Barren Extensions

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joint work with

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Thanks to Carlos Di Prisco

My first remembrance of knowing Carlos is from the 2008 Erice meeting on Set Theoretic Topology, but we probably met before that.

I was certainly aware of his work many years before that.

We overlapped in Paris many summers working with Stevo, at the Fields Institute Semester in 2012, and many other times.

He is very generous, having given me a postdoc (Goyo Mijares), research ideas, and support over the years.

$L(\mathbb{R})$

The Solovay model is the $L(\mathbb{R})$ in a model after Levy collapsing an inaccessible cardinal to ω_1.

In the presence of large cardinals, the $L(\mathbb{R})$ of the universe is elementarily equivalent to a Solovay model.
(Woodin, building on Foreman-Magidor-Shelah)

In the Solovay model,

1. $\omega \rightarrow (\omega)^\omega$ holds.
2. Every uncountable set of reals contains a perfect set.
3. Many other nice regularity properties.

$L(\mathbb{R})[\mathcal{U}]$ where $\mathcal{U}$ is Ramsey

There has been quite a bit of work finding which properties of the Solovay model $L(\mathbb{R})$ persist in the model $L(\mathbb{R})[\mathcal{U}]$, where $\mathcal{U}$ is a forced Ramsey ultrafilter.

Barren extensions, by Henle, Mathias and Woodin. (1985)

Perfect-set properties in $L(\mathbb{R})[\mathcal{U}]$, by Di Prisco and Todorcevic. (1998)

Ramsey ultrafilters and countable-to-one uniformization, by Ketchersid, Larson and Zapletal. (2016)

$L(\mathbb{R})[\mathcal{U}]$ where $\mathcal{U}$ is Ramsey

Let $\mathcal{U}$ be a Ramsey ultrafilter forced by $([\omega]^\omega, \subseteq^*)$ over $L(\mathbb{R})$.

In *A barren extension*, Henle, Mathias and Woodin proved that assuming $\omega \rightarrow (\omega)^\omega$,

1. $L(\mathbb{R})$ and $L(\mathbb{R})[\mathcal{U}]$ have the same sets of ordinals.
2. Under an additional assumption, $([\omega]^\omega, \subseteq^*)$ preserves all strong partition cardinals.

In *Perfect-set properties in $L(\mathbb{R})[\mathcal{U}]$*, Di Prisco and Todorcevic proved a collection of results about this model, including the Perfect-Set Property and the Open Coloring Axiom.

How important is the Ramseyness of $\mathcal{U}$ to these results?

Extensions to $L(\mathbb{R})[\mathcal{U}]$ where $\mathcal{U}$ is not Ramsey

In joint work with Hathaway, we extend results of Henle, Mathias and Woodin to $L(\mathbb{R})[\mathcal{U}]$, where $\mathcal{U}$ is a non-Ramsey ultrafilter forced by any member of several classes of topological Ramsey spaces.

In *A notion of selective ultrafilter corresponding to topological Ramsey space* (2007), Mijares proved that the Perfect Set Property and OCA hold in $L(\mathbb{R})[\mathcal{U}]$, where $\mathcal{U}$ is an ultrafilter forced by almost any topological Ramsey space.

These ultrafilters can have rich Rudin-Keisler and Tukey structures below them, but by a result of Navarro Flores, they each have a Ramsey ultrafilter RK below.

A Barren Extension

$$\omega \to (\omega)^\omega$$

means that for each $c : [\omega]^\omega \to 2$, there is an $N \in [\omega]^\omega$ such that c is constant on $[N]^\omega$.

$\omega \to (\omega)^\omega$ fails under the Axiom of Choice but

- holds assuming $\mathrm{AD}_\mathbb{R}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, and in $L(\mathbb{R})$.

Thm. (Henle-Mathias-Woodin) Let M be a transitive model of ZF + $\omega \to (\omega)^\omega$ and let N be a forcing extension via $([\omega]^\omega, \subseteq^*)$. Then M and N have the same sets of ordinals; moreover every sequence in N of elements of M lies in M.

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Note: $([\omega]^\omega, \subseteq^*)$ forces a Ramsey ultrafilter.

Question: Which other σ-closed forcings adding ultrafilters have similar properties?

Ultrafilters with Weak Partition Relations

$$\mathcal{U} \to (\mathcal{U})^2_{l,t}$$

means that for each $X \in \mathcal{U}$ and $c : [X]^2 \to l$, there is a $U \subseteq X$ in $\mathcal{U}$ such that c takes at most t colors on $[U]^2$.

The least t such that for all l, $\mathcal{U} \to (\mathcal{U})^2_{l,t}$ is the Ramsey degree of $\mathcal{U}$, denoted $t(\mathcal{U})$.

Examples

$\mathcal{P}(\omega)/\mathrm{Fin}$, equiv. $([\omega]^\omega, \subseteq^*)$, forces a Ramsey ultrafilter $\mathcal{U}$: $t(\mathcal{U}) = 1$.

A forcing of Laflamme produces a weakly Ramsey ultrafilter $\mathcal{U}_1$: $t(\mathcal{U}_1) = 2$.

(Laflamme) There is a hierarchy forcings $\mathbb{P}_\alpha$ $(\alpha < \omega_1)$ which produce ultrafilters $\mathcal{U}_\alpha$. For $k < \omega$, $t(\mathcal{U}_k) = k + 1$.

Examples

(Navarro Flores): For each $k \geq 1$, $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ forces an ultrafilter $\mathcal{G}_k$ with $t(\mathcal{G}_k) = \sum_{i<k} 3^i$. (Blass for $k = 2$)

(Blass): n-square forcing produces an ultrafilter with $t(\mathcal{U}) = 5$.

(Baumgartner-Taylor): For $k \geq 2$, $\mathbb{Q}_k$ produces a k-arrow/not $(k+1)$-arrow ultrafilter $\mathcal{A}_k$: $\mathcal{A}_k \to (\mathcal{A}_k, k)^2$ but $\mathcal{A}_k \not\to (\mathcal{A}_k, k+1)^2$. $t(\mathcal{A}_k) = 3$.

(D.-Mijares-Trujillo): Fraïssé classes can be used to generalize the previous two constructions to produce ultrafilters with various Ramsey degrees. Their Rudin-Keisler structures can be as complex as Fraïssé classes. For $k \geq 2$, k-hypercube forcing produces an ultrafilter $\mathcal{V}_k$ with

$$t(\mathcal{V}_k) = 1 + \sum_{i<k} 3^i$$

Examples

Milliken-Taylor ultrafilters are ultrafilters, $\mathcal{M}_k$, on base set FIN_k, $k \geq 1$, such that each member of $\mathcal{M}_k$ contains an infinite block sequence in FIN_k.

Moreover, for each coloring $c : \mathrm{FIN}_k \to l$, there is an infinite block sequence $X \in \mathcal{M}_k$ such that c is constant on $[X]$.

These ultrafilters are stronger than Ramsey ultrafilters.

These and related forcings contain topological Ramsey spaces as dense subsets.

Barren Extensions

Thm. (D.-Hathaway) Assume M satisfies ZF and either $\mathrm{AD}_{\mathbb{R}}$ or $\mathrm{AD}^{+} + V = L(\mathcal{P}(\mathbb{R}))$. Let $\mathcal{U}$ be any of the above ultrafilters forced over M. Then $M[\mathcal{U}]$ has the same sets of ordinals as M. Moreover it has no new sequences from any ordinal into M.

Remark 1. M can be $L(\mathbb{R})$ as either a Solovay model, or in V with large cardinals.

Remark 2. This theorem holds for many other ultrafilters as well. The main tool is topological Ramsey spaces (dense inside these forcings) endowed with σ-closed quasi orders with the same separative quotient as the original partial order, which behave similarly to $([\omega]^{\omega}, \subseteq^{*})$.

The Essence of this HMW Theorem

$\mathbb{P} = \langle P, \leq, \leq^* \rangle$ is strongly coarsened if

1. $\forall x, y \in P, \quad x \leq y \longrightarrow x \leq^* y$, and
2. $\forall x \in P \;\; \forall y \leq^* x \;\; \exists z \leq x$ such that $z =^* y$.

For $x \in P$, let $[x] = \{y \in P : y \leq x\}$ and $[x]^* = \{y \in P : y \leq^* x\}$.

Examples: $([\omega]^\omega, \subseteq, \subseteq^*)$.

$([\omega]^\omega, \subseteq, \subseteq^{\mathcal{I}})$ where $\mathcal{I}$ is a σ-closed ideal on $\mathcal{P}(\omega)$.

Topological Ramsey space $(\mathcal{R}, \leq, r)$ with an additional σ-closed partial order $\leq^*$ coarsening $\leq$ where $(\mathcal{R}, \leq)$ and $(\mathcal{R}, \leq^*)$ have the same separative quotient.

Left-Right Axiom - key properties of $([\omega]^\omega, \subseteq, \subseteq^*)$

A strongly coarsened poset $\mathbb{P} = \langle P, \leq, \leq^* \rangle$ satisfies the Left-Right Axiom (LRA) iff there are functions $\mathrm{L} : P \to P$ and $\mathrm{R} : P \to P$ such that the following are satisfied:

1. $\forall x \in P, \ \ \mathrm{L}(x), \mathrm{R}(x) \leq^* x$.
2. $\forall x \in P \ \ \exists y, z \leq x$ such that $\mathrm{L}(y) =^* \mathrm{R}(z)$ and $\mathrm{R}(y) =^* \mathrm{L}(z)$.
3. For each $p, x, y \in P$ with $x, y \leq p$, there is $z \leq p$ such that
   a) $\mathrm{L}(z) \leq^* x$
   b) $\mathrm{L}(\mathrm{R}(z)) \leq^* x$
   c) $\mathrm{R}(\mathrm{R}(z)) \leq^* y$.

Remark. This is slightly watered down, but gives the main idea. All of the ultrafilters mentioned above come from forcings which satisfy the LRA.

Barren Extensions - general theorem

Thm. (D.-Hathaway) Let M be a transitive model of ZF. Suppose $\mathbb{P} = \langle P, \leq, \leq^* \rangle \in M$ is a strongly coarsened poset satisfying

1. the Left-Right Axiom, and
2. for each $x \in P$ and every coloring $c : [x] \to 2$, there is some $y \leq x$ such that $c \restriction [y]$ is constant.

Let N be a forcing extension of M via $\langle P, \leq^* \rangle$. Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.

Remark. Condition (2) is like $\omega \to (\omega)^\omega$. It holds for topological Ramsey spaces assuming (a) $\mathrm{AD}_\mathbb{R}$ or (b) $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, or (c) $M = L(\mathbb{R})$ assuming large cardinals in V.

Cor. Let $L(\mathbb{R})$ be a Solovay model. Suppose $\langle \mathcal{R}, \leq, r \rangle$ is a topological Ramsey space, $\leq^*$ is a σ-closed coarsening of $\leq$ which has the same separative quotient as $\leq$, and $\mathcal{R}$ has the property of Independent Sequencing or is a high or infinite dimensional Ellentuck space. Then $L(\mathbb{R})$ and $L(\mathbb{R})[\mathcal{U}]$ have the same sets of ordinals, where $\mathcal{U}$ is the ultrafilter on base set $\mathcal{AR}_1$ forced by $\langle \mathcal{R}, \leq^* \rangle$.

Part II: Persistence of Strong Partition Cardinals

Strong Partition Cardinals Preserved by $([\omega]^\omega, \subseteq^*)$

$$\kappa \to (\kappa)^\lambda_\mu$$

means that for each $c : [\kappa]^\lambda \to \mu$, there is some $K \in [\kappa]^\kappa$ such that c is constant on $[K]^\lambda$.

Strong partition cardinals were revived by Kleinberg in 1970, when he showed that they are consistent with ZF.

Thm. (Henle-Mathias-Woodin) (ZF + EP + LU) Suppose

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^\lambda_\mu$, and
3. there is a surjection from $[\omega]^\omega$ onto $[\kappa]^\kappa$.

Then $\kappa \to (\kappa)^\lambda_\mu$ holds in the extension via $([\omega]^\omega, \subseteq^*)$.

EP and LU

A subset $A \subseteq [\omega]^\omega$ is invariant if $(p \in A \text{ and } p' =^* p) \longrightarrow p' \in A$.

For $a \in [\omega]^{<\omega}$ and $p \in [\omega]^\omega$, let

$$[a, p] = \{q \in [\omega]^\omega : a \sqsubset q \wedge q \subseteq p\}$$

$X \subseteq [\omega]^\omega$ is Completey Ramsey (CR) if $\forall \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that

$$\text{(a) } [a, q] \subseteq X \quad \text{or} \quad \text{(b) } [a, q] \cap X = \emptyset.$$

$X \subseteq [\omega]^\omega$ is CR^+ if $\forall\, \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that (a) holds;
X is CR^- if $\forall\, \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that (b) holds.

EP: The intersection of any well-ordered collection of CR^+ sets is CR^+.

LU: For any relation $R \subseteq [\omega]^\omega \times \mathcal{P}(\omega)$ such that $\forall p\ \exists y\ R(p, y)$, the set $\{x : R \text{ is uniformized on } [x]^\omega\}$ is CR^+.

Preserving Strong Partition Cardinals over $L(\mathbb{R})$

Thm. (Henle-Mathias-Woodin) (AD + $V = L(\mathbb{R})$)
If $0 < \lambda = \omega \cdot \lambda \leq \kappa$, $2 \leq \mu < \kappa$, and $\kappa \to (\kappa)^\lambda_\mu$, then

$$L(\mathbb{R})[\mathcal{U}] \models \kappa \to (\kappa)^\lambda_\mu,$$

where $\mathcal{U}$ is the Ramsey ultrafilter forced by $([\omega]^\omega, \subseteq^*)$ over $L(\mathbb{R})$.

Remark. AD + $V = L(\mathbb{R})$ imply LU, EP, and (3) in the above rendition of this theorem.

Extension to Topological Ramsey Spaces

Topological Ramsey spaces are triples $(\mathcal{R}, \le, (r_n)_{n<\omega})$, where $\le$ is a partial order and r is a finite approximation map; basic open sets are of the form

$$[a, p] = \{q \in \mathcal{R} : \exists n < \omega\, (a = r_n(p)) \text{ and } q \le p\}.$$

A subset $X \subseteq \mathcal{R}$ is (Completely) Ramsey if for each $\emptyset \ne [a, p]$ there is some $q \in [a, p]$ such that

$$\text{(a) } [a, q] \subseteq X \quad \text{ or else } \quad \text{(b) } [a, q] \cap X = \emptyset.$$

The defining property of a topological Ramsey space is that all subsets with the property of Baire are Ramsey. (This abstracts $\omega \to (\omega)^\omega$.)

The Ellentuck space $\mathcal{E} = ([\omega]^\omega, \subseteq, (r_n)_{n<\omega})$ has approximation maps $r_n(x) = \{x_i : i < n\}$, where $\{x_i : i < \omega\}$ is the enumeration of $x \in [\omega]^\omega$.

Abstractions of EP and LU

The structure of topological Ramsey spaces, as roughly ω-sequences of finite structures, often produces many of the same properties as the forcing $([\omega]^\omega, \subseteq^*)$.

$X \subseteq \mathcal{R}$ is invariant R^+ if

1. invariant: $(p \in X$ and $p' =^* p) \longrightarrow p' \in X$, and
2. R^+: $\forall p \in \mathcal{R}\ \exists q \leq p$ such that $[q] \subseteq X$.

Let $\mathbb{P} = \langle \mathcal{R}, \leq, \leq^* \rangle$.

EP($\mathbb{P}$): Given any well-ordered sequence $\langle C_\alpha \subseteq P : \alpha < \kappa \rangle$ of invariant R^+ sets, the intersection of the sequence is again invariant R^+.

Abstractions of CR^+, CR^-, $\omega \to (\omega)^\omega$, EP, and LU

$\mathrm{LU}^*(\mathbb{P})$: Uniformization relative to some invariant cube $[p]^*$ for relations $R \subseteq \mathcal{R} \times {}^\omega 2$.

$\mathrm{LCU}(\mathbb{P})$: Continuous uniformization for relations $R \subseteq \mathcal{R} \times {}^\omega 2$ relative to some cube $[p]$.

Similar to Todorcevic's Ramsey Uniformization Theorem for relations on $[\omega]^\omega \times X$ where X is a Polish space.

Prop. (D.-Hathaway) Assume either $\mathrm{AD}_\mathbb{R}$ or $\mathrm{AD}^+ + V = L(\mathbb{R}(\mathcal{R}))$. Let $\langle \mathcal{R}, \leq, r \rangle$ be a topological Ramsey space. Then every subset of $\mathcal{R}$ is Ramsey. Hence, also $\mathrm{LCU}(\mathcal{R}, \leq)$ holds.

Preserving Strong Partition Cardinals - general theorem

Thm. (D.-Hathaway) Suppose $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ is a coarsened poset such that EP($\mathbb{P}$) and LU($\mathbb{P}$) hold, and each $=^*$-equivalence class is countable. Assume that every subset of X is Ramsey and

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^{\lambda}_{\mu}$,
3. there is a surjection from ${}^{\omega}2$ onto $[\kappa]^{\kappa}$.

Then $\langle X, \leq \rangle$ forces $\kappa \to (\kappa)^{\lambda}_{\mu}$.

Preserving Strong Partition Cardinals - simple version

Thm. (D.-Hathaway) Assume either $\mathrm{AD}_{\mathbb{R}}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\mathbb{P} = \langle \mathcal{R}, \leq, \leq^*, r \rangle$ be a coarsened topological Ramsey space, where the $=^*$-equivalence classes are countable. Then forcing with $\langle \mathcal{R}, \leq \rangle$ preserves $\kappa \to (\kappa)^\lambda_\mu$ whenever

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^\lambda_\mu$.

Remark. The ultrafilters mentioned previously all preserve strong partition cardinals, except possibly those forced by $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$.

A key step in our results is the following:

Lemma. (D-H) Assume either 1) $\mathrm{AD}_{\mathbb{R}}$ or 2) $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\langle \mathcal{R}, \leq, r \rangle$ be a topological Ramsey space. Then every subset of $\mathcal{R}$ is Ramsey.

The proof uses that the Mathias-like forcing for a topological Ramsey space has the Prikry and Mathias properties, which was proved by Di Prisco, Mijares and Nieto in 2017.

References

Di Prisco-Mijares-Nieto, *Local Ramsey theory: An abstract approach*, MLQ (2017).

Di Prisco-Llopis-Todorcevic, *Parametrized partitions of products of finite sets*, Combinatorica (2004).

Di Prisco-Todorcevic, *Perfect-set properties in $L(\mathbb{R})[\mathcal{U}]$*, Advances in Math. (1998)

Dobrinen-Hathaway, *Barren extensions*, Submitted.

Henle-Mathias-Woodin, *A barren extension*, Lecture Notes in Math., 1130, Springer (1985).

Mijares, *A notion of selective ultrafilter corresponding to topological Ramsey spaces*, MLQ (2007).

Related References

Dobrinen, *High dimensional Ellentuck spaces and initial chains in the Tukey structure of non-p-points*, JSL (2016).

Dobrinen, *Infinite dimensional Ellentuck spaces and Ramsey-classification theorems*, JML (2016).

Dobrinen-Mijares-Trujillo, *Topological Ramsey spaces from Fraïssé classes and initial Tukey structures*, AFML (2017).

Dobrinen-Navarro Flores, *Ramsey degrees of ultrafilters, pseudointersection numbers, and the tools of topological Ramsey spaces* 25 pp. Submitted.

Dobrinen-Todorcevic, *Ramsey-Classification Theorems and their applications in the Tukey theory of ultrafilters, Part 1*, TAMS (2014).

Dobrinen-Todorcevic, *Ramsey-Classification Theorems and their applications in the Tukey theory of ultrafilters, Part 2*, TAMS (2015).

Todorcevic, *Introduction to Ramsey spaces*, Princeton University Press, (2010).

Happy Birthday Carlos!

and many more!