Strong coding trees
and
Ramsey theory on infinite structures

Natasha Dobrinen
University of Denver

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Subtitle: How forcing helps solve some problems in combinatorics.
Ramsey’s Theorems

**Finite Ramsey Theorem.** Given $k, m, r \geq 1$, there is an $n \geq m$ such that given a coloring $c : [n]^k \to r$, there is an $X \subseteq n$ of size $m$ such that $c$ is constant on $[X]^k$.

$$(\forall k, m, r \geq 1) \ (\exists n \geq m) \ n \to (m)^k_r$$

**Infinite Ramsey’s Theorem.** (finite dimensional) Given $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite subset $X \subseteq \omega$ such that $c$ is constant on $[X]^k$.

$$(\forall k, r \geq 1) \ \omega \to (\omega)^k_r$$

**Graph Interpretation:** $k$-hypergraphs.
A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is **Ramsey** if for each $X \in [\omega]^\omega$, there is a $Y \in [X]^\omega$ such that either $[Y]^\omega \subseteq \mathcal{X}$ or else $[Y]^\omega \cap \mathcal{X} = \emptyset$.

**Nash-Williams Theorem.** (1965) Clopen sets are Ramsey.

**Galvin-Prikry Theorem.** (1973) Borel sets are Ramsey.

**Silver Theorem.** (1970) Analytic sets are Ramsey.

**Ellentuck Theorem.** (1974) Sets with the property of Baire in the Ellentuck topology are Ramsey.

$$\omega \rightarrow^* (\omega)^\omega$$
Fraïssé Classes and Their Limits

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The **Fraïssé limit** of a Fraïssé class $\mathcal{K}$, denoted $\mathrm{Flim}(\mathcal{K})$ or $\mathcal{K}$, is (up to isomorphism) the ultrahomogeneous structure with $\mathrm{Age}(\mathcal{K}) = \mathcal{K}$.

**Examples.**
- Finite linear orders $\mathcal{LO}$; $\mathrm{Flim}(\mathcal{LO}) = \mathbb{Q}$.
- Finite graphs $\mathcal{G}$; $\mathrm{Flim}(\mathcal{G}) = \text{Rado graph}$. 
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Finite graphs $\mathcal{G}$; $\text{Flim}(\mathcal{G}) = \text{Rado graph}$. 
For structures $A, B$, write $A \leq B$ iff $A$ embeds into $B$. 

A Fra"{i}ss"{e} class $K$ has the Ramsey property if $(\forall A \leq B \in K) (\forall r \geq 1) \text{Flim}(K) \rightarrow (B^r)$ 

Some classes of finite structures with the Ramsey property: linear orders, complete graphs, Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting $k$-cliques, ordered metric spaces, and many others.
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Small Ramsey degrees:
- Bounds but not one color.
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Example: Colorings of Subgraphs

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Figure: A

Figure: A copy of A in B
More copies of A into B
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(KPT 2005) For $A \in \mathcal{K}$, $T(A, \mathcal{K})$ is the least number $T$, if it exists, such that for each $k \geq 1$ and any coloring of the copies of $A$ in $\mathbb{K}$, there is a substructure $\mathbb{K}' \leq \mathbb{K}$, isomorphic to $\mathbb{K}$, in which the copies of $A$ have no more than $T$ colors.
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$\mathbb{K}$ has finite big Ramsey degrees if $T(A, \mathcal{K})$ is finite, for each $A \in \mathcal{K}$. 

Motivation.

Problem 11.2 in (KPT 2005) and (Zucker 2019).
Infinite Structural Ramsey Theory (finite dimensional)

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Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- \( \mathbb{Q}_n \) and the directed graphs \( S(2), S(3) \). (Laflamme, NVT, Sauer 2010)
- The random \( k \)-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)
- Profinite graphs. (Huber-Geschke-Kojman, and Zheng 2018)
- Profinite \( k \)-clique-free graphs. (Dobrinen, Wang 2019)
(KPT 2005) Given $\mathcal{K} = \text{Flim}(\mathcal{K})$ and some natural topology on $\mathbb{I}_\mathcal{K} := \binom{\mathcal{K}}{\mathcal{K}}$, $\mathcal{K} \rightarrow^* (\mathcal{K})^\mathcal{K}$ means that all “definable” subsets of $\mathbb{I}_\mathcal{K}$ are Ramsey.
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Infinite Dimensional Structural Ramsey Theory

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**Examples.** The Baire space $[\omega]^\omega = \mathbb{I}_\omega$.

Any topological Ramsey space. But most known ones are not ultrahomogeneous structures.

(Dobrinen) The rationals, the Rado graph, and (to be checked) the Henson graphs.
Any Fraïssé class with small Ramsey degrees has Fraïssé limit with finite big Ramsey degrees and an infinite dimensional Ramsey theorem.
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(2) Milliken’s Ramsey theorem for strong trees, and variants.
Rationals. \((\mathbb{Q}, <)\) can be coded by \(2^{<\omega}\).
Using Trees to Code Binary Relational Structures

**Rationals.** $(\mathbb{Q}, <)$ can be coded by $2^{<\omega}$.

**Graphs.** Let $A$ be a graph with vertices $\langle v_n : n < N \rangle$. A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes $A$ if and only if for each pair $m < n < N$,

$$v_n E v_m \iff t_n(|t_m|) = 1.$$ 

The number $t_n(|t_m|)$ is called the **passing number** of $t_n$ at $t_m$. 
For $t \in 2^{<\omega}$, the length of $t$ is $|t| = \text{dom}(t)$. 
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$T \subseteq 2^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{ t \upharpoonright l : t \in T, \ l \in L \}$.
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For $t \in T$, the height of $t$ is $\text{ht}_T(t) = \text{o.t.}\{ u \in T : u \subseteq t \}$. 
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$T(n) = \{t \in T : \text{ht}_T(t) = n\}$. 
Strong Trees

For \( t \in 2^{< \omega} \), the length of \( t \) is \(|t| = \text{dom}(t)\).

\( T \subseteq 2^{< \omega} \) is a tree if \( \exists L \subseteq \omega \) such that \( T = \{ t \restriction l : t \in T, \ l \in L \} \).

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For \( t \in T \), \( \text{Succ}_T(t) = \{ u \restriction (|t| + 1) : u \in T \text{ and } u \supseteq t \} \).
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$S \subseteq T$ is a strong subtree of $T$ iff for some $\{m_n : n < N\}$ ($N \leq \omega$),

1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < N$, $s \in S(n)$ and $u \in \text{Succ}_T(s)$, there is exactly one $s' \in S(n + 1)$ extending $u$. 

Strong Trees
Example: A Strong Subtree $T \subseteq 2^{<\omega}$

The nodes in $T$ are of lengths 0, 1, 3, 6, \ldots
Example: A Strong Subtree $U \subseteq 2^{<\omega}$

The nodes in $U$ are of lengths 1, 4, 5, \ldots.
A Ramsey Theorem for Strong Trees

A \( k \)-strong tree is a finite strong tree with \( k \) levels.
A Ramsey Theorem for Strong Trees

A $k$-strong tree is a finite strong tree with $k$ levels.

**Thm.** (Milliken 1979) Let $T \subseteq 2^{<\omega}$ be a strong tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and $c$ be a coloring of all $k$-strong subtrees of $T$ into $r$ colors. Then there is a strong subtree $S \subseteq T$ such that all $k$-strong subtrees of $S$ have the same color.
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The main tool for Milliken’s theorem is the Halpern-Läuchli Theorem for colorings on products of trees.

Harrington devised a “forcing proof” of Halpern-Läuchli Theorem. This is very important to our approach to Ramsey theory on Fraïssé limits.
Big Ramsey degrees of the Rado graph

- Vertices have big Ramsey degree $1$. (Henson 1971)
- Edges have big Ramsey degree $\geq 2$. (Erdős-Hajnal-Pósa 1975)
- Edges have big Ramsey degree exactly $2$. (Pouzet-Sauer 1996)
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**Idea:** Since the Rado graph is bi-embeddable with the graph coded by all nodes in $2^{<\omega}$, use Milliken’s Theorem and later take out a copy of the Rado graph to deduce upper bounds for its big Ramsey degrees.
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- Actual big Ramsey degrees found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).
$K_k$ denotes a complete graph on $k$ vertices, also called a $k$-clique.

The $k$-clique-free Henson graph, $\mathcal{H}_k$, is the Fraïssé limit of the Fraïssé class of finite $K_k$-free graphs.

Thus, $\mathcal{H}_k$ is the ultrahomogenous $K_k$-free graph which is universal for all $k$-clique-free graphs on countably many vertices.

Henson graphs were constructed by Henson in 1971.
Henson Graphs: History of Results

• For each \( k \geq 3 \), \( \mathcal{H}_k \) is weakly indivisible. (Henson 1971)

• The Fraïssé class of finite ordered \( K_k \)-free graphs has the Ramsey property. (Nešetřil-Rödl 1977/83)

• \( \mathcal{H}_3 \) is indivisible. (Komjáth-Rödl 1986)

• For all \( k \geq 4 \), \( \mathcal{H}_k \) is indivisible. (El-Zahar-Sauer 1989)

• Edges have big Ramsey degree 2 in \( \mathcal{H}_3 \). (Sauer 1998)

• For each \( k \geq 3 \), \( \mathcal{H}_k \) has finite big Ramsey degrees. (Dobrinen 2017 and 2019)
Problem for Henson graphs: no Milliken theorem, and no nicely definable structure which is bi-embeddable with $\mathcal{H}_k$.

**Question.** How do you make a tree that codes a $K_k$-free graph which branches enough to carry some Ramsey theory?
New Methods

Problem for Henson graphs: no Milliken theorem, and no nicely definable structure which is bi-embeddable with $H_k$.

**Question.** How do you make a tree that codes a $K_k$-free graph which branches enough to carry some Ramsey theory?

**Key Ideas** in the proof that Henson graphs have finite big Ramsey degrees include

1. Trees with coding nodes.
2. Use forcing mechanism to obtain (in ZFC) new Milliken-style theorems for trees with coding nodes.
Develop strong $\mathcal{H}_k$-coding trees which code $\mathcal{H}_k$. 
Structure of Proof

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   These are analogues of Milliken’s strong trees able to handle forbidden $k$-cliques.
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III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_k$. 
A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c \rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and $c$ is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on $N$.

$c : N \to T$ is injective, and $m < n < N \implies |c(m)| < |c(n)|$.

$c(n)$ is the $n$-th coding node in $T$, usually denoted $c_n^T$. 
Let $k \geq 3$ be fixed.

Order the vertices of $\mathcal{H}_k$ in order-type $\omega$ as $\langle v_n : n < \omega \rangle$.

Let the $n$-th coding node, $c_n$, code the $n$-th vertex.
**Strong $K_3$-Free Tree**

Figure: A strong triangle-free tree $S_3$ densely coding $\mathcal{H}_3$
Strong $K_4$-Free Tree

Figure: A strong $K_4$-free tree $S_4$ densely coding $H_4$
Note: A collection of coding nodes \( \{c_{n_i} : i < k\} \) in \( T \) codes a \( k \)-clique iff \( i < j < k \implies c_{n_j}(|c_{n_i}|) = 1 \).
Bottom-up Approach

**Note:** A collection of coding nodes \( \{c_{n_i} : i < k \} \) in \( T \) codes a \( k \)-clique iff \( i < j < k \rightarrow c_{n_j}(|c_{n_i}|) = 1 \).

A tree \( T \) with coding nodes \( \langle c_n : n < N \rangle \) satisfies the \( K_k \)-Free Branching Criterion (\( k \)-FBC) if for each non-maximal node \( t \in T \), \( t^\prec 0 \in T \) and

\[ (*) \quad t^\prec 1 \text{ is in } T \text{ iff adding } t^\prec 1 \text{ as a coding node to } T \text{ would not code a } k \text{-clique with coding nodes in } T \text{ of shorter length}. \]
Henson’s Criterion for building $\mathcal{H}_k$

Henson gave a criterion for building $\mathcal{H}_k$, interpreted to our setting here:

A tree with coding nodes satisfies (A)\( k \)iff

(i) $T$ satisfies the $K_k$-Free Criterion.

(ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of finite subsets of $\omega$ such that for each $i < \omega$, $\max(F_i) < i - 1$, and each finite subset of $\omega$ appears as $F_i$ for infinitely many indices $i$. Given $i < \omega$, if for each subset $J \subseteq F_i$ of size $k - 1$, $\{ c_j : j \in J \}$ does not code a $(k - 1)$-clique, then there is some $n \geq i$ such that for all $j < i$, $c_n(l_j) = 1$ iff $j \in F_i$.

Thm. (D.) Suppose $T$ is a tree with no maximal nodes satisfying the $K_k$-Free Branching Criterion, and the set of coding nodes dense in $T$. Then $T$ satisfies (A)\( k \) tree, and hence codes $\mathcal{H}_k$. 
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Figure: A strong $K_4$-free tree $S_4$ densely coding $H_4$
We want to obtain a Ramsey theorem that says, “Given a coloring for a finite antichain $A$ of coding nodes inside a strong coding tree $T$, there is a subtree $S$ of $T$ which is ‘isomorphic’ to $T$ in which all ‘copies’ of $A$ have the same color.”
Almost sufficient

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Solution: Skew the levels of interest.
Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$
Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$
Defining the Space of Strong Coding Trees

Let \( k \geq 3 \) be fixed, and let \( a \in [3, k] \). A level set \( X \subseteq T_k \) with nodes of length \( \ell_X \), has a pre-\( a \)-clique if there are \( a - 2 \) coding nodes in \( T_k \) coding an \( (a - 2) \)-clique, and each node in \( X \) has passing number 1 by each of these coding nodes.
Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level set $X \subseteq T_k$ with nodes of length $\ell_X$, has a pre-$a$-clique if there are $a - 2$ coding nodes in $T_k$ coding an $(a - 2)$-clique, and each node in $X$ has passing number 1 by each of these coding nodes.

**The Point.** Pre-$a$-cliques for $a \in [3, k]$ code entanglements that affect how nodes in $X$ can extend inside $T$. 
A level set $U$ with a pre-3-clique

The yellow node is a coding node in $T_k$ not in $U$. 
A level set $X$ with a pre-3-clique

The yellow node is a coding node in $\mathbb{T}_k$ not in $X$. 
A level set $Y$ with a pre-4-clique

The yellow node is a coding node in $T_k$ not in $Y$. 
A level set $Z$ with a pre-4-clique

The yellow node is a coding node in $\mathbb{T}_k$ not in $Z$. 
Two subtrees $S$ and $T$ of $\mathbb{T}_k$ are strongly isomorphic iff there is a strong similarity map $f : S \rightarrow T$ which preserves maximal new pre-cliques in each interval.
The Space of Strong $\mathcal{H}_k$-Coding Trees $\mathcal{T}_k$

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Idea: Strong isomorphisms preserve

1. the structure of the trees with respect to tree and lexicographic orders
2. placement of coding nodes
3. passing numbers at levels of coding nodes
4. whether or not an interval has new pre-cliques.
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The members of $\mathcal{T}_k$ are called strong $\mathcal{H}_k$-coding trees.
Subtrees and Extension Lemmas

Provide guarantees for when a finite subtree of a strong coding tree $T$ can be extended within $T$ to a desired configuration:
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A subtree $A \subseteq T$ is **valid** if all pre-cliques in $A$ are witnessed by coding nodes in $A$ and $\max(A)$ is free in $T$.
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A series of “Extension Lemmas” guarantee when level sets can be extended as wished to new configurations within a given $T \in \mathcal{T}_k$. 
Part II: Ramsey Theorem for Strictly Similar Finite Antichains

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(b) Weave together to obtain an analogue of Milliken’s Theorem for “Strictly Witnessed” finite trees.

(c) New notion of envelope to move from Strictly Witnesses finite trees to any finite antichain of coding nodes.
Ramsey Theorem for Strictly Similar Antichains

Thm. (D.) Let $Z$ be a finite antichain of coding nodes in a strong $\mathcal{H}_k$-coding tree $T \in \mathcal{T}_k$, and suppose $h$ colors of all subsets of $T$ which are strictly similar to $Z$ into finitely many colors. Then there is an strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of $S$ strictly similar to $Z$ have the same $h$ color.
Some Examples of Strict Similarity Types for \( k = 3 \)

Let \( G \) be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding \( G \).
$G$ a graph with three vertices and no edges

A tree $A$ coding $G$
$G$ a graph with three vertices and no edges

$B$ codes $G$ and is strictly similar to $A$. 
The tree $C$ codes $G$

$C$ is not strictly similar to $A$.
The tree $D$ codes $G$

$D$ is not strictly similar to either $A$ or $C$. 
The tree $E$ codes $G$ and is not strictly similar to $A - D$
The tree $F$ codes $G$ and is strictly similar to $E$.
Envelopes and Witnessing Coding Nodes

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We now give some examples of envelopes.
$H$ codes a non-edge

$H$ is its own envelope.
I codes a non-edge

I is not its own envelope.
An envelope $E(I)$

An envelope of $I$. 
The antichain $E$ from before
The coding nodes $w_0, \ldots, w_3$ make an envelope of $E$. 
The tree $F$ from before is strictly similar to $E$
$E(F)$ is strictly similar to $E(E)$

The **coding nodes** $w_0, \ldots, w_3$ make an envelope of $F$. 
Some upper bounds for big Ramsey degrees in $\mathcal{H}_k$

\[
T(K_2, \mathcal{H}_3) = 2
\]
\[
T(K_2, \mathcal{H}_4) \leq 6
\]
\[
T(K_2, \mathcal{H}_5) \leq 88
\]
\[
T(\bar{K}_2, \mathcal{H}_3) \leq 7
\]
\[
T(\bar{K}_2, \mathcal{H}_4) \leq 58
\]

**Conjecture:** The number of incremental strict similarity types of antichains coding a finite graph $G \in \mathcal{K}_k$ is the big Ramsey degree $T(G, \mathcal{H}_k)$. 
Although trees with coding nodes were invented to handle forbidden cliques, it turns out they are good at coding relational structures with or without forbidden substructures.
Say $\mathcal{X} \subseteq [\omega]^{\omega}$ is **completely Ramsey (CR)** if for each nonempty $[s, A]$, there is a $B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$. 
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**Thm.** (Galvin-Prikry 1973) Every Borel subset of the Baire space is completely Ramsey.

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**Question.** (KPT 2005) Which Fraïssé structures have infinite dimensional Ramsey theory for definable subsets?
We approach this using trees with coding nodes.

By Devlin’s theorem, one must fix a strong similarity type coding the rationals into $2^{<\omega}$, and restrict to all subtrees with the same strong similarity type.
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**Thm.** (D.) Let $T_Q \subseteq 2^{<\omega}$ be a fixed tree with coding nodes coding a copy of the rationals in order type $\omega$, with no terminal nodes. Let $T_Q$ be the collection of all strongly similar subtrees of $T_Q$. Then $T_Q$ is a topological Ramsey space, hence has an analogue of Ellentuck’s theorem.
Infinite Dimensional Ramsey Theory for \( \mathbb{Q} \)

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**Thm.** (D.) Let \( T_\mathbb{Q} \subseteq 2^{<\omega} \) be a fixed tree with coding nodes coding a copy of the rationals in order type \( \omega \), with no terminal nodes. Let \( T_\mathbb{Q} \) be the collection of all strongly similar subtrees of \( T_\mathbb{Q} \). Then \( T_\mathbb{Q} \) is a topological Ramsey space, hence has an analogue of Ellentuck’s theorem.

This should also hold (modulo checking) for antichains in \( 2^{<\omega} \) coding the rationals. If true, this will recover Devlin’s result.
Infinite Dimensional Ramsey Theory for the Rado Graph

By Laflamme, Sauer and Vuksanovic’s theorem, one must fix a strong similarity type coding the Rado graph into $2^{<\omega}$, and restrict to all subtrees with the same strong similarity type.
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By Laflamme, Sauer and Vuksanovic’s theorem, one must fix a strong similarity type coding the Rado graph into $2^{<\omega}$, and restrict to all subtrees with the same strong similarity type.

Let $\mathbb{T}_R$ be $2^{<\omega}$ with coding nodes which are dense in $2^{<\omega}$.

$\mathcal{T}_R$ consists of all trees with coding nodes $(T, \omega; \subseteq, <, c^T)$, where

1. $T$ is a strong subtree of $2^{<\omega}$; and
2. The strong tree isomorphism $\varphi : \mathbb{T}_R \rightarrow T$ has the property that for each $n < \omega$, $\varphi(c(n)) = c^T(n)$. 


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The members of \( \mathcal{T}_R \) are called strong Rado coding trees.
A Strong Rado Coding Tree $\mathcal{T}_R$
A Strong Rado Coding Tree $T \in T_{\mathcal{R}}$
Give $\mathcal{T}_\mathcal{R}$ the topology inherited as a subspace of the Cantor space.

**Thm.** (D.) Every Borel subset of $\mathcal{T}_\mathcal{R}$ has the Ramsey property.
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So there is a topological space of Rado graphs which has infinite dimensional Ramsey theory.
Current Directions and Future Goals

1. Extend methods to other ultrahomogeneous structures with forbidden configurations.
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4. (KPT) What is the correspondence between infinite dimensional Ramsey theory and topological dynamics?
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Claim:
If a Fra¨ıss´e class has “flexible amalgamation” (no forbidden configurations) and its ordered version has the Ramsey property, then its Fra¨ıss´e limit has finite big Ramsey degrees. This is work in progress.
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Thank you for your kind attention!
References


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