Ramsey theory on infinite graphs

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Infinite Ramsey's Theorem

Infinite Ramsey's Theorem. (Ramsey, 1930) Given $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite subset $X \subseteq \omega$ such that c is constant on $[X]^k$.

$$(\forall k, r \geq 1) \;\; \omega \to (\omega)^k_r$$

Graph Interpretation: For $k \geq 1$, given a complete k-hypergraph on infinitely many vertices and a coloring of the k-hyperedges into finitely many colors, there is an infinite complete sub-hypergraph in which all k-hyperedges have the same color.

Extensions of Ramsey's Theorem to the Rado graph

The **Rado graph** is the random graph on infinitely many vertices.

Fact. (Henson 1971) The Rado graph is **indivisible**: Given any coloring of vertices into finitely many colors, there is a subgraph isomorphic to the original in which all vertices have the same color.

But when it comes to coloring edges in the Rado graph,

Theorem. (Erdős, Hajnal, Pósa 1975) There is a coloring of the edges of the Rado graph into two colors such that every subgraph which is again Rado has edges of both colors.

So an exact analogue of Ramsey's theorem for the Rado graph breaks. But how badly, and what is really going on?

"Big" Ramsey Degrees

Let $\mathcal{R}$ denote the Rado graph and G be a finite graph.

The **big Ramsey degree of G in $\mathcal{R}$** is the smallest number T (if it exists) such that for any coloring of all copies of G in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \le \mathcal{R}$, with $\mathcal{R}' \cong \mathcal{R}$, such that the copies of G in $\mathcal{R}'$ take no more than T colors. This number is denoted $T(\mathrm{G}, \mathcal{R})$.

$$(\forall k \ge 2) \;\; \mathcal{R} \to (\mathcal{R})^{\mathrm{G}}_{k, T(\mathrm{G},\mathcal{R})}$$

Erdős-Hajnal-Pósa's result says that $T(\mathrm{Edge}, \mathcal{R}) \ge 2$.

Pouzet and Sauer (1996) showed that, in fact, $T(\mathrm{Edge}, \mathcal{R}) = 2$.

Theorem. Each finite graph G has finite big Ramsey degree in the Rado graph. Moreover, these numbers $T(\mathrm{G}, \mathcal{R})$ can be computed. (Sauer 2006, Laflamme-Sauer-Vuksanovich 2006, J. Larson 2008)

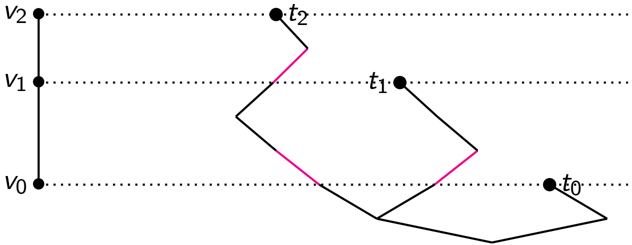
Ideas behind finite big Ramsey degrees for the Rado graph

The lexicographic order of Erdős-Hajnal-Pósa can be visualized via trees.

Let G be a graph with vertices $\langle v_n : n < N \rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes G if and only if for each pair $m < n < N$,

$$v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1.$$

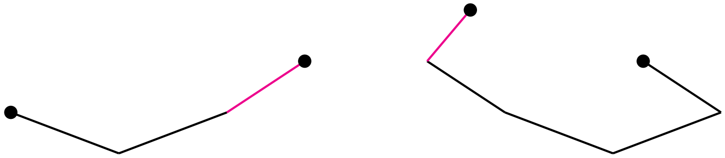
The number $t_n(|t_m|)$ is he **passing number** of t_n at t_m.



The nodes t_0, t_1, t_2 code the path $v_0 \ E \ v_1 \ E \ v_2$.

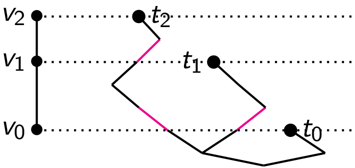
Similarity types in the lexicographic order

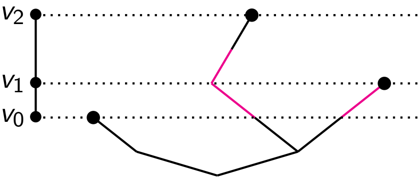
The **similarity type** of a set of binary sequences coding a graph takes into account tree structure and passing numbers.



These are distinct **similarity types** each coding an edge.

Some similarity types coding a path of length 2





Ideas behind finite big Ramsey degrees of Rado graph

Let a finite graph G and a finitary coloring of all copies of G in $\mathcal{R}$ be given.

Let T be the set of all binary sequences of finite length.

Let $\mathcal{U}$ be the graph coded by T: For $s, t \in T$ with $|s| < |t|$,
$v_s \, E \, v_t \Leftrightarrow t(|s|) = 1$. If $|s| = |t|$ then $v_s \not\!E \, v_t$. (s codes the vertex v_s.)

This graph $\mathcal{U}$ is universal for countable graphs.

Apply Milliken's Ramsey theorem for strong trees; get a subtree $T' \subseteq T$ with one color per similarity type of antichain coding G.

Pull out a Rado graph $\mathcal{R}'$ coded by an antichain in T'. This $\mathcal{R}'$ has with one color per strong similarity type for G.

Other structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$, the dense local order $\mathbf{S(2)}$, and $\mathbf{S(3)}$. (Laflamme, NVT, Sauer 2010)
- The random k-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)
- Profinite graphs. (Huber-Geschke-Kojman, and Zheng 2018)

Connections: big Ramsey degrees and topological dynamics

Motivation. Problem 11.2 in (KPT 2005) and (Zucker 2019).

(Kechris, Pestov, Todorcevic 2005) The KPT Correspondence: A Fraïssé class $\mathcal{K}$ has the Ramsey property iff $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ is extremely amenable.

(Zucker 2019) Characterized universal completion flows of $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ whenever $\mathrm{Flim}(\mathcal{K})$ admits a big Ramsey structure (big Ramsey degrees with a coherence property).

k-Clique-Free Random Graphs

The k-clique-free Henson graph, $\mathcal{H}_k$, is the ultrahomogenous k-clique-free graph which is universal for all k-clique-free graphs on countably many vertices.

Henson graphs are the k-clique-free analogues of the Rado graph.

They were constructed by Henson in 1971.

Henson Graphs: History of Results

- For each $k \geq 3$, $\mathcal{H}_k$ is weakly indivisible (Henson, 1971).

- The Fraïssé class of finite ordered K_k-free graphs has the Ramsey property. (Nešetřil-Rödl, 1977/83)

- $\mathcal{H}_3$ is indivisible. (Komjáth-Rödl, 1986)

- For all $k \geq 4$, $\mathcal{H}_k$ is indivisible. (El-Zahar-Sauer, 1989)

- Edges have big Ramsey degree 2 in $\mathcal{H}_3$. (Sauer, 1998)

There progress halted. Why?

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Henson graphs have finite big Ramsey degrees

Theorem. (D.) Let $k \geq 3$. For each finite k-clique-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{G}_k)$ such that for any coloring of all copies of A in $\mathcal{H}_k$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_k$, with $\mathcal{H} \cong \mathcal{H}_k$, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{G}_k)$ colors.

Structure of Proof

Proof Strategy:

I. Develop trees with coding nodes to represent $\mathcal{H}_k$.
Idea: Subtrees isomorphic to the original will also code $\mathcal{H}_k$.

II. Prove a Ramsey Theorem for certain kinds of finite antichains in these trees.
This is where set theory is used.

III. Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_k$.

Trees with Coding Nodes

A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c\rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

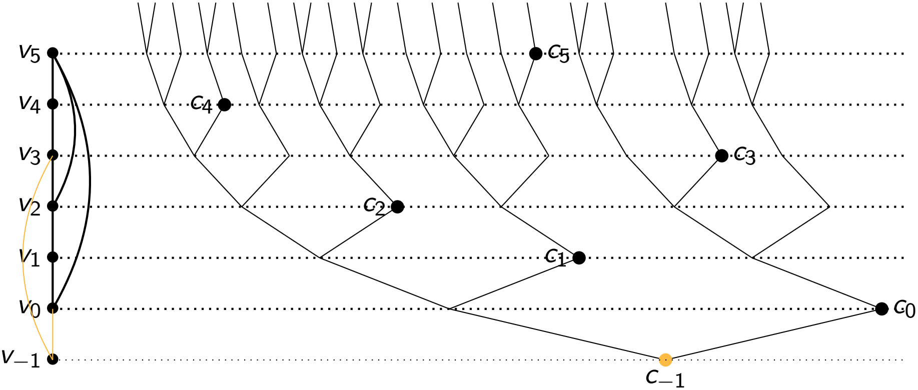
$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n-th coding node in T, usually denoted c^T_n.

$c(n)$ codes the n-th vertex of some ordered graph. They keep track of when nodes should not split.

Strong K_3-Free Tree densely coding $\mathcal{H}_3$



Each level codes the finite partial types over finite graph coded so far.

Strong K_4-Free Tree

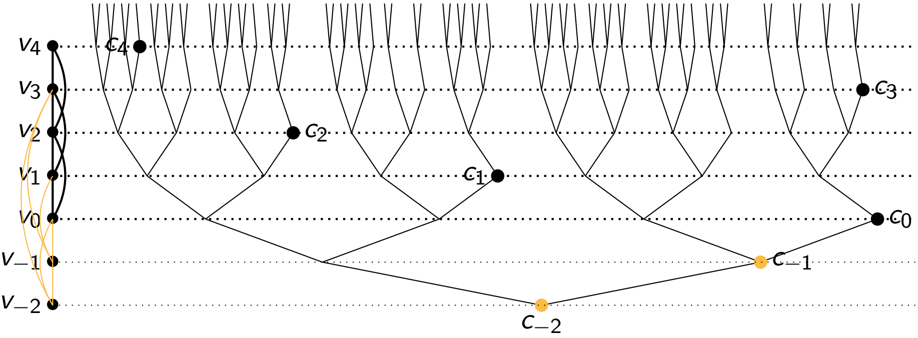


Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

Each level codes the finite partial types over finite graph coded so far.

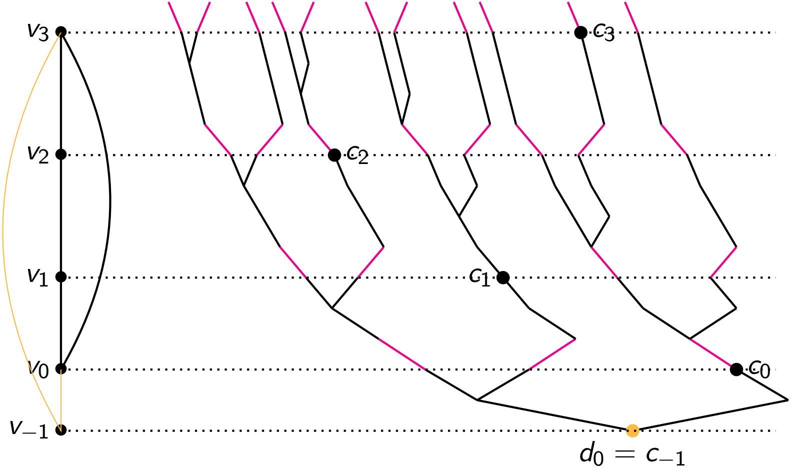
Almost sufficient

One can develop almost all the Ramsey theory one needs on strong K_k-free trees

except for vertex colorings: there is a bad coloring of coding nodes.

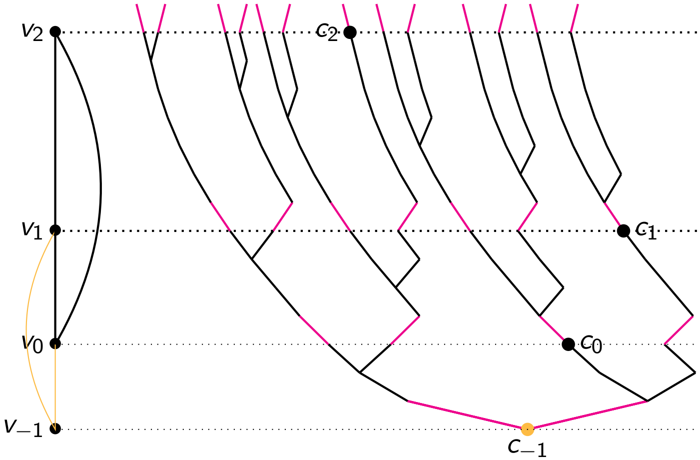
Solution: Skew the levels of interest.

Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



A subtree $T \subseteq \mathbb{T}_3$ is a **strong coding tree** if it is stably isomorphic to $\mathbb{T}_3$.

Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$



A subtree $T \subseteq \mathbb{T}_4$ is a **strong coding tree** if it is stably isomorphic to $\mathbb{T}_4$.

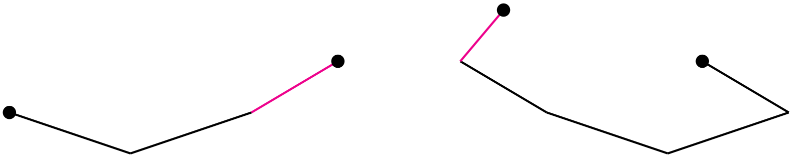
Ramsey Theorem for Strictly Similar Antichains

Theorem. (D.) Let G be a finite graph, and color all copies of G in the Henson graph $\mathcal{H}_k$ ($k \geq 3$) into finitely many colors. Transfer the coloring to antichains in $\mathbb{T}_k$ which code G. Then there is a subtree $T \subseteq \mathbb{T}_k$ which codes $\mathcal{H}_k$ and in which all strictly similar antichains coding G have the same color.

This gives an upper bound for the big Ramsey degree of G in $\mathcal{H}_k$.

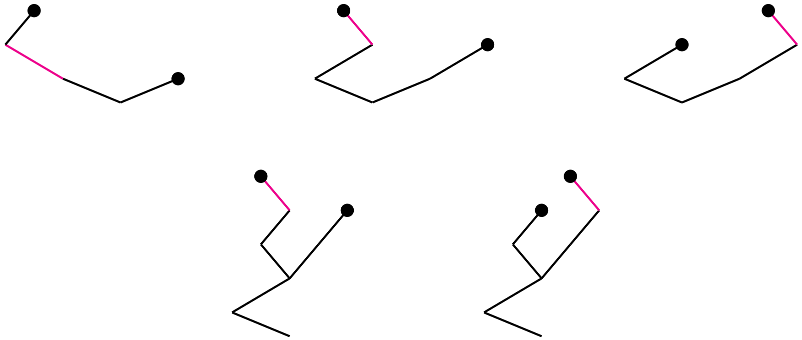
The proof of this theorem uses the method of forcing to do an unbounded search for a finite object. Builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



$T(Edge, \mathcal{H}_3) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have big Ramsey degree 5 in $\mathcal{H}_3$ (D.)



Strict similarity types code information about how the copy of a finite graph G sits within the original ordered Henson graph.

Although trees with coding nodes were invented in order to work with forbidden k-cliques, they turn out to be useful for the Rado graph itself to find infinite dimensional Ramsey theory.

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is Ramsey if each non-empty open set $\mathcal{O} \subseteq [\omega]^\omega$ contains another non-empty open subset $\mathcal{O}' \subseteq \mathcal{O}$ such that either $\mathcal{O}' \subseteq \mathcal{X}$ or else $\mathcal{O}' \cap \mathcal{X} = \emptyset$.

Nash-Williams Theorem. (1965) Clopen sets are Ramsey.

Galvin-Prikry Theorem. (1973) Borel sets are Ramsey.

Silver Theorem. (1970) Analytic sets are Ramsey.

Ellentuck Theorem. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \to_* (\omega)^\omega$$

Infinite Dimensional Structural Ramsey Theory

(KPT 2005) Given $\mathbb{K} = \mathsf{Flim}(\mathcal{K})$ and some natural topology on $\binom{\mathbb{K}}{\mathbb{K}}$,

$$\mathbb{K} \to_* (\mathbb{K})^{\mathbb{K}}$$

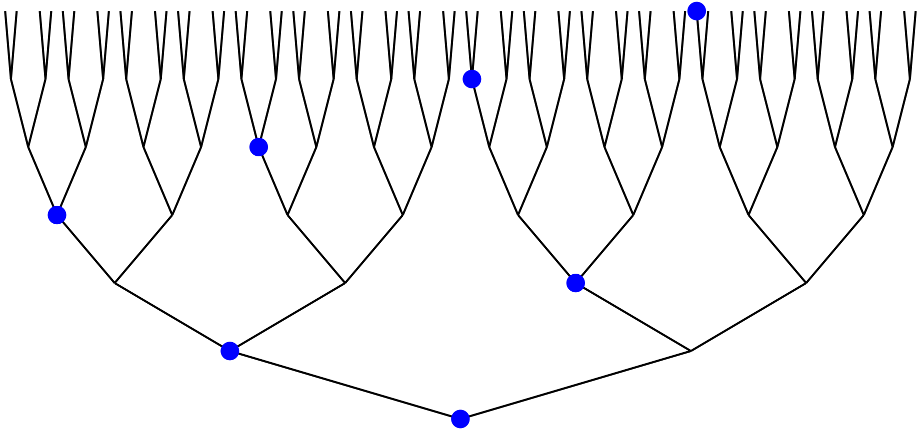
means that all "definable" subsets of $\binom{\mathbb{K}}{\mathbb{K}}$ are Ramsey.

Problem 11.2 in (KPT 2005). Develop infinite dimensional Ramsey theory for Fraïssé structures.

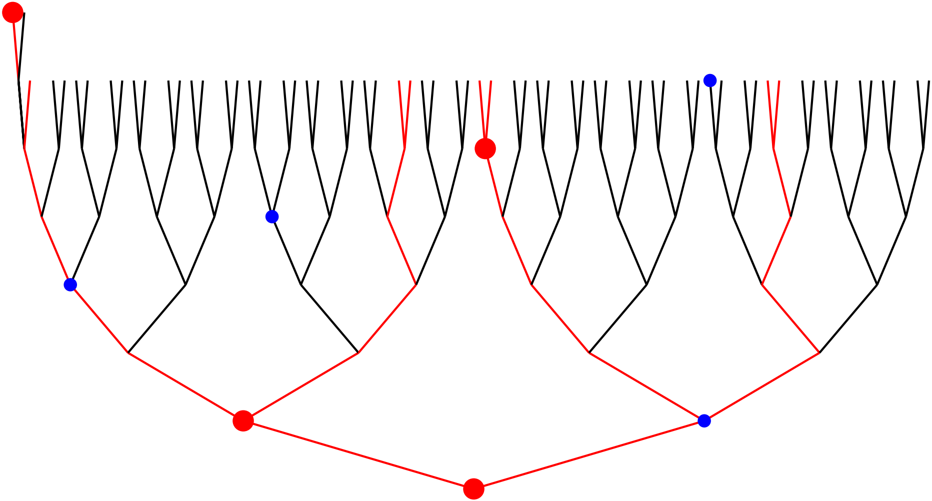
Remark. Very little known. Any topological Ramsey space has definable sets being Ramsey, but members of such spaces are usually not Fraïssé limits.

Theorem. (D.) There is a natural topological space of Rado graphs in which every Borel subset is Ramsey.

Strong Rado Coding Tree $\mathbb{T}_{\mathcal{R}}$



A Strong Rado Coding Tree $T \in \mathcal{T}_{\mathcal{R}}$



Thm. (D.) Every Borel subset of $\mathcal{T}_{\mathcal{R}}$ is Ramsey. That is, if $\mathcal{X} \subseteq \mathcal{T}_{\mathcal{R}}$ is Borel, then

$$(*) \qquad \forall [s, A] \ \ \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset.$$

So there is a topological space of Rado graphs which has infinite dimensional Ramsey theory.

Proof Ideas.

1. Show that all open sets are Ramsey.
2. Show that complements of Ramsey sets are Ramsey.
3. Show that Ramsey sets are closed under countable unions.

The catch is (1) and (3). We use a forcing argument utilizing methods from our work on the big Ramsey degrees of the Henson graphs.

Remarks, Questions and Future Directions

What other universal and/or ultrahomogeneous structures have finite big Ramsey degrees or infinite dimensional Ramsey theory?

Trees with coding nodes and forcing arguments were developed to work with forbidden k-cliques, but have shown to be useful for infinite dimensional Ramsey theory of the Rado graph. For what other structures will they be of aid?

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