Infinite dimensional Ramsey theory of homogeneous structures: a progress report

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A question of Kechris, Pestov and Todorcevic

What homogeneous structures carry infinite dimensional Ramsey theory?

An infinite structure **K** is homogeneous if every isomorphism between two finite induced substructures of **K** extends to an automorphism of **K**.

Finite Dimensional Ramsey Theory

Finite Ramsey's Theorem. Given $1 \leq k < m$ and $2 \leq r$, there is an $m < n$ such that for any coloring $c : [n]^k \to r$, there is an $M \in [n]^m$ such that c is constant on $[M]^k$; that is,

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Infinite Ramsey's Theorem. Given $1 \le k$, $2 \le r$, and a coloring $c : [\omega]^k \to r$, there is an $M \in [\omega]^\omega$ such that c is constant on $[M]^k$; that is,

$$\omega \longrightarrow (\omega)^k_r$$

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is Ramsey if each for $N \in [\omega]^\omega$, there is an $M \in [N]^\omega$ such that

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Nash-Williams. (1965) Clopen sets are Ramsey.

Galvin–Prikry. (1973) Borel sets are Ramsey.

Silver. (1970) Analytic sets are Ramsey.

Ellentuck. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \longrightarrow_* (\omega)^\omega$$

Finite Structural Ramsey Theory

A Fraïssé class $\mathcal{K}$ has the Ramsey Property if whenever $\mathbf{A} \le \mathbf{B}$ are structures in $\mathcal{K}$, given $r \ge 2$ there is a $\mathbf{C} \in \mathcal{K}$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_r$$

Examples: The classes of finite linear orders, finite ordered graphs, finite ordered hypergraphs, finite ordered triangle-free graphs, and many other classes have the Ramsey Property.

Big Ramsey degrees

Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ be its Fraïssé limit. We say that $\mathcal{K}$ has finite big Ramsey degrees if for each $\mathbf{A} \in \mathcal{K}$, there is a $T \geq 1$ such that given any $r \geq 2$ and any coloring of $\binom{\mathbf{K}}{\mathbf{A}}$ into r colors, there is a subcopy $\mathbf{K}'$ of $\mathbf{K}$ such that $\binom{\mathbf{K}'}{\mathbf{A}}$ takes no more than T colors.

This is as good as it can get for structural analogues of the Infinite Ramsey Theorem.

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- Very little known. In Luminy 2019, I showed how certain spaces of Rado graphs carry Galvin-Prikry analogues.
- Todorcevic pointed out that ideally such an infinite dimensional Ramsey theorem should recover exact big Ramsey degrees.
- Today, I show how to do this for the Rado graph and the rationals, as well as a general method for extending it to binary relational homogeneous structures satisfying SDAP^+.

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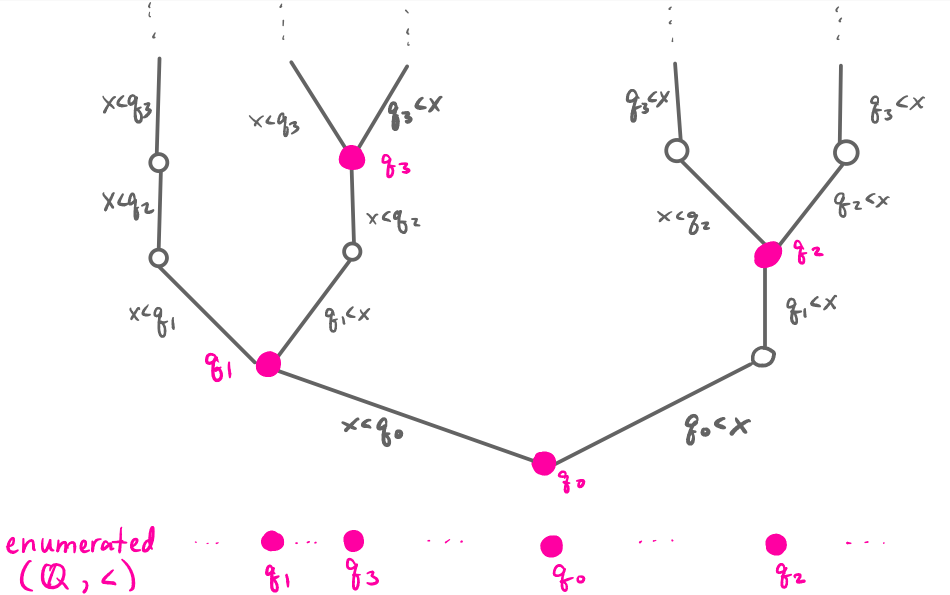
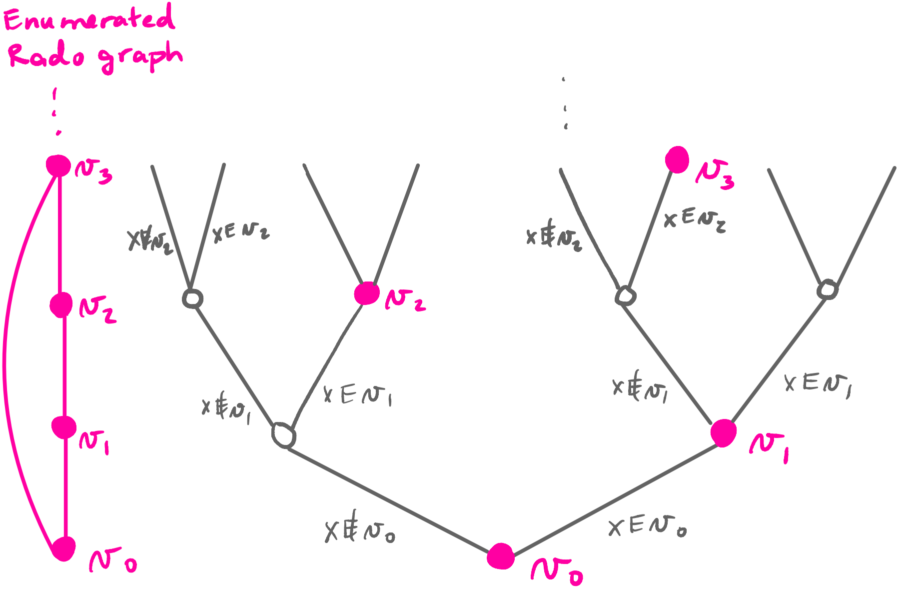
Why this is a 'progress report': Subject to triple-checking, the same methods will recover Galvin-Prikry theorems whose Nash-Williams analogues recover exact big Ramsey degrees for all structures satisfying SDAP^+. More on that later.

The set-up for the proof begins with coding trees of 1-types representing enumerated homogeneous structures.

Coding tree of 1-types for the rationals



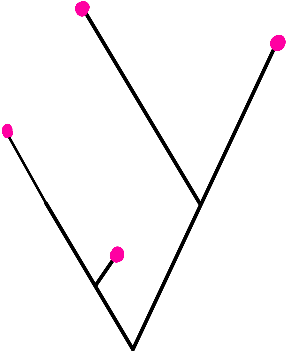
Coding tree of 1-types for the Rado graph

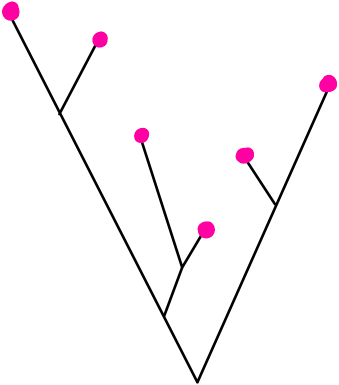


Diagonal Antichains

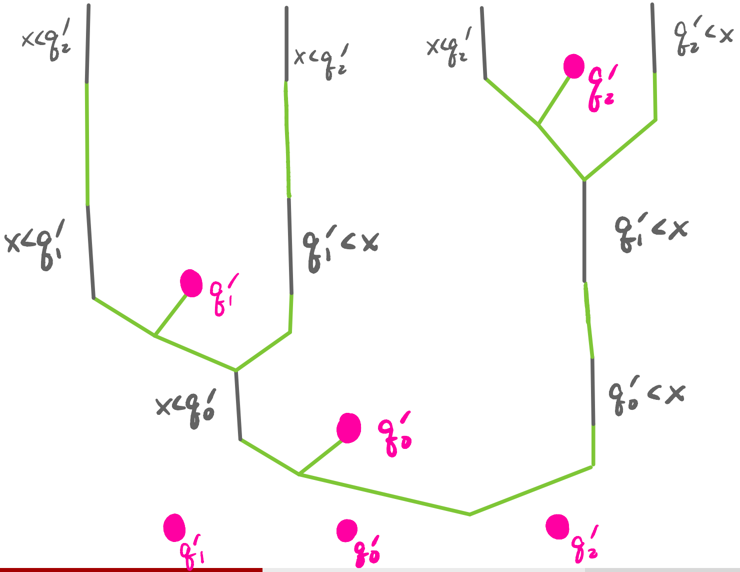
An antichain is diagonal if its meet closure has at each level at most one splitting node or one maximal node, and splitting degree is two.

A structure **A** is a diagonal antichain if its representation in the tree of 1-types is a diagonal antichain.

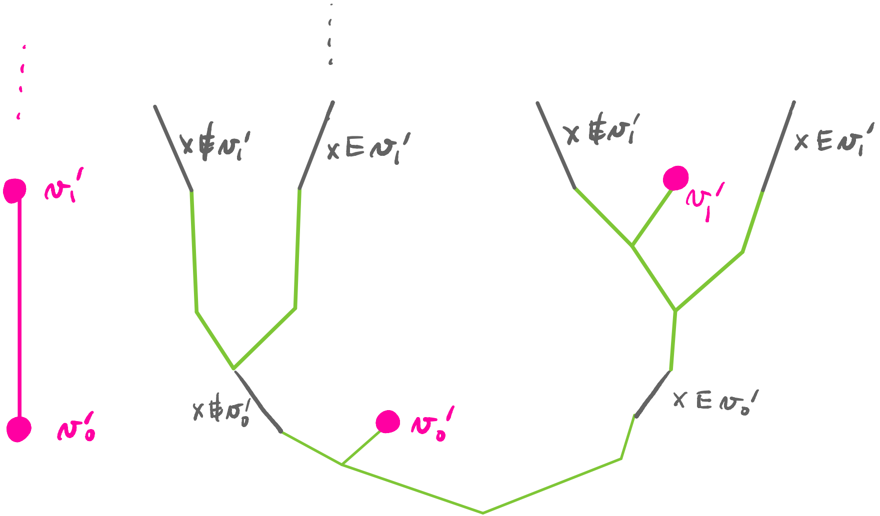




Diagonal Antichain representing $\mathbb{Q}$



Diagonal Antichain representing the Rado graph



Similarity of Diagonal Antichains

Given an antichain D, let $D^{\wedge}$ denote the meet closure of D.

Two diagonal antichains D and E in the tree of 1-types are similar if there is a bijection $f : D \to E$ such that

1. Given $v_m, v_n \in D$, $\{f(v_m), f(v_n)\}$ satisfy exactly the same relations as $\{v_m, v_n\}$; and
2. The induced map $\tilde{f} : D^{\wedge} \to E^{\wedge}$ preserves meets, relative lengths, initial segments, and lexicographic order.

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The theory of big Ramsey degrees for $\mathbb{Q}$ (Devlin), the Rado graph (Laflamme–Sauer–Vuksanovic), and more generally homogeneous structures satisfying SDAP$^+$ with finitely many relations of arity at most two (Coulson–D.–Patel) necessitate working with diagonal antichains.

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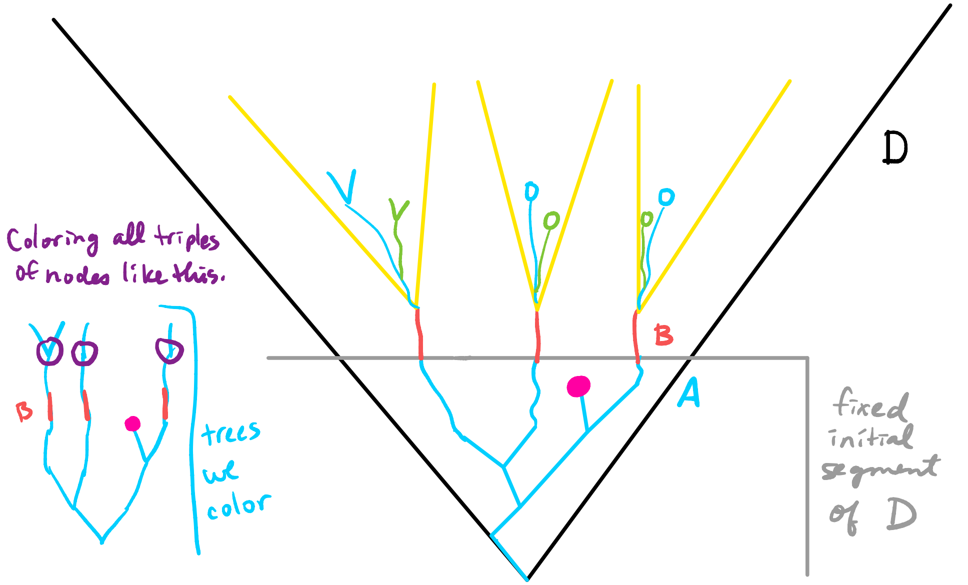
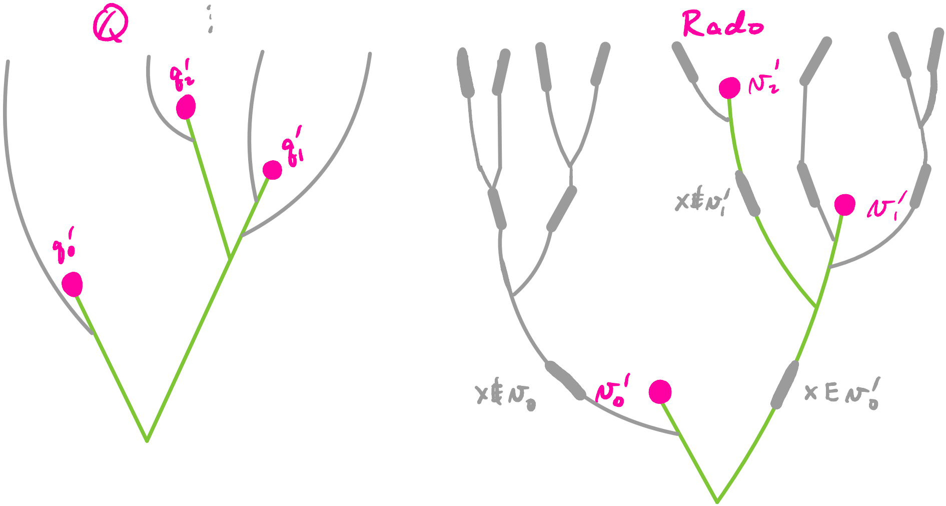
(3) uses a forcing argument to make a souped-up Pigeonhole to help with fusion arguments.

CR* and forcing a strong pigeonhole



Recovering exact big Ramsey degrees



Examples of envelopes

Substructure Free Amalgamation Property

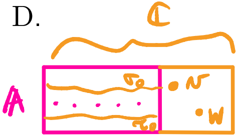
A Fraïssé class $\mathcal{K}$ satisfies SFAP if $\mathcal{K}$ has free amalgamation and the following holds: Given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$ such that

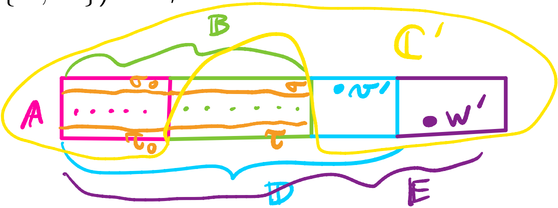
(1) $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, $\{v, w\}$;

(2) $\mathbf{A}$ is a substructure of $\mathbf{B}$ and σ and τ are 1-types over $\mathbf{B}$ with $\sigma \restriction \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \restriction \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$; and

(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say v', such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over D.





Examples

The following Fraïssé classes satisfy SFAP:

- graphs, ordered graphs, graphs with finitely many edge relations
- n-partite graphs
- hypergraphs
- free amalgamation relational classes omitting 3-irreducible substructures
- free superpositions of finitely many SFAP classes

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The SDAP$^+$ is a weakening of SFAP for disjoint amalgamation classes.

The following Fraïssé classes have limits satisfying SDAP$^+$:

- $\mathbb{Q}$, $\mathbb{Q}_{\mathbb{Q}}$, $\mathbb{Q}_{\mathbb{Q}_{\mathbb{Q}}}$,..., $\mathbb{Q}_n$, and variations
- unrestricted relational structures with finitely many relations of any arity and ordered versions (e.g. generic tournament)
- SFAP classes with an additional linear order

Stronger Theorems

Theorem.* (D.) Suppose $\mathcal{K}$ has finitely many binary relations and satisfies SFAP, and let $\mathbf{K}$ denote the Fraïssé limit of $\mathcal{K}$ or its ordered version.

Then for each diagonal antichain D coding $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, the Baire space $\mathcal{B}(D)$ of similar substructures has the property the collection of all CR* subsets of $\mathcal{T}$ contains all Borel subsets of $\mathcal{T}$.

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(**) A similar theorem should work for SDAP$^+$ more generally; there is a particular constraint I need to check.

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- Higher arities? Strong amalgamation classes?

Thank you for your attention!