Logic and Combinatorics

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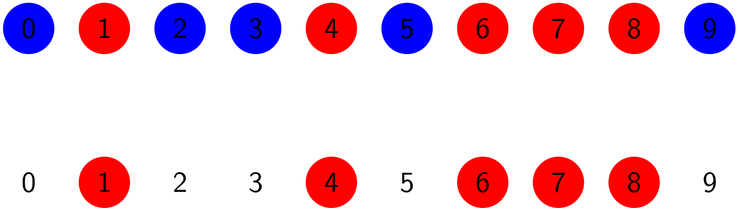
Pigeonhole Principle

If $n > m$ and there are n pigeons and m holes, and each pigeon is put in a hole, then at least one of the holes must contain at least two pigeons.

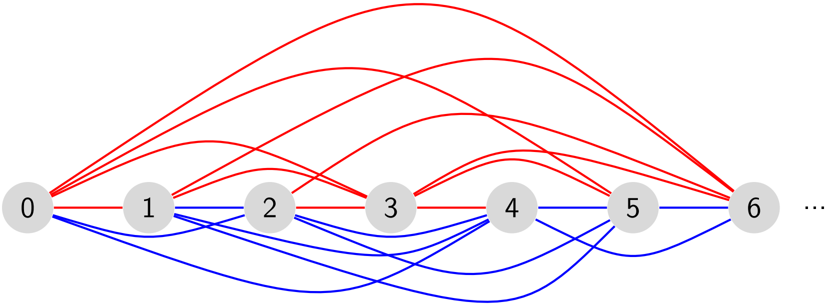
Infinite Pigeonhole Principle

Given a coloring of all the natural numbers $\mathbb{N}$ into red and blue, there is an infinite subset of the natural numbers all of the same color.



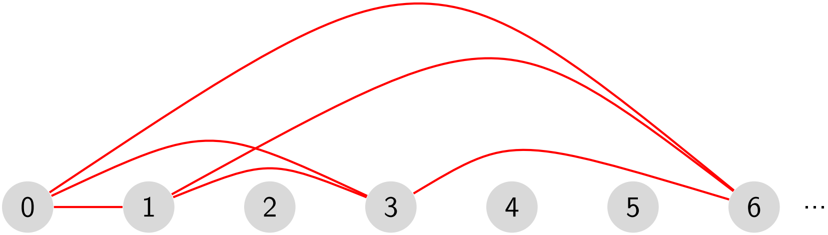
Ramsey's Theorem for Pairs

Given a 2-coloring of the edges of a complete graph on ω vertices,



Ramsey's Theorem for Pairs

There is an infinite complete subgraph such that all edges have the same color.



General Ramsey's Theorem

Ramsey's Theorem. Given any $k, l \geq 1$ and a coloring on the collection of all k-element subsets of $\mathbb{N}$ into l colors, there is an infinite set M of natural numbers such that each k-element subset of M has the same color.

Finite Ramsey's Theorem. Given any $k, l, n \geq 1$, there is a number r such that for any set X of size r and any coloring of the k-element subsets of X into l colors, there is a subset $Y \subseteq X$ of size n such that all k-element subsets of Y have the same color.

Ramsey's Theorem and Logic

Ramsey's Theorem appeared in his paper, *On a problem of formal logic*, which was motivated by Hilbert's Entscheidungsproblem:

Find a procedure for determining whether any given (first-order logic) formula is valid.

Ramsey applied his theorem to solve this problem for formulas with only universal quantifiers in front (Π_1).

It is remarkable that Ramsey could solve this instance, since in general, the Entsheidungsproblem is unsolvable (Church and Turing).

Ramsey's Theorem has been extended to

- graphs
- trees
- other structures
- sets and structures of uncountable size

We will present some highlights of connections between logic and Ramsey theory over the last 90 years.

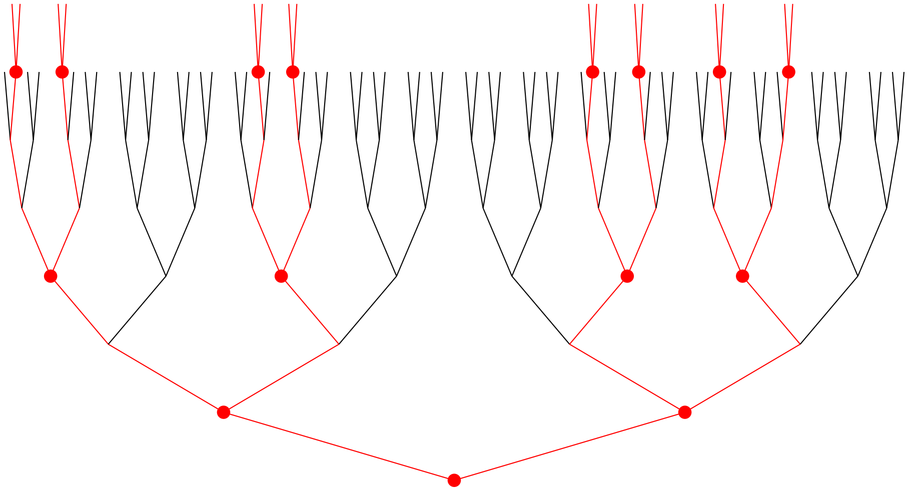
Axiom of Choice vs Boolean Prime Ideal Theorem

A major question in the early to mid-20th Century was the strength of the Axiom of Choice versus the Boolean Prime Ideal Theorem.

Halpern and Lévy proved that AC is stronger than BPI over ZF. (1967)

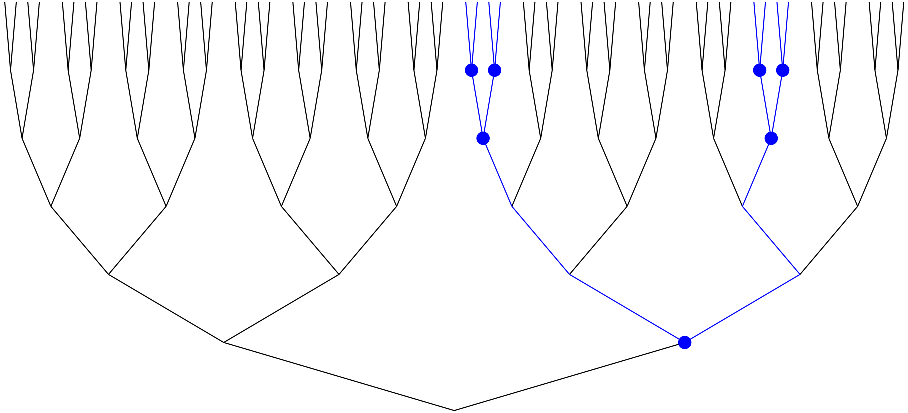
Their proof uses a Ramsey theorem on trees due to Halpern and Läuchli (1966).

Example: A Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $0, 1, 3, 6, \ldots$

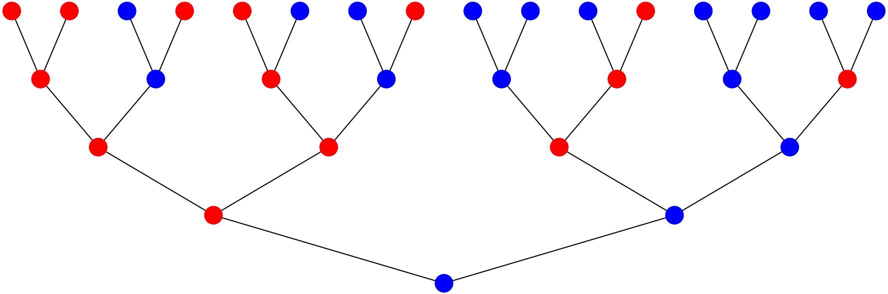
Example: A Strong Subtree $U \subseteq 2^{<\omega}$



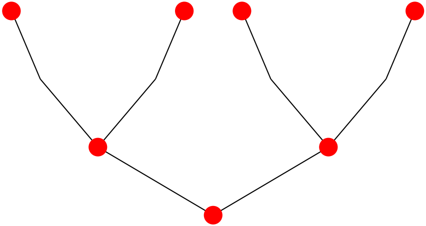
The nodes in U are of lengths $1, 4, 5, \ldots$.

Halpern-Läuchli Theorem for one tree

Let $T \subseteq 2^{<\omega}$ be an infinite strong tree and suppose the nodes in T are colored red and blue. Then there is an infinite strong subtree $S \subseteq T$ in which all the nodes have the same color.



A Monochromatic Strong Subtree Isomorphic to $2^{\leq 2}$



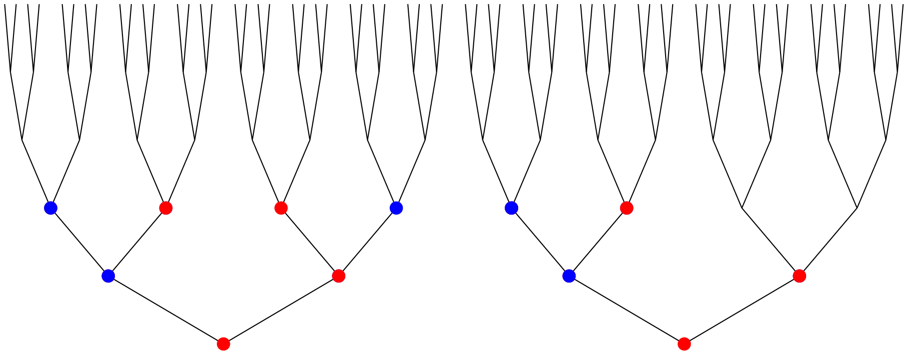
Halpern-Läuchli Theorem - strong tree version

Notation: $$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n)$$

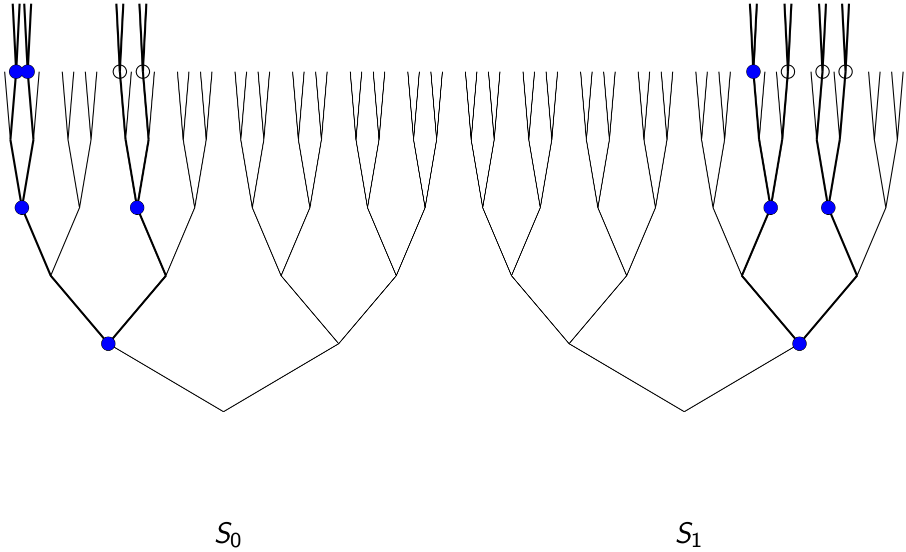
Theorem. (Halpern-Läuchli, 1966) Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes and let $r \geq 2$. Given a coloring $c : \bigotimes_{i<d} T_i \to r$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that c is constant on $\bigotimes_{i<d} S_i$.

We now give some examples of colorings of level products of two trees $T_0 = T_1 = 2^{<\omega}$, and show visually what the Halpern-Läuchli Theorem does.

Coloring Products of Level Sets: $T_0(0) \times T_1(0)$



HL gives Strong Subtrees with 1 color for level products



Application to Products of Rationals

Thm. (Laver, 1984) Given $d < \omega$ and a coloring of $\mathbb{Q}^d$ into finitely many colors, there are $X_i \subseteq \mathbb{Q}$, $i < d$, isomorphic to $\mathbb{Q}$ such that $X_0 \times \cdots \times X_{d-1}$ takes at most $d!$ many colors.

Proof of Halpern-Läuchli uses Logic

The original proof uses symbolic logic.

A proof due to Harrington uses the set-theoretic technique of forcing.

This method of proof has proved to be a good starting point for many extensions of this theorem.

A Ramsey Theorem for Strong Trees

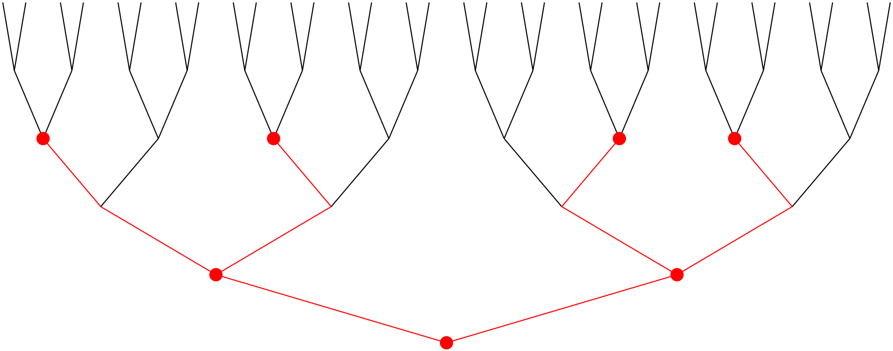
The next theorem is proved by induction using repeated applications of the Halpern-Läuchli Theorem.

Thm. (Milliken 1979) Let T be a finitely branching tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

We give examples for $T = 2^{<\omega}$ and $k = 3$.

Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

Given a coloring c of all 3-strong trees in $2^{<\omega}$ into red and blue:



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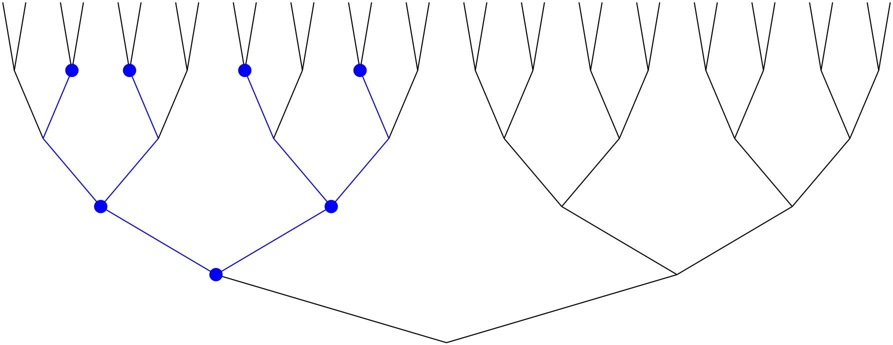

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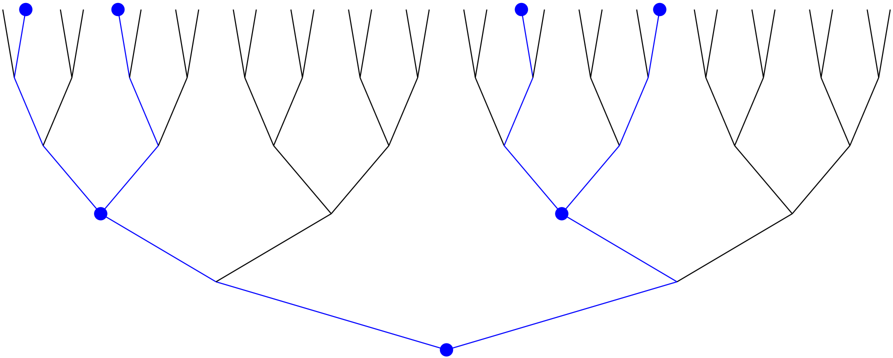
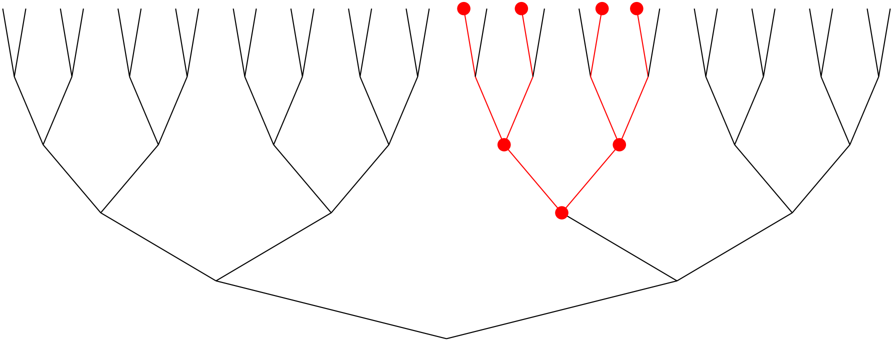


Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$



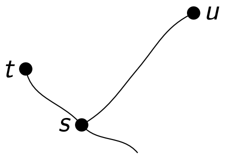
Milliken's Theorem guarantees a strong subtree in which all 3-strong subtrees have the same color.

Do analogues of the Infinite Ramsey Theorem hold for infinite structures?

The Rationals as a Linearly Ordered Structure

For $x, y \in 2^{<\omega}$, define $x \lhd y$ iff one of the following holds:

1. $x <_{\text{lex}} y$,
2. $x \sqsubset y$ and $y(|x|) = 1$, or
3. $y \sqsubset x$ and $x(|y|) = 0$.



In this picture, $t \lhd s \lhd u$.

Note: $(2^{<\omega}, \lhd) \cong (\mathbb{Q}, <)$.

Sierpiński's result viewed in trees

Thm. (Sierpiński) There is a coloring $c : [\mathbb{Q}]^2 \to 1$ such that for each subset $\mathbb{Q}' \subseteq \mathbb{Q}$ which forms a dense linear order without endpoints, both colors occur on $[\mathbb{Q}']^2$.

Given a pair of nodes s, t in $2^{<\omega}$ with $|s| < |t|$, let

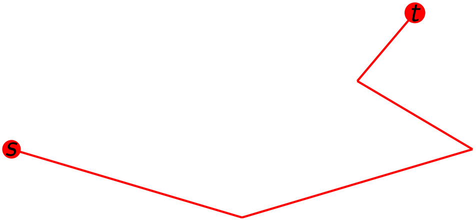
$$c(\{s,t\}) = \begin{cases} 0 & \text{if } s \lhd t \\ 1 & \text{if } t \lhd s \end{cases}$$

Given any subset $S \subseteq 2^{<\omega}$ for which $(S, \lhd) \cong (\mathbb{Q}, <)$, both colors will persist in S.

Thm. (Galvin) Given any coloring of pairs of rationals into finitely many colors, there is a subset which is again a dense linear order in which at most two colors are used.

Given $s, t \in 2^{<\omega}$ with $|s| < |t|$, a strong tree envelope is a 3-strong tree which contains s and t and has nodes of lengths $|s \wedge t|, |s|, |t|$.

Example 1: $|s| < |t|$ and $s \lhd t$



A strong tree envelope of s and t

Thm. (Galvin) Given any coloring of pairs of rationals into finitely many colors, there is a subset which is again a dense linear order in which at most two colors are used.

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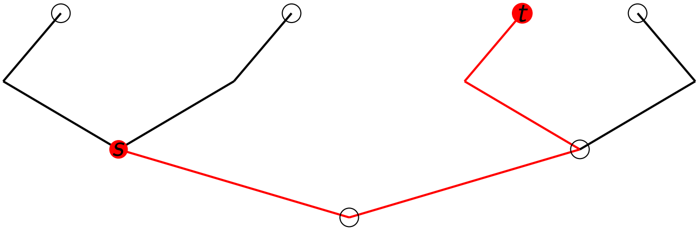
Example 1: $|s| < |t|$ and $s \triangleleft t$

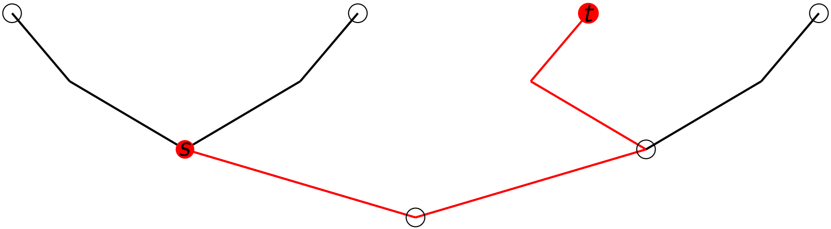


Another strong tree envelope of s and t

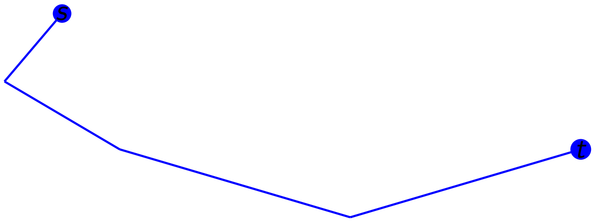
Thm. (Galvin) Given any coloring of pairs of rationals into finitely many colors, there is a subset which is again a dense linear order in which at most two colors are used.

Given $s, t \in 2^{<\omega}$ with $|s| < |t|$, a strong tree envelope is a 3-strong tree which contains s and t and has nodes of lengths $|s \wedge t|, |s|, |t|$.

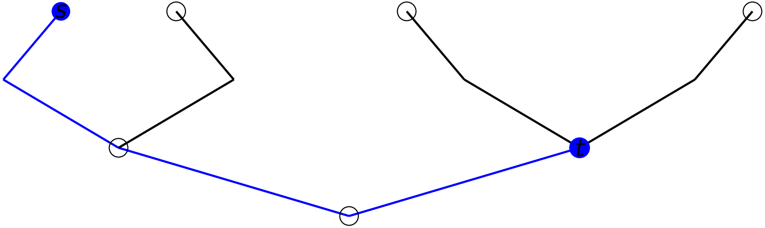
Example 1: $|s| < |t|$ and $s \lhd t$



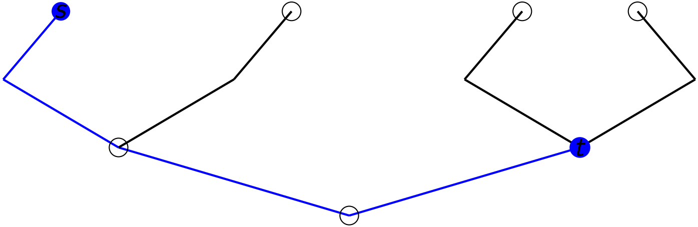
Example 2: $|s| < |t|$ with $t \lhd s$



Example 2: A strong tree envelope



Example 2: Another strong tree envelope



Coloring Pairs of Rationals

1. Let c be a coloring of $[\mathbb{Q}]^2$ into finitely many colors.
2. Transfer the coloring to pairs of nodes in $2^{<\omega}$. There are two strong similarity types for pairs.
3. Fix one strong similarity type. For each pair of nodes s, t of that type, color all 3-strong trees containing s and t with the color $c(\{s, t\})$.
4. Apply Milliken's Theorem to 3-strong trees. Get one color for all pairs with that similarity type.
5. Repeat for the second strong similarity type.
6. Take a strongly diagonal antichain $\mathbb{A} \subseteq 2^{<\omega}$ such that $(\mathbb{A}, \triangleleft) \cong (\mathbb{Q}, <)$.

$(\mathbb{Q}, <)$ has an approximate Infinite Ramsey Theorem

Thm. (Laver (bounds, unpublished), Devlin (exact bounds) 1979)
Given $k \geq 2$, there is a number $T(k, \mathbb{Q})$ such that for each coloring of the k-element subsets of $\mathbb{Q}$ into finitely many colors, there is a copy Q of $\mathbb{Q}$ in which no more than $T(k, \mathbb{Q})$ colors occur.
These are actually tangent numbers.

So $(\mathbb{Q}, <)$ does not have the exact analogue of Ramsey's Theorem for $\mathbb{N}$.

But this structure still behaves quite nicely in that finite bounds exist.
These bounds $T(k, \mathbb{Q})$ are called the big Ramsey degrees of k in $\mathbb{Q}$.

What about Ramsey theory on other structures?

Graphs and Ordered Graphs

Graphs are sets of vertices with edges between some of the pairs of vertices.

An ordered graph is a graph whose vertices are linearly ordered.

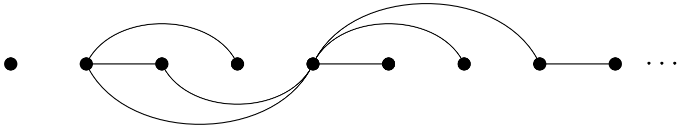


Figure: An ordered graph B

Embeddings of Graphs

An ordered graph A embeds into an ordered graph B if there is a one-to-one mapping of the vertices of A into some of the vertices of B such that each edge in A gets mapped to an edge in B, and each non-edge in A gets mapped to a non-edge in B.

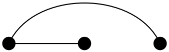


Figure: A

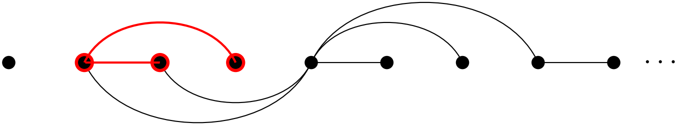
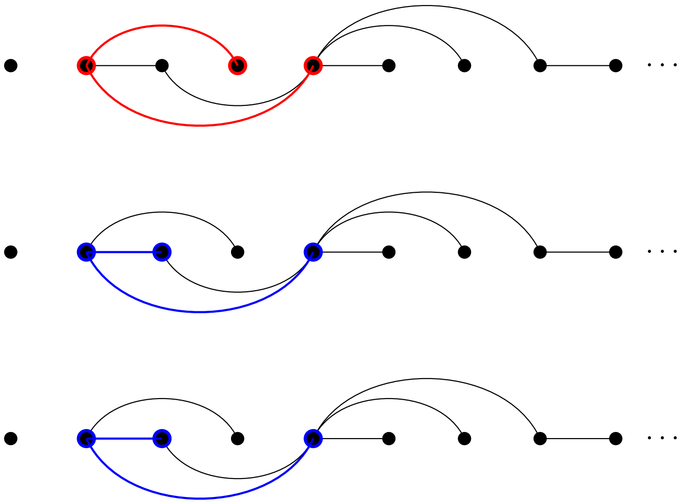
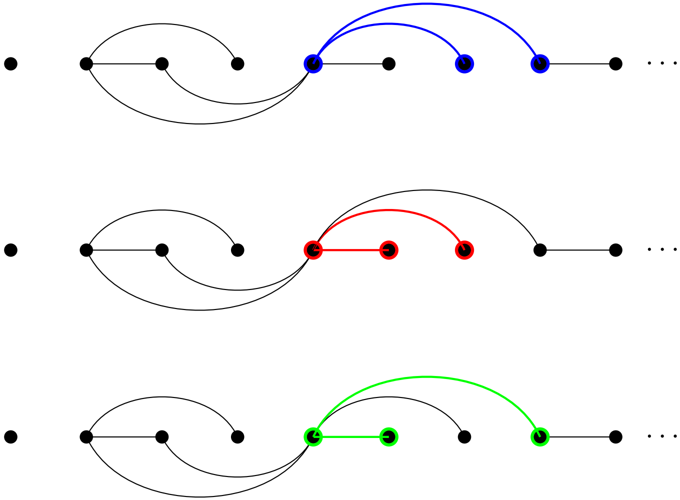


Figure: A copy of A in B

More copies of $\mathbb{A}$ into $\mathbb{B}$



Still more copies of A into B



Different Types of Colorings on Graphs

Let G be a given graph.

Vertex Colorings: The vertices in G are colored.

Edge Colorings: The edges in G are colored.

Colorings of *n*-cycles: All *n*-cycles in G are colored.

Colorings of A: Given a finite graph A, all copies of A which occur in G are colored.

Ramsey Theorem for Finite Ordered Graphs

Thm. (Nešetřil/Rödl) For any finite ordered graphs A and B such that $\mathrm{A} \leq \mathrm{B}$, there is a finite ordered graph C such that for each coloring of all the copies of A in C into red and blue, there is a $\mathrm{B}' \leq \mathrm{C}$ which is a copy of B such that all copies of A in B' have the same color.

In symbols, given any $f : \binom{\mathrm{C}}{\mathrm{A}} \to 2$, there is a $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that f takes only one color on all members of $\binom{\mathrm{B}'}{\mathrm{A}}$.

The Rado Graph $\mathcal{R} = (R, E)$

The Rado graph is the random graph on countably many vertices.

The Rado graph is indivisible: Given any partition of the vertices into finitely many pieces, one piece contains a copy of $\mathcal{R}$.

Thm. (Sauer 2006, Laflamme-Sauer-Vuksanovic 2006)
Every finite graph has a finite big Ramsey degree. Given any finite graph A, there is a number $T(\mathrm{A})$ such that for any coloring of all copies of A in $\mathcal{R}$ into finitely many colors, there is a subcopy of $\mathcal{R}$ in which the copies of A take no more than $T(\mathrm{A})$ colors.

Actual degrees were found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

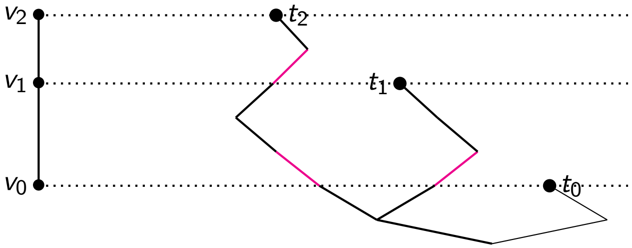
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



The Big Ramsey Degrees for the Random Graph

The finite Ramsey degree for a given finite graph A is the number of different types of diagonal trees coding A.

The proof uses Milliken's Theorem.

Big Ramsey Degrees of Infinite Structures

Let $\mathcal{S}$ be an infinite structure. For a finite substructure $A \leq \mathcal{S}$, let $T(A, \mathcal{S})$ denote the least number, if it exists, such that for each coloring of the copies of A in $\mathcal{S}$ into finitely many colors, there is a substructure $\mathcal{S}'$ isomorphic to $\mathcal{S}$ in which the copies of A take no more than $T(A, \mathcal{S})$ colors.

(Kechris, Pestov, Todorcevic, 2005) $\mathcal{S}$ has finite big Ramsey degrees if for each finite $A \leq \mathcal{S}$, $T(A, \mathcal{S})$ exists.

Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S}(\mathbf{2})$, $\mathbf{S}(\mathbf{3})$. (Laflamme, NVT, Sauer 2010)
- The random k-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)

Ramsey Theory and Topological Dynamics

(Kechris, Pestov, Todorcevic 2005) The KPT Correspondence:
A Fraïssé class $\mathcal{K}$ has the Ramsey property iff $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ is extremely amenable.

(Zucker 2019) Characterized universal completion flows of $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ whenever $\mathrm{Flim}(\mathcal{K})$ admits a big Ramsey structure (big Ramsey degrees with a coherence property).

A class $\mathcal{K}$ of finite structures is a Fraïssé class if it is hereditary, has the Joint Embedding Property, and the Amalgamation Property.

$\mathrm{Flim}(\mathcal{K})$ is a homogeneous countable structure into which each member of $\mathcal{K}$ embeds.

k-Clique-Free Random Graphs = Henson Graphs

For $k \geq 3$, a k-clique, denoted K_k, is a complete graph on k vertices.

$\mathcal{H}_k$, the k-clique-free Henson graph, is the homogenous K_k-free graph which is universal for all k-clique-free graphs on countably many vertices.

Henson graphs are the k-clique-free analogues of the Rado graph.
They were constructed by Henson in 1971.

Henson Graphs: History of Results

- For each $k \geq 3$, $\mathcal{H}_k$ is weakly indivisible (Henson, 1971).

- The Fraïssé class of finite ordered K_k-free graphs has the Ramsey property. (Nešetřil-Rödl, 1977/83)

- $\mathcal{H}_3$ is indivisible. (Komjáth-Rödl, 1986)

- For all $k \geq 4$, $\mathcal{H}_k$ is indivisible. (El-Zahar-Sauer, 1989)

- Edges have big Ramsey degree 2 in $\mathcal{H}_3$. (Sauer, 1998)

There progress halted. Why?

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Ramsey Theory for Henson Graphs

Theorem. (D.) Let $k \geq 3$. For each finite k-clique-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{G}_k)$ such that for any coloring of all copies of A in $\mathcal{H}_k$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_k$, with $\mathcal{H} \cong \mathcal{H}_k$, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{G}_k)$ colors.

Structure of Proof

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for strictly similar finite antichains.

This is an analogue of Milliken's Theorem for strong trees - the proof uses forcing for a ZFC result. It also requires a new notion of envelope.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Similar to the end of Sauer's proof.

Strong K_3-Free Tree

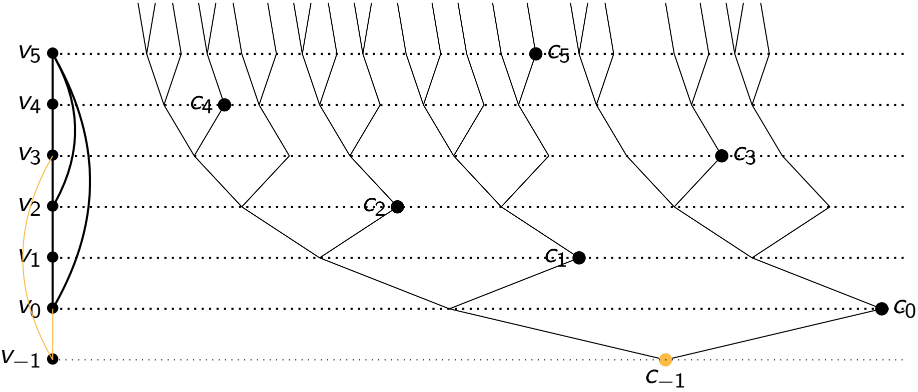


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

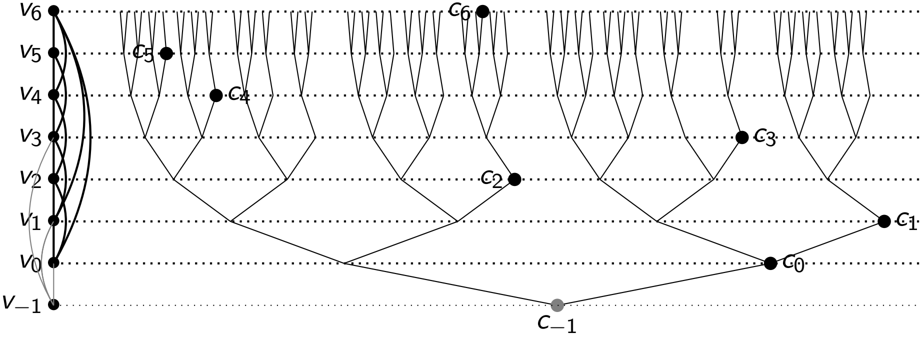


Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

Future Directions

The method of forcing was used to conduct a bounded search in ZFC to construct subtrees with one color per strong similarity type coding a finite k-clique tree graph.

Using forcing aided us in solving the problem of big Ramsey degrees for Henson graphs.

These methods look promising for the further development of Ramsey theory on homogeneous structures.

References

Dobrinen, *The Ramsey theory of the universal homogeneous triangle-free graph*, JML (202*) (To Appear).

Dobrinen, *The Ramsey theory of the Henson graphs* (2019) (Preprint).

Dobrinen, *Borel of Rado graphs and Ramsey's theorem* (2019) (Submitted).

Ellentuck, *A new proof that analytic sets are Ramsey*, JSL (1974).

Erdős-Rado, *A partition calculus in set theory*, Bull. AMS (1956).

Galvin-Prikry, *Borel sets and Ramsey's Theorem*, JSL (1973).

Halpern-Läuchli, *A partition theorem*, TAMS (1966).

Halpern-Lévy, *The Boolean prime ideal theorem does not imply the axiom of choice*, Axiomatic Set Theory (1967).

Henson, *A family of countable homogeneous graphs*, Pacific Jour. Math. (1971).

References

Laflamme-Sauer-Vuksanovic, *Canonical partitions of universal structures*, Combinatorica (2006).

Larson, J. *Counting canonical partitions in the Random graph*, Combinatorica (2008).

Larson, J. *Infinite combinatorics*, Handbook of the History of Logic (2012).

Laver, *Products of infinitely many perfect trees*, Jour. London Math. Soc. (1984).

Kechris-Pestov-Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis (2005).

Milliken, *A Ramsey theorem for trees*, Jour. Combinatorial Th., Ser. A (1979).

Nešetřil-Rödl, *Partitions of finite relational and set systems*, Jour. Combinatorial Th., Ser. A (1977).

References

Nešetřil-Rödl, *Ramsey classes of set systems*, Jour. Combinatorial Th., Ser. A (1983).

Nguyen Van Thé, *Big Ramsey degrees and divisibility in classes of ultrametric spaces*, Canadian Math. Bull. (2008).

Pouzet-Sauer, *Edge partitions of the Rado graph*, Combinatorica (1996).

Sauer, *Edge partitions of the countable triangle free homogeneous graph*, Discrete Math. (1998).

Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica (2006).

Todorcevic, *Introduction to Ramsey spaces* (2010).

Zucker, *Big Ramsey degrees and topological dynamics*, Groups Geom. Dyn. (2019).