Barren Extensions

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joint work with

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Overview

In their paper, "A barren extension," Henle, Mathias and Woodin proved that assuming $\omega \to (\omega)^\omega$,

1. Forcing with $([\omega]^\omega, \subseteq^*)$ adds no new sets of ordinals.
2. Under an additional assumption, $([\omega]^\omega, \subseteq^*)$ preserves all strong partition cardinals.

In joint work with Hathaway, we extend these results to a large collection of σ-closed forcings which add ultrafilters with weak partition properties.

These ultrafilters can have rich Rudin-Keisler and Tukey structures below them, with a Ramsey ultrafilter at the bottom.

Part I: Barren Extensions

Infinite Dimensional Ramsey Theorem

$\omega \to (\omega)^\omega$ means that for each $c : [\omega]^\omega \to 2$, there is an $N \in [\omega]^\omega$ such that c is constant on $[N]^\omega$.

$\omega \to (\omega)^\omega$ fails under the Axiom of Choice but holds

- assuming $\mathrm{AD}_{\mathbb{R}}$ (Prikry, Mathias).
- assuming $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ (Cabal).
- in the $L(\mathbb{R})$ of $V^{\mathrm{Coll}(\omega,<\kappa)}$, where κ is strongly inaccessible (Mathias).
- in $L(\mathbb{R})$ in the presence of a supercompact in V (Shelah-Woodin).

A Barren Extension

Thm. (Henle-Mathias-Woodin) Let M be a transitive model of ZF + $\omega \to (\omega)^\omega$ and let N be a forcing extension via $([\omega]^\omega, \subseteq^*)$. Then M and N have the same sets of ordinals; moreover every sequence in N of elements of M lies in M.

Note: $([\omega]^\omega, \subseteq^*)$ forces a Ramsey ultrafilter.

Question: Which other σ-closed forcings adding ultrafilters have similar properties?

Ultrafilters with Weak Partition Relations

$$\mathcal{U} \to (\mathcal{U})^2_{l,t}$$

means that for each $X \in \mathcal{U}$ and $c : [X]^2 \to l$, there is a $U \subseteq X$ in $\mathcal{U}$ such that c takes at most t colors on $[U]^2$.

The least t such that for all l, $\mathcal{U} \to (\mathcal{U})^2_{l,t}$ is the Ramsey degree of $\mathcal{U}$, denoted $t(\mathcal{U})$.

Examples

$\mathcal{P}(\omega)/\text{Fin}$, equiv. $([\omega]^\omega, \subseteq^*)$, forces a Ramsey ultrafilter $\mathcal{U}$: $t(\mathcal{U}) = 1$.

A forcing of Laflamme produces a weakly Ramsey ultrafilter $\mathcal{U}_1$: $t(\mathcal{U}_1) = 2$.

(Laflamme) There is a hierarchy forcings $\mathbb{P}_\alpha$ $(\alpha < \omega_1)$ which produce ultrafilters $\mathcal{U}_\alpha$. For $k < \omega$, $t(\mathcal{U}_k) = 2^k$.

Ultrafilters with Weak Partition Relations

(Navarro Flores): For each $k \geq 1$, $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ forces an ultrafilter $\mathcal{G}_k$ with $t(\mathcal{G}_k) = \sum_{i<k} 3^i$. (Blass for $k = 2$)

Blass' n-square forcing produces an ultrafilter with $t(\mathcal{U}) = 5$.

(Baumgartner-Taylor): For $k \geq 2$, $\mathbb{Q}_k$ produces a k-arrow/not $(k+1)$-arrow ultrafilter $\mathcal{A}_k$: $\mathcal{A}_k \to (\mathcal{A}_k, k)^2$ but $\mathcal{A}_k \not\to (\mathcal{A}_k, k+1)^2$.

(D.-Mijares-Trujillo): Fraïssé classes can be used to generalize the previous two constructions to produce ultrafilters with various Ramsey degrees. Their Rudin-Keisler structures can be as complex as Fraïssé classes.

Many of these Ramsey degrees were computed in (D.-Navarro Flores).

Barren Extensions

Thm. (D.-Hathaway) Assume M is a model of ZF + $\mathrm{AD}_{\mathbb{R}}$ or ($\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$), or $M = L(\mathbb{R})$ is the Solovay model or there is a supercompact cardinal in V.

Let $\mathcal{U}$ be any of the above ultrafilters forced over M. Then $M[\mathcal{U}]$ has the same sets of ordinals as M. Moreover it adds no new functions from any ordinal to M.

Remark. This theorem holds for many other ultrafilters as well, including stable ordered union. The main tool is topological Ramsey spaces (dense inside these forcings), because they have infinite dimensional Ramsey theorems similar to $\omega \to (\omega)^\omega$, under the above assumptions on M.

The Essence of this HMW Theorem

$\mathbb{P} = \langle P, \leq, \leq^* \rangle$ is strongly coarsened if

1. $\forall x, y \in P, \quad x \leq y \longrightarrow x \leq^* y$, and
2. $\forall x \in P \;\; \forall y \leq^* x \;\; \exists z \leq x$ such that $z =^* y$.

For $x \in P$, let $[x] = \{y \in P : y \leq x\}$ and $[x]^* = \{y \in P : y \leq^* x\}$.

Examples: $([\omega]^\omega, \subseteq, \subseteq^*)$

More generally, $([\omega]^\omega, \subseteq, \subseteq^{\mathcal{I}})$ where $\mathcal{I}$ is a σ-closed ideal on $\mathcal{P}(\omega)$.

For many topological Ramsey spaces $(\mathcal{R}, \leq, r)$, there is a naturally related σ-closed partial order $\leq^*$ which strongly coarsens $\leq$.

Left-Right Axiom - key properties of $([\omega]^\omega, \subseteq, \subseteq^*)$

A strongly coarsened poset $\mathbb{P} = \langle P, \leq, \leq^* \rangle$ satisfies the Left-Right Axiom (LRA) iff there are functions $\mathrm{L} : P \to P$ and $\mathrm{R} : P \to P$ such that the following are satisfied:

1. $\forall x \in P,\ \ \mathrm{L}(x), \mathrm{R}(x) \leq^* x$.
2. $\forall x \in P\ \ \exists y, z \leq x$ such that $\mathrm{L}(y) =^* \mathrm{R}(z)$ and $\mathrm{R}(y) =^* \mathrm{L}(z)$.
3. For each $p, x, y \in P$ with $x, y \leq p$, there is $z \leq p$ such that
    a) $\mathrm{L}(z) \leq^* x$
    b) $\mathrm{L}(\mathrm{R}(z)) \leq^* x$
    c) $\mathrm{R}(\mathrm{R}(z)) \leq^* y$.

Remark. All of the partial orders mentioned on slides 6 and 7 contain dense subsets forming Ramsey spaces which satisfy the LRA.

Barren Extensions - general theorem

Thm. (D.-Hathaway) Let M be a transitive model of ZF. Suppose $\mathbb{P} = \langle P, \leq, \leq^* \rangle \in M$ is a strongly coarsened poset satisfying

1. the Left-Right Axiom, and
2. for each $x \in P$ and every coloring $c : [x]^* \to 2$, there is some $y \leq^* x$ such that $c \restriction [y]$ is constant.

Let N be a forcing extension of M via $\langle P, \leq^* \rangle$. Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.

Remark. Condition (2) is like $\omega \to (\omega)^\omega$.

Part II: Preservation of Strong Partition Cardinals

Strong Partition Cardinals Preserved by $([\omega]^\omega, \subseteq^*)$

$\kappa \to (\kappa)^\lambda_\mu$ means that for each $c : [\kappa]^\lambda \to \mu$, there is a $K \in [\kappa]^\kappa$ such that c is constant on $[K]^\lambda$.

Thm. (Henle-Mathias-Woodin) (ZF + EP + LU) Suppose

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^\lambda_\mu$, and
3. there is a surjection from $[\omega]^\omega$ onto $[\kappa]^\kappa$.

Then $\kappa \to (\kappa)^\lambda_\mu$ holds in the extension via $([\omega]^\omega, \subseteq^*)$.

EP and LU

A subset $A \subseteq [\omega]^\omega$ is invariant if $(p \in A$ and $p' =^* p) \longrightarrow p' \in A$.

For $a \in [\omega]^{<\omega}$ and $p \in [\omega]^\omega$, let

$$[a, p] = \{q \in [\omega]^\omega : a \sqsubset q \wedge q \subseteq p\}$$

$X \subseteq [\omega]^\omega$ is Completey Ramsey (CR) if $\forall \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that

$$\text{(a) } [a, q] \subseteq X \quad \text{or} \quad \text{(b) } [a, q] \cap X = \emptyset.$$

$X \subseteq [\omega]^\omega$ is CR^+ if $\forall\, \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that (a) holds;
X is CR^- if $\forall\, \emptyset \neq [a, x]\ \exists q \in [a, x]$ such that (b) holds.

EP: The intersection of any well-ordered collection of CR^+ sets is CR^+.

LU: For any relation $R \subseteq [\omega]^\omega \times \mathcal{P}(\omega)$ such that $\forall p\ \exists y\ R(p, y)$, the set $\{x : R$ is uniformized on $[x]^\omega\}$ is CR^+.

Preserving Strong Partition Cardinals over $L(\mathbb{R})$

Thm. (Henle-Mathias-Woodin) (AD + $V = L(\mathbb{R})$)
If $0 < \lambda = \omega \cdot \lambda \leq \kappa$, $2 \leq \mu < \kappa$, and $\kappa \to (\kappa)^\lambda_\mu$, then

$$L(\mathbb{R})[\mathcal{U}] \models \kappa \to (\kappa)^\lambda_\mu,$$

where $\mathcal{U}$ is the Ramsey ultrafilter forced by $([\omega]^\omega, \subseteq^*)$ over $L(\mathbb{R})$.

Remark. AD + $V = L(\mathbb{R})$ imply LU, EP, and (3) in the previous rendition of this theorem.

Extension to Topological Ramsey Spaces

Topological Ramsey spaces are triples $(\mathcal{R}, \leq, (r_n)_{n<\omega})$, where $\leq$ is a partial order and r is a finite approximation map; basic open sets are of the form

$$[a, p] = \{q \in \mathcal{R} : \exists n < \omega \, (a = r_n(p)) \text{ and } q \leq p\}.$$

A subset $X \subseteq \mathcal{R}$ is (Completely) Ramsey if for each $\emptyset \neq [a, p]$ there is some $q \in [a, p]$ such that

$$\text{(a) } [a, q] \subseteq X \quad \text{ or else } \quad \text{(b) } [a, q] \cap X = \emptyset.$$

The defining property of a topological Ramsey space is that all subsets with the property of Baire are Ramsey.

The Ellentuck space $\mathcal{E} = ([\omega]^\omega, \subseteq, (r_n)_{n<\omega})$ has approximation maps $r_n(x) = \{x_i : i < n\}$, where $\{x_i : i < \omega\}$ is the enumeration of $x \in [\omega]^\omega$.

Abstractions of EP and LU

The structure of topological Ramsey spaces, as roughly ω-sequences of finite structures, often produces many of the same properties as the forcing $([\omega]^\omega, \subseteq^*)$.

$X \subseteq \mathcal{R}$ is invariant R^+ if

1. invariant: $(p \in X \text{ and } p' =^* p) \longrightarrow p' \in X$, and
2. R^+: $\forall p \in \mathcal{R}\ \exists q \leq p$ such that $[q] \subseteq X$.

Let $\mathbb{P} = \langle \mathcal{R}, \leq, \leq^* \rangle$.

$\mathrm{EP}(\mathbb{P})$: Given any well-ordered sequence $\langle C_\alpha \subseteq P : \alpha < \kappa \rangle$ of invariant R^+ sets, the intersection of the sequence is again invariant R^+.

Abstractions of CR^+, CR^-, $\omega \to (\omega)^\omega$, EP, and LU

$\mathrm{LU}^*(\mathbb{P})$: Uniformization relative to some invariant cube $[p]^*$ for relations $R \subseteq \mathcal{R} \times {}^\omega 2$.

$\mathrm{LCU}(\mathbb{P})$: Continuous uniformization for relations $R \subseteq \mathcal{R} \times {}^\omega 2$ relative to some cube $[p]$.

Similar to Todorcevic's Ramsey Uniformization Theorem for relations on $[\omega]^\omega \times X$ where X is a Polish space.

Prop. (D.-Hathaway) Assume either $\mathrm{AD}_\mathbb{R}$ or $\mathrm{AD}^+ + V = L(\mathbb{R}(\mathcal{R}))$. Let $\langle \mathcal{R}, \leq, r \rangle$ be a topological Ramsey space. Then every subset of $\mathcal{R}$ is Ramsey. Hence, also $\mathrm{LCU}(\mathcal{R}, \leq)$ holds.

Preserving Strong Partition Cardinals - general theorem

Thm. (D.-Hathaway) Suppose $\mathbb{P} = \langle X, \leq, \leq^* \rangle$ is a coarsened poset such that EP($\mathbb{P}$) and LU($\mathbb{P}$) hold, and each $=^*$-equivalence class is countable. Assume that every subset of X is Ramsey and

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \to (\kappa)^\lambda_\mu$,
3. there is a surjection from ${}^\omega 2$ onto $[\kappa]^\kappa$.

Then $\langle X, \leq \rangle$ forces $\kappa \to (\kappa)^\lambda_\mu$.

Preserving Strong Partition Cardinals - simple version

Thm. (D.-Hathaway) Assume either $\mathrm{AD}_{\mathbb{R}}$ or $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\mathbb{P} = \langle \mathcal{R}, \leq, \leq^*, r \rangle$ be a coarsened topological Ramsey space, where the $=^*$-equivalence classes are countable. Then forcing with $\langle \mathcal{R}, \leq \rangle$ preserves $\kappa \rightarrow (\kappa)^\lambda_\mu$ whenever

1. $0 < \lambda = \omega \cdot \lambda \leq \kappa$ and $2 \leq \mu < \kappa$,
2. $\kappa \rightarrow (\kappa)^\lambda_\mu$.

Remark. The ultrafilters mentioned previously all preserve strong partition cardinals, except possibly those forced by $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$.

A key step in our results is the following:

Lemma. (D-H) Assume either 1) $\mathrm{AD}_{\mathbb{R}}$ or 2) $\mathrm{AD}^{+} + V = L(\mathcal{P}(\mathbb{R}))$. Let $\langle \mathcal{R}, \leq, r \rangle$ be a topological Ramsey space. Then every subset of $\mathcal{R}$ is Ramsey.

The proof uses that the Mathias-like forcing for a topological Ramsey space has the Prikry and Mathias properties, which was proved by Di Prisco, Mijares and Nieto in 2017.

Also in that paper, DMN proved that ultrafilters forced by Ramsey spaces have complete combinatorics, extending Todorcevic's result that every Ramsey ultrafilter is generic for $([\omega]^{\omega}, \subseteq^{*})$ over $L(\mathbb{R})$ in the presence of large cardinals.

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