Exact big Ramsey degrees via coding trees

Natasha Dobrinen

University of Denver

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This talk presents results from two papers:

[1] *Exact big Ramsey degrees via coding trees*,
by Balko, Chodounský, Dobrinen, Hubička, Vena, and Zucker

[2] *Fraïssé classes with simply characterized big Ramsey structures*,
by Coulson, Dobrinen, and Patel

In [1], exact big Ramsey degrees are characterized for Fraïssé limits of all free amalgamation classes with relations of arity at most two.

In [2], a strengthening of strong amalgamation is shown to produce exact big Ramsey degrees which have a simple characterization, for relations of arity at most two. For relations of any arity, this property also implies indivisibility.

Both papers use coding trees to obtain their results.

Infinite Ramsey's Theorem

Ramsey's Theorem. Given any $k, \ell \geq 1$ and a coloring $c : [\omega]^k \to \ell$, there is an infinite set $M \in [\omega]^\omega$ that $c \restriction [M]^k$ is monochromatic.

What about Ramsey theory on infinite structures?

Big Ramsey Degrees

Def. [KPT] Given an infinite structure $\mathbf{S}$ and a finite substructure $\mathbf{A} \le \mathbf{S}$, let $T(\mathbf{A}, \mathbf{S})$ denote the least T (if it exists) such that for any integer $\ell \ge 1$, given any coloring of the copies of $\mathbf{A}$ in $\mathbf{S}$ into ℓ colors, there is a substructure $\mathbf{S}'$ of $\mathbf{S}$, isomorphic to $\mathbf{S}$, such that all copies of $\mathbf{A}$ in $\mathbf{S}'$ take no more than T colors.

Big Ramsey Degrees

Def. [KPT] Given an infinite structure $\mathbf{S}$ and a finite substructure $\mathbf{A} \leq \mathbf{S}$, let $T(\mathbf{A}, \mathbf{S})$ denote the least T (if it exists) such that for any integer $\ell \geq 1$, given any coloring of the copies of $\mathbf{A}$ in $\mathbf{S}$ into ℓ colors, there is a substructure $\mathbf{S}'$ of $\mathbf{S}$, isomorphic to $\mathbf{S}$, such that all copies of $\mathbf{A}$ in $\mathbf{S}'$ take no more than T colors.

$$\forall \ell \geq 1, \quad \mathbf{S} \rightarrow (\mathbf{S})^{\mathbf{A}}_{\ell, T(\mathbf{A},\mathbf{S})}$$

When it exists, $T(\mathbf{A}, \mathbf{S})$ is called the big Ramsey degree of $\mathbf{A}$ in $\mathbf{S}$.

$\mathbf{S}$ has finite big Ramsey degrees if $T(\mathbf{A}, \mathbf{S})$ exists for each finite substructure $\mathbf{A} \leq \mathbf{S}$. Exact big Ramsey degrees means there is a characterization from which the degrees can be computed.

Structures with finite big Ramsey degrees: Brief History

- Infinite complete k-hypergraph: All BRD $= 1$. (Ramsey 1929)
- $T(2, \mathbb{Q}) \geq 2$ (Sierpiński 1933). $T(2, \mathbb{Q}) = 2$ (Galvin unpub.)
- $(\mathbb{Q}, <)$: BRD computed. (Devlin 1979) (upper bounds by Laver unpub.)
- K_3-free generic graph $\mathbf{G}_3$: $T(1, \mathbf{G}_3) = 1$. (Komjáth, Rödl 1986)
- K_n-free generic graph $\mathbf{G}_n$, $n \geq 3$: $T(1, \mathbf{G}_n) = 1$. (El-Zahar, Sauer 1989)
- Rado graph, etc.: Exact BRD. (Laflamme, Sauer, Vuksanović 2006). BRD computed. (J. Larson 2008)
- Countable ultrametric Urysohn space: BRD computed. (Nguyen VanThé 2008)
- $\mathbb{Q}_n$ and the dense local order $\mathbf{S}(\mathbf{2})$. BRD computed. (Laflamme, Nguyen Van Thé, Sauer 2010)

Structures with finite big Ramsey degrees

- K_n-free generic graphs $\mathbf{G}_n$ $n \geq 3$: Finite BRD
  (Dobrinen 2020 and 2019*)
  developed method of coding trees and related forcings

- Generic 3-regular hypergraph: Finite BRD
  (Balko, Chodounský, Hubička, Konečný, Vena 2019)
  used product Milliken theorem

- Universal structures, some metric spaces: Finite BRD (Mašulović 2020)
  used category theory

- $\mathbf{S}(n)$ for all $n \geq 2$: BRD computed. (Barbosa 2020*)
  used category theory

- FAP classes with relations of artity ≤ 2: Finite BRD (Zucker 2020*)
  used coding trees and forcing, developed abstract approach

$*$ means arvix date for papers not yet published

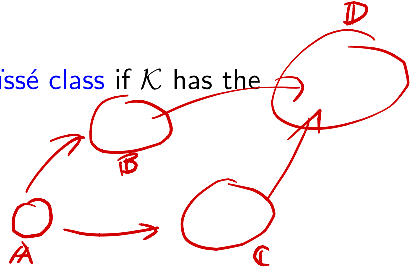
Structures with finite big Ramsey degrees

- Partial order, metric spaces, etc.: Finite BRD. (Hubička 2020*)
  used parameter words, first forcing-free proof for $\mathbf{G}_3$
- Fraïssé structures with SDAP$^+$: Exact BRD.
  (Coulson, Dobrinen, Patel 2020*)
  develop coding trees of 1-types, first envelope-free proof
- FAP classes with relations of artity ≤ 2: Exact BRD.
  (Balko, Chodounský, Dobrinen, Hubička, Konečny, Vena, Zucker 2021*)
  used upper bounds from (Zucker 2020*)
- Generic partial order: Exact BRD.
  (Balko, Chodounský, Dobrinen, Hubička, Konečny, Vena, Zucker 2021*)
  used upper bounds from (Hubička 2020*)
- Big Ramsey degrees and forbidden cycles: Finite BRD.
  (Balko, Chodounský, Hubička, Konečny, Nešetřil, and Vena, 2021*)
  used upper bounds from (Hubička 2020*)

Fraïssé Classes

A class $\mathcal{K}$ of finite structures is a Fraïssé class if $\mathcal{K}$ has the

- hereditary property
- joint embedding property
- amalgamation property

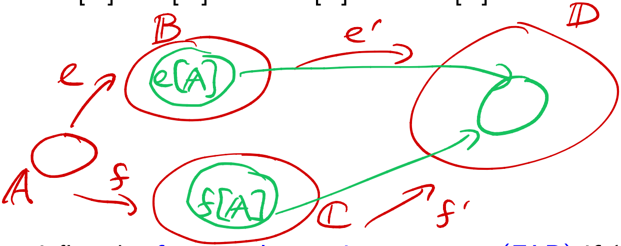


A structure **K** is ultrahomogeneous if each isomorphism between two finite substructures of **K** is extendable to an automorphism of **K**.

Given a Fraïssé class $\mathcal{K}$, there is a unique (up to isomorphism) countably infinite structure $\mathbf{K} := \mathrm{Flim}(\mathcal{K})$ which is (ultra)homogeneous and universal for $\mathcal{K}$. We call **K** a Fraïssé structure.

SAP and FAP

A Fraïssé class $\mathcal{K}$ satisfies the strong amalgamation property (SAP) if given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $e : \mathbf{A} \to \mathbf{B}$ and $f : \mathbf{A} \to \mathbf{C}$, there is some $\mathbf{D} \in \mathcal{K}$ and embeddings $e' : \mathbf{B} \to \mathbf{D}$ and $f' : \mathbf{C} \to \mathbf{D}$ such that $e' \circ e = f' \circ f$, and $e'[B] \cap f'[C] = e' \circ e[A] = f' \circ f[A]$.



We say that $\mathcal{K}$ satisfies the free amalgamation property (FAP) if $\mathcal{K}$ satisfies the SAP and moreover, $\mathbf{D}$ can be chosen so that $\mathbf{D}$ has no additional relations other than those inherited from $\mathbf{B}$ and $\mathbf{C}$.

Exact big Ramsey degrees for FAP classes, with relations of arity ≤ 2

Thm. (Balko, Chodounský, Dobrinen, Hubička, Vena, and Zucker)
Let $\mathcal{K}$ be a relational Fraïssé class $\mathcal{K}$ with FAP and relations of arity at most two, and let $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$. We characterize the exact big Ramsey degrees of $\mathbf{K}$.

This theorem has as its starting point the upper bounds proved by Zucker in 2020. His proof of upper bounds uses coding trees and forcing.

Coding trees on $k^{\mathrm{u}} \times k^{<\omega}$

Given a relational language $\mathcal{L} = \{U_i : i < k^{\mathrm{u}}\} \cup \{R_i : i < k\}$, where U_i is unary and R_i is binary. By adding symbols if necessary, we can assume

1. $\forall a \in \mathbf{A},\ \exists!\ i < k^{\mathrm{u}}$ s.t. $U_i^{\mathbf{A}}(a)$ holds.
2. $\forall a \in \mathbf{A},\ \forall i < k,\ \neg R_i^{\mathbf{A}}(a, a)$.
3. $\forall a \neq b \in \mathbf{A},\ \exists!\ i < k$ s.t. $R_i^{\mathbf{A}}(a, b)$.
4. $\exists$ an order 2 bijection Flip: $k \to k$ s.t. $R_i^{\mathbf{A}}(a, b) \leftrightarrow R_{\mathrm{Flip}(i)}^{\mathbf{A}}(b, a)$.

Let $\mathbf{K}$ be an enumerated Fraïssé structure in language $\mathcal{L}$.

unary relations.

The coding tree CT for $\mathbf{K}$ is $T = k^{\mathrm{u}} \times k^{<\omega}$ along with a coding function $c^{\mathbf{K}} : \mathbf{K} \to T$, where the coding nodes represent the vertices of $\mathbf{K}$, and relations are represented via the index of the binary relation.

binary

Example: Coding tree for $\mathcal{H}_3$ $\mathbb{G}_3$

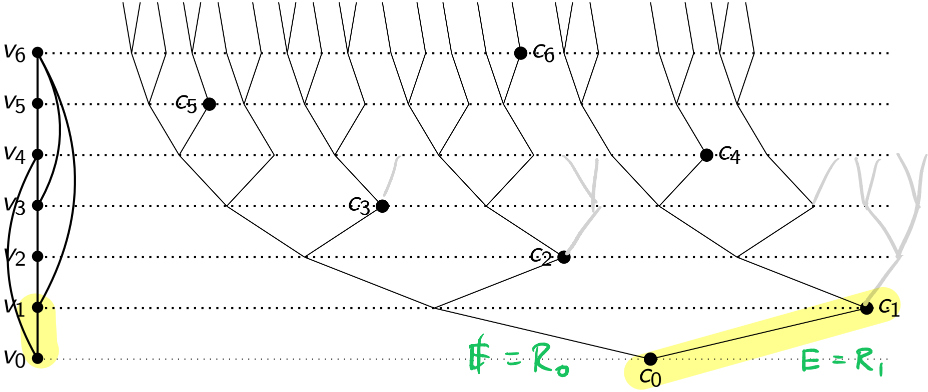
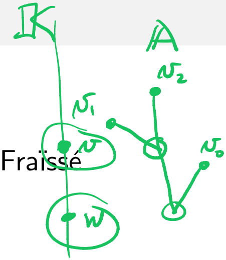


Figure: Coding tree for homogeneous triangle-free graph $\mathbf{G}_3$

Main Ideas of the Characterization



Diagonal diaries characterise big Ramsey degrees for FAP Fraïssé structures.

Given $\mathbf{A} \le \mathbf{K}$, a diagonal diary for $\mathbf{A}$ involves an antichain of coding nodes $A \subseteq CT$ representing $\mathbf{A}$ such that

1. the meet-closure of A, $A^\wedge$, has splitting degree two,
2. at each level of $A^\wedge$ there is at most one coding node or splitting node,
3. between splitting and coding node levels, any 'interesting event' is minimal.

Exact BRD for triangle-free graphs

Let CT be the coding tree for an enumerated homogeneous triangle-free graph $\mathbf{G}_3$.

An interesting event is a pair of nodes $s, t \in CT$ such that s and t both code an edge with a common vertex of $\mathbf{G}_3$, but the least vertex of $\mathbf{G}_3$ with which s codes an edge differs from the least vertex of $\mathbf{G}_3$ with which s codes an edge.

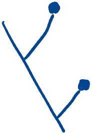
Let $\mathbf{A} \in \mathcal{G}_3$ be given. The big Ramsey degree of $\mathbf{A}$ in $\mathbf{G}_3$ is the number of different ways $\mathbf{A}$ can be represented by diagonal antichains of coding nodes in CT and the first occurrences of interesting events. That is, the number of different diagonal diaries for $\mathbf{A}$.

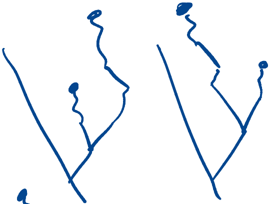
Examples of exact big Ramsey degrees for $\mathbf{G}_3$

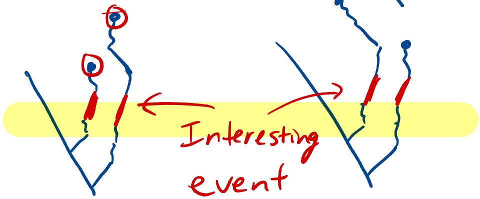
Edges

no interesting events

Non-Edges







SDAP$^+$ implies simply characterized big Ramsey degrees

Theorem. (Coulson, Dobrinen, Patel) Suppose $\mathcal{K}$ is Fraïssé relational class with finitely many relations satisfying SDAP$^+$. Let $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$.

1. If all relations are of arity ≤ 2, given $\mathbf{A} \in \mathcal{K}$, the big Ramsey degree of $\mathbf{A}$, $T(\mathbf{A}, \mathbf{K})$, equals the number of similarity types diagonal antichains of coding nodes in $\mathbb{S}(\mathbf{K})$ representing a copy of $\mathbf{A}$.
2. For relations of any arity, $\mathbf{K}$ is indivisible.

Proof uses forcing. No envelopes needed ever.

Diagonal antichains are antichains which have meet-closures with branching degree 2, and such that distinct nodes from among the branching and coding nodes have distinct lengths. If there are unary relations but no transitive relations, the unaries are separated at the base level of the tree.

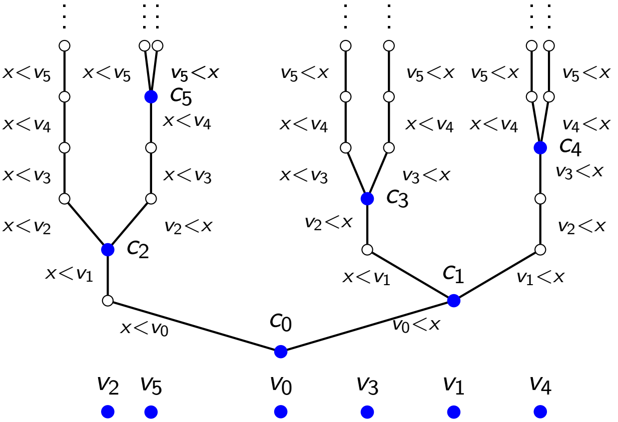
Coding trees of 1-types

In (CDP), we moved from trees on $k^{<\omega}$ to trees of 1-types over initial structures of a fixed enumerated Fraïssé structure.

Let $\mathbf{K}$ be the Fraïssé limit of a given Fraïssé class $\mathcal{K}$ with enumerated vertices $\langle v_n : n < \omega \rangle$. Let $\mathbf{K}_n$ denote $\mathbf{K} \restriction \{v_i : i < n\}$.

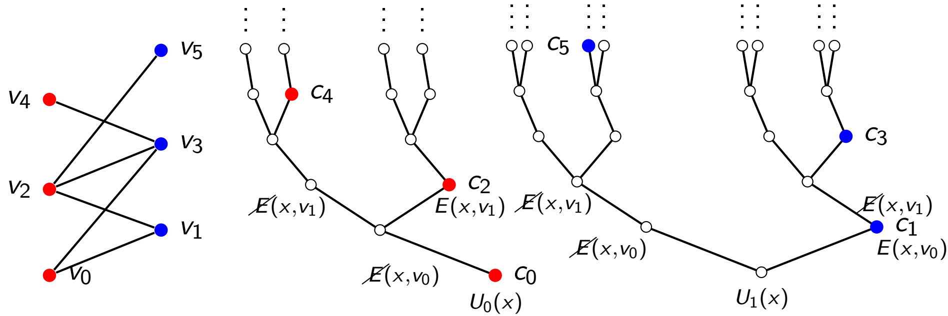
The coding tree of 1-types $\mathbb{S}(\mathbf{K})$ is the set of all q.f. complete 1-types over initial segments of $\mathbf{K}$ along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ such that $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is simply inclusion.

Coding Tree of 1-types for $(\mathbb{Q}, <)$



$c_0 = \emptyset$. $c_1 = \{(v_0 < x)\}$. $c_2 = \{(x < v_0), (x < v_1)\}$.
$c_3 = \{(v_0 < x), (x < v_1), (v_2 < x)\}$.

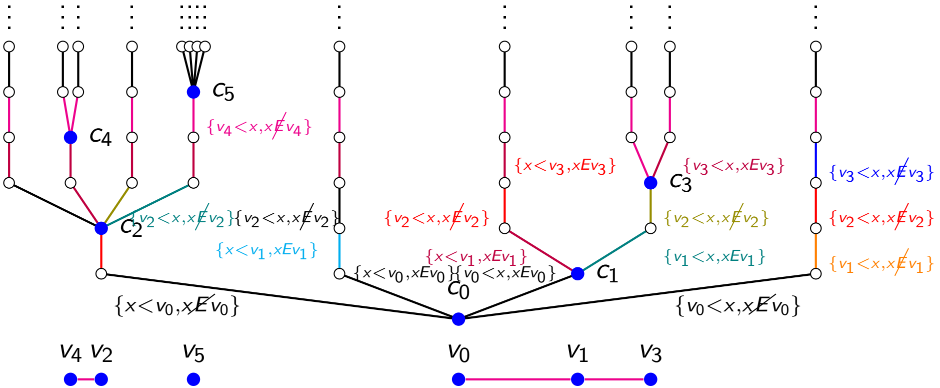
Coding tree of 1-types for the generic bipartite graph



$c_0 = \{U_0(x)\}$. $c_1 = \{U_1(x), E(x, v_0)\}$.
$c_2 = \{U_0(x)\}, \neg E(x, v_0), E(x, v_1)\}$.
$c_3 = \{U_1(x), E(x, v_0), \neg E(x, v_1), E(x, v_2)\}$.

Coding tree of 1-types for $\mathbb{Q}_{\mathbb{Q}}$

Language $\mathcal{L} = \{<, E\}$. The equivalence classes are convex.



$c_0 = \emptyset$. $c_1 = \{v_0 < x, xEv_0\}$. $c_2 = \{x < v_0, x \not E v_0, x < v_1, x \not E v_1\}$.

Substructure Free Amalgamation Property

A Fraïssé class $\mathcal{K}$ satisfies SFAP if $\mathcal{K}$ has free amalgamation, and given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$, suppose

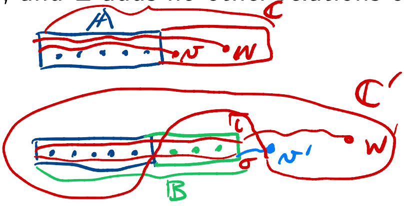
(1) $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, $\{v, w\}$;

(2) $\mathbf{A}$ is a substructure of $\mathbf{B}$ and σ and τ are 1-types over $\mathbf{B}$ with $\sigma \upharpoonright \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \upharpoonright \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$; and

(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say v', such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \upharpoonright (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over D.

Any pair of 1-types can be extended in any way and still extend to a copy of $\mathbb{C}$



Substructure Disjoint Amalgamation Property^{+}

The SDAP is a weakening of SFAP for strong amalgamation classes.

SDAP^{+} is SDAP + additionally

1. There is a diagonal coding tree within $\mathbb{S}(\mathbf{K})$ representing $\mathbf{K}$.
2. A property on how splitting nodes can be extended.
   (In many cases this is trivially satisfied.)

SFAP implies SDAP^{+}.

Any SFAP class with a superimposed linear order satisfies SDAP^{+}.

Examples

The following Fraïssé classes satisfy SFAP:

- graphs, ordered graphs, graphs with finitely many edge relations
- *n*-partite graphs
- hypergraphs
- free amalgamation relational classes omitting 3-irreducible substructures
- free superpositions of finitely many SFAP classes

The following Fraïssé classes have limits satisfying SDAP$^+$:

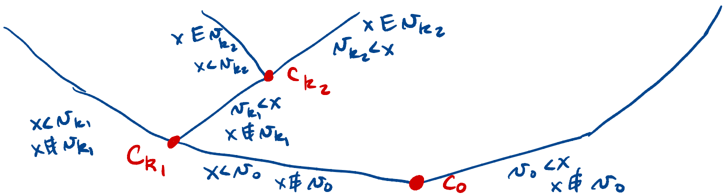
- linear orders, possibly with equivalence relations
- $\mathbb{Q}_{\mathbb{Q}}$, $\mathbb{Q}_{\mathbb{Q}_{\mathbb{Q}}}$, ..., and variations
- unrestricted relational structures with finitely many relations of any arity and ordered versions (e.g. generic tournament)
- Fraïssé classes with SFAP with an additional linear order

Examples of characterized big Ramsey degrees

The rationals Just antichains of coding nodes.
in c.t.'s of 1-types.

$\mathbb{Q}_{\mathbb{Q}}$ Just antichains of coding nodes in this
diagonal coding tree.

Diagonal Coding Subtree



THANK YOU!

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