Borel partitions of Rado graphs are Ramsey

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15th International Luminy Workshop in Set Theory

September 23–27, 2019

A question of Kechris, Pestov and Todorcevic (paraphrased)

What infinite structures carry infinite dimensional Ramsey theory?

Finite Dimensional Ramsey Theory

Ramsey's Theorem. (Ramsey, 1929) Given $k \geq 1$ and a coloring $c : [\omega]^k \to 2$, there is an infinite subset $M \subseteq \omega$ such that c is constant on $[M]^k$.

$$\forall k, \quad \omega \to (\omega)^k$$

This is called **finite dimensional** because the objects being colored are finite sets.

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that

$$[N]^\omega \subseteq \mathcal{X} \text{ or } [N]^\omega \cap \mathcal{X} = \emptyset.$$

Nash-Williams Theorem. (1965) Clopen sets are Ramsey.

Galvin-Prikry Theorem. (1973) Borel sets are Ramsey.

Silver Theorem. (1970) Analytic sets are Ramsey.

Ellentuck Theorem. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \to_* (\omega)^\omega$$

Ellentuck Theorem

The **Ellentuck topology** is generated by basic open sets of the form

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Ellentuck Theorem. (1974) Given any $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire with respect to the Ellentuck topology,

$(*)$ $\quad \forall [s, A] \ \exists B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$.

$(*)$ is called **completely Ramsey** in Galvin-Prikry and **Ramsey** in Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**:
These are spaces whose members are infinite sequences, with a topology induced by finite heads and infinite tails, and in which every subset with the property of Baire satisfies $(*)$.

A KPT Question

Problem 11.2 in (KPT 2005). Develop infinite dimensional Ramsey theory for Fraïssé structures.

Given $\mathbb{K} = \mathrm{Flim}(\mathcal{K})$ for some Fraïssé class $\mathcal{K}$, and some natural topology on $\binom{\mathbb{K}}{\mathbb{K}}$, are all "definable" sets Ramsey?

$$\mathbb{K} \to_* (\mathbb{K})^{\mathbb{K}}?$$

That is, can the Galvin-Prikry or Ellentuck Theorems be extended to spaces whose points represent homogeneous structures?

Very little known. Topological Ramsey spaces have infinite dimensional Ramsey theory, but $\mathbb{N}$ as a set and the rationals as a linear order are the only Fraïssé structures modeled by a Ramsey space.

KPT Subquestion

The **Rado graph** is the Fraïssé limit of the class of finite graphs.
It is ultrahomogeneous and universal for countable graphs.

Question. Is there an analogue of Galvin-Prikry, Silver, or Ellentuck for the Rado graph?

Is there a way to topologize all subcopies of the Rado graph so that all definable sets have the Ramsey property?

Main Theorem (D.)

There is a natural topological space of Rado graphs in which every Borel subset is Ramsey.

In details: There is a subspace $\mathcal{R}$ of the Baire space in which each point represents a Rado graph so that for any Borel $\mathcal{X} \subseteq \mathcal{R}$ and each Rado graph $R \in \mathcal{R}$, there is a subgraph $R' \leq R$ in $\mathcal{R}$ such that collection of all subgraphs of R' in $\mathcal{R}$ is either contained in or disjoint from $\mathcal{X}$.

Necessary concession: restrict to one strong similarity type

Theorem. (Abramson-Harrington 1978 and Nešetřil-Rödl 1977/83) The class of all finite ordered graphs has the Ramsey property.

Let **R** denote the Rado graph.

Theorem. (Laflamme, Sauer, Vuksanovic 2006) For each finite graph G, there is a number $T(G)$ such that

$$(\forall k \geq 1) \ \ \mathbf{R} \rightarrow (\mathbf{R})^{G}_{k,T(G)}$$

$T(G)$ is exactly the number of **strong similarity types** of codings of G in the binary tree $2^{<\omega}$.

So to get a positive answer to KPT Question for the Rado graph, we must restrict to copies of the Rado graph which all have the same strong similarity type.

What is a strong similarity type?

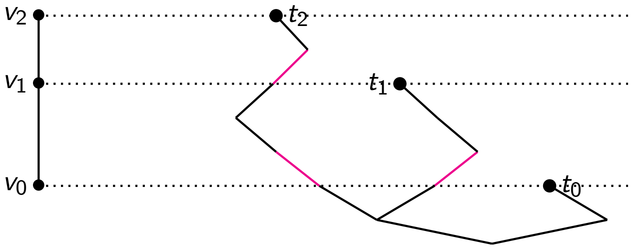
It has to do with using trees to code graphs.

Coding Graphs in $2^{<\omega}$

Let A be a graph with vertices $\langle v_n : n < N\rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Strong Similarity

Let $S, T \subseteq 2^{<\omega}$ be meet-closed. $f : S \to T$ is a **strong similarity** of S to T if f is a bijection and for all nodes $s, t, u, v \in S$, the following hold:

1. f preserves initial segments: $s \wedge t \subseteq u \wedge v$ if and only if $f(s) \wedge f(t) \subseteq f(u) \wedge f(v)$.
2. f preserves meets: $f(s \wedge t) = f(s) \wedge f(t)$.
3. f preserves relative lengths: $|s \wedge t| < |u \wedge v|$ if and only if $|f(s) \wedge f(t)| < |f(u) \wedge f(v)|$.
4. f preserves passing numbers at levels of meets and maximal nodes.

S and T are **strongly similar** exactly when there is a strong similarity map between S and T.

Goal

We want to make a topological space in which each point represents a Rado graph and such that every Borel subset is Ramsey.

Known: Strong trees and Milliken's Theorem help get big Ramsey degrees for the Rado graph.

Strong Subtrees of $2^{<\omega}$

For $t \in 2^{<\omega}$, the length of t is $|t| = \mathrm{dom}(t)$.

$T \subseteq 2^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

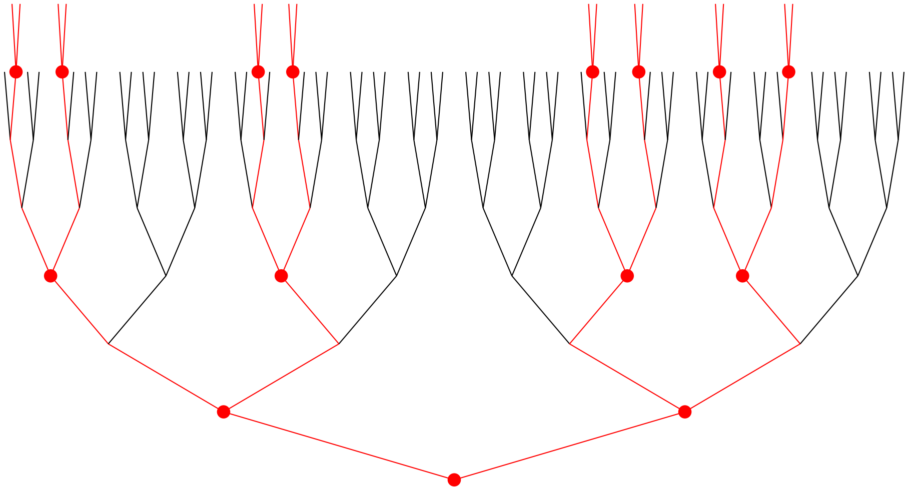
For $t \in T$, the height of t is $\mathrm{ht}_T(t) = \text{o.t.}\{u \in T : u \subset t\}$.

$T(n) = \{t \in T : \mathrm{ht}_T(t) = n\}$.

$S \subseteq T$ is a strong subtree of T iff for some $\{m_n : n < N\}$ $(N \leq \omega)$,

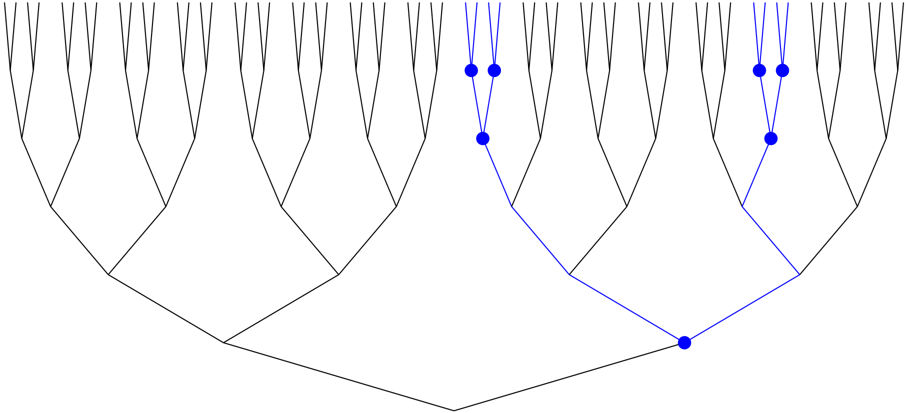
1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < N$, $s \in S(n)$ and immediate successor u of s in T, there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $0, 1, 3, 6, \ldots$

Example: A Strong Subtree $U \subseteq 2^{<\omega}$



The nodes in U are of lengths $1, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees, simple version

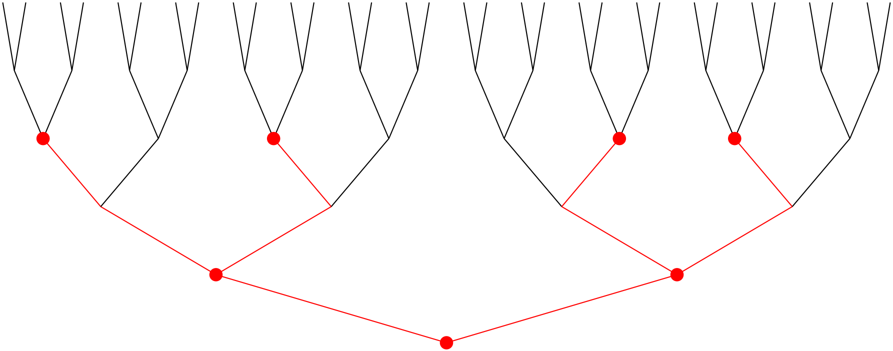
Thm. (Milliken 1979) Let $T \subseteq 2^{<\omega}$ be a strong tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

A ***k*-strong tree** is a finite strong tree where all terminal nodes have height $k - 1$.

We give some examples for $T = 2^{<\omega}$.

Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

Given a coloring c of all 3-strong trees in $2^{<\omega}$ into red and blue:



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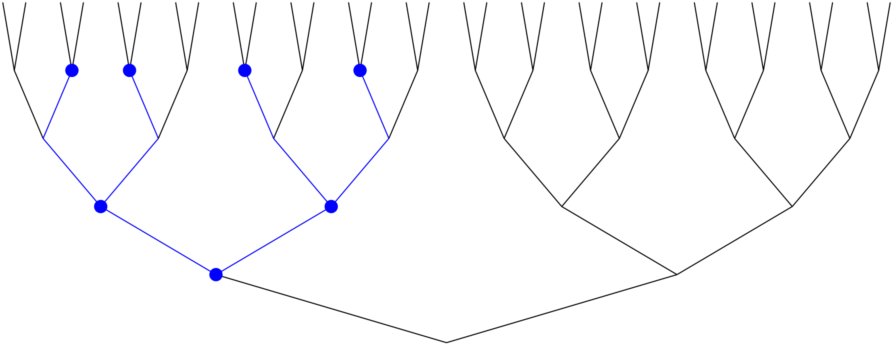
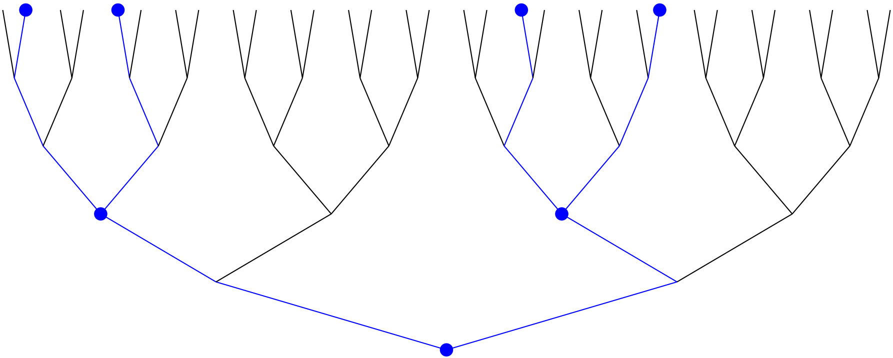
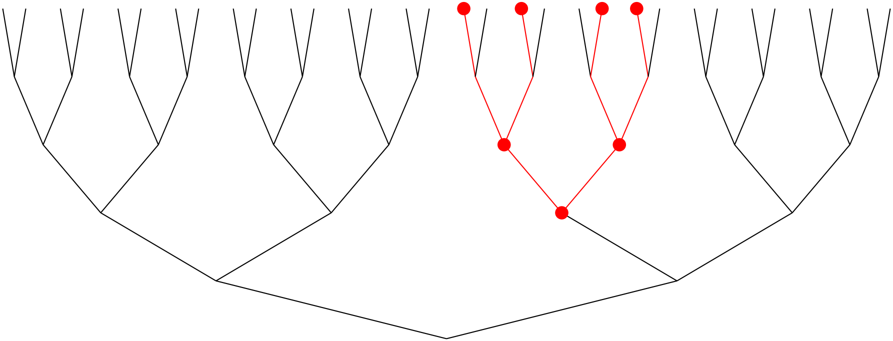


Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$



Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$



Milliken's Theorem guarantees a strong subtree in which all 3-strong subtrees have the same color.

Upper bounds for big Ramsey degrees of the Rado graph are obtained as follows:

1. The graph coded by the nodes in $2^{<\omega}$ is universal.
2. Fix a finite graph G and color all copies of G in $2^{<\omega}$.
3. Apply Milliken's Theorem to strong subtree envelopes for copies of G.
4. Obtain a strong subtree S which has one color per strong similarity type of G.
5. Take an antichain in S which codes the Rado graph.

Problems for infinite dimensional Ramsey theory:

If we simply work with strong trees, there is no way to ensure what sub-Rado graph is being coded by the subtree.

Once we take the antichain coding the Rado graph, there is no way to do further Ramsey theory using Milliken's Theorem.

Trees with Coding Nodes

A **tree with coding nodes** is a structure $\langle T, N; \subseteq, <, c\rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n**-th coding node in** T, usually denoted c_n^T.

Trees with coding nodes were developed to code graphs with forbidden cliques, to prove that Henson graphs have finite big Ramsey degrees. But they turned out to be useful for the problem of infinite dimensional Ramsey theory of the Rado graph.

The Space of Strong Rado Coding Trees $(\mathcal{T}_{\mathbf{R}}, \leq, r)$

Let $\mathbf{R}$ be a Rado graph with vertices $\langle v_n : n < \omega \rangle$.

Define $\mathbb{T}_{\mathbf{R}} = (2^{<\omega}, \omega; \subseteq, <, c)$, where for each $n < \omega$, $c(n)$ represents v_n.

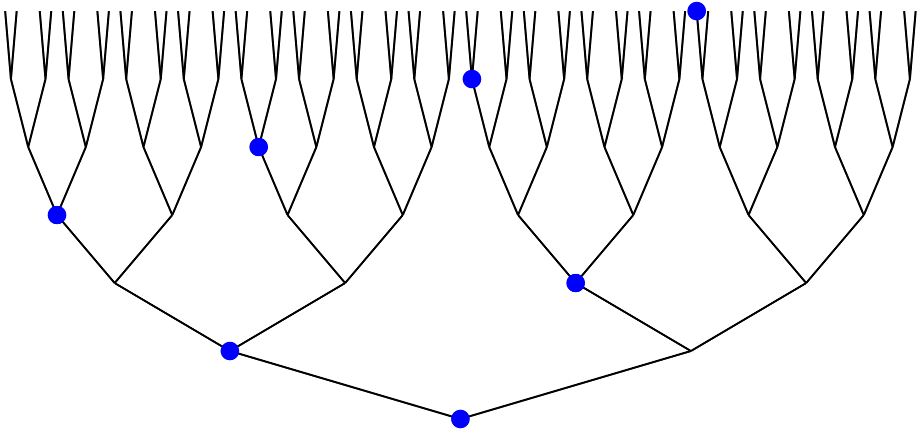
$\mathcal{T}_{\mathbf{R}}$ consists of all trees with coding nodes $(T, \omega; \subseteq, <, c^T)$, where

1. T is a strong subtree of $2^{<\omega}$; and
2. The strong tree isomorphism $\varphi : \mathbb{T}_{\mathbf{R}} \to T$ has the property that for each $n < \omega$, $\varphi(c(n)) = c^T(n)$.

The members of $\mathcal{T}_{\mathbf{R}}$ are called **strong Rado coding trees**. They represent all subgraphs of $\mathbf{R}$ which are strongly similar to $\mathbf{R}$.

We shall fix the Rado graph $\mathbf{R}$ coded by the coding nodes in the following tree.

Strong Rado Coding Tree $\mathbb{T}_{\mathbf{R}}$

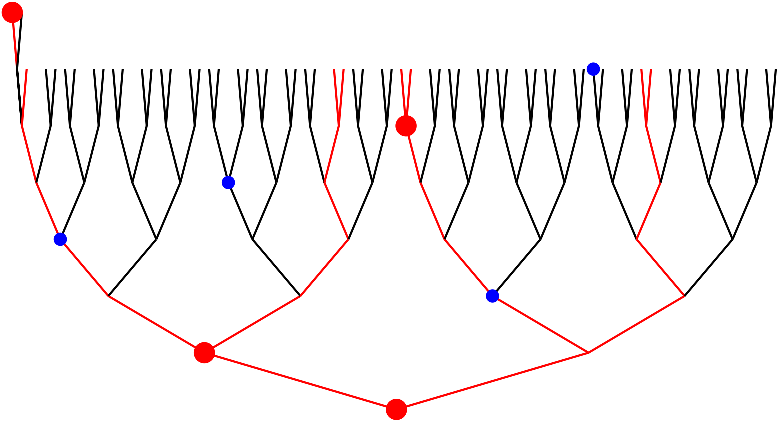


Strong Similarity for Trees with Coding Nodes

For $S \subseteq \mathbb{T}_{\mathbf{R}}$, $f : \mathbb{T}_{\mathbf{R}} \to S$ is a **strong similarity** if f is a bijection and for all nodes $s, t, u, v \in S$, the following hold:

1. f preserves initial segments.
2. f preserves meets.
3. f preserves relative lengths.
4. f preserves coding nodes: f maps the n-th coding node in $\mathbb{T}_{\mathbf{R}}$ to the n-th coding node in S.
5. f preserves passing numbers at coding nodes: If c is a coding node in $\mathbb{T}_{\mathbf{R}}$ and t is a node in $\mathbb{T}_{\mathbf{R}}$ with $|t| > |c|$, then $f(t)(|f(c)|) = t(|c|)$.

A Typical Strong Rado Coding Tree $T \in \mathcal{T}_{\mathbf{R}}$



$\mathcal{T}_{\mathbf{R}}$ is the collection of all subtrees of $\mathbb{T}_{\mathbf{R}}$ which are strongly similar to $\mathbb{T}_{\mathbf{R}}$.

Main Theorem (stronger version)

Thm. (D.) Every Borel subset of $\mathcal{T}$ is completely Ramsey. That is, if $\mathcal{X} \subseteq \mathcal{T}$ is Borel, then

$(*)$ $\quad \forall[s, A] \;\; \exists B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$.

$\mathcal{R}$ is the collection of all Rado graphs subgraphs of **R** coded by the coding nodes of some member of $\mathcal{T}$.

$\mathcal{R}$ = all Rado subgraphs of **R** with induced trees strongly similar to **R**.

$\mathcal{R}$ forms a subspace of $[\omega]^\omega$. For $R \in \mathcal{R}$, define

$$\mathcal{R}(R) = \{R' \in \mathcal{R} : R' \leq R\}.$$

Thm. (D.) Every Borel subset of $\mathcal{R}$ is Ramsey: Given a Borel set $\mathcal{X} \subseteq \mathcal{R}$ and $R \in \mathcal{R}$, there is some $R' \leq R$ such that

$$\mathcal{R}(R') \subseteq \mathcal{X} \;\text{ or }\; \mathcal{R}(R) \cap \mathcal{X} = \emptyset.$$

Proof Ideas.

1. Show that all open sets are completely Ramsey.
2. Show that complements of Ramsey sets are completely Ramsey.
3. Show that completely Ramsey sets are closed under countable unions.

The catch is (1) and (3). We use a forcing argument utilizing methods from our work on the big Ramsey degrees of the Henson graphs.

Hypotheses for Main Lemma. Given $T \in \mathcal{T}$, $D = r_n(T)$, and A an initial segment of some member of $\mathcal{T}$ with $\max(A) \subseteq \max(D)$: Let

$$A^+ = A \cup \{s^\frown i : s \in \max(A) \text{ and } i \in \{0,1\}\}.$$

Let B denote the subset of A^+ which will be end-extended to $(k+1)$-st approximations which are colored, where k is given according to

Case (a). $k \geq 1$ and $A = r_k(A)$ and $B = A^+$.

Case (b). $\max(A)$ has at least one node, and each member of $\max(A)$ has exactly one extension in B. Let k satisfy $2^k = \text{card}(\max(A))$.

Define $r_{k+1}[B, T]^* = \{C \in \mathcal{AT}_{k+1}(T) : \max(C) \sqsupseteq \max(B)\}$.

Lemma. (D.) Let $h : r_{k+1}[B, T]^* \to 2$ be a coloring. Then there is a strong Rado coding tree $S \in [D, T]$ such that h is monochromatic on $r_{k+1}[B, S]^*$.

Lemma. (D.) Let $h : r_{k+1}[B, T]^* \to 2$ be a coloring. Then there is a strong Rado coding tree $S \in [D, T]$ such that h is monochromatic on $r_{k+1}[B, S]^*$.

$$[B, T]^* = \{T \in \mathcal{T} : \max(r_k(T)) \sqsupseteq \max(B) \text{ and } T \leq S\},$$

where k comes from Case (a) or (b).

A subset $\mathcal{X} \subseteq \mathcal{T}$ is **CR*** if for each nonempty $[B, T]^*$, there is an $S \in [B, T]^*$ such that either $[B, S]^* \subseteq \mathcal{X}$ or else $[B, S]^* \cap \mathcal{X} = \emptyset$.

The Lemma is used both to show that

1. open sets in $\mathcal{T}$ are CR* and
2. to do fusion arguments for showing that countable unions of CR* sets are CR*, because $\mathcal{T}_R$ does not satisfy Todorcevic's Axiom **A.3**(2) for topological Ramsey spaces.

Forcing as unbounded searches of finite sets

The proof uses a simplified version of the forcing in (D. 2017 and 2019), building on Harrington's proof of the Halpern-Läuchli Theorem.

Given $U \in \mathcal{T}(T)$, define

$$\mathrm{Ext}_U(B) = \{\max(C) : C \in r_{k+1}[B, T]^* \text{ and } C \subseteq U\}.$$

Define $h' : \mathrm{Ext}_T(B) \to 2$ by $h'(X) = h(r_k(A) \cup X)$.

Let $d+1$ be the number of nodes in $\max(B)$. Let $s_0, \dots, s_d$ enumerate the nodes in $\max(B)$ so that the coding node in in each $X \in \mathrm{Ext}_T(B)$ extends s_d.

In Cases (a) and (b), $d + 1 = 2^k$, as any $C \in \mathcal{AT}_{k+1}$ has 2^k maximal nodes.

Let L denote the collection of all $l < \omega$ for which some member of $\mathrm{Ext}_T(B)$ has nodes of length l.

For $i \leq d$, let $T_i = \{t \in T : t \supseteq s_i\}$.

Let $\kappa = \beth_{2d}^+$, so that $\kappa \rightarrow (\aleph_1)^{2d}_{\aleph_0}$ (Erdős-Rado).

The following forcing notion $\mathbb{P}$ adds κ many paths through each T_i, $i < d$, and one path through T_d.

However, as our goal is to find a tree $S \in [D, T]$ for which h is monochromatic on $r_{k+1}[B, S]^*$, the forcing will be applied in finite increments to construct S, without ever moving to a generic extension.

Define $\mathbb{P}$ to consist of all finite functions p of the form

$$p : (d \times \vec{\delta}_p) \cup \{d\} \to \bigcup_{i \le d} T_i \upharpoonright l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $l_p \in L$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$ for each $i < d$, and $p(d)$ is *the* coding node in $T \upharpoonright l_p$ extending s_d.

The partial ordering on $\mathbb{P}$ is defined as follows: $q \le p$ if and only if $l_q \ge l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, $q(d) \supseteq p(d)$, and
$q(i, \delta) \supseteq p(i, \delta)$ for each $(i, \delta) \in d \times \vec{\delta}_p$.

Like Harrington's 'Forcing' Proof of Halpern-Läuchli

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i:

$$\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{P}, \text{ and } (i,\alpha) \in \mathrm{dom}(p)\}.$$

Note: If $(i,\alpha) \in \mathrm{dom}(p)$, then $p \Vdash \dot{b}_{i,\alpha} \restriction l_p = p(i,\alpha)$.

Let $\dot{b}_d = \{\langle p(d), p\rangle : p \in \mathbb{P}\}$.

Let $\dot{L}_d$ be a $\mathbb{P}$-name for the set of lengths of coding nodes in $\dot{b}_d$, and note that $\mathbb{P}$ forces that $\dot{L}_d \subseteq L$.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}_d$.

For $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \ldots, \dot{b}_{d-1,\alpha_{d-1}, \dot{b}_d} \rangle$.

For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that

1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
2. $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

Apply the Erdős-Rado Theorem to obtain disjoint infinite sets $K_i \subseteq \kappa$, $i < d$, so that

Lemma. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Let $t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$ and $t_d^* = p_{\vec{\alpha}}(d)$ for any $\vec{\alpha} \in \prod_{i<d} K_i$.

These are guaranteed good starting nodes to build our tree.

Build a Rado tree $S \in [D, T]$ so that the coloring h will be monochromatic on $r_{k+1}[B, S]^*$.

Let n be the integer such that $D \in \mathcal{AT}_n$.

Let $M = \{m_j : j < \omega\}$ be the strictly increasing enumeration of those integers $m > n$ such that for each $F \in r_m[D, T]$, the coding node in $\max(F)$ extends s_d.

For each integers in M, use the forcing to find the next level of S so that the members of $r_{k+1}[B, S]^*$ will have the same h-color.

For integers not in M, choose the next level of S manually.

Main Theorem (strongest version)

The collection of CR^* subsets of $\mathcal{T}$ contains all Borel subsets of $\mathcal{T}$.

Why only Borel and not Property of Baire?

Similarly to strong coding trees developed for the big Ramsey degrees of the Henson graphs, the collection of strong Rado trees form a space satisfying all four of Todorcevic's Axioms for topological Ramsey spaces, except for A.3(2) (Amalgamation).

A "forced" Halpern-Läuchli-style theorem provides a means for fusion arguments in the style of Galvin-Prikry, but is not sufficient for Ellentuck's arguments.

Remarks, Questions and Future Directions

Rem 1. We could fix any strong similarity type of a tree with coding nodes coding the Rado graph and get a space of Rado graphs in which every Borel set is Ramsey.

Rem 2. Trees with coding nodes and these forcing arguments were developed to work with forbidden k-cliques, but have shown to be useful for infinite dimensional Ramsey theory of the Rado graph.

Rem 3. A similar fusion lemma for Henson graphs follows from my work on their big Ramsey degrees, so they will have a similar theorem.

Question. What other Fraïssé structures have infinite dimensional Ramsey theory?

Question. Is there a topological Ramsey space of Rado graphs?

References

Dobrinen, *Borel of Rado graphs and Ramsey's theorem* (2019) 25 pp (Submitted).

Dobrinen, *The Ramsey theory of the universal homogeneous triangle-free graph* (2017) 65 pp (Submitted).

Dobrinen, *Ramsey theory of the Henson graphs* (2019) 68 pp (Preprint).

Ellentuck, *A new proof that analytic sets are Ramsey*, JSL (1974).

Galvin-Prikry, *Borel sets and Ramsey's Theorem*, JSL (1973).

Halpern-Läuchli, *A partition theorem*, TAMS (1966).

Laflamme-Sauer-Vuksanovic, *Canonical partitions of universal structures*, Combinatorica (2006).

Kechris-Pestov-Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis (2005).

Milliken, *A Ramsey theorem for trees*, Jour. Combinatorial Th., Ser. A (1979).

Thank you for your attention!

Thanks to NSF grant DMS-1600781 for supporting this research.