Borel partitions of a space of Rado graphs are Ramsey

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Finite Dimensional Ramsey Theory

Infinite Ramsey's Theorem. (Ramsey, 1929) Given $k \geq 1$ and a coloring $c : [\omega]^k \to 2$, there is an infinite subset $X \subseteq \omega$ such that c is constant on $[X]^k$.

$$\forall k, \;\; \omega \to (\omega)^k$$

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is Ramsey if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that either $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Nash-Williams Theorem. (1965) Clopen sets are Ramsey.

Galvin-Prikry Theorem. (1973) Borel sets are Ramsey.

Silver Theorem. (1970) Analytic sets are Ramsey.

Ellentuck Theorem. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \rightarrow_* (\omega)^\omega$$

Ellentuck's Theorem

The Ellentuck topology is generated by basic open sets of the form

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Ellentuck Theorem. (1974) Given any $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire with respect to the Ellentuck topology,

$(*)$ $\quad \forall [s, A] \ \exists B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$.

The Ellentuck space is the prototype for topological Ramsey spaces:

These are spaces whose members are infinite sequences, with a topology induced by finite heads and infinite tails, and in which every subset with the property of Baire satisfies $(*)$.

A KPT Question

Problem 11.2 in (KPT 2005). Develop infinite dimensional Ramsey theory for Fraïssé structures.

Given $\mathbb{K} = \mathrm{Flim}(\mathcal{K})$ for some Fraïssé class $\mathcal{K}$, and some natural topology on $\binom{\mathbb{K}}{\mathbb{K}}$, are all "definable" sets Ramsey?

$$\mathbb{K} \to_* (\mathbb{K})^{\mathbb{K}}?$$

That is, can the Galvin-Prikry or Ellentuck Theorems be extended to spaces whose points represent homogeneous structures?

Very little known. Topological Ramsey spaces have infinite dimensional Ramsey theory, but the rationals are the only Fraïssé structure modeled by a tRs.

KPT Subquestion

Question. Is there an analogue of Galvin-Prikry, Silver, or Ellentuck for the Rado graph?

Is there a way to topologize all subcopies of the Rado graph so that all definable sets have the Ramsey property?

Theorem. (D.) There is a natural topological space of Rado graphs in which every Borel subset is Ramsey.

Ramsey Theory for Colorings of Finite Graphs

Theorem. (Abramson-Harrington 1978 and Nešetřil-Rödl 1977/83) The class of all finite ordered graphs has the Ramsey property.

Let $\mathbb{R}$ denote the Rado graph.

Theorem. (Laflamme, Sauer, Vuksanovic 2006) For each finite graph G, there is a number $T(G)$ such that

$$(\forall k \geq 1) \ \ \mathbb{R} \rightarrow (\mathbb{R})^{G}_{k,T(G)}$$

The number $T(G)$ is exactly the number of strong similarity types of codings of G in the binary tree $2^{<\omega}$.

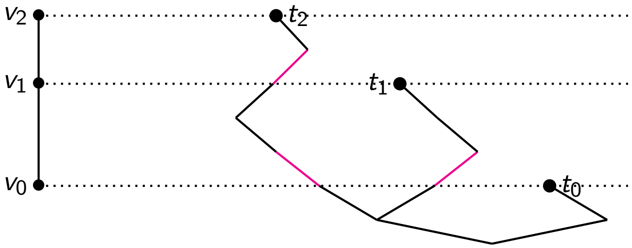
So to get a positive answer to KPT Question for the Rado graph, we need to restrict to copies of the Rado graph which all have the same strong similarity type.

Coding Graphs in $2^{<\omega}$

Let A be a graph with vertices $\langle v_n : n < N \rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Trees with Coding Nodes

Strong trees can code Rado graphs, but subtrees do not carry enough information to tell us which subcopy of the Rado graph they represent.

A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c\rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n-th coding node in T, usually denoted c_n^T.

The Space of Strong Rado Coding Trees $(\mathcal{T}_R, \leq, r)$

Let R be a Rado graph with vertices $\langle v_n : n < \omega \rangle$.

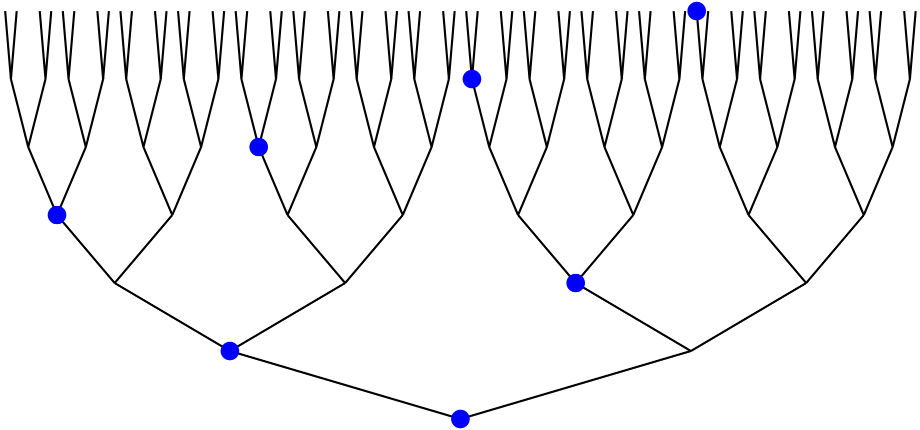
Define $\mathbb{T}_R = (2^{<\omega}, \omega; \subseteq, <, c)$, where for each $n < \omega$, $c(n)$ represents v_n.

$\mathcal{T}_R$ consists of all trees with coding nodes $(T, \omega; \subseteq, <, c^T)$, where

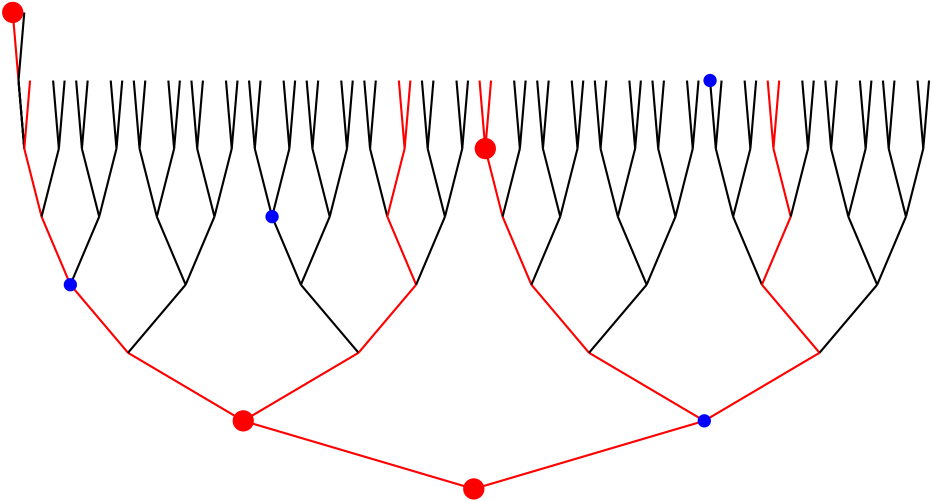
1. T is a strong subtree of $2^{<\omega}$; and
2. The strong tree isomorphism $\varphi : \mathbb{T}_R \rightarrow T$ has the property that for each $n < \omega$, $\varphi(c(n)) = c^T(n)$.

The members of $\mathcal{T}_R$ are called strong Rado coding trees. They represent all subgraphs of R which are strongly similar to R.

Strong Rado Coding Tree $\mathbb{T}_R$



A Strong Rado Coding Tree $T \in \mathcal{T}_R$



Thm. (D.) Every Borel subset of $\mathcal{T}_R$ is Ramsey. That is, if $\mathcal{X} \subseteq \mathcal{T}_{\mathcal{R}}$ is Borel, then

$$(*) \qquad \forall [s, A] \;\; \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset.$$

Let $\mathcal{R}$ be the collection of all Rado graphs coded by some member of $\mathcal{T}_R$. That is, $\mathcal{R}$ is the space of all Rado subgraphs of R with the same strong similarity type as R.

Thm. (restated) $\mathcal{R}$ forms a subspace of $[\omega]^\omega$, and every Borel subset of $\mathcal{R}$ is Ramsey.

So there is a topological space of Rado graphs which has infinite dimensional Ramsey theory, and by LSV, any such space must have a restriction to one strong similarity type.

Proof Ideas.

1. Show that all open sets are Ramsey.
2. Show that complements of Ramsey sets are Ramsey.
3. Show that Ramsey sets are closed under countable unions.

The catch is (1) and (3). We use a forcing argument utilizing methods from our work on the big Ramsey degrees of the Henson graphs.

Hypotheses for Main Lemma. Given $T \in \mathcal{T}_R$, $D = r_n(T)$, and A an initial segment of some member of $\mathcal{T}_R$ with $\max(A) \subseteq \max(D)$: Let

$$A^+ = A \cup \{s^\frown i : s \in \max(A) \text{ and } i \in \{0,1\}\}.$$

Let B denote the subset of A^+ which will be extended to $(k+1)$-st approximations which are colored. Define

$$r_{k+1}[B, T]^* = \{C \in \mathcal{AT}_{k+1}(T) : \max(C) \sqsupseteq \max(B)\}.$$

Lemma. (D.) Let $h : r_{k+1}[B, T]^* \to 2$ be a coloring. Then there is a strong Rado coding tree $S \in [D, T]$ such that h is monochromatic on $r_{k+1}[B, S]^*$.

Lemma. (D.) Let $h : r_{k+1}[B, T]^* \to 2$ be a coloring. Then there is a strong Rado coding tree $S \in [D, T]$ such that h is monochromatic on $r_{k+1}[B, S]^*$.

Rem. The proof uses forcing in the style of (D. 2017 and 2019), building on Harrington's proof of the Halpern-Läuchli Theorem.

The Lemma is used both to show that

1. open sets in $\mathcal{T}_R$ are completely Ramsey and
2. to do fusion arguments for showing that countable unions of cR sets are cR, because $\mathcal{T}_R$ does not satisfy Todorcevic's Axiom **A.3**(2) for topological Ramsey spaces.

Why only Borel and not Property of Baire?

Thm. (D.) Every Borel subset of $\mathcal{T}_R$ has the Ramsey property.

Similarly to strong coding trees developed for the big Ramsey degrees of the Henson graphs, the collection of strong Rado trees form a space satisfying all of Todorcevic's Axioms for topological Ramsey spaces, except for A.3(2) (Amalgamation Axiom).

A "forced" Halpern-Läuchli-style theorem provides a means for fusion arguments in the style of Galvin-Prikry, but is not sufficient for Ellentuck's arguments.

Remarks, Questions and Future Directions

Rem 1. We could fix any strong similarity type of a tree with coding nodes coding the Rado graph and get a space of Rado graphs in which every Borel set is Ramsey.

Rem 2. Trees with coding nodes and these forcing arguments were developed to work with forbidden k-cliques, but have shown to be useful for infinite dimensional Ramsey theory of the Rado graph.

Rem 3. A similar fusion lemma for Henson graphs follows from my work on their big Ramsey degrees, so they will have a similar theorem.

Question. What other Fraïssé structures have infinite dimensional Ramsey theory?

Question. Is there a topological Ramsey space of Rado graphs?

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