Ramsey Theory on Infinite Structures

Natasha Dobrinen

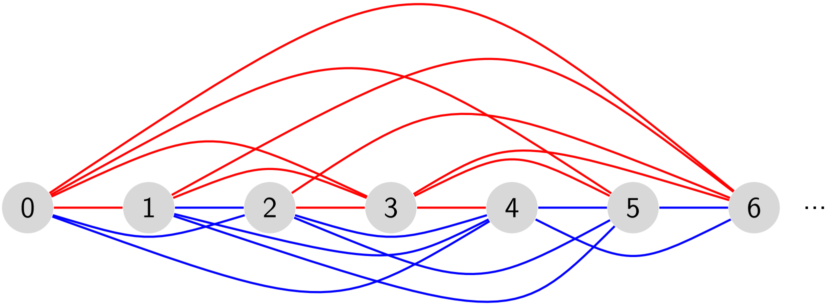
University of Denver

SEALS 2021

Research supported by DMS-1901753

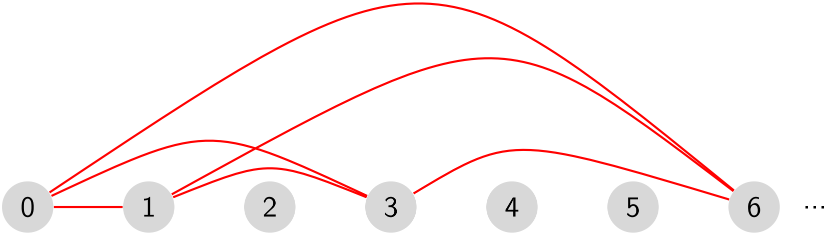
Ramsey's Theorem for Pairs

Given a 2-coloring of the edges of a complete graph on ω vertices,



Ramsey's Theorem for Pairs

There is an infinite complete subgraph such that all edges have the same color.



General Ramsey's Theorem

Ramsey's Theorem. Given any $k, l \geq 1$ and a coloring on the collection of all k-element subsets of ω into l colors, there is an infinite set M of natural numbers such that each k-element subset of M has the same color.

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Ramsey's theorem can be thought of as a coloring theorem on a complete k-hypergraph on ω many vertices.

Ramsey theory on $(\mathbb{Q}, <)$

Thm. (Sierpiński) There is a coloring $c : [\mathbb{Q}]^2 \to 2$ such that for each subset $\mathbb{Q}' \subseteq \mathbb{Q}$ which forms a dense linear order without endpoints, both colors occur on $[\mathbb{Q}']^2$.

Sierpiński's coloring: Let $\prec$ well-order $\mathbb{Q}$ in order-type ω. Given a pair $\{p, q\} \in [\mathbb{Q}]^2$, with $p < q$, define

$$c(\{p,q\}) = \begin{cases} 0 & \text{if } p \prec q \\ 1 & \text{if } q \prec p \end{cases}$$

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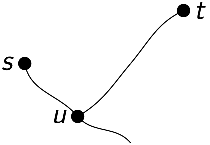
Thm. (Galvin) Given any coloring of pairs of rationals into finitely many colors, there is a subset which is again a dense linear order in which at most two colors are used.

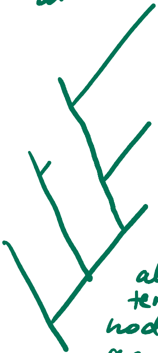
Ramsey theory on $\mathbb{Q}$, coded in $2^{<\omega}$

For $s, t \in 2^{<\omega}$, define $s \triangleleft t$ iff one of the following holds:

1. $s <_{\text{lex}} t$, or
2. $s \sqsubset t$ and $t(|s|) = 1$, or
3. $t \sqsubset s$ and $s(|t|) = 0$.

A diagonal antichain.





In this picture, $s \triangleleft u \triangleleft t$.

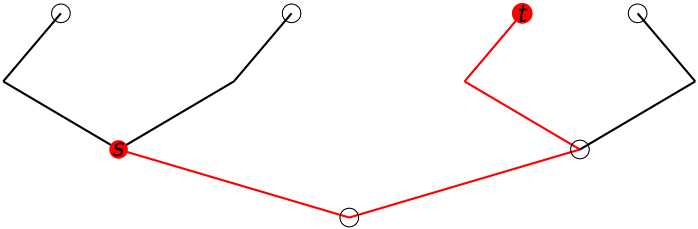
Note: $(2^{<\omega}, \triangleleft) \cong (\mathbb{Q}, <)$. Also, any diagonal antichain $A \subseteq 2^{<\omega}$ is linearly ordered in order-type ω.

all terminal nodes and splitting nodes have different lengths

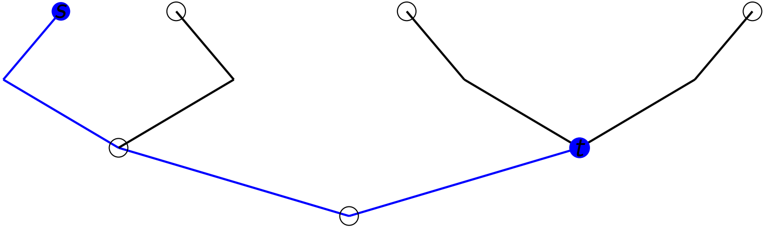
A strong tree envelope of s and t

Given incomparable $s, t \in 2^{<\omega}$ with $|s| < |t|$, a strong tree envelope is a finite strong tree which contains s and t and has nodes of lengths $|s \wedge t|, |s|, |t|$.

Example 1: $|s| < |t|$ and $s \triangleleft t$



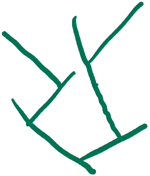
Example 2: $|s| < |t|$ with $t \lhd s$ and a strong tree envelope



Sierpiński's and Galvin's results: Re-interpreted inside $2^{<\omega}$

1. Let c be a coloring of $[\mathbb{Q}]^2$ into finitely many colors.
2. Transfer the coloring to pairs of nodes in $2^{<\omega}$.
3. Fix one similarity type. For each pair of nodes s, t of that type, color all 3-strong trees containing s and t with the color $c(\{s, t\})$.
4. Apply Milliken's Theorem to 3-strong trees. Get one color for all pairs with that similarity type.
5. Repeat for the second similarity type.
6. Take a diagonal antichain $\mathbb{A} \subseteq 2^{<\omega}$ such that $(\mathbb{A}, \triangleleft) \cong (\mathbb{Q}, <)$.

($\mathbb{A} \subseteq 2^{<\omega}$ is a diagonal antichain if any two nodes in the meet closure of $\mathbb{A}$ have different lengths, and all nodes except splitting nodes extend left.)



Ex of a diag. anti-chain

$(\mathbb{Q}, <)$ has an approximate Infinite Ramsey Theorem

Thm. (Laver (upper bounds, unpub.), Devlin (exact bounds) 1979)
Given $k \geq 2$, there is a number $T(k, \mathbb{Q})$ such that for each coloring of the k-element subsets of $\mathbb{Q}$ into finitely many colors, there is a subcopy of $\mathbb{Q}$ in which no more than $T(k, \mathbb{Q})$ colors occur.
These are actually tangent numbers.

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So $(\mathbb{Q}, <)$ does not have the exact analogue of Ramsey's Theorem for $\mathbb{N}$.

But $(\mathbb{Q}, <)$ still behaves quite nicely in that finite bounds exist. These bounds $T(k, \mathbb{Q})$ are called the big Ramsey degrees of k in $\mathbb{Q}$.

What about Ramsey theory on other infinite structures?

Big Ramsey Degrees

Given an infinite structure $\mathbf{S}$ and a finite substructure $\mathbf{A} \leq \mathbf{S}$, we let $T(\mathbf{A}, \mathbf{S})$ denote the least T (if it exists) such that for any integer $\ell \geq 1$, given any coloring of $\binom{\mathbf{S}}{\mathbf{A}}$ into ℓ colors, there is a substructure $\mathbf{S}'$ of $\mathbf{S}$, isomorphic to $\mathbf{S}$, such that $\binom{\mathbf{S}'}{\mathbf{A}}$ takes no more than T colors.

$$\forall \ell \geq 1, \quad \mathbf{S} \to (\mathbf{S})^{\mathbf{A}}_{\ell, T(\mathbf{A},\mathbf{S})}$$

When it exists, $T(\mathbf{A}, \mathbf{S})$ is called the big Ramsey degree of $\mathbf{A}$ in $\mathbf{S}$.

We say that $\mathbf{S}$ has finite big Ramsey degrees if $T(\mathbf{A}, \mathbf{S})$ exists for each finite substructure $\mathbf{A} \leq \mathbf{S}$.

Structures with finite big Ramsey degrees: Brief History

- Infinite complete k-hypergraph: All BRD $= 1$. (Ramsey 1929)
- $T(2, \mathbb{Q}) \geq 2$ (Sierpiński 1933). $T(2, \mathbb{Q}) = 2$ (Galvin unpub.)
- $T(\text{Edge, Rado}) \geq 2$ (Erdős, Hajnal, Pósa 1975)
- The rationals: BRD computed. (Devlin 1979)
- The K_3-free generic graph $\mathcal{H}_3$: $T(1, \mathcal{H}_3) = 1$. (Komjáth, Rödl 1986)
- The K_n-free generic graph $\mathcal{H}_n$: $T(1, \mathcal{H}_n) = 1$. (El-Zahar, Sauer 1989)
- $T(\text{Edge, Rado}) = 2$. (Pouzet, Sauer 1996)
- $T(\text{Edge}, \mathcal{H}_3) = 2$ (Sauer 1998)
- The Rado graph, etc.: BRD characterized. (Laflamme, Sauer, Vuksanović 2006). BRD computed. (J. Larson 2008)
- The countable ultrametric Urysohn space: BRD computed. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S(2)}$, $\mathbf{S(3)}$. BRD computed. (Laflamme, Nguyen Van Thé, Sauer 2010)

Structures with finite big Ramsey degrees: Recent Work

- The k-clique-free generic graph $\mathcal{H}_k$: Finite BRD (Dobrinen 2020 and 2019*)
  developed method of coding trees and related forcings
- 3-regular hypergraphs: Finite BRD (Balko, Chodounský, Hubička, Konečný, Vena 2019)
  developed a new Milliken-style theorem
- Universal structures, some metric spaces: Finite BRD (Mašulović 2020)
  used category theory
- $\mathbf{S}(n)$ for all $n \geq 2$: BRD calculated. (Barbosa 2020*)
  used category theory

Structures with finite big Ramsey degrees: Current Work

- Binary relational structures with free amalgamation, omitting finitely many irreducible substructures: Finite BRD (Zucker 2020*)
  used coding trees and forcing, and developed abstract approach

- Partial order, metric spaces, etc.: Finite BRD. (Hubička 2020*)
  used parameter words, first forcing-free proof for $\mathcal{H}_3$

- Fraïssé structures with SDAP$^+$: BRD characterized.
  (Coulson, Dobrinen, Patel 2020*)
  develop coding trees of 1-types, first envelope-free proof

- Binary with finitely many forbidden irreducible substructures: BRD characterized.
  (Balko, Chodounsky, Dobrinen, Hubička, Konečny, Vena, Zucker 2021*)
  various approaches

Other extensions in the works.

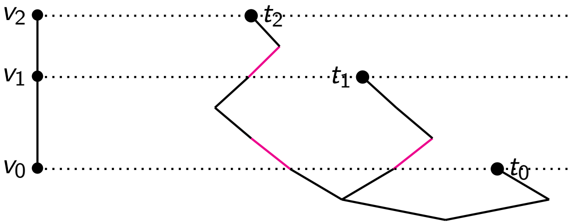
Fraïssé classes with simply characterized big Ramsey degrees

The beginning: Coding graphs using trees

Laflamme, Sauer, Vuksanović, building on Erdős, Hajnal, Posá.

Let **A** be a graph with enumerated vertices $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes **A** if and only if for each pair $m < n < N$, $v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1$. $t_n(|t_m|)$ is called the passing number of t_n at t_m.



The set $\{t_0, t_1, t_2\}$ is an example of a diagonal antichain.

Similarity Types of Diagonal Antichains

Fix $k \geq 2$. An antichain $A \subseteq k^{<\omega}$ is diagonal if the meet closure $A^{\wedge}$ of A has splitting nodes with degree 2, the lengths among the splitting and terminal nodes are all distinct, and all passing numbers at splitting nodes are 0 (except for the right branch).

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Let A and B be antichains in $k^{<\omega}$. We say that A and B are similar ($A \sim B$) iff there is a map $f : A^\wedge \to B^\wedge$ satisfying the following:

1. f is a bijection preserving lexicographic order.
2. f preserves meets, and hence splitting nodes.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves passing numbers at terminal nodes.

Fraïssé classes with simply characterized BRD

Theorem. (LSV) All Fraïssé structures on finitely many binary relations with a universal constraint set have big Ramsey degrees characterized by similarity types of diagonal antichains.

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(LSV) results apply to the Rado graph, generic ordered graph, generic tournament, random graphs with finitely many edge relations, etc.

Proof uses trees $k^{<\omega}$, for some finite k, which codes a universal structure into which the Fraïssé class embeds, envelopes, Milliken's theorem, and a diagonal antichain coding the Fraïssé structure.

Fraïssé classes with simply characterized BRD

Work by Coulson, D., and Patel was motivated by the following questions:

(Sauer, BIRS 2018) Can the forcing (done by D. for Henson graphs) be done directly on the Fraïssé structures?

(D.) For which Fraïssé classes can we prove finite big Ramsey degrees with reasonably simple forcing arguments (as compared with the Henson graphs)?

Coding trees of 1-types

In (CDP), we move from trees on $k^{<\omega}$ to trees of 1-types over initial structures of a fixed Fraïssé structure.

Let $\mathbf{K}$ be the Fraïssé limit of a given Fraïssé class $\mathcal{K}$ with enumerated vertices $\langle v_n : n < \omega \rangle$. Let $\mathbf{K}_n$ denote $\mathbf{K} \restriction \{v_i : i < n\}$.

The coding tree of 1-types $\mathbb{S}(\mathbf{K})$ is the set of all complete 1-types over initial segments of $\mathbf{K}$ along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ such that $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is simply inclusion.

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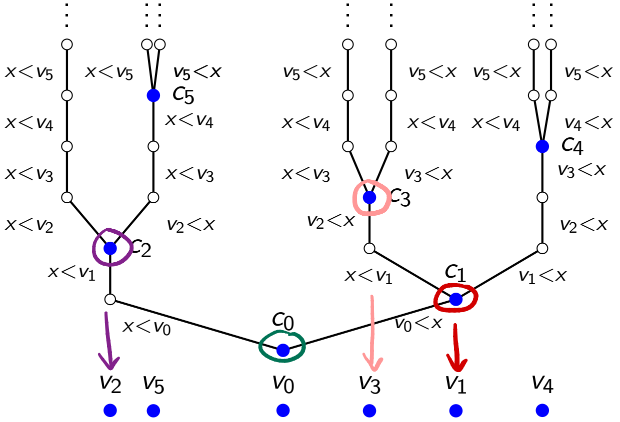
Members of $\mathbb{S}(\mathbf{K})$ can be thought of as sequences of partial 1-types:

$s(0)$ is the 1-type over the empty structure such that $s(0) \subseteq s$.

For $1 \leq i \leq n$, $s(i)$ is the set of formulas in $s \restriction \mathbf{K}_i$ that are not in $s \restriction \mathbf{K}_{i-1}$.

Each $s \in \mathbb{S}(\mathbf{K})$ determines a unique sequence $\langle s(i) : i < |s| \rangle$, where $\{s(i) : i < |s|\}$ forms a partition of s.

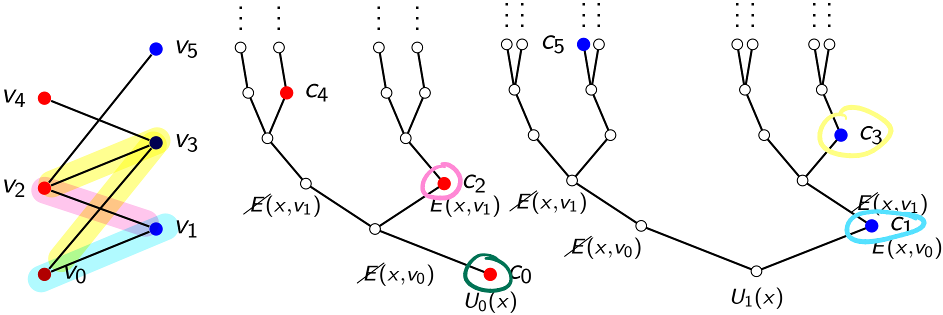
Coding Tree of 1-types for $(\mathbb{Q}, <)$



Copy of $(\mathbb{Q}, <)$ represented by the coding nodes

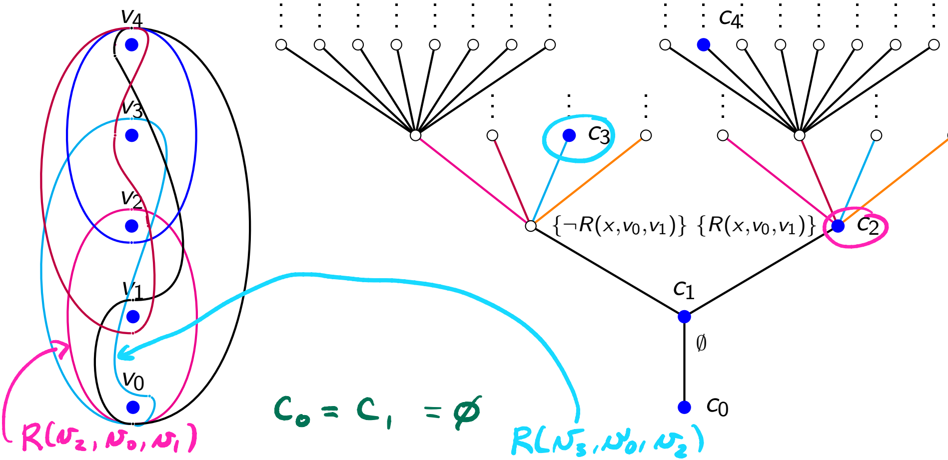
$c_0 = \emptyset$. $c_1 = \{(v_0 < x)\}$. $c_2 = \{(x < v_0), (x < v_1)\}$.
$c_3 = \{(v_0 < x), (x < v_1), (v_2 < x)\}$.

Coding tree of 1-types for the generic bipartite graph



$c_0 = \{U_0(x)\}$. $c_1 = \{U_1(x), E(x, v_0)\}$.
$c_2 = \{U_0(x)\cancel{\}}, \neg E(x, v_0), E(x, v_1)\}$.
$c_3 = \{U_1(x), E(x, v_0), \neg E(x, v_1), E(x, v_2)\}$.

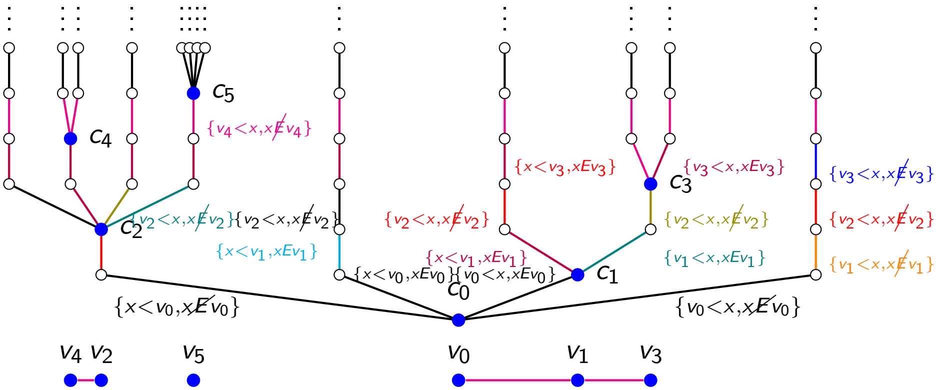
Coding tree of 1-types for the generic 3-regular hypergraph



$c_2 = \{R(x, v_0, v_1)\}$. $c_3 = \{\neg R(x, v_0, v_1), R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$.

Coding tree of 1-types for $\mathbb{Q}_{\mathbb{Q}}$

Language $\mathcal{L} = \{<, E\}$. The equivalence classes are convex.



$c_0 = \emptyset$. $c_1 = \{v_0 < x, xEv_0\}$. $c_2 = \{x < v_0, x\not{E}v_0, x < v_1, x\not{E}v_1\}$.

Substructure Free Amalgamation Property

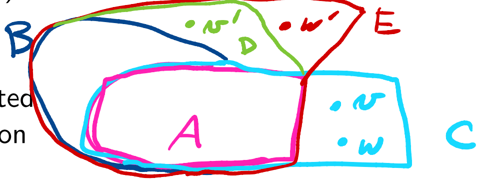
A Fraïssé class $\mathcal{K}$ satisfies SFAP if $\mathcal{K}$ has free amalgamation, and given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$, the following holds: Suppose

(1) $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$;

(2) $\mathbf{A}$ is a substructure of $\mathbf{B}$ and σ and τ are 1-types over $\mathbf{B}$ with $\sigma \restriction \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \restriction \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$; and

(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say v', such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over D.



We also formulate a weaker related property for disjoint amalgamation classes called SDAP$^+$.

Passing Types and Similarity

Given $s, t \in \mathbb{S}$ with $|s| < |t|$, $t(|s|)$ is the set of all formulas in $t \restriction \mathbf{K}_{|s|}$, $t(|s|)$ is the passing type of t at s.

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Given $A \subseteq \mathbb{S}$, $t, c_n \in \mathbb{S}$ with $|c_n| < |t|$, $t(c_n; A)$ denotes the set of those formulas in $t(|c_n|)$ in which all parameters are from among vertices represented by coding nodes in A with length less than $|c_n|$, along with v_n.

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Given $A, B \subseteq \mathbb{S}$ and m, n, write $s(c_m; A) \sim t(c_n; B)$ whenever there is a bijection between the coding nodes in A of length less than $|c_m|$ and the coding nodes in B of length less than $|c_n|$, and the order-preserving bijection between those vertices takes $s(c_m; A)$ to $t(c_n; B)$.

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Fact. If A and B have coding nodes $\langle c_n^A : n < N \rangle$ and $\langle c_n^B : n < N \rangle$, and for all $m < n < N$, $c_n^A(c_m^A; A) \sim c_n^B(c_m^B; B)$, then the subtructures of $\mathbf{K}$ represented by A and B are order-isomorphic.

Similarity Maps in Coding Trees of 1-types

Let S and T be meet-closed subsets of $\mathbb{S}$. A function $f : S \to T$ is a similarity map from S to T if for all nodes $s, t \in \mathbb{S}$, the following hold:

1. f is a bijection which preserves the lexicographic order in $\mathbb{S}$.
2. f preserves meets, and hence splitting nodes.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves coding nodes and their parameter-free formulas.
6. f preserves relative passing types at coding nodes: $s(c_n^S; S) \sim f(s)(c_n^T; T)$, for each n such that $|c_n^S| < |s|$.

We write $S \sim T$ whenever there exists a similarity map from S to T.

SDAP$^+$ implies simply characterized big Ramsey degrees

Theorem. (CDP) Suppose $\mathcal{K}$ is Fraïssé relational class with finitely many relations satisfying SFAP, or just SDAP$^+$. Given $\mathbf{A} \in \mathcal{K}$, the big Ramsey degree of $\mathbf{A}$, $T(\mathbf{A}, \mathbf{K})$, equals the number of similarity types diagonal antichains of coding nodes in $\mathbb{S}(\mathbf{K})$ representing a copy of $\mathbf{A}$.

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Diagonal antichains are antichains which have meet-closures with branching degree 2, and such that distinct nodes from among the splitting and coding nodes have distinct lengths.

$\mathrm{SDAP}^+ \Longrightarrow$ simply characterized BRD: Proof Ideas

1. If $\mathcal{K}$ satisfies SFAP, or just SDAP^+, then there is a diagonal subtree $\mathbb{T}$ of $\mathbb{S}(\mathbf{K})$ which again codes a copy of $\mathbf{K}$.

2. Do forcing arguments over $\mathbb{T}$. Given $\mathbf{A} \in \mathcal{K}$, show that $T(\mathbf{A}, \mathbf{K})$ is bounded above by the number of similarity types of diagonal antichains coding $\mathbf{A}$. Note: The forcing just conducts an unbounded search for a finite object - we never pass to a generic extension.

3. Take an antichain $\mathbb{D}$ of coding nodes in $\mathbb{T}$ which represents a copy of $\mathbf{K}$. Prove that for each $\mathbf{A} \in \mathcal{K}$, all similarity types of antichains in $\mathbb{D}$ coding $\mathbf{A}$ persist in any subset of $\mathbb{D}$ which again codes a copy of $\mathbf{K}$.

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Fun Fact: This approach bypasses any need for envelopes.

Fraïssé classes with simply characterized BRD

Theorem. (CDP) The following Fraïssé classes have simply characterized big Ramsey degrees, as they satisfy SFAP or SDAP$^+$:

(1) The following Fraïssé classes satisfy SFAP:
   (a) Forb($\mathcal{F}$), where $\mathcal{F}$ is a finite set of finite 3-irreducible structures in a finite relational language.
   (b) Unrestricted Fraïssé classes (generalizing those in (LSV) to higher arities).
   (c) n-partite graphs, for any $n \geq 2$.

(2) The following Fraïssé limits satisfy SDAP$^+$:
   (a) $\mathbb{Q}$, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$, $(\mathbb{Q}_{\mathbb{Q}})_n$, the main reducts of $\mathbb{Q}$.
   (b) Fraïssé limits of ordered Fraïssé classes satisfying SFAP.

Fraïssé classes with not-as-simply characterized big Ramsey degrees

k-Clique-Free Generic Graphs = Henson Graphs

For $k \geq 3$, a k-clique, denoted K_k, is a complete graph on k vertices.

$\mathcal{G}_k$ is the Fraïssé class of all finite K_k-free graphs.

$\mathcal{H}_k$, the k-clique-free Henson graph, is the Fraïssé limit of $\mathcal{G}_k$.

Henson graphs are the k-clique-free analogues of the Rado graph.
They were constructed by Henson in 1971.

Henson Graphs: History of Results

- For each $k \geq 3$, $\mathcal{H}_k$ is weakly indivisible. (Henson, 1971)
- The Fraïssé class of finite ordered K_k-free graphs has the Ramsey property. (Nešetřil-Rödl, 1977/83)
- $\mathcal{H}_3$ is indivisible. (Komjáth-Rödl, 1986)
- For all $k \geq 4$, $\mathcal{H}_k$ is indivisible. (El-Zahar-Sauer, 1989)
- Edges have big Ramsey degree 2 in $\mathcal{H}_3$. (Sauer, 1998)
- For each $k \geq 3$, $\mathcal{G}_k$ has Finite BRD (Dobrinen, 2020 and 2019*)
- Given finitely many binary relations and $\mathcal{F}$ a finite set of finite irreducible structures, Forb($\mathcal{F}$) has Finite BRD. (Zucker 2020*)

Binary relational structures with forbidden irreducible substructures: Exact big Ramsey degrees

A structure **F** is irreducible if any two vertices in **F** are in some **F**-relation.

In a forthcoming paper, Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena, and Zucker characterize the exact big Ramsey degrees for binary relational free amalgamation classes with finitely many forbidden irreducible substructures:

Theorem. (BCDHKVZ) Let $\mathcal{L}$ be a finite binary relational language. Let $\mathcal{F}$ be a set of finitely many finite irreducible $\mathcal{L}$-structures, and let Forb($\mathcal{F}$) be the Fraïssé class of finite $\mathcal{L}$ structures **A** such that not member of $\mathcal{F}$ embeds into **A**. Then the big Ramsey degrees of Forb($\mathcal{F}$) are characterized.

Example: Coding tree for $\mathcal{H}_3$

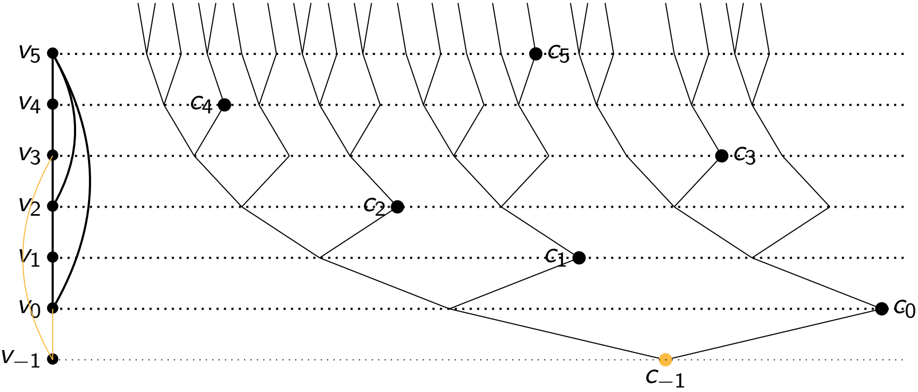
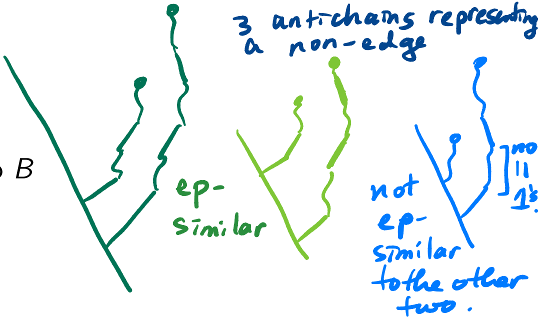


Figure: A coding tree of 1-types $\mathbb{S}_3$ coding $\mathcal{H}_3$

Exact BRD for triangle-free graphs

An essential linked pair is a pair of nodes $s, t \in \mathbb{S}_3$ such that s and t both code an edge with a common vertex of $\mathcal{H}_3$, but the least vertex of $\mathcal{H}_3$ with which s codes an edge differs from the least vertex of $\mathcal{H}_3$ with which s codes an edge.

Two trees $A, B \subseteq \mathbb{S}_3$ are ep-similar if A and B are similar, and the similarity map from A to B preserves the order in which new essential linked pairs appear.



Theorem. (BCDHKVZ) Let $\mathbf{A} \in \mathcal{G}_3$ be given. The big Ramsey degree of of $\mathbf{A}$ in $\mathcal{H}_3$ is the number of ep-similiarity types of diagonal antichains of coding nodes in $\mathbb{S}_3$ representing a copy of $\mathbf{A}$.

Current Methods, and Future Directions

Current Methods:

- Milliken's Theorem and variations, (no forcing): (Devlin), (Laflamme, Sauer, Vuksanović), (Laflamme,Nguyen Van Thé, Sauer), (BCHKV).
- Coding trees (using forcing on diagonal subtrees): (D), (CDP).
- Coding trees (using forcing on Milliken-style trees): (Zucker), (BCDHKVZ).
- Category theory: (Barbosa), (Mašulović).
- Parameter words: (Hubička)

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Future directions:

- BRD for free amalgamation more generally.
- BRD for strong amalgamation more generally.
- Infinite dimensional Ramsey theory: Rado graph done in (D. 2019*), Structures with SDAP$^+$ - in preparation (D. 2021*). Others?

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