $s, t \in \mathcal{F}|C$ with $f(s) = f(t)$, and let $m = \max(\text{depth}_C(s), \text{depth}_C(t))$. $f(s) = f(t)$ implies that for all $l \leq m$, C mixes $s \cap r_l(C)$ and $t \cap r_l(C)$. We shall show by induction that for all $l \leq m$, $\varphi(s \cap r_l(C)) = \varphi(t \cap r_l(C))$. For

$l = 0$, $\varphi(s \cap r_l(C)) = \varphi(\emptyset) = \varphi(t \cap r_l(C))$. Suppose $l < m$ and $\varphi(s \cap r_l(C)) = \varphi(t \cap r_l(C))$. If $s \cap C(l) = t \cap C(l) = \emptyset$, then $\varphi(s \cap r_{l+1}(C)) = \varphi(s \cap r_l(C)) = \varphi(t \cap r_l(C)) = \varphi(t \cap r_{l+1}(C))$.

If both $s \cap C(l) \neq \emptyset$ and $t \cap C(l) \neq \emptyset$, then $s(i) \in \mathcal{H}^2(i)|C/(r_i(s), r_j(t))$ and $t(j) \in \mathcal{H}^2(j)|C/(r_i(s), r_j(t))$ where i, j are such that $s(i) \subseteq C(l)$ and $t(j) \subseteq C(l)$. By the inductive hypothesis, $\varphi(r_i(s)) = \varphi(r_j(t))$ because $r_i(s) = s \cap r_l(C)$ and $r_j(t) = t \cap r_l(C)$ are mixed by C . Since $f(s) = f(t)$, C mixes $r_i(s) \cup s(i) = s \cap r_{l+1}(C)$ and $r_j(t) \cup t(j) = t \cap r_{l+1}(C)$. By Claim 17, $\varphi_{r_i(s)}(s(i)) = \varphi_{r_j(t)}(t(j))$.

Therefore

$$\begin{aligned} \varphi(s \cap r_{l+1}(C)) &= \varphi(r_i(s)) \cup \varphi_{r_i(s)}(s(i)), \\ &= \varphi(r_j(t)) \cup \varphi_{r_j(t)}(t(j)), \\ &= \varphi(t \cap r_{l+1}(C)). \end{aligned} \tag{5.3.23}$$

Finally, suppose that $s \cap C(l) \neq \emptyset$ and $t \cap C(l) = \emptyset$. Let i be such that $s(i) \subseteq C(l)$. If $T_{r_i(s)} \neq T_{\langle \rangle}$, then $t \cap r_{l+1}(C)$ must be a proper initial segment of t ; otherwise, we would have $\varphi(t) = \varphi(t \cap r_{l+1}(C)) = \varphi(t \cap r_l(C)) = \varphi(s \cap r_l(C)) \sqsubset \varphi(s)$, contradicting Claim 19. Let j be such that $r_j(t) = t \cap r_{l+1}(C)$. The $j < |t|$. C mixes $r_{i+1}(s) = (s \cap r_l(C)) \cup s(i)$ and $r_{j+1}(t) = (t \cap r_l(C)) \cup t(j)$; so $\varphi_{r_i(s)}(s(i)) = \varphi_{r_j(t)}(t(j))$, by Claim 17. But this contradicts the facts that $T_{r_i(s)} \neq T_{\langle \rangle}$, $s(i) \subseteq C(l)$

and $t(j) \cap C(l) = \emptyset$. It follows that $T_{r_i(s)}$ must be $T_{\langle \rangle}$; hence

$$\begin{aligned}
\varphi(s \cap r_{l+1}(C)) &= \varphi(r_i(s)) \cup \varphi_{r_i(s)}(s(i)), \\
&= \varphi(r_i(s)) \cup \varphi_{T_{\langle \rangle}}(s(i)), \\
&= \varphi(r_j(t)) \cup \{\langle \rangle\}, \\
&= \varphi(t \cap r_{l+1}(C)).
\end{aligned} \tag{5.3.24}$$

Likewise, if $s \cap C(l) = \emptyset$ and $t \cap C(l) \neq \emptyset$, we find that $\varphi(s \cap r_{l+1}(C)) = \varphi(t \cap r_{l+1}(C))$. \square

It remains to show that φ witnesses that R is canonical. By definition, φ is inner, and by Claim 19, φ is Nash-Williams. By Claims 18 and 20, we have that for each $a, b \in \mathcal{F}|C$, aRb if and only if $\varphi(a) = \varphi(b)$. Thus, we only need to show that φ is maximal among all inner Nash-Williams maps φ' on $\mathcal{F}|C$ which also represents the equivalence relation R .

Lemma 5.3.13. *Suppose $X \leq C$ and φ' is an inner function on $\mathcal{F}|X$ which represents R . Then there is a $Y \leq X$ such that for each $t \in \mathcal{F}|Y$, for each $i < |t|$, there is a tree $S_{r_i(t)} \subseteq T_{r_i(t)}$ such that the following hold:*

- (1) *For each $s \in \mathcal{F}|Y$ for which $s \sqsupset r_i(t)$, $\varphi'(s) \cap s(i) = \pi_{S_{r_i(t)}}(s(i))$.*
- (2) *$\varphi'(t) = \bigcup \{\pi_{S_{r_i(t)}}(t(i)) : i < |t|\} \subseteq \varphi(t)$.*

Thus, φ is \subseteq -maximal among all inner functions φ' on $\mathcal{F}|C$ which represent R .

Proof. Let $X \leq C$ and φ' satisfy the hypotheses. Note that φ' is inner and also

represents the equivalence relation R . Fix $t \in \mathcal{F}$, $i < |t|$ and $X' \leq X$. Let

$$\mathcal{F}_0 = \{s \in \mathcal{F} : r_i(t) \sqsubseteq s \text{ and } \exists S \in \mathcal{T}(i), \varphi'(s) \cap s = \pi_S(s(i))\} \quad (5.3.25)$$

$$\mathcal{F}_1 = \{s \in \mathcal{F} : r_i(t) \sqsubseteq s \text{ and } \forall S \in \mathcal{T}(i), \varphi'(s) \cap s \neq \pi_S(s(i))\} \quad (5.3.26)$$

Then $\mathcal{F}_0 \cup \mathcal{F}_1 = \{s \in \mathcal{F} : r_i(t) \sqsubseteq s\} \subseteq \mathcal{F}$. Since \mathcal{F} is a front we find that $\mathcal{F}_0 \cup \mathcal{F}_1$ is a Nash-Williams family. By Theorem 1.2.7 $\mathcal{F}_0 \cup \mathcal{F}_1$ is Ramsey. Thus, there is an $X'' \leq X'$ such that either $\mathcal{F}_0|X'' = \emptyset$ or $\mathcal{F}_1|X'' = \emptyset$.

Toward a contradiction suppose that $\mathcal{F}_0|X'' = \emptyset$. Then for all $s \in \mathcal{F}|X''$ such that $r_i(t) \sqsubseteq s$ we find that, for all $S \in \mathcal{T}(i)$, $\varphi'(s) \cap s(i) \neq \pi_S(s(i))$. But then for all $s \in \mathcal{F}|X''$, $\varphi'(s) \not\sqsubseteq s$, which is a contradiction since φ' is inner on X'' . Thus, it must be the case that $\mathcal{F}_1|X'' = \emptyset$.

Hence the following holds, there is a tree $S_{r_i(t)} \in \mathcal{T}(i)$ such that for each $s \in \mathcal{F}$ extending $r_i(t)$ with $s \setminus r_i(t) \in \text{Ext}(X'')$, $\varphi'(s) \cap s(i) = \pi_{S_{r_i(t)}}(s(i))$. By Lemma 5.3.1, there is a $Y \leq X$ such that for each $t \in \mathcal{F}|Y$ and each $i < |t|$, there is a tree $S_{r_i(t)}$ satisfying (1). Thus, for each $t \in \mathcal{F}|Y$,

$$\begin{aligned} \varphi'(t) &= \bigcup_{i < |t|} \varphi'(t) \cap t(i), \\ &= \bigcup \{\pi_{S_{r_i(t)}}(t(i)) : i < |t|\}. \end{aligned} \quad (5.3.27)$$

Note that for each $S_{r_i(t)}$ it must be the case that it is contained within $T_{r_i(t)}$, the tree associated with $E_{r_i(t)}$ -mixing of immediate extensions of $r_i(t)$; otherwise, there would exist $u, v \in \mathcal{H}^2(i)|Y/r_i(t)$ such that $r_i(t) \cup u$ and $r_i(t) \cup v$ are mixed. However, all extensions of them have different φ' values, which would contradict that φ' induces

the same equivalence relation as f . Thus, for each $t \in \mathcal{F}|Y$,

$$\varphi'(t) = \bigcup \{\pi_{S_{r_i(t)}}(t(i)) : i < |t|\} \subseteq \bigcup \{\pi_{T_{r_i(t)}}(t(i)) : i < |t|\} = \varphi(t). \quad (5.3.28)$$

□

By Lemma 5.3.13, R is canonical on $\mathcal{F}|C$, which finishes the proof of the theorem. □

Remark 5.3.14. The following arguments are based on identical results obtained for the topological Ramsey space \mathcal{R}_1 by Dobrinen and Todorćević in [17]. The map φ from Theorem 5.3.11 has the following property. One can thin to a Z such that

(*) for each $s \in \mathcal{F}|Z$, there is a $t \in \mathcal{F}$ such that $\varphi(s) = \varphi(t) = s \cap t$.

This is not the case for any smaller inner map φ' , by Lemma 5.3.13. Suppose φ' is an inner map representing R , φ' satisfies the conclusions of Lemma 5.3.13 on $\mathcal{F}|Y$, and there is an $s \in \mathcal{F}|Y$ for which $\varphi'(s) \subsetneq \varphi(s)$. Then there is some $i < |s|$ for which the tree $S_{r_i(s)} \subsetneq T_{r_i(s)}$. This implies that $\varphi'(s) \subsetneq \varphi(s)$ for every $s \in \mathcal{F}|Y$ such that $r_i(t) \sqsubset s$. Recall that $\varphi'(s) = \varphi'(t)$ if and only if $\varphi(s) = \varphi(t)$; and in this case, $\varphi(s) \cap \varphi(t) \subseteq t \cap s$. It follows that for any t for which $\varphi'(s) = \varphi'(t)$, $\varphi'(t) \cap \varphi'(s)$ will always be a proper subset of $t \cap s$. Thus, φ is the minimal inner map for which property (*) holds.

It may also be of interest to note that for φ' inner and $s \in \mathcal{F}|Z$ from Lemma 5.3.13, if $i < |s|$ is maximal such that $T_{r_i(s)} \neq T_{\langle \rangle}$, then i is also maximal such that $S_{r_i(s)} \neq T_{\langle \rangle}$, and moreover, $S_{r_i(s)} = T_{r_i(s)}$.

Example 5.3.15. Let \mathcal{F} be the analogue of the Schreier barrier for \mathcal{H}^2 . That is, let

$$\mathcal{F} = \{s \in \mathcal{AH}^2 : |s| = \text{depth}_{T_1 \otimes T_1}(s(0))\}. \quad (5.3.29)$$

Let R be the equivalence relation on \mathcal{F} , where sRt if and only if $|s| = |t|$ and $t(|t| - 1) = s(|s| - 1)$. The map from φ Theorem 5.3.11 for R has the property that $\varphi(t) \cap t(0) = t(0)$ for all $t \in \mathcal{F}$.

The following map φ' is inner Nash-Williams and also represents the equivalence relation R . Let $\varphi'(t) = t(|t| - 1)$, for each t in \mathcal{F} . Since for all $t \in \mathcal{F}$, $t(0) \in \varphi(t) \setminus \varphi'(t)$ we find that $\varphi'(t) \subsetneq \varphi(t)$ for all $t \in \mathcal{F}$. However φ' does not satisfy (*).

Definition 5.3.16. Let $1 \leq n < \omega$ be fixed. An equivalence relation R on \mathcal{AH}_n^2 is canonical if and only if there are trees $T(0) \in \mathcal{T}(0), T(1) \in \mathcal{T}(1), \dots, T(n-1) \in \mathcal{T}(n)$ such that for all $a, b \in \mathcal{AH}_n^2$,

$$aRb \Leftrightarrow \forall i < n (\pi_{T(i)}(a(i)) = \pi_{T(i)}(b(i))). \quad (5.3.30)$$

Remark 5.3.17. For each $1 \leq n < \omega$, there are $\prod_{i=1}^n (4^i + 1)$ canonical equivalence relations on \mathcal{AH}_n^2 . Each i^{th} component of the product has $|\mathcal{T}(i)| = 4^i + 1$ possibilities.

The next canonization theorem can be proved directly; however, we shall follow [17] and prove it by a short application of Theorem 5.3.11.

Theorem 5.3.18. *Let $1 \leq n < \omega$. Given any $A \in \mathcal{H}^2$ and any equivalence relation R on $\mathcal{AH}_n^2|A$, there is a $D \leq A$ such that R is canonical on $\mathcal{AH}_n^2|D$.*

Proof. Let $1 \leq n < \omega$ and R be an equivalence relation on \mathcal{AH}_n^2 . Let $f : \mathcal{AH}_n^2 \rightarrow \omega$ be any function which induces the equivalence relation R . Let $C \leq A$ be ob-

tained from Theorem 5.3.11. Then for each $s \in \mathcal{AH}_n^2|C$, there is a sequence $\langle T_{r_i(s)} : i < n \rangle$ of trees, where $T_{r_i(s)} \in \mathcal{T}(i)$, satisfying the following. For each $s, t \in \mathcal{AH}_n^2|C$, $f(s) = f(t)$ if and only if $\bigcup_{i < n} \pi_{T_{r_i(s)}}(s(i)) = \bigcup_{i < n} \pi_{T_{r_i(t)}}(t(i))$. We shall use the fact that \mathcal{H}^2 is a topological Ramsey space to obtain $D \leq C$ such that for all $s, t \in \mathcal{AH}_n^2|D$ and all $i < n$, $T_{r_i(s)} = T_{r_i(t)}$. By Theorem 5.3.11, for all $s, t \in \mathcal{H}^2|C$, $T_{r_0(s)} = T_\emptyset = T_{r_0(t)}$, so let $X_0 = C$ and $T(0) = T_{r_0(s)}$ for any $s \in \mathcal{AH}^2|D$. Given $i < n - 1$, X_i , and $T(i)$, then for each $T \in \mathcal{T}(i + 1)$, define

$$\mathcal{X}_T = \{X \leq C : T_{r_{i+1}(X)} = T\}. \quad (5.3.31)$$

Notice that \mathcal{X}_T is an open set since $\mathcal{X}_T = \bigcup_{Y \in \mathcal{X}_T} [r_{i+1}(Y), Y]$. It follows that \mathcal{X}_T is Ramsey since \mathcal{H}^2 is a topological Ramsey space. Since $\mathcal{T}(i + 1)$ is finite we can apply the Ramsey property finitely many times to find $X_{i+1} \leq X_i$ such that for each $T \in \mathcal{T}(i + 1)$, either $[\emptyset, X_{i+1}] \subseteq \mathcal{X}_T$ or $[\emptyset, X_{i+1}] \cap \mathcal{X}_T = \emptyset$.

Toward a contradiction suppose that for all $T \in \mathcal{T}(i + 1)$, $[\emptyset, X_{i+1}] \cap \mathcal{X}_T = \emptyset$. Then for all $T \in \mathcal{T}(i + 1)$, $T_{r_{i+1}(X_{i+1})} \neq T$, contradiction. Thus, there must be at least one $T \in \mathcal{T}(i + 1)$ such that $[\emptyset, X_{i+1}] \subseteq \mathcal{X}_T$. If T' is also such that $[\emptyset, X_{i+1}] \subseteq \mathcal{X}_{T'}$, then $T = T_{r_{i+1}(X)} = T'$. Thus, there is a unique $T \in \mathcal{T}(i + 1)$ such that $[\emptyset, X_{i+1}] \subseteq \mathcal{X}_T$. Let $T(i + 1) = T$. For all $s \in \mathcal{AH}_n^2|X_{i+1}$, there exists $W \leq X_{i+1}$ such that $r_i(W) = r_i(s)$; therefore, $T_{r_i(s)} = T_{r_i(w)} = T(i + 1)$.

Let $D = X_{n-1}$. Then for all $s, t \in \mathcal{AH}_n^2|D$,

$$\begin{aligned}
f(s) = f(t) &\Leftrightarrow \varphi(s) = \varphi(t), \\
&\Leftrightarrow \forall i < n, \pi_{T_{r_i(s)}}(s(i)) = \pi_{T_{r_i(t)}}(t(i)), \\
&\Leftrightarrow \forall i < n, \pi_{T(i)}(s(i)) = \pi_{T(i)}(t(i)), \\
&\Leftrightarrow \forall i < n, s(i)E_{T(i)}t(i).
\end{aligned} \tag{5.3.32}$$

Thus, the equivalence relation induced by f is canonical on $\mathcal{AH}_n^2(D)$. \square

5.4 An application to the Tukey theory of ultrafilters

In this section, we apply the canonical Ramsey theory for \mathcal{H}^2 to the Tukey theory of ultrafilters and show that it is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure.

We begin by reminding the reader of the basic definitions of the Tukey theory of ultrafilters (see Section 1.4). Suppose that \mathcal{U} and \mathcal{V} are ultrafilters. A function f from \mathcal{U} to \mathcal{V} is *cofinal* if every cofinal subset of (\mathcal{U}, \supseteq) is mapped by f to a cofinal subset of (\mathcal{V}, \supseteq) . We say that \mathcal{V} is *Tukey reducible to \mathcal{U}* and write $\mathcal{V} \leq_T \mathcal{U}$ if there exists a cofinal map $f : \mathcal{U} \rightarrow \mathcal{V}$. If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ then we write $\mathcal{V} \equiv_T \mathcal{U}$ and say that \mathcal{U} and \mathcal{V} are *Tukey equivalent*. The relation \equiv_T is an equivalence relation and \leq_T is a partial order on its equivalence classes. The equivalence classes are also called *Tukey types*.

In the proof of the Ellentuck Theorem for \mathcal{H}^2 , we showed that \mathcal{H}^2 is closed in the subspace topology it inherits from $(\mathcal{AH}^2)^\omega$. A sequence $(X_n)_{n < \omega}$ of elements of \mathcal{H}^2 converges to an element $X \in \mathcal{H}^2$ if and only if for each $k < \omega$ there is an $N < \omega$ such

that for each $n \geq N$, $r_k(X_n) = r_k(X)$. A function $f : \mathcal{R} \rightarrow \mathcal{P}(\omega)$ is *continuous* if and only if for each convergent sequence $(X_n)_{n < \omega}$ in \mathcal{H}^2 with $X_n \rightarrow X$, we also have $f(X_n) \rightarrow f(X)$ in the topology obtained by identifying $\mathcal{P}(\omega)$ with 2^ω . A function $f : \mathcal{C} \rightarrow \mathcal{V}$ with $\mathcal{C} \subseteq \mathcal{H}^2$ is said to be *continuous* if it is continuous with respect to the topologies on \mathcal{C} and \mathcal{V} taken as subspaces of $(\mathcal{A}\mathcal{H}^2)^\omega$ and 2^ω , respectively. The next definition is a generalization of the notion of basic Tukey reductions for an ultrafilter on ω , Definition 3 (3) of Dobrinen in [12], to ultrafilters on \mathcal{H}^2 .

Remark 5.4.1. Suppose that $\mathcal{C} \subseteq \mathcal{H}^2$ generates an ultrafilter \mathcal{U} on $[T_1 \otimes T_1]$. The maps sending $X \mapsto [X]$ and $Y \mapsto cl(Y)$ are cofinal maps from (\mathcal{C}, \geq) to (\mathcal{U}, \supseteq) , and from (\mathcal{U}, \supseteq) to (\mathcal{C}, \geq) , respectively. Thus, for any ultrafilter on ω , $\mathcal{V} \leq_T \mathcal{U}$ if and only if there is a cofinal map from (\mathcal{C}, \geq) to (\mathcal{V}, \supseteq) . We abuse notation and let $\mathcal{V} \leq_T \mathcal{C}$, denote that there is a cofinal map from (\mathcal{C}, \geq) to (\mathcal{V}, \supseteq) .

Definition 5.4.2. Assume that $\mathcal{C} \subseteq \mathcal{H}^2$ generates an ultrafilter on $[T_1 \otimes T_1]$. \mathcal{C} has *basic Tukey reductions* if whenever \mathcal{V} is a nonprincipal ultrafilter on ω and $f : \mathcal{C} \rightarrow \mathcal{V}$ is a monotone cofinal map, there is an $X \in \mathcal{C}$, a continuous monotone map $f' : \mathcal{C} \rightarrow \mathcal{V}$ and a function $\check{f} : \mathcal{A}\mathcal{H}^2 \rightarrow [\omega]^{<\omega}$ such that

1. $f \upharpoonright (\mathcal{C} \upharpoonright X)$ is continuous.
2. f' extends $f \upharpoonright (\mathcal{C} \upharpoonright X)$ to \mathcal{C} .
3. (a) For each $k < \omega$ and each $s \in \mathcal{A}\mathcal{H}^2$, if $\text{depth}_{T_1 \otimes T_1}(s) \leq k$ then $\check{f}(s) \subseteq k$;
(b) $s \sqsubseteq t \in \mathcal{A}\mathcal{H}^2$ implies that $\check{f}(s) \sqsubseteq \check{f}(t)$;
(c) For each $Y \in \mathcal{C}$, $f'(Y) = \bigcup_{k < \omega} \check{f}(r_k(Y))$; and
(d) \check{f} is monotonic, that is, if $s, t \in \mathcal{A}\mathcal{H}^2$ with $s \leq_{\text{fin}} t$, then $\check{f}(s) \subseteq \check{f}(t)$.

The next proposition provides an important application of the notion of basic Tukey reductions for \mathcal{C} and helps reduce the characterization of the ultrafilters on ω Tukey reducible to (\mathcal{C}, \geq) to the study of canonical equivalence relations for fronts on \mathcal{C} . It is the generalization of Proposition 5.5 from [17] to our current setting.

Definition 5.4.3. If $\mathcal{C} \subseteq \mathcal{H}^2$ and $\mathcal{F} \subseteq \mathcal{AH}^2$ then we will say that \mathcal{F} is a front on \mathcal{C} if and only if for each $C \in \mathcal{C}$, there exists $s \in \mathcal{F}$ such that $s \sqsubseteq X$.

Proposition 5.4.4. Suppose that $\mathcal{C} \subseteq \mathcal{H}^2$ generates an ultrafilter on $[T_1 \otimes T_1]$. If \mathcal{C} has basic Tukey reductions and \mathcal{V} is a nonprincipal ultrafilter on ω with $\mathcal{C} \geq_T \mathcal{V}$, then there is a front \mathcal{F} on \mathcal{C} and a function $f : \mathcal{F} \rightarrow \omega$ such that for each $Y \in \mathcal{V}$ there exists $X \in \mathcal{C}$ such that $f(\mathcal{F}|X) \subseteq Y$. Moreover, if $\mathcal{C} \upharpoonright \mathcal{F}$ is a base for an ultrafilter on \mathcal{F} then $\mathcal{V} = f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$.

Proof. Suppose that \mathcal{C} and \mathcal{V} are given and satisfy the assumptions of the proposition. By Fact 6 of Dobrinen and Todorćević from [16], there is a monotone cofinal map $g : (\mathcal{C}, \geq) \rightarrow (\mathcal{V}, \supseteq)$. Since \mathcal{C} has basic Tukey reductions there is a continuous monotone cofinal map $g' : \mathcal{C} \rightarrow \mathcal{V}$ and a function $\check{g} : \mathcal{AH}^2 \rightarrow [\omega]^{<\omega}$ satisfying (1)-(3) in the definition of basic Tukey reductions. Let \mathcal{F} consist of all $r_n(Y)$ such that $Y \in \mathcal{C}$ and n is minimal such that $\check{g}(r_n(Y)) \neq \emptyset$. By the properties of \check{g} , $\min(\check{g}(r_n(Y))) = \min(g(Y))$. If $s, t \in \mathcal{F}$ then there are $S, T \in \mathcal{H}^2$ and $n, n' < \omega$ such that $s = r_n(S)$, $t = r_{n'}(T)$, and n and n' are the least n and n' such that $\check{g}(r_n(S)) \neq \emptyset$ and $\check{g}(r_{n'}(T)) \neq \emptyset$. If $s \sqsubseteq t$ then n' would not be minimal. Thus, $s \not\sqsubseteq t$. Since s and t were arbitrary, \mathcal{F} is a front on \mathcal{C} . Define a new function $f : \mathcal{F} \rightarrow \omega$ by $f(b) = \min(\check{g}(b))$, for each $b \in \mathcal{F}$.

Since g' is a monotone cofinal map, the g' -image of \mathcal{C} in \mathcal{V} is a base for \mathcal{V} . From the construction of f , we see that for each $X \in \mathcal{C}$, $f(\mathcal{F}|X) = \{f(a) : a \in \mathcal{F}|X\} \subseteq$

$g'(X)$. Therefore for each $Y \in \mathcal{V}$ there exists $X \in \mathcal{C}$ such that $f(\mathcal{F}|X) \subseteq Y$. We remind the reader of Fact 5.4 from [17].

Fact 5.4.5 ([17]). Suppose \mathcal{V} and \mathcal{U} are proper ultrafilters on the same countable base set, and for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that $U \subseteq V$. Then $\mathcal{U} = \mathcal{V}$.

Suppose that $\mathcal{C} \upharpoonright \mathcal{F}$ generates an ultrafilter on \mathcal{F} and let $\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle$ denote the ultrafilter it generates. Then the Rudin-Keisler image $f(\langle \mathcal{F} \upharpoonright \mathcal{C} \rangle)$ forms an ultrafilter on ω with base $\{f(\mathcal{F}|X) : X \in \mathcal{C}\}$. Hence, Fact 5.4.5 implies that $f(\langle \mathcal{F} \upharpoonright \mathcal{C} \rangle) = \mathcal{V}$. \square

The next two facts are the analogues of Fact 5.2 and Fact 5.3, respectively from [17] for \mathcal{H}^2 and follow immediately from Theorem 2.6.7.

Fact 5.4.6. If $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$, then for each front \mathcal{F} on \mathcal{C} and each $\mathcal{G} \subseteq \mathcal{F}$, there is a $X \in \mathcal{C}$ such that either $\mathcal{F}|X \subseteq \mathcal{G}$, or else $\mathcal{F}|X \cap \mathcal{G} = \emptyset$.

Fact 5.4.7. Suppose $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$. If \mathcal{C}' is any cofinal subset of \mathcal{C} , and $\mathcal{F} \subseteq \mathcal{A}\mathcal{H}^2$ is any front on \mathcal{C}' , then $\mathcal{C}' \upharpoonright \mathcal{F}$ generates an ultrafilter on \mathcal{F} .

Proposition 5.4.8. *Assume that $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$. Suppose \mathcal{C} has basic Tukey reductions and \mathcal{V} is a nonprincipal ultrafilter on ω with $\mathcal{C} \geq_T \mathcal{V}$. Then there is a front \mathcal{F} on \mathcal{C} and a function $f : \mathcal{F} \rightarrow \omega$ such that $\mathcal{V} = f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$.*

Proof. Suppose that \mathcal{V} is Tukey reducible to some Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$ generated by $\mathcal{C} \subseteq \mathcal{H}^2$. Assume that \mathcal{C} has Basic Tukey reductions. Theorem 5.4.4

and Fact 5.4.7 imply that there is a front \mathcal{F} on \mathcal{C} and a function $f : \mathcal{F} \rightarrow \omega$ such that $\mathcal{V} = f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$. \square

The next theorem will be used later in conjunction with the previous proposition to identify initial structures in the Tukey types of ultrafilters.

Lemma 5.4.9. *If $\mathcal{C} \subseteq \mathcal{H}^2$ generates a selective for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$ then \mathcal{C} has basic Tukey reductions.*

Proof. Since \mathcal{C} generates a p-point on $[T_1 \otimes T_1]$, the proof of Theorem 20 of Dobrinen and Todorčević from [16] is general enough to conclude that \mathcal{C} has basic Tukey reductions. \square

The next example forms the prototype for the remaining proofs and results in this section.

Example 5.4.10. Dobrinen and Todorčević in [17] show that it is consistent with ZFC that the two-element Boolean algebra appears as an initial Tukey structure. Dobrinen and Todorčević first use Theorem 20 from [16], every p-point has basic monotone reductions, to show that all ultrafilters Tukey reducible to a Ramsey for \mathcal{R}_1 ultrafilter \mathcal{U}_1 are of the form $f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$ for some front on \mathcal{C} . Then localized versions of a canonization theorem for equivalence relations on fronts on \mathcal{C} are shown to hold, and for each $n < \omega$, the authors identify p-point ultrafilters \mathcal{Y}_{n+1} on the base set $\mathcal{R}_1(n)$ generated by $\mathcal{C} \upharpoonright \mathcal{R}_1(n)$. It is then shown that $\mathcal{Y}_1 <_{RK} \mathcal{Y}_2 <_{RK} \dots$, the localized canonization theorem and the structure of the canonical relations imply that all ultrafilters which are Tukey reducible to \mathcal{U}_1 are isomorphic to an ultrafilter of $\vec{\mathcal{W}}$ -trees, where $\hat{\mathcal{S}} \setminus \mathcal{S}$ is a well-founded tree, $\vec{\mathcal{W}} = (\mathcal{W}_s : s \in \hat{\mathcal{S}} \setminus \mathcal{S})$, and each \mathcal{W}_s is isomorphic to \mathcal{Y}_{n+1} for some $n < \omega$ or isomorphic to \mathcal{U}_0 . Then the theory of

uniform fronts is used to show that each ultrafilter generated by a $\vec{\mathcal{W}}$ -tree is isomorphic to a countable Fubini product from among the ultrafilters \mathcal{Y}_n , $n < \omega$. This is then used to show that the Tukey structure of the nonprincipal ultrafilters on ω Tukey reducible to \mathcal{U}_1 is isomorphic to the two element Boolean algebra.

The next theorem constitutes a localized version of the canonical Ramsey theory for \mathcal{H}^2 .

Theorem 5.4.11. *If $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$ then for any front \mathcal{F} on \mathcal{R} and any equivalence relation R on \mathcal{F} , there exists a $C \in \mathcal{C}$ such that R is canonical on $\mathcal{F}|C$.*

Proof. Theorem 2.6.7 implies that \mathcal{C} satisfies a localized version of the Ellentuck theorem. Thus the proof of 5.2.2 and the proof of Theorem 5.3.11 for \mathcal{H}^2 can be localized to \mathcal{C} . Hence for any front \mathcal{F} on \mathcal{R} and any equivalence relation R on \mathcal{F} , there exists a $C \in \mathcal{C}$ such that R is canonical on $\mathcal{F}|C$. \square

Next we define a collection of p-point ultrafilters needed to identify the Tukey structure of Tukey types of ultrafilters which are Tukey reducible to a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$.

Fact 5.4.12. *If $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$ then for each $n < \omega$, $\mathcal{C} \upharpoonright \mathcal{H}^2(n) = \{\mathcal{H}^2(n)|C : C \in \mathcal{C}\}$ generates an ultrafilter on $\mathcal{H}^2(n)$.*

Definition 5.4.13. *Suppose that $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$. For each $n < \omega$, let \mathcal{D}_{n+1} denote the ultrafilter on $\mathcal{H}^2(n)$ generated by $\mathcal{C} \upharpoonright \mathcal{H}^2(n)$. Additionally, we let*

$$\mathcal{D}_0 = \pi_{T_{\langle \rangle}}(\mathcal{D}_1) \ \& \ \mathcal{D}_{\langle \rangle} = \pi_{T_{\langle \rangle}}(\mathcal{D}_1). \quad (5.4.1)$$

Recall that for each $n < \omega$, $\mathcal{T}(n)$ denotes the collection of trees of the form $T_{\langle \rangle}, T_{\langle 0 \rangle}$ and $T(I, J)$ with $I, J \subseteq n + 1$ and either I or J is nonempty. For each $n < \omega$ and each $T(I, J) \in \mathcal{T}(n)$, let

$$\mathcal{D}_{I,J} = \pi_{T(I,J)}(\mathcal{D}_{n+1}). \quad (5.4.2)$$

The next proposition describes the configuration, with respect to the Rudin-Keisler ordering, of the ultrafilters $\pi_T(\mathcal{D}_{i+1})$ with $i < \omega$ and π_T a projection map on $\mathcal{H}^2(i + 1)$, i.e. $T \in \mathcal{T}(i + 1)$.

Proposition 5.4.14. *Suppose that $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$.*

1. \mathcal{D}_0 is a Ramsey ultrafilter and \mathcal{D}_1 is not a weakly Ramsey ultrafilter.
2. For each $n < \omega$, $\mathcal{D}_{n+1} = \mathcal{D}_{n+1, n+1}$.
3. For each $n < \omega$ and each $I, J \subseteq n + 1$ with either I or J nonempty, $\pi_{T(I,J)}(\mathcal{D}_{n+1})$ is a p -point.
4. For each $n < \omega$ and each $I, J \subseteq n + 1$ with either I or J nonempty, $\pi_{T(I,J)}(\mathcal{D}_{n+1}) \cong \mathcal{D}_{|I|, |J|}$.
5. For $(i, j), (k, l) \in \omega \times \omega$ with $(i, j), (k, l) \neq (0, 0)$,

$$\mathcal{D}_{i,j} \leq_{RK} \mathcal{D}_{k,l} \Leftrightarrow i \leq k \ \& \ j \leq l.$$

Proof. (1) $\mathcal{D}_0 = \pi_{T_{\langle 0 \rangle}}(\mathcal{D}_1)$ is a Ramsey ultrafilter since $\{\pi''_{T_{\langle 0 \rangle}} \mathcal{H}^2(0) \upharpoonright C : C \in \mathcal{H}\}$ is identical to the Ellentuck space. Note that the base for \mathcal{D}_1 is $\mathcal{H}^2(0)$. A typical element of $\mathcal{H}^2(0)$ is a tree of the form $\{\langle \rangle, \langle k \rangle, \langle k, (i, j) \rangle\}$ with $k(k+1)/2 \leq i, j \leq$

$(k+1)(k+2)/2$. Define the map $F : [\mathcal{H}^2(0)]^2 \rightarrow 3$ as follows

$$F(\{\{\langle \rangle, \langle k \rangle, \langle k, (i, j) \rangle\}, \{\langle \rangle, \langle l \rangle, \langle l, (n, m) \rangle\}\}) = \begin{cases} 0 & \text{if } n = i, \\ 1 & \text{if } n \neq i \ \& \ m = j, \\ 2 & \text{otherwise.} \end{cases}$$

The map F has the property that for all $A \in \mathcal{H}^2$, F does not omit a color on $A|\mathcal{H}^2(0)$.

Thus \mathcal{D}_1 is not a weakly Ramsey ultrafilter.

(2) By definition, for each $n < \omega$, \mathcal{D}_{n+1} and $\mathcal{D}_{n+1, n+1}$ are exactly the same.

(3) Suppose that $n < \omega$ and $\pi_{T(I, J)}$ is a projection map with domain $\mathcal{H}^2(n)$ and either I or J nonempty. Let $m = \max(|I|, |J|) + 1$. Since $I, J \subseteq n$, we find that m is less than or equal to n . Let $X \in \mathcal{C}$ and consider the set $\mathcal{G} = \{x \in \mathcal{A}\mathcal{H}_{n+1}^2 : \exists y \in \mathcal{H}^2(n)|X, \pi_{T(I, J)}(y) = \pi_{T(|I|, |J|)}(x(m))\}$. Since \mathcal{C} generates a Ramsey for \mathcal{H}^2 ultrafilter, there exists $Y \leq X, Y \in \mathcal{C}$ such that $\mathcal{G} \cap \mathcal{A}\mathcal{H}_{n+1}^2|Y = \emptyset$ or $\mathcal{A}\mathcal{H}_{n+1}^2|Y \subseteq \mathcal{G}$. Since there exists $z \in \mathcal{A}\mathcal{H}_{n+1}^2|Y$ such that $\pi_{T(I, J)}(Y(n)) = \pi_{T(|I|, |J|)}(z(m))$ it must be the case that $\mathcal{A}\mathcal{H}_{n+1}^2|Y \subseteq \mathcal{G}$. Hence, for each $X \in \mathcal{C}$ there exists $Y \in [\emptyset, X] \cap \mathcal{C}$ such that $\pi_{T(I, J)}''\mathcal{H}^2(m)|Y \subseteq \pi_{T(|I|, |J|)}''\mathcal{H}^2(n)|X$. By Fact 5.4.5 it follows that $\pi_{T(I, J)}(\mathcal{D}_n) = \pi_{T(|I|, |J|)}(\mathcal{D}_m) = \mathcal{D}_{|I|, |J|}$.

(4) Suppose that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ is a sequence of sets in $\pi_{T(I, J)}(\mathcal{D}_n)$ with $T(I, J) \in \mathcal{T}(n)$. Then there exists a sequence $C_0 \geq C_1 \geq C_2 \geq \dots$ of elements of \mathcal{C} such that for each $i < \omega$, $\pi_{T(I, J)}''(\mathcal{H}(n)|C_i) \subseteq X_i$. Since every Ramsey for \mathcal{H}^2 ultrafilter is also a selective for \mathcal{H}^2 ultrafilter, there exists $C \in \mathcal{C}$ such that for each $i < \omega$, $C \setminus r_i(C) \subseteq C_i$. Since each element of $\mathcal{A}\mathcal{H}^2$ is finite, it follows that for each $i < \omega$, $\pi_{T(I, J)}''(\mathcal{H}^2(n)|C) \subseteq^* \pi_{T(I, J)}''(\mathcal{H}^2(n)|C_i)$. Therefore $\pi_{T(I, J)}(\mathcal{D}_n)$ is a p-point.

(5) (\Leftarrow) Suppose that (i, j) and (k, l) are given and not equal to $(0, 0)$. Addition-

ally, suppose that $i \leq k$ and $j \leq l$. Let $n = \max\{i, j\} + 1$ and $m = \max\{k, l\} + 1$. Then $n \geq m$, $\mathcal{D}_{i,j} = \pi_{T(i,j)}(\mathcal{D}_n)$ and $\mathcal{D}_{k,l} = \pi_{T(k,l)}(\mathcal{D}_m)$. By (2), $\pi_{T(i,j)}(\mathcal{D}_m) = \mathcal{D}_{i,j}$. Notice that for all $s \in \mathcal{H}^2(m)$, $\pi_{T(i,j)}(s) = \pi_{T(i,j)} \circ \pi_{T(k,l)}(s)$. Thus for each $X \in \mathcal{C}$, $\pi_{T(i,j)}'' \mathcal{H}^2(m)|X = \pi_{T(i,j)} \circ \pi_{T(k,l)}'' \mathcal{H}^2(m)|X$. By Fact 5.4.5 it follows that $\mathcal{D}_{i,j} = \pi_{T(i,j)}(\mathcal{D}_m) = \pi_{T(i,j)} \circ \pi_{T(k,l)}(\mathcal{D}_m) = \pi_{T(i,j)}(\mathcal{D}_{k,l})$. Therefore $\mathcal{D}_{i,j} \leq \mathcal{D}_{k,l}$.

(5) (\Rightarrow) Suppose that (i, j) and (k, l) are given and not equal to $(0, 0)$. Additionally, suppose that $\mathcal{D}_{i,j} \leq \mathcal{D}_{k,l}$. Let $m = \max(k, l) + 1$.

Claim 21. *For each nonprincipal ultrafilter \mathcal{V} on ω with $\mathcal{V} \leq_{RK} \mathcal{D}_{k,l}$, either $\mathcal{V} \cong \mathcal{D}_0$ or there exists (p, q) not equal to $(0, 0)$ such that $p \leq k$, $q \leq l$ and $\mathcal{V} \cong \mathcal{D}_{p,q}$. Furthermore, if $\mathcal{V} \cong \mathcal{D}_{k,l}$ then $(p, q) = (k, l)$.*

Proof. Suppose that \mathcal{V} is a nonprincipal ultrafilter on ω such that $\mathcal{V} \leq \mathcal{D}_{k,l}$. Then there is a function $\theta : \pi_{T(k,l)}'' \mathcal{H}^2(m) \rightarrow \omega$ such that $\theta(\mathcal{D}_{k,l}) = \mathcal{V}$. Since $\theta \circ \pi_{T(k,l)} : \mathcal{H}^2(m) \rightarrow \omega$, Theorem 5.4.11 implies that there exists an $X \in \mathcal{C}$ and a projection map π_T on $\mathcal{H}^2(m)$ such that for all $y, z \in \mathcal{H}^2(m) \upharpoonright X$,

$$\theta \circ \pi_{T(k,l)}(y) = \theta \circ \pi_{T(k,l)}(z) \text{ if and only if } \pi_T(y) = \pi_T(z). \quad (5.4.3)$$

Equation (5.4.3) shows that $\pi_T(\mathcal{D}_m) \cong \theta \circ \pi_{T(k,l)}(\mathcal{D}_m) = \mathcal{V}$. By (4) either $\pi_T(\mathcal{D}_m) = \mathcal{D}_0$ or there exist (p, q) not equal to $(0, 0)$ such that $p \leq k$, $q \leq l$ and $\pi_T(\mathcal{D}_m) = \mathcal{D}_{p,q}$.

Next suppose that $\mathcal{V} \cong \mathcal{D}_{k,l}$. Then there exists $Y \in \mathcal{C}$ such that $Y \leq X$ and θ is injective on $\pi_{T(k,l)}''(\mathcal{H}^2(m)|Y)$. Suppose that $T = T(P, Q)$ with P or Q nonempty. Let $p = |P|$ and $q = |Q|$. If $p < k$ or $q < l$ then there exists $s, t \in \mathcal{H}^2(m)|Y$ such that $\pi_T(s) = \pi_T(t)$ and $\pi_{T(k,l)}(s) \neq \pi_{T(k,l)}(t)$. However, this contradicts the fact that θ is injective on $\pi_{T(k,l)}''(\mathcal{H}^2(m)|Y)$. Thus, $p \not< k$ and $q \not< l$. Since $T \in \mathcal{T}(m)$, $(p, q) = (k, l)$. \square

Since $\mathcal{D}_{i,j} \leq_{RK} \mathcal{D}_{k,l}$, the previous claim shows that $i \leq k$ and $j \leq l$. \square

In order to state the next theorem we need to introduce the notion of an ultrafilter of $\vec{\mathcal{U}}$ -trees.

Definition 5.4.15. Let S be a tree and $\vec{\mathcal{U}} = (\mathcal{U}_s : s \in S)$ be a sequence of nonprincipal ultrafilters such that for each $s \in S$, \mathcal{U}_s is an ultrafilter on the set of immediate successors of s in S . A $\vec{\mathcal{U}}$ -tree is a subtree T of S with the property that for all $t \in T$,

$$\{s \in S : s \text{ is an immediate successor of } t \text{ and } s \in T\} \in \mathcal{U}_t.$$

The collection of maximal nodes of $\vec{\mathcal{U}}$ -trees, that is $\{[T] : T \text{ is a } \vec{\mathcal{U}}\text{-tree}\}$, generates an ultrafilter on $[S]$ which we denote by $[\vec{\mathcal{U}}]$ and call *the ultrafilter of $\vec{\mathcal{U}}$ -trees*.

The proof of the next theorem is nearly identical to the proof of Theorem 5.10 in [17]. We omit any proofs of results which follow the exact same argument as their counterparts in the proof of Theorem 5.10 in [17].

Theorem 5.4.16. *Suppose that $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter. If \mathcal{V} is a nonprincipal ultrafilter and $\mathcal{C} \geq_T \mathcal{V}$, then \mathcal{V} is isomorphic to an ultrafilter of $\vec{\mathcal{W}}$ -trees, where $\hat{S} \setminus S$ is a well-founded tree, $\vec{\mathcal{W}} = (\mathcal{W}_s : s \in \hat{S} \setminus S)$, and each \mathcal{W}_s is isomorphic to \mathcal{D}_0 or $\mathcal{D}_{i,j}$ for some $(i, j) \in \omega \times \omega$ with $(i, j) \neq (0, 0)$.*

Proof. Suppose that \mathcal{C} and \mathcal{V} are given and satisfy the assumptions of the theorem. By Proposition 5.4.4 and Lemma 5.4.11 there is a front \mathcal{F} on \mathcal{C} , a function $f : \mathcal{F} \rightarrow \omega$, and a $C \in \mathcal{C}$ such that the following hold:

1. The equivalence relation induced by f on $\mathcal{F}|C$ is canonical.
2. $\mathcal{V} = f(\langle \mathcal{C} \upharpoonright \mathcal{F} \rangle)$.

Let $\mathcal{S} = \{\varphi(t) : t \in \mathcal{F}|C\}$ where φ induces the canonical relation equivalent to the relation induced by f on $\mathcal{F}|C$. The filter \mathcal{W} on the base set \mathcal{S} generated by $\varphi(\mathcal{C} \upharpoonright \mathcal{F})$, is an ultrafilter, and $\mathcal{W} \cong \mathcal{V}$. We omit the proof of this fact since it follows from exactly the same argument as its counterpart in the proof of Theorem 5.10 in [17].

We abuse notation and let \mathcal{F} denote $\mathcal{F}|C$ and \mathcal{C} denote $\mathcal{C}|C$. Let $\mathcal{S} = \{\varphi(t) : t \in \mathcal{F}\}$ and let $\hat{\mathcal{S}}$ denote the collection of all initial segments of elements of \mathcal{S} . $\hat{\mathcal{S}}$ forms a tree with no infinite branches under the ordering \sqsubseteq . Define \mathcal{W}_s to be the filter generated by the sets $\{\varphi_{r_j(t)}(u) : u \in \mathcal{H}^2(j)|X/t\}$, $X \in \mathcal{C}$ for $t \in \mathcal{F}$ such that $s \sqsubseteq \varphi(t)$ and $j < |t|$ maximal such that $\varphi(r_j(t)) = s$. We omit the proof of the next claim since its proof follows an identical argument as its counterpart in the proof of Theorem 5.1 in [17].

Claim 22. *For each $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$, \mathcal{W}_s is an ultrafilter which is generated by the collection of $\{\varphi_{r_j(t)}(u) : u \in \mathcal{H}^2(j)|X\}$, $X \in \mathcal{C}$, for any $t \in \mathcal{F}$ and $j < |t|$ maximal such that $\varphi(r_j(t)) = s$.*

The proof of the next claim differs from its counterpart in the proof of Theorem 5.1 in [17]. Thus, we include the proof.

Claim 23. *For each $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$, \mathcal{W}_s is isomorphic to \mathcal{D}_0 for $\mathcal{D}_{i,j}$ for some $(i,j) \in \omega \times \omega \setminus \{(0,0)\}$.*

Proof. Fix $t \in \mathcal{F}$ and $j < |t|$ with j maximal such that $\varphi(r_j(t)) = s$. Suppose that $\varphi_{r_j(t)} = \pi_{T_{(0)}}$. Then for each $X \in \mathcal{C}$, $\{\varphi_{r_j(t)}(u) : u \in \mathcal{H}^2(j)|X\} = \pi_{T_{(0)}}(\mathcal{H}^2(j)|X) \in \mathcal{D}_0$. Since \mathcal{W}_s is nonprincipal, $\mathcal{W}_s = \mathcal{D}_0$, by Fact 5.4.5. If $\varphi_{r_j(t)} = \pi_{T_{(I,J)}}$ with I or J nonempty then for each $X \in \mathcal{C}$, $\{\varphi_{r_j(t)}(u) : u \in$

$\mathcal{H}^2(j)|X/t\} \subseteq \{\pi_{T(I,J)}(u) : u \in \mathcal{H}^2(j)|X\} \in \pi_{T(I,J)}(\mathcal{D}_{j+1})$. Thus, by Fact 5.4.5, $\mathcal{W}_s = \pi_{T(I,J)}(\mathcal{D}_{j+1})$. By Proposition 5.4.14 (4), $\mathcal{W}_s \cong \mathcal{D}_{|I|,|J|}$. \square

The proof of the next claim is omitted since it follows the same argument as its counterpart in the proof of Claim 5.15 in [17].

Claim 24. \mathcal{W} is the ultrafilter generated by $\vec{\mathcal{W}}$ -trees, where $\vec{\mathcal{W}} = (\mathcal{W}_s : s \in \hat{\mathcal{S}} \setminus \mathcal{S})$.

The previous claims show that \mathcal{V} is isomorphic to the ultrafilter \mathcal{W} on the base \mathcal{S} generated by the $\vec{\mathcal{W}}$ -trees, where for each $s \in \hat{\mathcal{S}} \setminus \mathcal{S}$, \mathcal{W}_s is isomorphic to \mathcal{D}_0 or $\mathcal{D}_{i,j}$ for some (i, j) not equal to $(0, 0)$. \square

We finish this section by identifying initial structures in the Tukey ordering of ultrafilters which are Tukey reducible to some Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$.

Theorem 5.4.17. *Suppose that $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$. The Rudin-Keisler ordering of the p-points Tukey reducible to \mathcal{C} is isomorphic to $\omega \times \omega$ under the product ordering.*

Proof. By the correspondence of ultrafilters of $\vec{\mathcal{U}}$ -trees and iterated Fubini products (see Fact 15 and Fact 16 in [12]), the previous theorem implies that every ultrafilter $\mathcal{V} \leq_T \mathcal{C}$ is isomorphic to some Fubini iterate of p-points from among \mathcal{D}_0 and $\mathcal{D}_{i,j}$ where $(i, j) \in \omega \times \omega$ but not equal to $(0, 0)$. Since the Fubini product of ultrafilters is never a p-point, we find that the only p-points Tukey reducible to \mathcal{C} are the p-points \mathcal{D}_0 and $\mathcal{D}_{i,j}$ for some (i, j) in $\omega \times \omega$ not equal to $(0, 0)$. Thus the map

$$(i, j) \mapsto \begin{cases} \mathcal{D}_0 & \text{if } (i, j) = (0, 0), \\ \mathcal{D}_{i,j} & \text{otherwise,} \end{cases} \quad (5.4.4)$$

from $\omega \times \omega$ to the p-points Tukey reducible to \mathcal{C} is a bijection. Proposition 5.4.14 (5) implies that this map is an isomorphism of partial orders. Figure 5.4 provides a graph of the Rudin-Kesler structure of the p-points Tukey reducible to \mathcal{D}_1 . \square

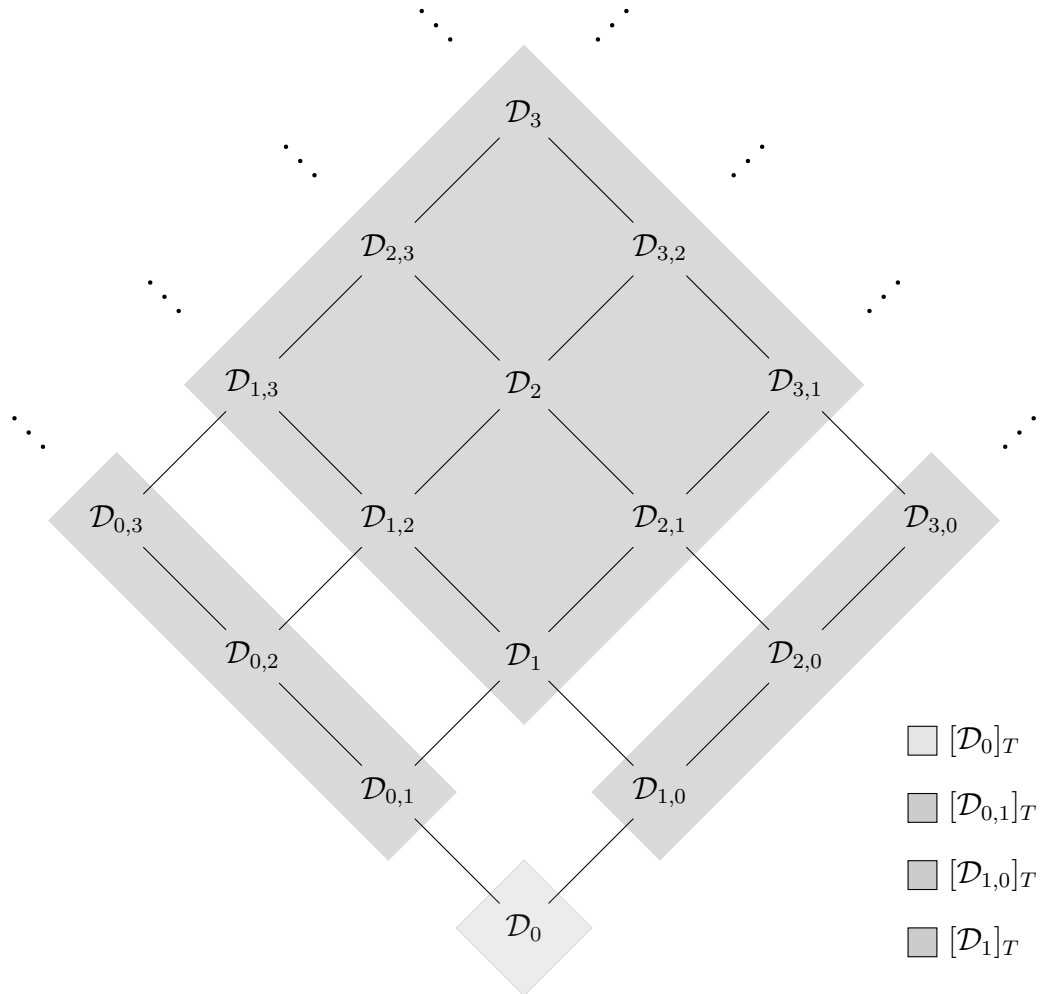


Figure 5.4: Rudin-Kesler structure of the p-point ultrafilters within the Tukey types of nonprincipal ultrafilters Tukey reducible to \mathcal{D}_1

The next theorem is the analogue of Theorem 5.18 in [17] for \mathcal{R}_1 .

Theorem 5.4.18. *Suppose $\mathcal{C} \subseteq \mathcal{H}^2$ generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$. If \mathcal{V} is an ultrafilter on ω and \mathcal{V} is Tukey reducible to \mathcal{C} then one of the following*

holds,

1. $\mathcal{V} \equiv_T \mathcal{D}_\emptyset$,
2. $\mathcal{V} \equiv_T \mathcal{D}_0$,
3. $\mathcal{V} \equiv_T \mathcal{D}_{0,1}$,
4. $\mathcal{V} \equiv_T \mathcal{D}_{1,0}$ or
5. $\mathcal{V} \equiv_T \mathcal{D}_{1,1}$.

Proof. Let \mathcal{V} and \mathcal{C} be given and suppose that they satisfy the hypotheses of the theorem. If \mathcal{V} is principal then $\mathcal{V} \equiv \mathcal{D}_\emptyset$. Thus, without loss of generality, we assume \mathcal{V} is nonprincipal. By Theorem 5.4.17, \mathcal{V} is isomorphic to $\mathcal{D}_{i,j}$ with either i or j nonzero, or \mathcal{V} is isomorphic to \mathcal{D}_0 . Next we consider three cases depending on whether i or j is zero.

Suppose that $i = 0$ and $j > 0$. By Proposition 5.4.14, $\mathcal{D}_{0,1} \leq \mathcal{D}_{0,j}$. Thus $\mathcal{D}_{0,1} \leq_T \mathcal{D}_{0,j}$. Define $g : \mathcal{C} \upharpoonright \pi''_{T(0,1)} \mathcal{H}^2(1) \rightarrow \pi''_{T(0,j)} \mathcal{H}^2(j)$ by

$$g(\pi''_{T(0,1)} \mathcal{H}^2(1)|X) = \pi_{T(0,j)} \mathcal{H}^2(j)|X. \quad (5.4.5)$$

The function g is well-defined on a cofinal subset of $\mathcal{D}_{0,1}$, since from the the image of $\mathcal{H}^2(1)|X$ under $\pi_{T(0,1)}$ one can reconstruct the projection of the tree X to its second-coordinate which is all that is needed to compute $\pi_{T(0,j)} \mathcal{H}^2(j)|X$. The function g is a monotone cofinal map from a cofinal subset of $\mathcal{D}_{0,1}$ to a cofinal subset of $\mathcal{D}_{0,j}$. In other words, $\mathcal{D}_{0,1} \geq_T \mathcal{D}_{0,j}$. Therefore, $\mathcal{D}_{0,1} \equiv_T \mathcal{D}_{0,j}$.

Suppose that $i > 0$ and $j = 0$. By an identical argument using the first coordinate instead of the second coordinate, $\mathcal{D}_{1,0} \equiv_T \mathcal{D}_{i,0}$.

Suppose that $i > 0$ and $j > 0$. By Proposition 5.4.14, $\mathcal{D}_{1,1} \leq \mathcal{D}_{i,j}$. Thus $\mathcal{D}_{1,1} \leq_T \mathcal{D}_{i,j}$. Define $h : \mathcal{C} \upharpoonright \pi''_{T(1,1)} \mathcal{H}^2(1) \rightarrow \pi''_{T(i,j)} \mathcal{H}^2(\max(i, j) + 1)$ by

$$h(\pi''_{T(1,1)} \mathcal{H}^2(1)|X) = \pi_{T(i,j)} \mathcal{H}^2(\max(i, j) + 1)|X. \quad (5.4.6)$$

Notice that the function h is well-defined on a cofinal subset of $\mathcal{D}_{1,1}$, since from the set $\pi''_{T(1,1)} \mathcal{H}^2(1)|X$ one can reconstruct the tree X . The function h is a monotone cofinal map from a cofinal subset of $\mathcal{D}_{1,1}$ to a cofinal subset of $\mathcal{D}_{i,j}$. In other words, $\mathcal{D}_{1,1} \geq_T \mathcal{D}_{i,j}$. Therefore, $\mathcal{D}_{1,1} \equiv_T \mathcal{D}_{i,j}$. \square

Lemma 5.4.19. \mathcal{D}_0 , $\mathcal{D}_{0,1}$, $\mathcal{D}_{1,0}$ and \mathcal{D}_1 are pairwise not Tukey equivalent to one another.

Proof. Since \mathcal{H}^2 is a product of \mathcal{R}_1 with itself, $\mathcal{D}_{0,1}$ and $\mathcal{D}_{1,0}$ are Ramsey for \mathcal{R}_1 ultrafilters on $[T_1]$. By Theorem 5.18 from [17], $\mathcal{D}_{0,1}, \mathcal{D}_{1,0} \not\equiv_T \mathcal{D}_0$. Furthermore, Theorem 5.18 from [17] implies that if $\mathcal{D}_{0,1}$ and $\mathcal{D}_{1,0}$ are Tukey equivalent then they are also Rudin-Kiesler equivalent as there are only two nonisomorphic p-points Tukey reducible to a Ramsey for \mathcal{R}_1 ultrafilter. Since $\mathcal{D}_{0,1}$ and $\mathcal{D}_{1,0}$ are not Rudin-Keisler equivalent they are also not Tukey equivalent. Since there are four nonisomorphic p-points Tukey reducible \mathcal{D}_1 , Theorem 5.18 from [17] shows that \mathcal{D}_1 is not Tukey equivalent to $\mathcal{D}_{0,1}$, $\mathcal{D}_{1,0}$ or \mathcal{D}_0 . \square

We are now in a position to prove the main theorem of this chapter.

Theorem 5.4.20. *It is consistent with ZFC that the four-element Boolean algebra appears as an initial Tukey structure.*

Proof. By Theorem 2.4.5, forcing with (\mathcal{H}^2, \leq^*) adjoins a subset of $\mathcal{C} \subseteq \mathcal{H}^2$ that generates a Ramsey for \mathcal{H}^2 ultrafilter on $[T_1 \otimes T_1]$. Recall that the four-element

Boolean algebra as a partial order is isomorphic to 2×2 under the product ordering. Let g be the map from 2×2 to the Tukey-types of nonprincipal ultrafilters Tukey reducible to \mathcal{C} given by

$$g(i, j) = \begin{cases} [\mathcal{D}_0]_T & \text{if } (i, j) = (0, 0), \\ [\mathcal{D}_{i,j}]_T & \text{otherwise.} \end{cases} \quad (5.4.7)$$

By the previous theorem, g is a surjection. Proposition 5.4.14 (5) shows that g is order-preserving with respect to the Tukey reducibility relation. The previous lemma verifies that g is injective. Therefore, the Tukey types of nonprincipal ultrafilters Tukey reducible to \mathcal{C} is isomorphic to 2×2 under the product ordering, *i.e.* the four-element Boolean algebra. Thus, it is consistent with ZFC, that the four-element Boolean algebra appears as an initial Tukey structure. The diagram in Figure 5.5 is a graph of the Tukey types of nonprincipal ultrafilter Tukey reducible to \mathcal{C} . \square

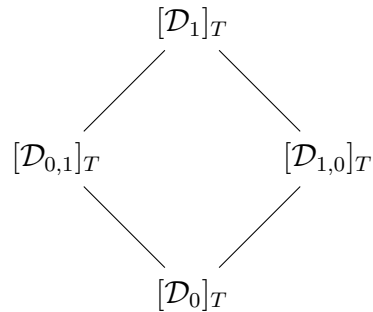


Figure 5.5: Hasse diagram of the Tukey classes of nonprincipal ultrafilters Tukey reducible to \mathcal{D}_1

The previous two theorems suggest the following two questions of Dobrinen

Question 5.4.21. What are the possible initial Tukey structures for ultrafilters on a countable base set?

Question 5.4.22. What are the possible initial Rudin-Keisler structures for ultrafilters on a countable base set?

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