Strong coding trees and Ramsey theory on infinite structures

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Subtitle: How forcing helps solve some problems in combinatorics.

Ramsey's Theorems

Finite Ramsey Theorem. Given $k, m, r \geq 1$, there is an $n \geq m$ such that given a coloring $c : [n]^k \to r$, there is an $X \subseteq n$ of size m such that c is constant on $[X]^k$.

$$(\forall k, m, r \geq 1) \ \ (\exists n \geq m) \ \ n \to (m)^k_r$$

Infinite Ramsey's Theorem. (finite dimensional) Given $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite subset $X \subseteq \omega$ such that c is constant on $[X]^k$.

$$(\forall k, r \geq 1) \ \ \omega \to (\omega)^k_r$$

Graph Interpretation: k-hypergraphs.

Infinite Dimensional Ramsey Theory

A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is Ramsey if for each $X \in [\omega]^\omega$, there is a $Y \in [X]^\omega$ such that either $[Y]^\omega \subseteq \mathcal{X}$ or else $[Y]^\omega \cap \mathcal{X} = \emptyset$.

Nash-Williams Theorem. (1965) Clopen sets are Ramsey.

Galvin-Prikry Theorem. (1973) Borel sets are Ramsey.

Silver Theorem. (1970) Analytic sets are Ramsey.

Ellentuck Theorem. (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \to_* (\omega)^\omega$$

Fraïssé Classes and Their Limits

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Finite graphs $\mathcal{G}$; $\mathrm{Flim}(\mathcal{G}) =$ Rado graph.

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Linear orders, complete graphs, Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

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Small Ramsey degrees: Bounds but not one color.

Example: Colorings of Subgraphs

An ordered graph A embeds into an ordered graph B if there is a one-to-one mapping of the vertices of A into some of the vertices of B such that each edge in A gets mapped to an edge in B, and each non-edge in A gets mapped to a non-edge in B.

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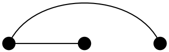

Figure: A

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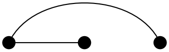


Figure: A

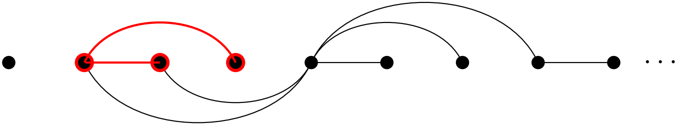
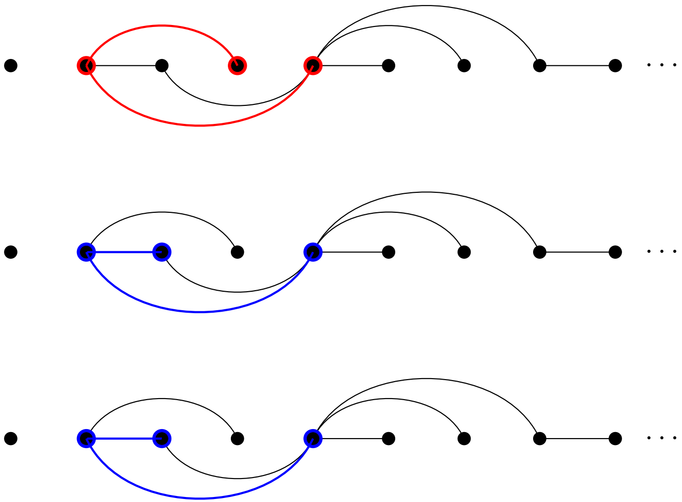


Figure: A copy of A in B

More copies of A into B



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(KPT 2005) For $A \in \mathcal{K}$, $T(A, \mathcal{K})$ is the least number T, if it exists, such that for each $k \geq 1$ and any coloring of the copies of A in $\mathbb{K}$, there is a substructure $\mathbb{K}' \leq \mathbb{K}$, isomorphic to $\mathbb{K}$, in which the copies of A have no more than T colors.

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Motivation. Problem 11.2 in (KPT 2005) and (Zucker 2019).

Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S(2)}$, $\mathbf{S(3)}$. (Laflamme, NVT, Sauer 2010)
- The random k-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)
- Profinite graphs. (Huber-Geschke-Kojman, and Zheng 2018)
- Profinite k-clique-free graphs. (Dobrinen, Wang 2019)

Infinite Dimensional Structural Ramsey Theory

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Examples. The Baire space $[\omega]^\omega = \mathbb{I}_\omega$.

Any topological Ramsey space. But most known ones are not ultrahomogeneous structures.

(Dobrinen) The rationals, the Rado graph, and (to be checked) the Henson graphs.

Bold Conjecture

Any Fraïssé class with small Ramsey degrees has Fraïssé limit with finite big Ramsey degrees and an infinite dimensional Ramsey theorem.

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(2) Milliken's Ramsey theorem for strong trees, and variants.

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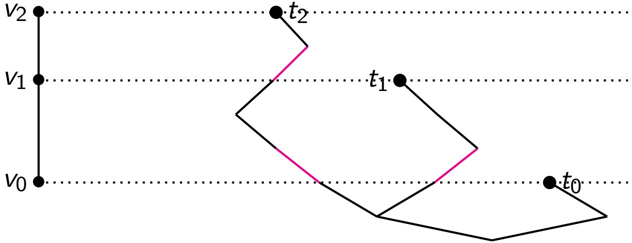
Graphs. Let A be a graph with vertices $\langle v_n : n < N \rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



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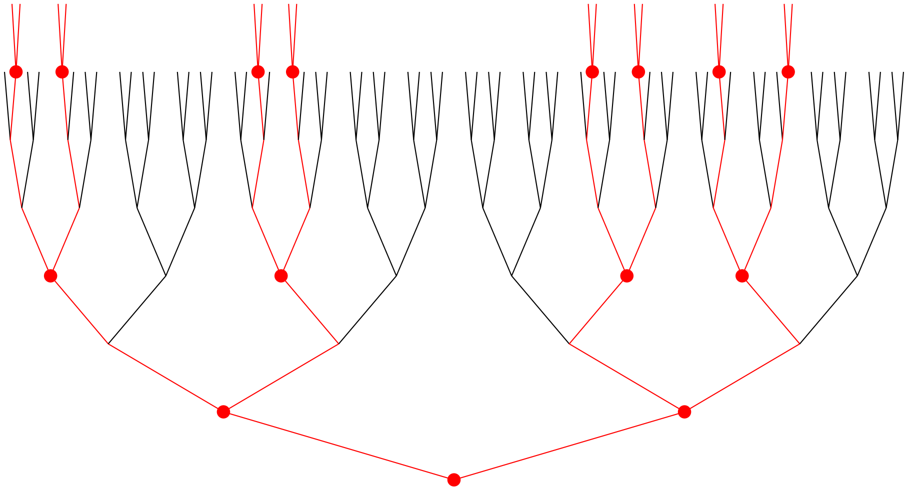
$S \subseteq T$ is a strong subtree of T iff for some $\{m_n : n < N\}$ ($N \leq \omega$),

1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < N$, $s \in S(n)$ and $u \in \text{Succ}_T(s)$,
   there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $0, 1, 3, 6, \ldots$

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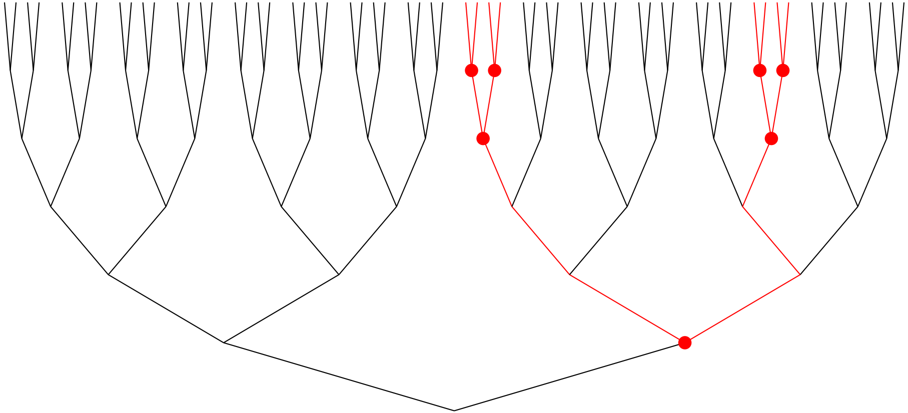

The nodes in U are of lengths $1, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees

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Thm. (Milliken 1979) Let $T \subseteq 2^{<\omega}$ be a strong tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

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The main tool for Milliken's theorem is the Halpern-Läuchli Theorem for colorings on products of trees.

Harrington devised a "forcing proof" of Halpern-Läuchli Theorem. This is very important to our approach to Ramsey theory on Fraïssé limits.

Big Ramsey degrees of the Rado graph

- Vertices have big Ramsey degree 1. (Henson 1971)
- Edges have big Ramsey degree ≥ 2. (Erdős-Hajnal-Pósa 1975)
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- Actual big Ramsey degrees found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

Henson Graphs $\mathcal{H}_k$: The k-Clique-Free Random Graph

K_k denotes a complete graph on k vertices, also called a k-clique.

The the k-clique-free Henson graph, $\mathcal{H}_k$, is the Fraïssé limit of the Fraïssé class of finite K_k-free graphs.

Thus, $\mathcal{H}_k$ is the ultrahomogenous K_k-free graph which is universal for all k-clique-free graphs on countably many vertices.

Henson graphs were constructed by Henson in 1971.

Henson Graphs: History of Results

- For each $k \geq 3$, $\mathcal{H}_k$ is weakly indivisible. (Henson 1971)

- The Fraïssé class of finite ordered K_k-free graphs has the Ramsey property. (Nešetřil-Rödl 1977/83)

- $\mathcal{H}_3$ is indivisible. (Komjáth-Rödl 1986)

- For all $k \geq 4$, $\mathcal{H}_k$ is indivisible. (El-Zahar-Sauer 1989)

- Edges have big Ramsey degree 2 in $\mathcal{H}_3$. (Sauer 1998)

- For each $k \geq 3$, $\mathcal{H}_k$ has finite big Ramsey degrees. (Dobrinen 2017 and 2019)

New Methods

Problem for Henson graphs: no Milliken theorem, and no nicely definable structure which is bi-embeddable with $\mathcal{H}_k$.

Question. How do you make a tree that codes a K_k-free graph which branches enough to carry some Ramsey theory?

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Key Ideas in the proof that Henson graphs have finite big Ramsey degrees include

(1) Trees with coding nodes.

(2) Use forcing mechanism to obtain (in ZFC) new Milliken-style theorems for trees with coding nodes.

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III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_k$.

Trees with Coding Nodes

A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c\rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n-th coding node in T, usually denoted c_n^T.

Top-down approach to Strong Coding Trees

Let $k \geq 3$ be fixed.

Order the vertices of $\mathcal{H}_k$ in order-type ω as $\langle v_n : n < \omega \rangle$.

Let the n-th coding node, c_n, code the n-th vertex.

Strong K_3-Free Tree

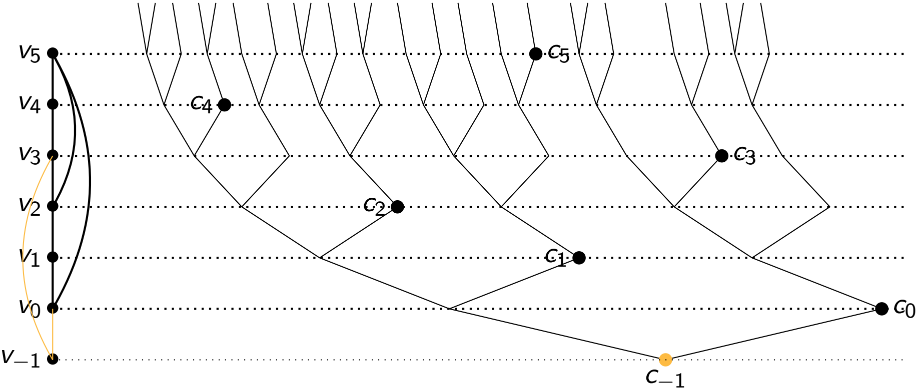
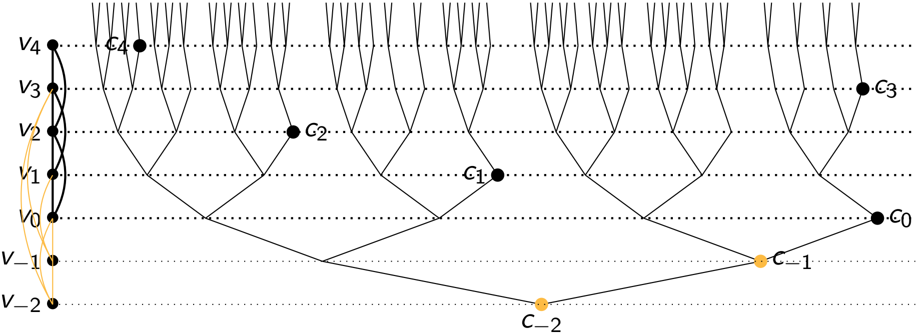


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree



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Bottom-up Approach

Note: A collection of coding nodes $\{c_{n_i} : i < k\}$ in T codes a k-clique iff $i < j < k \longrightarrow c_{n_j}(|c_{n_i}|) = 1$.

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A tree T with coding nodes $\langle c_n : n < N\rangle$ satisfies the K_k-Free Branching Criterion (k-FBC) if for each non-maximal node $t \in T$, $t^\frown 0 \in T$ and

(∗) $t^\frown 1$ is in T iff adding $t^\frown 1$ as a coding node to T would not code a k-clique with coding nodes in T of shorter length.

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A tree with coding nodes satisfies $(A_k)^{\mathtt{tree}}$ iff

(i) T satisfies the K_k-Free Criterion.

(ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i - 1$, and each finite subset of ω appears as F_i for infinitely many indices i. Given $i < \omega$, if for each subset $J \subseteq F_i$ of size $k-1$, $\{c_j : j \in J\}$ does not code a $(k-1)$-clique, then there is some $n \geq i$ such that for all $j < i$, $c_n(l_j) = 1$ iff $j \in F_i$.

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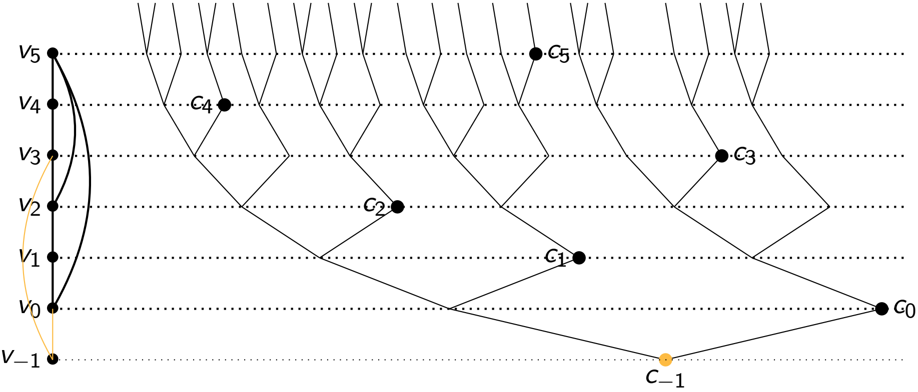
Thm. (D.) Suppose T is a tree with no maximal nodes satisfying the K_k-Free Branching Criterion, and the set of coding nodes dense in T. Then T satisfies $(A_k)^{\texttt{tree}}$, and hence codes $\mathcal{H}_k$.

Strong K_3-Free Tree



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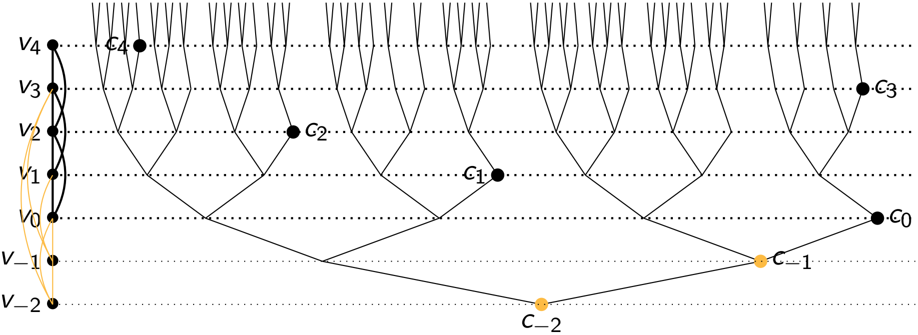
Strong K_4-Free Tree



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Almost sufficient

We want to obtain a Ramsey theorem that says, “Given a coloring for a finite antichain A of coding nodes inside a strong coding tree T, there is a subtree S of T which is ‘isomorphic’ to T in which all ‘copies’ of A have the same color.”

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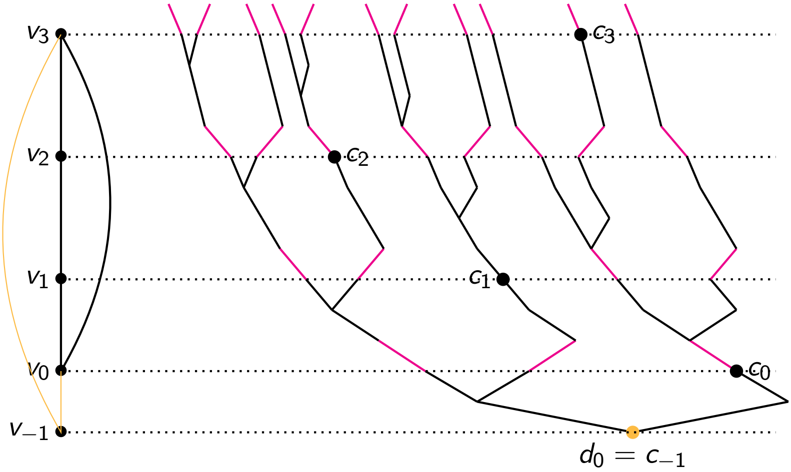
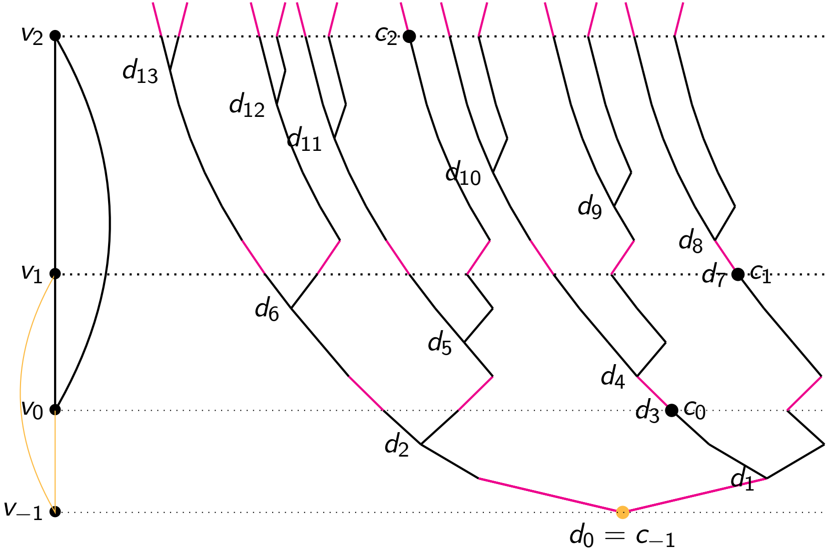
Solution: Skew the levels of interest.

Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$



Defining the Space of Strong Coding Trees

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level set $X \subseteq \mathbb{T}_k$ with nodes of length ℓ_X, has a pre-a-clique if there are $a - 2$ coding nodes in $\mathbb{T}_k$ coding an $(a - 2)$-clique, and each node in X has passing number 1 by each of these coding nodes.

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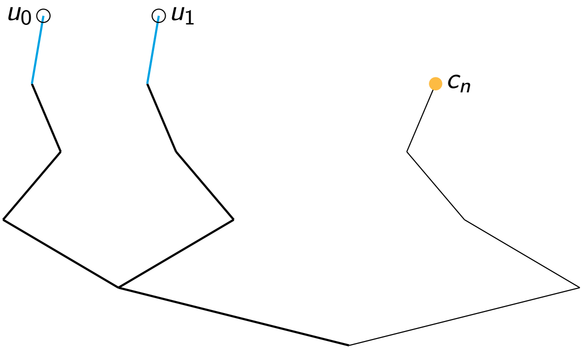
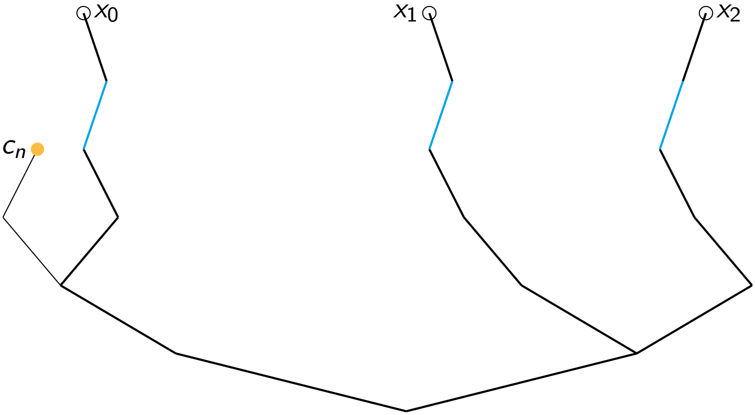
The Point. Pre-a-cliques for $a \in [3, k]$ code entanglements that affect how nodes in X can extend inside $\mathbb{T}$.

A level set U with a pre-3-clique



The yellow node is a coding node in $\mathbb{T}_k$ not in U.

A level set X with a pre-3-clique



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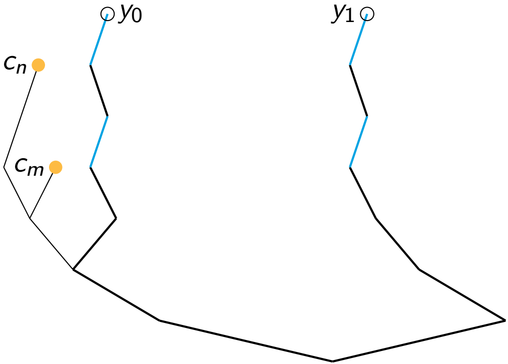
A level set Y with a pre-4-clique



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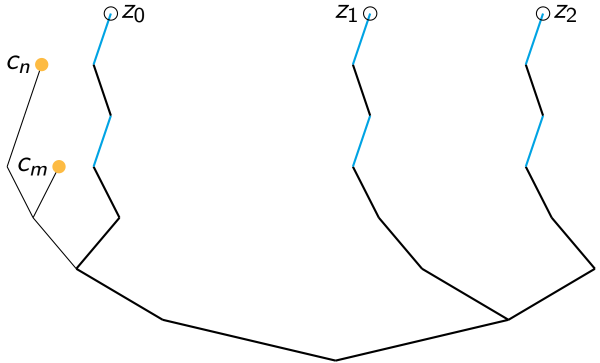

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The Space of Strong $\mathcal{H}_k$-Coding Trees $\mathcal{T}_k$

Two subtrees S and T of $\mathbb{T}_k$ are strongly isomorphic iff there is a strong similarity map $f : S \to T$ which preserves maximal new pre-cliques in each interval.

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A series of "Extension Lemmas" guarantee when level sets can be extended as wished to new configurations within a given $T \in \mathcal{T}_k$.

Part II: Ramsey Theorem for Strictly Similar Finite Antichains

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(c) New notion of envelope to move from Strictly Witnesses finite trees to any finite antichain of coding nodes.

Ramsey Theorem for Strictly Similar Antichains

Thm. (D.) Let Z be a finite antichain of coding nodes in a strong $\mathcal{H}_k$-coding tree $T \in \mathcal{T}_k$, and suppose h colors of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.

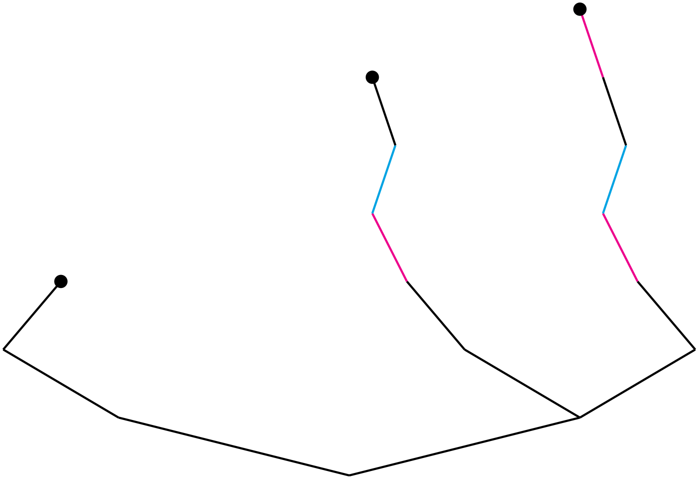
Some Examples of Strict Similarity Types for $k = 3$

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

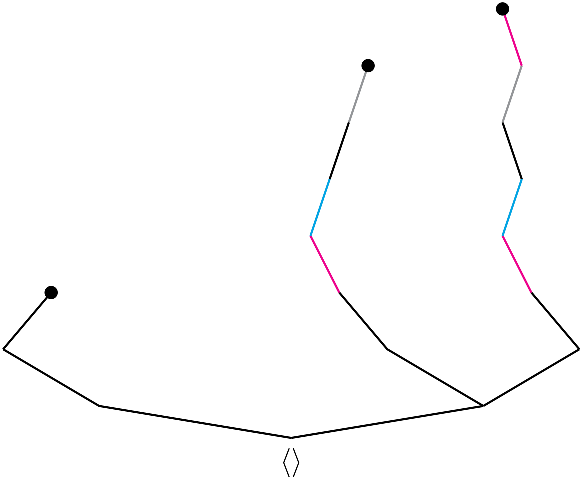
G a graph with three vertices and no edges

A tree A coding G



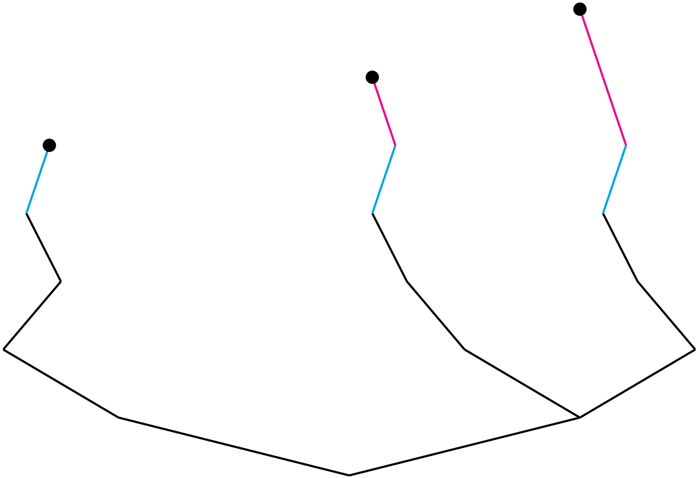
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



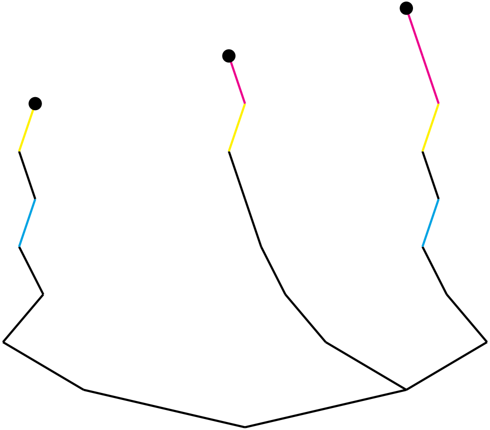
The tree C codes G

C is not strictly similar to A.

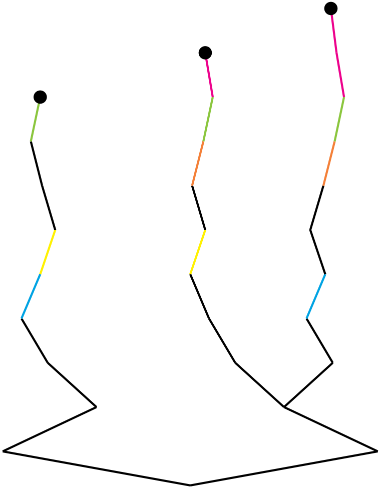


The tree D codes G

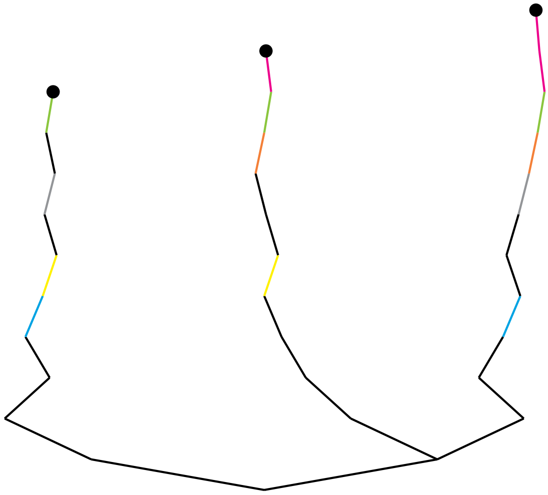
D is not strictly similar to either A or C.



The tree E codes G and is not strictly similar to A - D



The tree F codes G and is strictly similar to E



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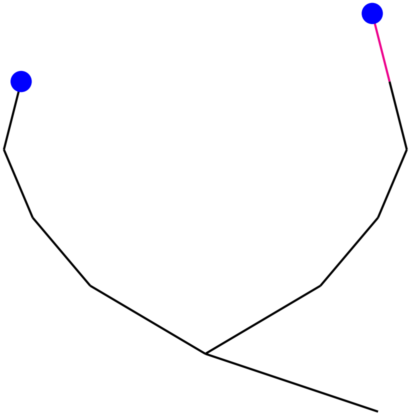
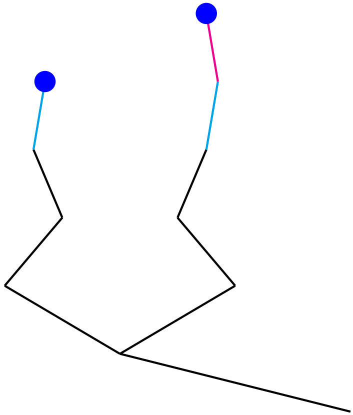
We now give some examples of envelopes.

H codes a non-edge



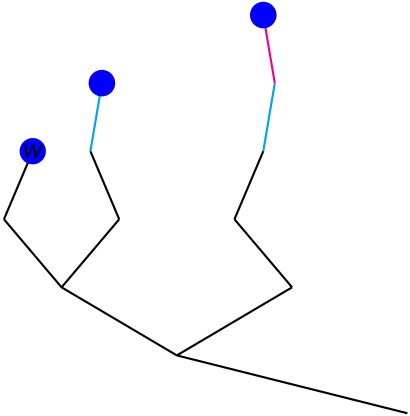
H is its own envelope.

I codes a non-edge



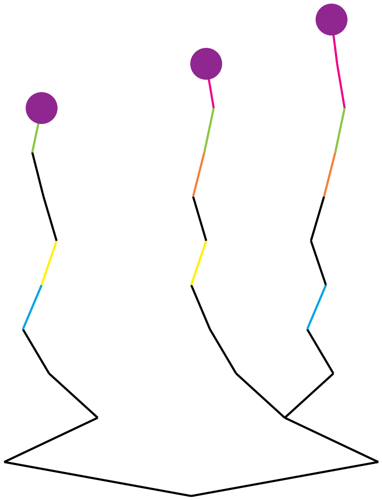
I is not its own envelope.

An Envelope $\mathbf{E}(I)$

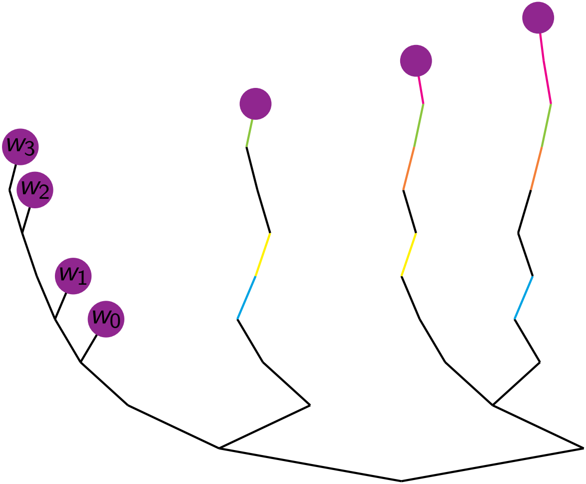


An envelope of I.

The antichain E from before

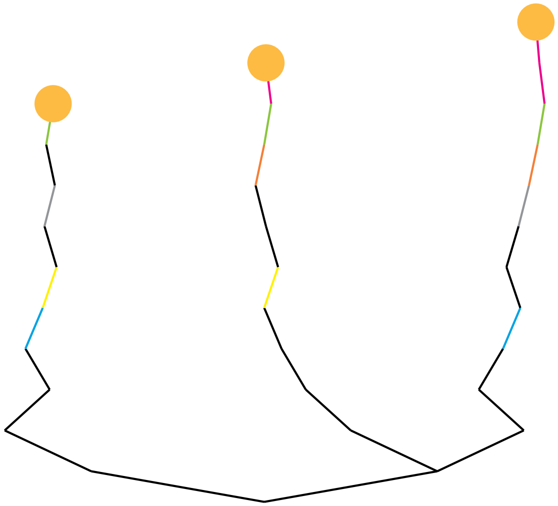


An envelope $\mathbf{E}(E)$

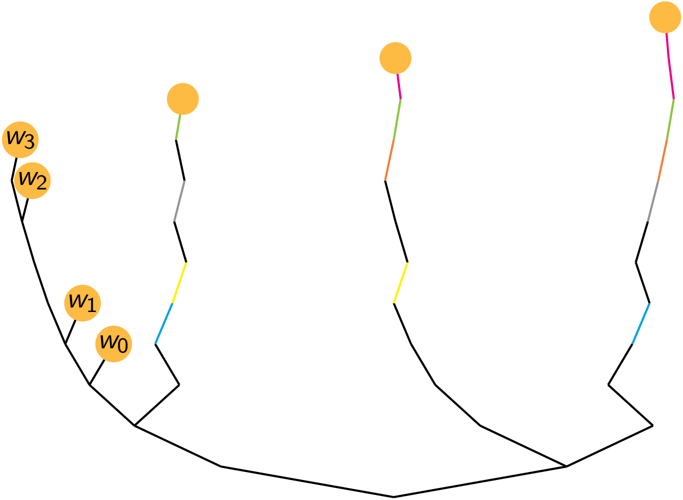


The coding nodes $w_0, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E



$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



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Some upper bounds for big Ramsey degrees in $\mathcal{H}_k$

$T(K_2, \mathcal{H}_3) = 2$

$T(K_2, \mathcal{H}_4) \leq 6$

$T(K_2, \mathcal{H}_5) \leq 88$

$T(\bar{K}_2, \mathcal{H}_3) \leq 7$

$T(\bar{K}_2, \mathcal{H}_4) \leq 58$

Conjecture: The number of incremental strict similarity types of antichains coding a finite graph $G \in \mathcal{K}_k$ **is** the big Ramsey degree $T(G, \mathcal{H}_k)$.

Although trees with coding nodes were invented to handle forbidden cliques, it turns out they are good a coding relational structures with or without forbidden substructures.

Infinite Dimensional Ramsey Theory of the Rado Graph

Say $\mathcal{X} \subseteq [\omega]^\omega$ is completely Ramsey (CR) if for each nonempty $[s, A]$, there is a $B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$.

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Question. (KPT 2005) Which Fraïssé structures have infinite dimensional Ramsey theory for definable subsets?

Infinite Dimensional Ramsey Theory for $\mathbb{Q}$

We approach this using trees with coding nodes.

By Devlin's theorem, one must fix a strong similarity type coding the rationals into $2^{<\omega}$, and restrict to all subtrees with the same strong similarity type.

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Thm. (D.) Let $\mathbb{T}_{\mathbb{Q}} \subseteq 2^{<\omega}$ be a fixed tree with coding nodes coding a copy of the rationals in order type ω, with no terminal nodes. Let $\mathcal{T}_{\mathbb{Q}}$ be the collection of all strongly similar subtrees of $\mathbb{T}_{\mathbb{Q}}$. Then $\mathcal{T}_{\mathbb{Q}}$ is a topological Ramsey space, hence has an analogue of Ellentuck's theorem.

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This should also hold (modulo checking) for antichains in $2^{<\omega}$ coding the rationals. If true, this will recover Devlin's result.

Infinite Dimensional Ramsey Theory for the Rado Graph

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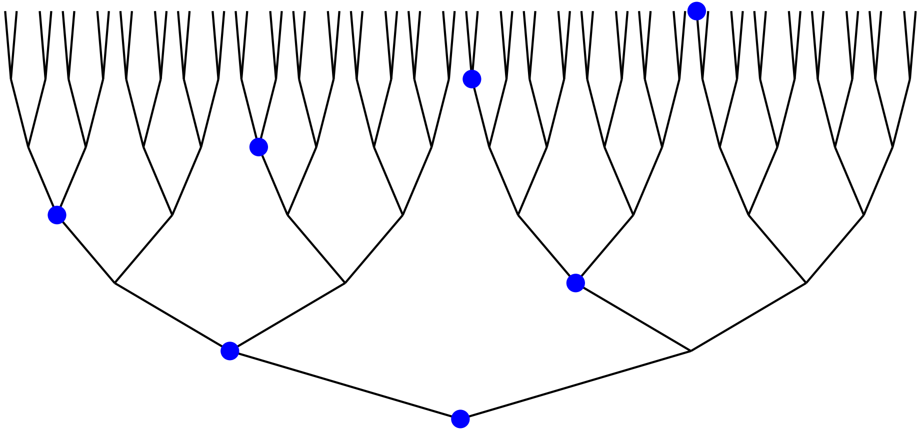
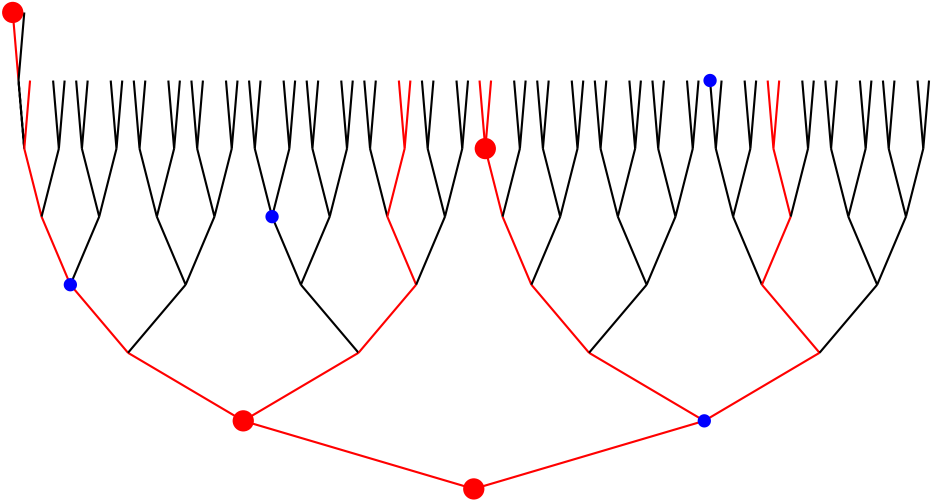
A Strong Rado Coding Tree $\mathbb{T}_{\mathcal{R}}$



A Strong Rado Coding Tree $T \in \mathcal{T}_{\mathcal{R}}$



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So there is a topological space of Rado graphs which has infinite dimensional Ramsey theory.

Current Directions and Future Goals

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   In-progress: Ultrahomogeneous partial order, metric spaces, bowtie-free graph, etc.

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4. (KPT) What is the correspondence between infinite dimensional Ramsey theory and topological dynamics?

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Thank you for your kind attention!

References

Dobrinen, *The Ramsey theory of the universal homogeneous triangle-free graph*, JML (To appear).

Dobrinen, *Ramsey theory of the Henson graphs* (2019) (Almost submitted).

Dobrinen, *Borel of Rado graphs and Ramsey's theorem* (2019) (Submitted).

Dobrinen-Wang, *Profinite k-clique-free graphs and big Ramsey degrees* (2019) (Preprint).

Ellentuck, *A new proof that analytic sets are Ramsey*, JSL (1974).

Erdős-Rado, *A partition calculus in set theory*, Bull. AMS (1956).

Galvin-Prikry, *Borel sets and Ramsey's Theorem*, JSL (1973).

Halpern-Läuchli, *A partition theorem*, TAMS (1966).

Henson, *A family of countable homogeneous graphs*, Pacific Jour. Math. (1971).

References

Laflamme-Nguyen Van Thé-Sauer, *Partition properties of the dense local order and a colored version of Milliken's Theorem*, Combinatorica (2010).

Laflamme-Sauer-Vuksanovic, *Canonical partitions of universal structures*, Combinatorica (2006).

Larson, J. *Counting canonical partitions in the Random graph*, Combinatorica (2008).

Larson, J. *Infinite combinatorics*, Handbook of the History of Logic (2012).

Laver, *Products of infinitely many perfect trees*, Jour. London Math. Soc. (1984).

Kechris-Pestov-Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis (2005).

Milliken, *A Ramsey theorem for trees*, Jour. Combinatorial Th., Ser. A (1979).

Nešetřil-Rödl, *Partitions of finite relational and set systems*, Jour. Combinatorial Th., Ser. A (1977).

References

Nešetřil-Rödl, *Ramsey classes of set systems*, Jour. Combinatorial Th., Ser. A (1983).

Nguyen Van Thé, *Big Ramsey degrees and divisibility in classes of ultrametric spaces*, Canadian Math. Bull. (2008).

Pouzet-Sauer, *Edge partitions of the Rado graph*, Combinatorica (1996).

Sauer, *Edge partitions of the countable triangle free homogeneous graph*, Discrete Math. (1998).

Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica (2006).

Todorcevic, *Introduction to Ramsey spaces* (2010).

Zucker, *Big Ramsey degrees and topological dynamics*, Groups Geom. Dyn. (2019).