Ramsey Theory on Trees and Applications to Infinite Graphs

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Outline: Lecture 1

(1) Ramsey theory on sets and structures

(2) Ramsey theory on the rationals and the Rado graph

(3) Milliken's Ramsey theorem for strong trees

(4) Trees coding sets of rationals and graphs

(5) Applications of Milliken's Theorem to big Ramsey degrees of the rationals and the Rado graph

   (a) Strong similarity types of trees

   (b) Strong tree envelopes

(6) Connection: structural Ramsey theory and topological dynamics

(7) The Halpern-Läuchli Theorem and its "forcing proof"

Outline: Lecture 2

(8) The question of big Ramsey degrees for infinite structures

(9) Overview of known results

(10) Henson graphs have finite big Ramsey degrees

(11) Techniques of the proof

   (a) Trees with coding nodes

   (b) Ramsey theorems for strong coding trees - "forcing proofs"

   (c) Strict similarity types and envelopes

(12) Future directions in big Ramsey degrees and infinite dimensional structural Ramsey theory

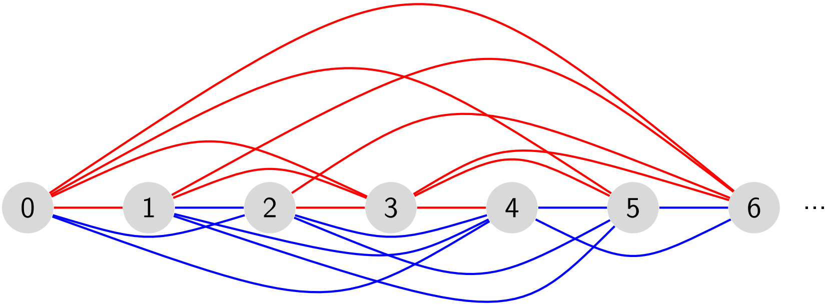
Infinite Ramsey's Theorem

Infinite Ramsey's Theorem. (Ramsey, 1929) Given $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite subset $X \subseteq \omega$ such that c is constant on $[X]^k$.

Graph Interpretation: For $k \geq 1$, given a complete k-hypergraph on infinitely many vertices and a coloring of the k-hyperedges into finitely many colors, there is an infinite complete sub-hypergraph in which all k-hyperedges have the same color.

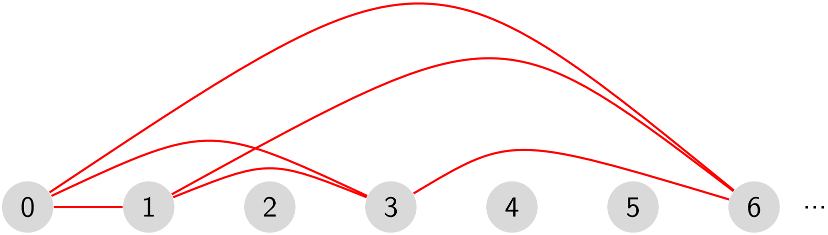
Example

Given a 2-coloring of the edges of a complete graph on ω vertices,



Example

There is an infinite complete subgraph such that all edges have the same color.



Finite Ramsey's Theorem and Logic

Ramsey deduced the finite version from the infinite version.

Finite Ramsey Theorem. (Ramsey, 1929) Given $k, m, r \geq 1$, there is an $n \geq m$ such that for each coloring of the k-element subsets of n into r colors, there is an $X \subseteq n$ of size m such that the coloring takes one color on the k-element subsets of X.

This theorem appears in Ramsey's paper, *On a problem of formal logic*, and is motivated by Hilbert's Entscheidungsproblem:

Find a procedure for determining whether any given formula is valid.

Ramsey applied his theorem to solve this problem for formulas with only universal quantifiers in front (Π_1).

Finite Structural Ramsey Theory

Note: Ramsey's theorems may be thought of as involving the class of complete graphs (or hypergraphs) on finitely many vertices, or the class of finite linear orders.

For structures A, B, write $A \leq B$ iff A embeds into B.

A class $\mathcal{K}$ of structures has the Ramsey property if for each pair $A \leq B$ in $\mathcal{K}$ and $r \geq 1$, there is some C in $\mathcal{K}$ such that for each coloring of the copies of A in C into r colors, there is a $B' \leq C$ isomorphic to B such that all copies of A in B' have the same color.

Some classes of finite structures with the Ramsey property:
Linear orders, complete graphs, Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Do analogues of the Infinite Ramsey Theorem hold for infinite structures?

Test Case: The Rationals as a Linear Ordering $(\mathbb{Q}, <)$

Fact. $(\mathbb{Q}, <)$ is indivisible: Given any partition of the rationals into finitely many pieces, one of the pieces contains a copy of the rationals.

Question. Given a coloring of pairs of rationals into two colors, can one find a subset $Q \subseteq \mathbb{Q}$ such that $(Q, <) \cong (\mathbb{Q}, <)$ and all pairsets in Q have the same color?

Answer. Not necessarily! Sierpiński designed the following example:

Let $\prec$ be a well-ordering of the rationals.
Define $c(\{p, q\}) = 0$ iff the two orders $\prec$ and $<$ agree on $\{p, q\}$.
Otherwise, $c(\{p, q\}) = 1$.

A modern proof uses a Ramsey theorem for strong trees.

Strong Subtrees of $2^{<\omega}$

For $t \in 2^{<\omega}$, the length of t is $|t| = \mathrm{dom}(t)$.

$T \subseteq 2^{<\omega}$ is a tree if $\exists L \subseteq \omega$ such that $T = \{t \restriction l : t \in T,\ l \in L\}$.

For $t \in T$, the height of t is $\mathrm{ht}_T(t) = \mathrm{o.t.}\{u \in T : u \subset t\}$.

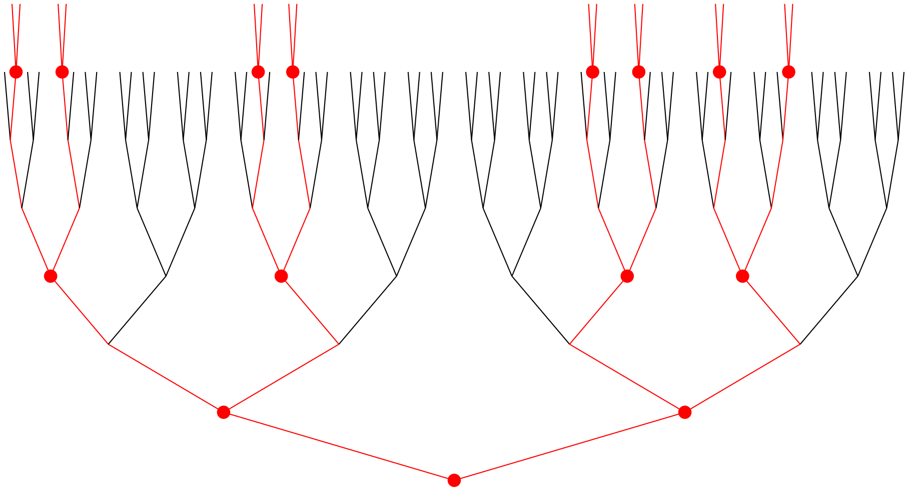
$T(n) = \{t \in T : \mathrm{ht}_T(t) = n\}$.

For $t \in T$, $\mathrm{Succ}_T(t) = \{u \restriction (|t| + 1) : u \in T \text{ and } u \supset t\}$.

$S \subseteq T$ is a strong subtree of T iff for some $\{m_n : n < N\}$ $(N \leq \omega)$,

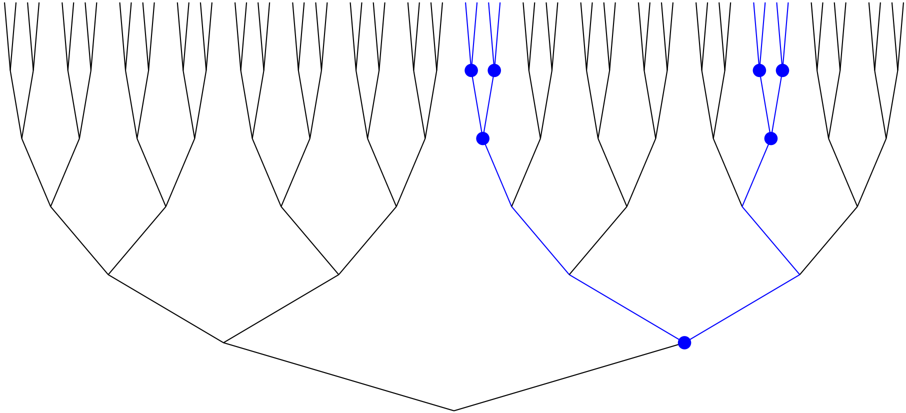
1. Each $S(n) \subseteq T(m_n)$, and
2. For each $n < N$, $s \in S(n)$ and $u \in \mathrm{Succ}_T(s)$, there is exactly one $s' \in S(n+1)$ extending u.

Example: A Strong Subtree $T \subseteq 2^{<\omega}$



The nodes in T are of lengths $0, 1, 3, 6, \ldots$

Example: A Strong Subtree $U \subseteq 2^{<\omega}$



The nodes in U are of lengths $1, 4, 5, \ldots$.

A Ramsey Theorem for Strong Trees

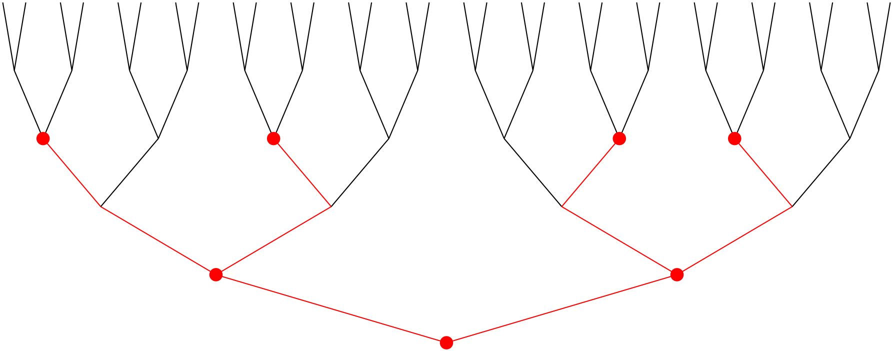
Thm. (Milliken 1979) Let $T \subseteq 2^{<\omega}$ be a strong tree with no terminal nodes. Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of T into r colors. Then there is a strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.

A k-strong tree is a finite strong tree where all terminal nodes have height $k-1$.

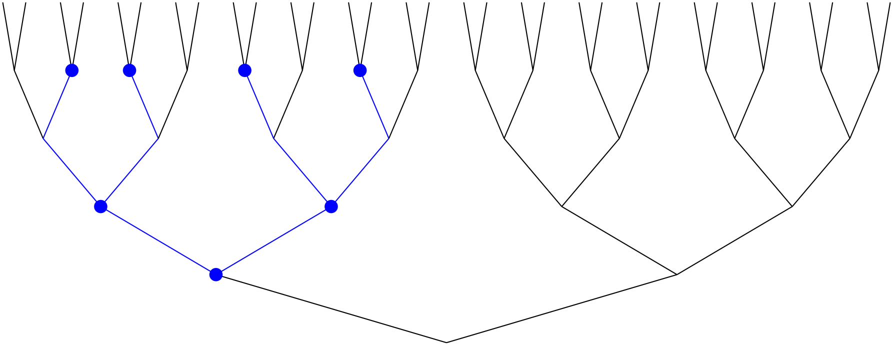
We give some examples for $T = 2^{<\omega}$.

Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

Given a coloring c of all 3-strong trees in $2^{<\omega}$ into red and blue:



Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$



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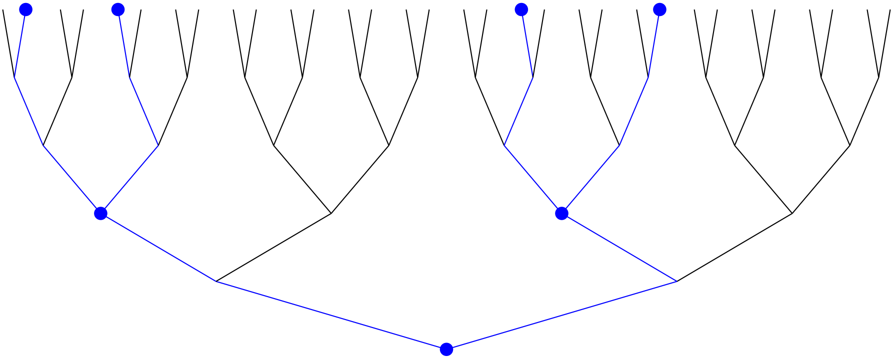
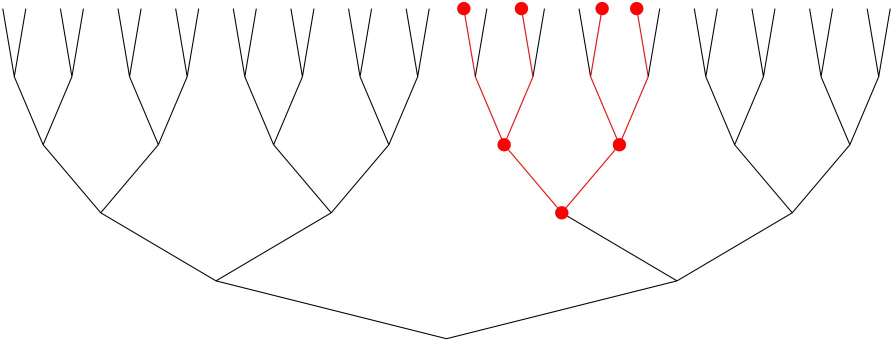


Milliken's Theorem for 3-Strong Subtrees of $T = 2^{<\omega}$

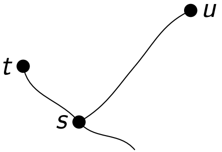


Milliken's Theorem guarantees a strong subtree in which all 3-strong subtrees have the same color.

The Rationals Coded in $2^{<\omega}$

For $x, y \in 2^{<\omega}$, define $x \triangleleft y$ iff one of the following holds:

1. $x <_{\text{lex}} y$,
2. $x \sqsubset y$ and $y(|x|) = 1$, or
3. $y \sqsubset x$ and $x(|y|) = 0$.



In this picture, $t \triangleleft s \triangleleft u$.

Note: $(2^{<\omega}, \triangleleft) \cong (\mathbb{Q}, <)$.

Sierpiński's result viewed in trees

Given a pair of nodes s, t in $2^{<\omega}$ with $|s| < |t|$, let

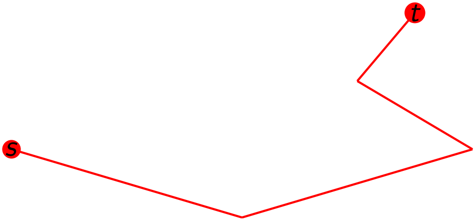
$$c(\{s,t\}) = \begin{cases} 0 & \text{if } s \lhd t \\ 1 & \text{if } t \lhd s \end{cases}$$

Given any subset $S \subseteq 2^{<\omega}$ for which $(S, \lhd) \cong (\mathbb{Q}, <)$, both colors will persist in S.

Thm. (Galvin) Given any coloring of pairs of rationals into finitely many colors, there is a subset which is again a dense linear order in which at most two colors are used.

Given $s, t \in 2^{<\omega}$ with $|s| < |t|$, a strong tree envelope is a 3-strong tree which contains s and t and has nodes of lengths $|s \wedge t|, |s|, |t|$.

Example 1: $|s| < |t|$ and $s \triangleleft t$



A strong tree envelope of s and t

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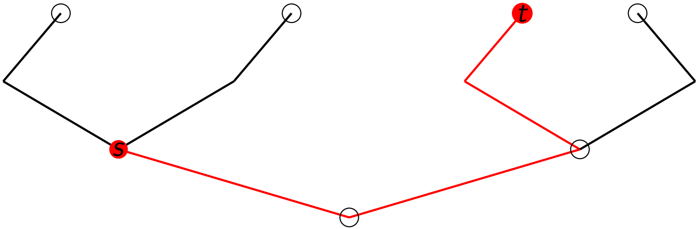
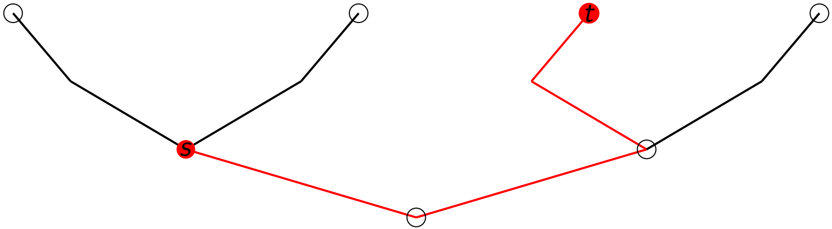


Another strong tree envelope of s and t

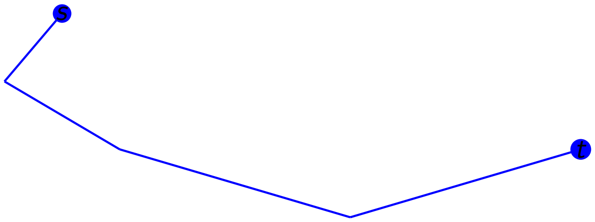
Thm. (Galvin) Given any coloring of pairs of rationals into finitely many colors, there is a subset which is again a dense linear order in which at most two colors are used.

Given $s, t \in 2^{<\omega}$ with $|s| < |t|$, a strong tree envelope is a 3-strong tree which contains s and t and has nodes of lengths $|s \wedge t|, |s|, |t|$.

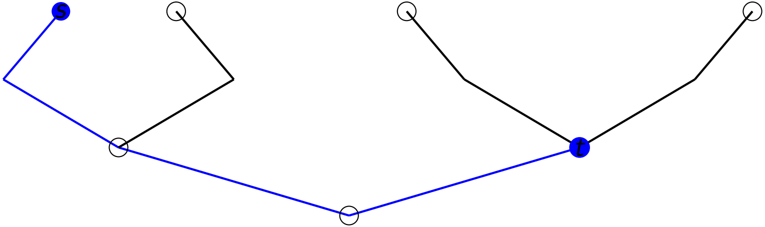
Example 1: $|s| < |t|$ and $s \lhd t$



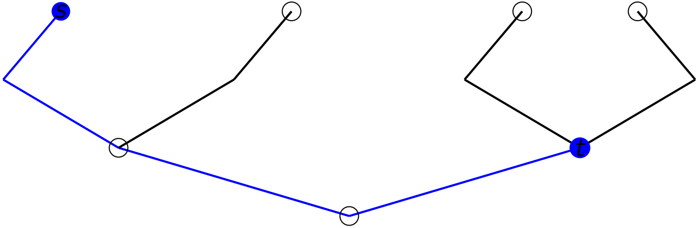
Example 2: $|s| < |t|$ with $t \triangleleft s$



Example 2: A strong tree envelope



Example 2: Another strong tree envelope



Strong Similarity Types

Two finite antichains $A, B \subseteq 2^{<\omega}$ are strongly similar iff they have the same cardinality, and the lexicographic preserving map from the tree induced by A to the tree induced by B is a tree isomorphism preserving passing numbers at levels of meets and maximal nodes.

Big Ramsey Degree for Pairs of Rationals is 2

1. Let c be a coloring of $[\mathbb{Q}]^2$ into finitely many colors.
2. Transfer the coloring to pairs of nodes in $2^{<\omega}$. There are two strong similarity types for pairs.
3. Fix one strong similarity type. For each pair of nodes s, t of that type, color all 3-strong trees containing s and t with the color $c(\{s, t\})$.
4. Apply Milliken's Theorem to 3-strong trees. Get one color for all pairs with that similarity type.
5. Repeat for the second strong similarity type.
6. Take a strongly diagonal antichain $\mathbb{A} \subseteq 2^{<\omega}$ such that $(\mathbb{A}, \triangleleft) \cong (\mathbb{Q}, <)$.

$(\mathbb{Q}, <)$ has an approximate Infinite Ramsey Theorem

Thm. (Laver (bounds, unpublished), Devlin (exact bounds) 1979)
Given $k \geq 2$, there is a number $T(k, \mathbb{Q})$ such that for each coloring of the k-element subsets of $\mathbb{Q}$ into finitely many colors, there is a copy Q of $\mathbb{Q}$ in which no more than $T(k, \mathbb{Q})$ colors occur.
These are actually tangent numbers.

So $(\mathbb{Q}, <)$ does not have the exact analogue of Ramsey's Theorem for $\mathbb{N}$.

But this structure still behaves quite nicely in that finite bounds exist.
These bounds $T(k, \mathbb{Q})$ are called the big Ramsey degrees of k in $\mathbb{Q}$.

Next, we look at Ramsey theory on the Rado graph.

The Rado Graph $\mathcal{R} = (R, E)$

The Rado graph is the homogeneous graph on countably many vertices which is universal for all countable graphs.

homogeneous: Any isomorphism between two finite subgraphs of $\mathcal{R}$ extends to an automorphism of $\mathcal{R}$.

universal: Each graph on countably many vertices embeds into $\mathcal{R}$.

The Rado graph is indivisible: Given any partition of the vertices into finitely many pieces, one piece contains a copy of $\mathcal{R}$.

However, Erdős, Hajnal and Posa found a two-valued edge coloring for which both colors persist on every subgraph isomorphic to $\mathcal{R}$.

Ramsey Theory on the Rado graph

First, some terminology:

Let G be a finite graph. $T(G, \mathcal{R})$ denotes the minimal number T such that given a coloring of the copies of G in $\mathcal{R}$ into finitely many colors, there is an induced subgraph $\mathcal{R}' \subseteq \mathcal{R}$ isomorphic to $\mathcal{R}$ in which the copies of G take no more than T colors.

$T(G, \mathcal{R})$ is called the big Ramsey degree of G in $\mathcal{R}$, if it exists.

Fact. (Folklore) Vertices have big Ramsey degree 1: $\mathcal{R}$ is indivisible.

Thm. (Erdős-Hajnal-Pósa 1975) Edges have big Ramsey degree ≥ 2.

Thm. (Pouzet-Sauer 1996) Edges have big Ramsey degree exactly 2.

Ramsey Theory on the Rado graph

Thm. (Sauer 2006, Laflamme-Sauer-Vuksanovic 2006)
Every finite graph has a finite big Ramsey degree.

Actual degrees were found structurally in (Laflamme-Sauer-Vuksanovic 2006) and computed in (J. Larson 2008).

Colorings of Finite Graphs

Example: The path of length 2 embeds into the graph B.

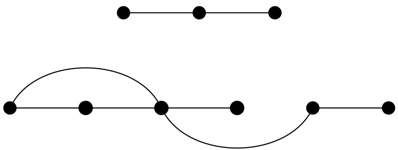
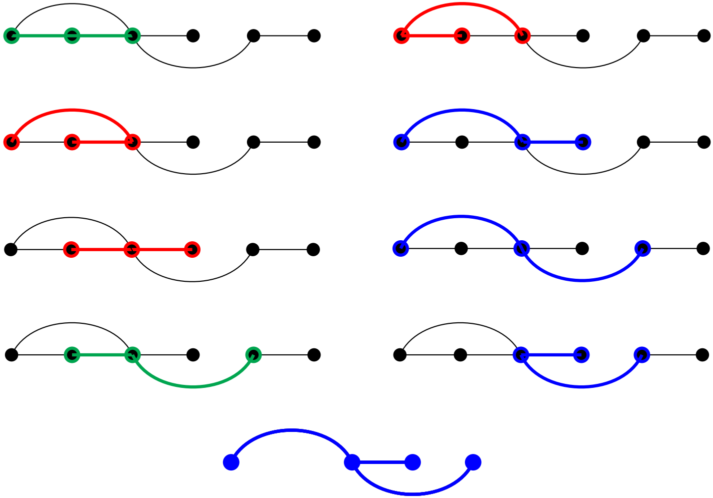


Figure: Graph B

Copies of the Path of Length 2 in B



A star with all paths blue.

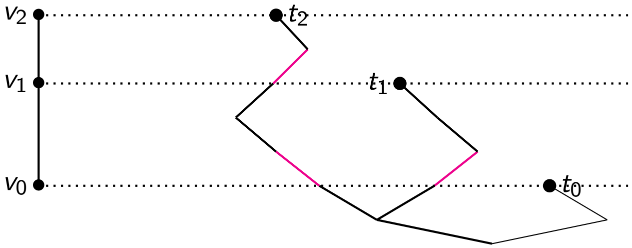
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

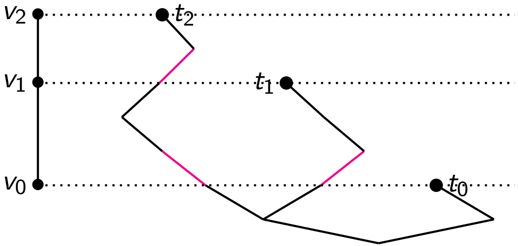
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Diagonal Trees Code Graphs

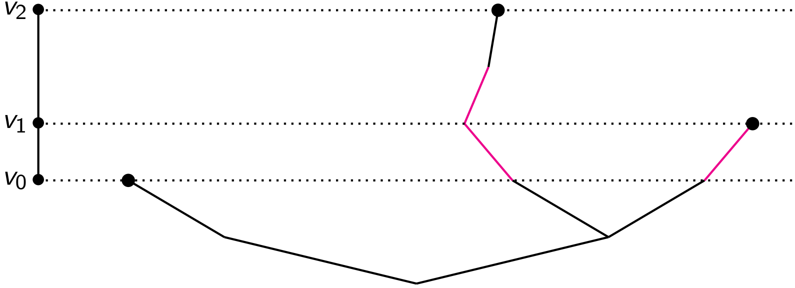
A tree T is diagonal if there is at most one meet or terminal node per level.

T is strongly diagonal if passing numbers at splitting levels are all 0 (except for the right extension of the splitting node).

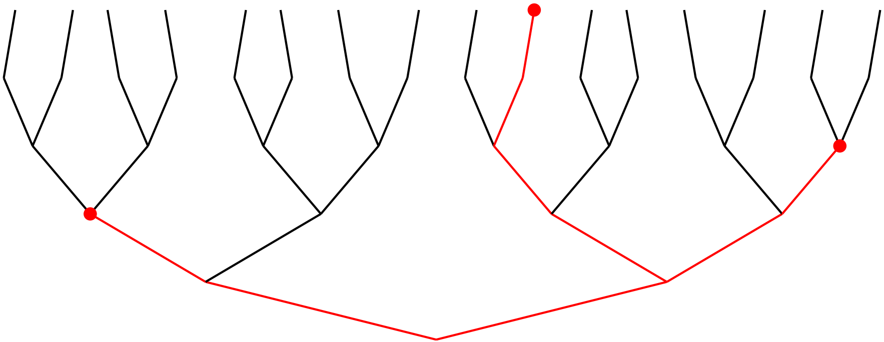


Every graph can be coded by the terminal nodes of a diagonal tree.

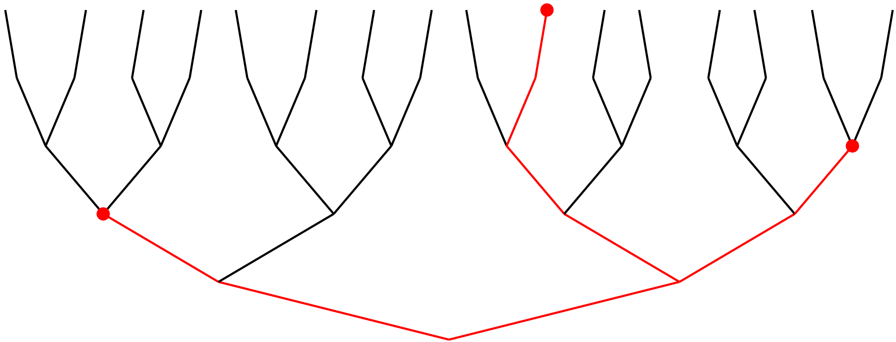
A Different Strongly Diagonal Tree Coding a Path



Strongly diagonal trees can be enveloped into strong trees



Another strong tree envelope



Outline of Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of strongly diagonal trees.
3. Each strongly diagonal tree can be enveloped into a finite strong tree.
4. Apply Milliken's Theorem finitely many times to obtain one color for each type.
5. Choose a strongly diagonal antichain coding the Rado graph.

Big Ramsey Degrees of Infinite Structures

Let $\mathcal{S}$ be an infinite structure. For a finite substructure $A \leq \mathcal{S}$, let $T(A, \mathcal{S})$ denote the least number, if it exists, such that for each coloring of the copies of A in $\mathcal{S}$ into finitely many colors, there is a substructure $\mathcal{S}'$ isomorphic to $\mathcal{S}$ in which the copies of A take no more than $T(A, \mathcal{S})$ colors.

(Kechris, Pestov, Todorcevic, 2005) $\mathcal{S}$ has finite big Ramsey degrees if for each finite $A \leq \mathcal{S}$, $T(A, \mathcal{S})$ exists.

Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)
- The rationals. (Devlin 1979)
- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)
- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S(2)}$, $\mathbf{S(3)}$. (Laflamme, NVT, Sauer 2010)
- The random k-clique-free graphs. (Dobrinen 2017 and 2019)
- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)

Ramsey Theory and Topological Dynamics

(Kechris, Pestov, Todorcevic 2005) The KPT Correspondence:
A Fraïssé class $\mathcal{K}$ has the Ramsey property iff $\text{Aut}(\text{Flim}(\mathcal{K}))$ is extremely amenable.

(Zucker 2019) Characterized universal completion flows of $\text{Aut}(\text{Flim}(\mathcal{K}))$ whenever $\text{Flim}(\mathcal{K})$ admits a big Ramsey structure (big Ramsey degrees with a coherence property).

A class $\mathcal{K}$ of finite structures is a Fraïssé class if it is hereditary, has the Joint Embedding Property, and the Amalgamation Property.

$\text{Flim}(\mathcal{K})$ is a homogeneous countable structure into which each member of $\mathcal{K}$ embeds.

Halpern-Läuchli Theorem - strong tree version

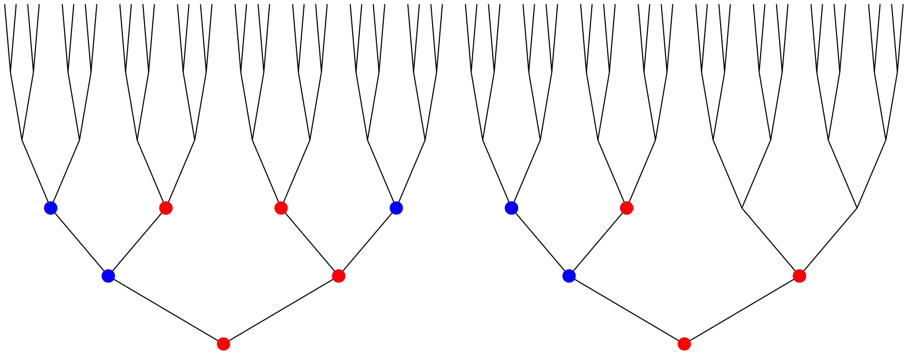
Notation: $$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n)$$

Theorem. (Halpern-Läuchli, 1966) Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes and let $r \geq 2$. Given a coloring $c : \bigotimes_{i<d} T_i \to r$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that c is constant on $\bigotimes_{i<d} S_i$.

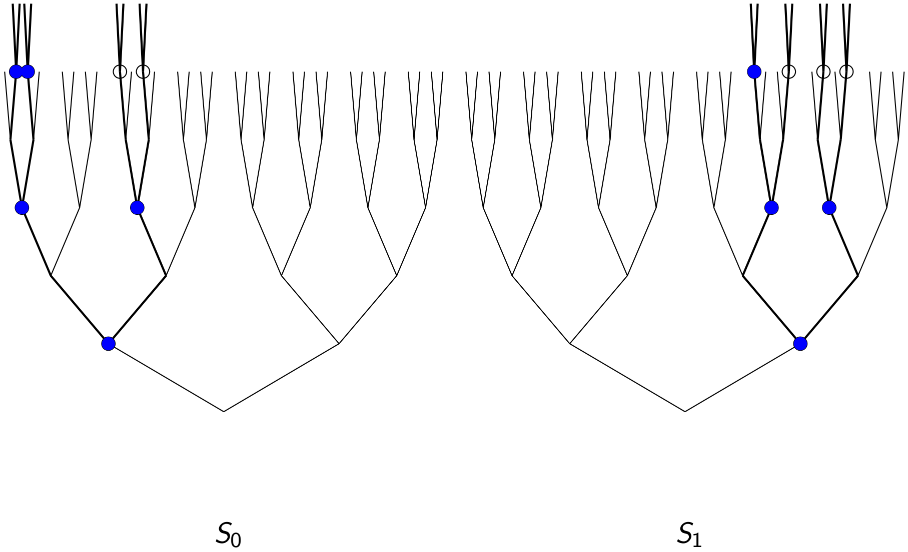
This was discovered as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (Halpern-Lévy, 1971) It is also the crux of Milliken's Theorem.

We now give some examples of colorings of level products of two trees $T_0 = T_1 = 2^{<\omega}$, and show visually what the Halpern-Läuchli Theorem does.

Coloring Products of Level Sets: $T_0(0) \times T_1(0)$



HL gives Strong Subtrees with 1 color for level products



Application to Products of Rationals

Thm. (Laver, 1984) Given $d < \omega$ and a coloring of $\mathbb{Q}^d$ into finitely many colors, there are $X_i \subseteq \mathbb{Q}$, $i < d$, isomorphic to $\mathbb{Q}$ such that $X_0 \times \cdots \times X_{d-1}$ takes at most $d!$ many colors.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

Harrington devised a proof of the Halpern-Läuchli Theorem that uses forcing methods, but never goes to a generic extension.

Fix $d \geq 2$ and let $T_i = 2^{<\omega}$ $(i < d)$ be finitely branching trees with no terminal nodes. Fix a coloring $c : \bigotimes_{i<d} T_i \to 2$.

Thm. (Erdős-Rado, 1956) For $r < \omega$ and μ an infinite cardinal,

$$\beth_r(\mu)^+ \to (\mu^+)_\mu^{r+1}$$

Let $\kappa = \beth_{2d}$. Then $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$.

Harrington's 'Forcing' Proof: The Forcing

The Forcing: $\mathbb{P}$ is the set of functions p of the form

$$p : d \times \vec{\delta}_p \to \bigcup_{i<d} T_i \restriction l_p$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $l_p < \omega$, and $\forall i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

$q \leq p$ iff $l_q \geq l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and $\forall (i,\delta) \in d \times \vec{\delta}_p$, $q(i,\delta) \supseteq p(i,\delta)$.

$\mathbb{P}$ adds κ branches through each tree T_i, $i < d$.

$\mathbb{P}$ is Cohen forcing adding κ new branches to each tree.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i:

$$\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{P}, \text{ and } (i,\alpha) \in \mathrm{dom}(p)\}.$$

Note: If $(i,\alpha) \in \mathrm{dom}(p)$, then $p \Vdash \dot{b}_{i,\alpha} \restriction l_p = p(i,\alpha)$.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on ω.

For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}\rangle$.

- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i,\alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

Harrington's 'Forcing' Proof: The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, \langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ \langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ \langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned} \tag{1}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

Harrington's 'Forcing' Proof: f gives fixed ranges and color

Note: $\mathrm{dom}(f) = [\kappa]^{2d}$ and $\mathrm{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Lem 1. There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle\langle t_{i,j} : j < k^*\rangle : i < d\rangle$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$,

$$\varepsilon_{\vec{\alpha}} = \varepsilon^*,\ k_{\vec{\alpha}} = k^*,\ \text{and } (\forall i < d)\,\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle.$$

Pf. Let $\iota \in \mathcal{I}$ be the identity function on $2d$. For any $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, there are $\vec{\theta}, \vec{\theta'} \in [K]^{2d}$ such that $\vec{\alpha} = \iota_e(\vec{\theta})$ and $\vec{\beta} = \iota_e(\vec{\theta'})$. Then $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta'})$ implies the conclusion. □

Harrington's 'Forcing' Proof: Same ordinals, same position

Lem 2. For $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, j' < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$, then $j = j'$.

Pf Idea. (sliding argument) Suppose $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$.

Let $\rho_i \in \{<, =, >\}$ be the relation such that $\alpha_i \, \rho_i \, \beta_i$, $(i < d)$.

Take $\iota \in \mathcal{I}$ so that for any $\vec{\zeta} \in [K]^{2d}$ and $i < d$, $\zeta_{\iota(2i)} \ \rho_i \ \zeta_{\iota(2i+1)}$.

Fix $\vec{\theta} \in [K']^{2d}$ such that $\iota_e(\vec{\theta}) = \vec{\alpha}$ and $\iota_o(\vec{\theta}) = \vec{\beta}$.

Take $\vec{\gamma} \in [K]^d$ such that $(\forall i < d) \ \alpha_i \, \rho_i \, \gamma_i$ and $\gamma_i \, \rho_i \, \beta_i$.

Take $\vec{\mu}, \vec{\nu} \in [K]^{2d}$ with $\iota_e(\vec{\mu}) = \vec{\alpha}$, $\iota_o(\vec{\mu}) = \iota_e(\vec{\nu}) = \vec{\gamma}$, and $\iota_o(\vec{\nu}) = \vec{\beta}$.

$\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ implies $\langle j, j' \rangle$ is in the last sequence in $f(\iota, \vec{\theta})$.

$f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$ implies $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\beta}}(j') = \delta_{\vec{\alpha}}(j) = \delta_{\vec{\gamma}}(j')$,

which implies $j = j'$. □

Harrington's 'Forcing' Proof: Set of compatible conditions

Main Lemma. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Pf. Suppose TAC $\exists \vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$ with $p_{\vec{\alpha}} \perp p_{\vec{\beta}}$.
By Lem 1, for each $i < d$ and $j < k^*$, $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j))$.
So $p_{\vec{\alpha}} \perp p_{\vec{\beta}}$ implies $\exists i < d$ and $j, j' < k^*$ with $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ but $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) \neq p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j'))$.
Note that $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j}$ and $p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j')) = t_{i,j'}$ imply $j \neq j'$.
But by Lem 2, $j \neq j'$ implies $\delta_{\vec{\alpha}}(j) \neq \delta_{\vec{\beta}}(j')$. $\rightarrow\leftarrow$ $\square$

By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i < d \rangle \in [k^*]^d$ such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$.
Then for each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} =: t_i^*.$$

Harrington's 'Forcing' Proof: The Construction

Build strong subtrees $S_i \subseteq T_i$ homogeneous for c: Let $\text{stem}(S_i) = t_i^*$.

Induction Assumption: $m \geq 1$, and we have constructed m-strong subtrees $\bigcup_{j<m} S_i(j)$ of T_i such that c takes color ε^* on $\bigcup_{j<m} \prod_{i<d} S_i(j)$.

Let X_i be the set of immediate extensions in T_i of the nodes in $S_i(m-1)$. Let $J_i \subseteq [K_i]^{|X_i|}$. Label the nodes in X_i as $\{q(i,\delta) : \delta \in J_i\}$. Let $\vec{J} = \prod_{i<d} J_i$. For each $\vec{\alpha} \in \vec{J}$ and $i < d$, $q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$. Let $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair (i, γ) with $\gamma \in \vec{\delta}_q \setminus J_i$, $\exists \vec{\alpha} \in \vec{J}$ and $\exists j' < k^*$ such that $\delta_{\vec{\alpha}}(j') = \gamma$. By Main Lemma, $\vec{\beta} \in \vec{J}$ and $\gamma \in \vec{\delta}_{\vec{\beta}}$ imply that $p_{\vec{\beta}}(i,\gamma) = p_{\vec{\alpha}}(i,\gamma) = t^*_{i,j'}$. Let $q(i,\gamma)$ be the leftmost extension of $t^*_{i,j'}$ in T. This defines q. Check that $q \in \mathbb{P}$.

Note that $q \leq p_{\vec{\alpha}}$, for all $\vec{\alpha} \in \vec{J}$.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

To construct $S_i(m)$, take $r \le q$ for which $r \Vdash$ "$\forall \vec{\alpha} \in \vec{J},\ c(\dot{b}_{\vec{\alpha}} \restriction l_r) = \varepsilon^*$".

Then it is simply true in the ground model that

$$c(\{r(i,\alpha_i) : i < d\}) = \varepsilon^*, \text{ for each } \vec{\alpha} \in \vec{J}.$$

For each $i < d$, we define $S_i(m) = \{r(i,\delta) : \delta \in J_i\}$. This set extends X_i.

Then c takes value ε^* on $\prod_{i<d} S_i(m)$.

Set $S_i = \bigcup_{m<\omega} S_i(m)$. c is monochromatic on $\bigotimes_{i<d} S_i$. $\square$ HL

Milliken's Ramsey Theorem for Strong Trees

The Halpern-Läuchli Theorem is the basis for

Thm. (Milliken 1979) Let $k \geq 1$, $r \geq 2$, and c be a coloring of all k-strong subtrees of $2^{<\omega}$ into r colors. Then there is a strong subtree $S \subseteq 2^{<\omega}$ such that all k-strong subtrees of S have the same color.

The proof is by induction on k using the Halpern-Läuchli Theorem.

Outline: Lecture 2

(8) The question of big Ramsey degrees for infinite structures

(9) Overview of known results

(10) Henson graphs have finite big Ramsey degrees

(11) Techniques of the proof

   (a) Trees with coding nodes

   (b) Ramsey theorems for strong coding trees - 'forcing proofs'

   (c) Strict similarity types and envelopes

(12) Future directions in big Ramsey degrees and infinite dimensional structural Ramsey theory

Big Ramsey Degrees of Infinite Structures

Let $\mathcal{S}$ be an infinite structure and A be a finite substructure. $T(A, \mathcal{S})$ denotes the least number, if it exists, such that for each coloring of the copies of A in $\mathcal{S}$ into finitely many colors, there is a substructure $\mathcal{S}'$ isomorphic to $\mathcal{S}$ in which the copies of A take no more than $T(A, \mathcal{S})$ colors.

(KPT 2005) $\mathcal{S}$ has finite big Ramsey degrees if for each finite $A \leq \mathcal{S}$, $T(A, \mathcal{S})$ exists.

Question. Which infinite structures have finite big Ramsey degrees?

Structures with finite big Ramsey degrees

- The infinite complete graph. (Ramsey 1929)

- The rationals. (Devlin 1979)

- The Rado graph, random tournament, and similar binary relational structures. (Sauer 2006)

- The countable ultrametric Urysohn space. (Nguyen Van Thé 2008)

- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S}(\mathbf{2})$, $\mathbf{S}(\mathbf{3})$. (Laflamme, NVT, Sauer 2010)

- The random k-clique-free graphs. (Dobrinen 2017 and 2019)

- Several more universal structures, including some metric spaces with finite distance sets. (Mašulović 2019)

Ramsey Theory and Topological Dynamics

(Kechris, Pestov, Todorcevic 2005) The KPT Correspondence:
A Fraïssé class $\mathcal{K}$ has the Ramsey property iff Aut(Flim($\mathcal{K}$)) is extremely amenable.

(Zucker 2019) Characterized universal completion flows of Aut(Flim($\mathcal{K}$)) whenever Flim($\mathcal{K}$) admits a big Ramsey structure (big Ramsey degrees with a coherence property).

A class $\mathcal{K}$ of finite structures is a Fraïssé class if it is hereditary, has the Joint Embedding Property, and the Amalgamation Property.

Flim($\mathcal{K}$) is a homogeneous countable structure into which each member of $\mathcal{K}$ embeds.

k-Clique-Free Random Graphs

For $k \geq 3$, a k-clique, denoted K_k, is a complete graph on k vertices.

$\mathcal{H}_k$, the k-clique-free Henson graph, is the homogenous K_k-free graph which is universal for all k-clique-free graphs on countably many vertices.

Henson graphs are the k-clique-free analogues of the Rado graph. They were constructed by Henson in 1971.

Henson Graphs: History of Results

- For each $k \geq 3$, $\mathcal{H}_k$ is weakly indivisible (Henson, 1971).

- The Fraïssé class of finite ordered K_k-free graphs has the Ramsey property. (Nešetřil-Rödl, 1977/83)

- $\mathcal{H}_3$ is indivisible. (Komjáth-Rödl, 1986)

- For all $k \geq 4$, $\mathcal{H}_k$ is indivisible. (El-Zahar-Sauer, 1989)

- Edges have big Ramsey degree 2 in $\mathcal{H}_3$. (Sauer, 1998)

There progress halted. Why?

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Ramsey Theory for Henson Graphs

Theorem. (D.) Let $k \geq 3$. For each finite k-clique-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{G}_k)$ such that for any coloring of all copies of A in $\mathcal{H}_k$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_k$, with $\mathcal{H} \cong \mathcal{H}_k$, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{G}_k)$ colors.

Structure of Proof

Proof Strategy:

I Develop notion of strong $\mathcal{H}_k$-coding tree to represent $\mathcal{H}_k$.

These are analogues of Milliken's strong trees able to handle forbidden k-cliques.

II Prove a Ramsey Theorem for strictly similar finite antichains.

This is an analogue of Milliken's Theorem for strong trees - the proof uses forcing for a ZFC result. It also requires a new notion of envelope.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Similar to the end of Sauer's proof.

Trees with Coding Nodes

A tree with coding nodes is a structure $\langle T, N; \subseteq, <, c\rangle$ in the language $\mathcal{L} = \{\subseteq, <, c\}$ where $\subseteq, <$ are binary relation symbols and c is a unary function symbol satisfying the following:

$T \subseteq 2^{<\omega}$ and $(T, \subseteq)$ is a tree.

$N \leq \omega$ and $<$ is the standard linear order on N.

$c : N \to T$ is injective, and $m < n < N \longrightarrow |c(m)| < |c(n)|$.

$c(n)$ is the n-th coding node in T, usually denoted c_n^T.

K_k-Free Branching Criterion

Note: A collection of coding nodes $\{c_{n_i} : i < k\}$ in T codes a k-clique iff $i < j < k \longrightarrow c_{n_j}(|c_{n_i}|) = 1$.

A tree T with coding nodes $\langle c_n : n < N \rangle$ satisfies the K_k-Free Branching Criterion (k-FBC) if for each non-maximal node $t \in T$, $t^\frown 0 \in T$ and

(∗) $t^\frown 1$ is in T iff adding $t^\frown 1$ as a coding node to T would not code a k-clique with coding nodes in T of shorter length.

Henson's Criterion for building $\mathcal{H}_k$

Henson gave a criterion for building $\mathcal{H}_k$, interpreted to our setting here:

A tree with coding nodes satisfies $(A_k)^{\mathtt{tree}}$ iff

(i) T satisfies the K_k-Free Criterion.

(ii) Let $\langle F_i : i < \omega \rangle$ be any enumeration of finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i - 1$, and each finite subset of ω appears as F_i for infinitely many indices i. Given $i < \omega$, if for each subset $J \subseteq F_i$ of size $k - 1$, $\{c_j : j \in J\}$ does not code a $(k-1)$-clique, then there is some $n \geq i$ such that for all $j < i$, $c_n(l_j) = 1$ iff $j \in F_i$.

Thm. (D.) Suppose T is a tree with no maximal nodes satisfying the K_k-Free Branching Criterion, and the set of coding nodes dense in T. Then T satisfies $(A_k)^{\mathtt{tree}}$, and hence codes $\mathcal{H}_k$.

Strong K_3-Free Tree

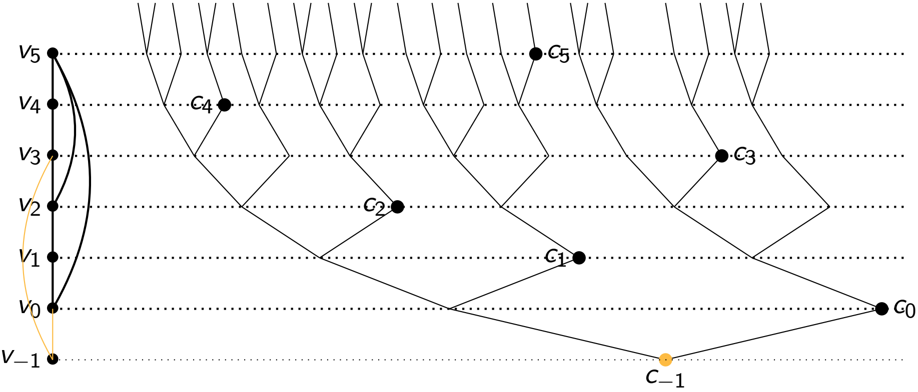


Figure: A strong triangle-free tree $\mathbb{S}_3$ densely coding $\mathcal{H}_3$

Strong K_4-Free Tree

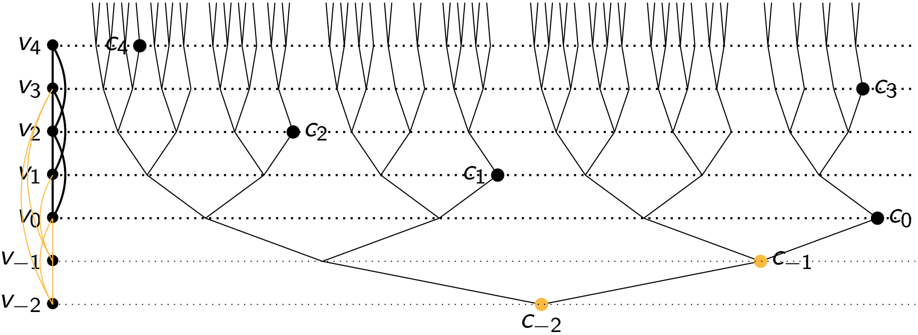


Figure: A strong K_4-free tree $\mathbb{S}_4$ densely coding $\mathcal{H}_4$

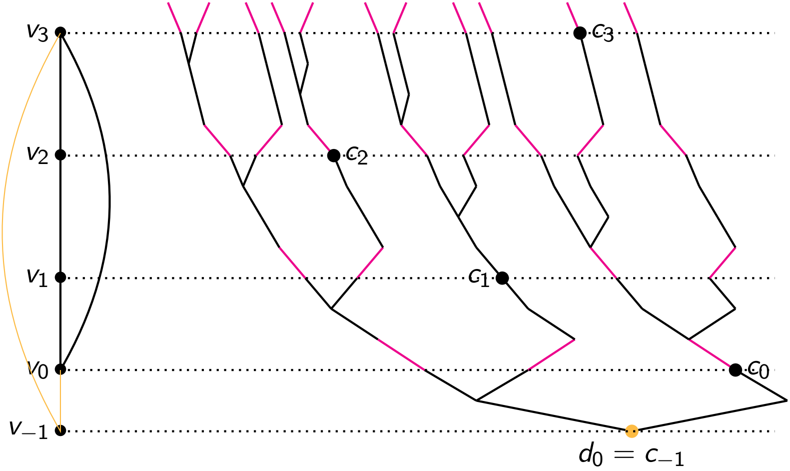
Almost sufficient

One can develop almost all the Ramsey theory one needs on strong K_k-free trees

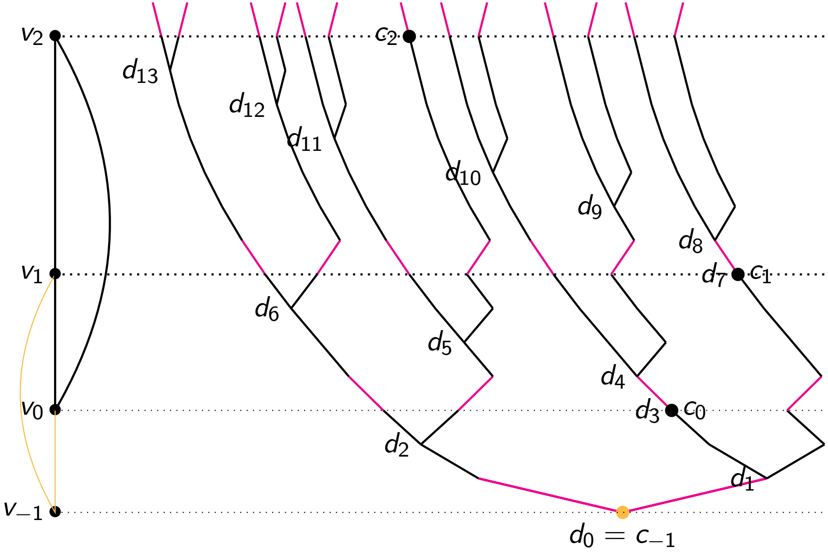
except for vertex colorings: there is a bad coloring of coding nodes.

Solution: Skew the levels of interest.

Strong $\mathcal{H}_3$-Coding Tree $\mathbb{T}_3$



Strong $\mathcal{H}_4$-Coding Tree, $\mathbb{T}_4$

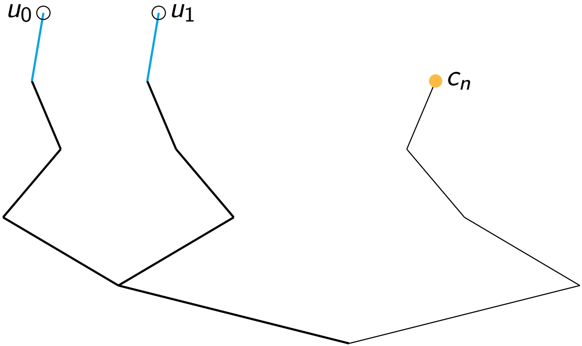


Pre-a-Clique: A key concept

Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level set $X \subseteq \mathbb{T}_k$ of size at least two, with nodes of length ℓ_X, has a pre-a-clique if there are $a - 2$ coding nodes in $\mathbb{T}_k$ coding an $(a - 2)$-clique, and each node in X has passing number 1 by each of these coding nodes.

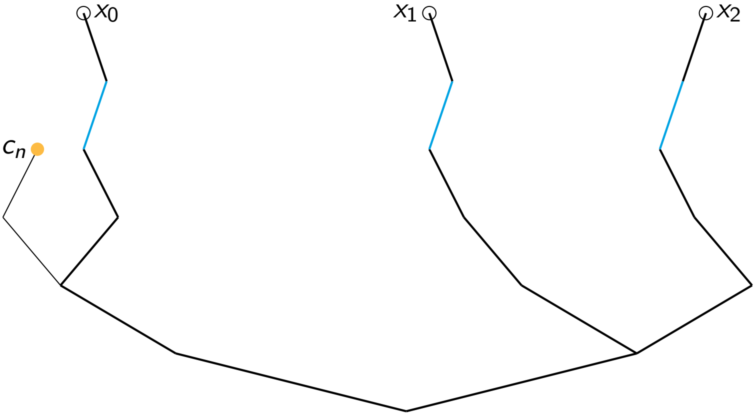
The Point. Pre-a-cliques for $a \in [3, k]$ code entanglements that affect how nodes in X can extend inside $\mathbb{T}$.

A level set U with a pre-3-clique



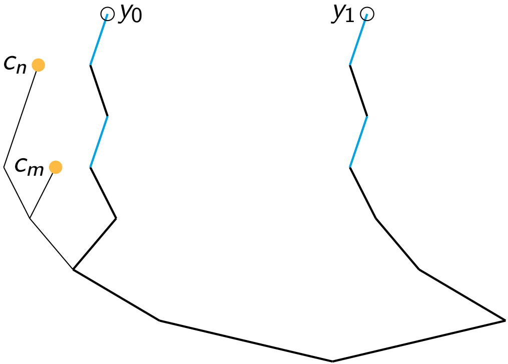
The yellow node is a coding node in $\mathbb{T}_k$ not in U.

A level set X with a pre-3-clique



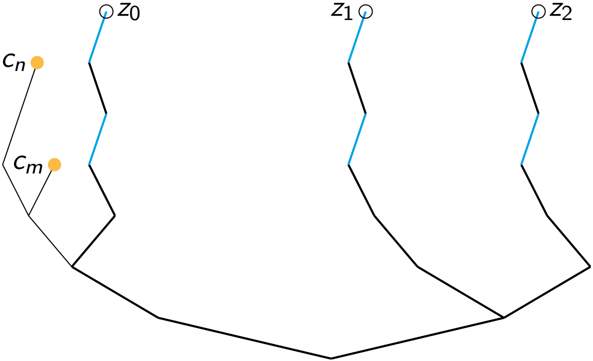
The yellow node is a coding node in $\mathbb{T}_k$ not in X.

A level set Y with a pre-4-clique



The yellow node is a coding node in $\mathbb{T}_k$ not in Y.

A level set Z with a pre-4-clique



The yellow node is a coding node in $\mathbb{T}_k$ not in Z.

Strong Similarity Map

Let $k \geq 3$ be given and let $S, T \subseteq \mathbb{T}_k$ be meet-closed subsets. A bijection $f : S \to T$ is a strong similarity map if for all nodes $s, t, u, v \in S$, the following hold:

1. f preserves lexicographic order.
2. f preserves meets, and hence splitting nodes.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves coding nodes.
6. f preserves passing numbers at coding nodes.

Two subtrees S and T of $\mathbb{T}_k$ are stably isomorphic iff there is a strong similarity map $f : S \to T$ which preserves maximal new pre-cliques in each interval. Such a map f is a stable isomorphism.

The Space of Strong $\mathcal{H}_k$-Coding Trees $\mathcal{T}_k$

$\mathcal{T}_k$ is the collection of all subtrees of $\mathbb{T}_k$ which are stably isomorphic to $\mathbb{T}_k$.

The members of $\mathcal{T}_k$ are called strong $\mathcal{H}_k$-coding trees.

Extension Lemmas provide conditions guaranteeing when a given finite subtree of a strong coding tree T can be extended within T as needed.

Part II: Ramsey Theorem for Strictly Similar Finite Antichains

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's 'forcing proof' of the Halpern-Läuchli Theorem.

(b) Then weave together to obtain an analogue of Milliken's Theorem.

(c) New notion of envelope.

Ramsey Theorem for Strictly Similar Antichains

Thm. Let Z be a finite antichain of coding nodes in a strong $\mathcal{H}_k$-coding tree $T \in \mathcal{T}_k$, and suppose h colors of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an strong $\mathcal{H}_k$-coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.

Strict similarity takes into account the tree structure and the order and intervals in which new pre-cliques appear.

Some Examples of Strict Similarity Types for $k = 3$

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

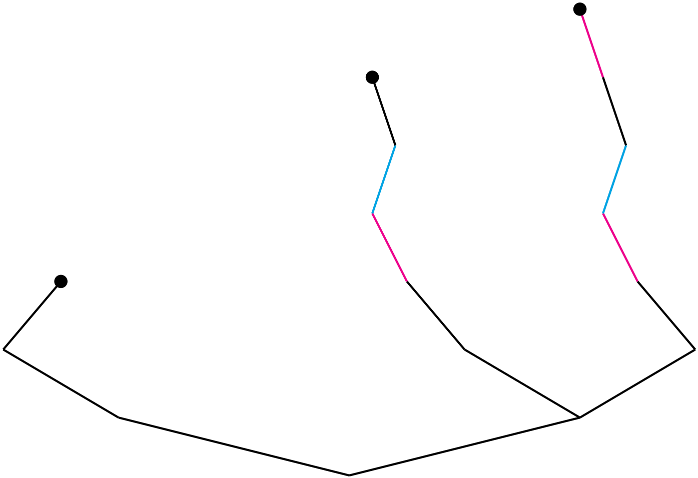
Some Examples of Strict Similarity Types for $k = 3$

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

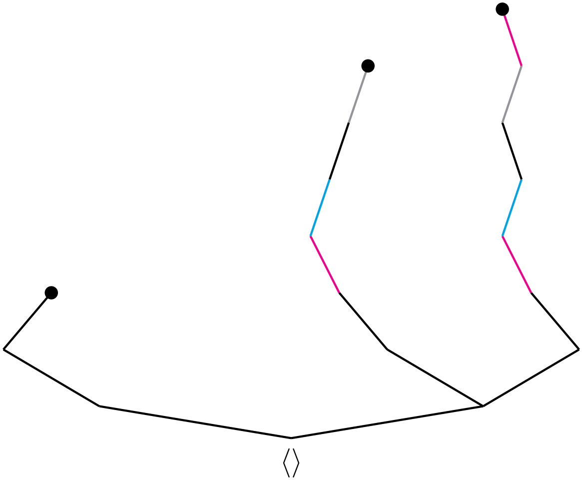
G a graph with three vertices and no edges

A tree A coding G



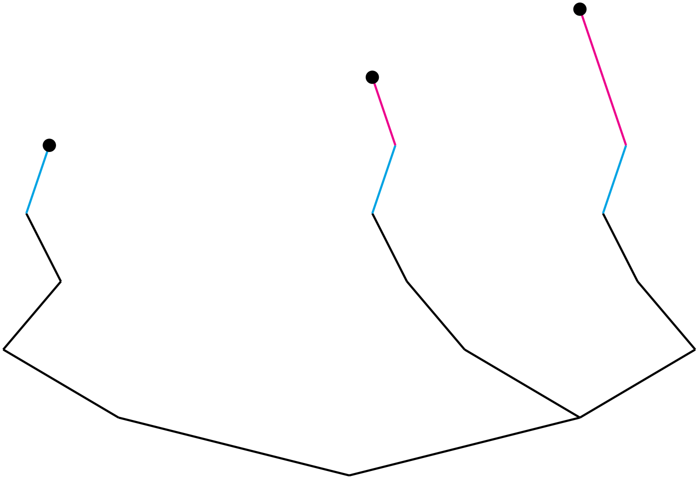
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



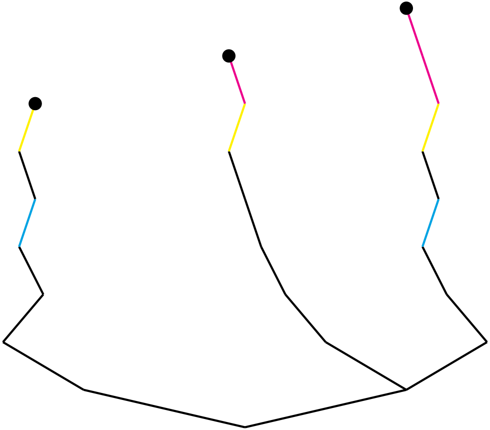
The tree C codes G

C is not strictly similar to A.

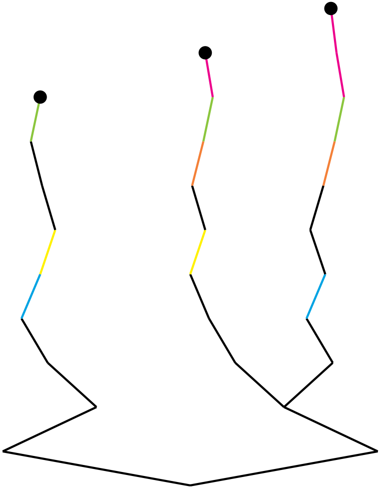


The tree D codes G

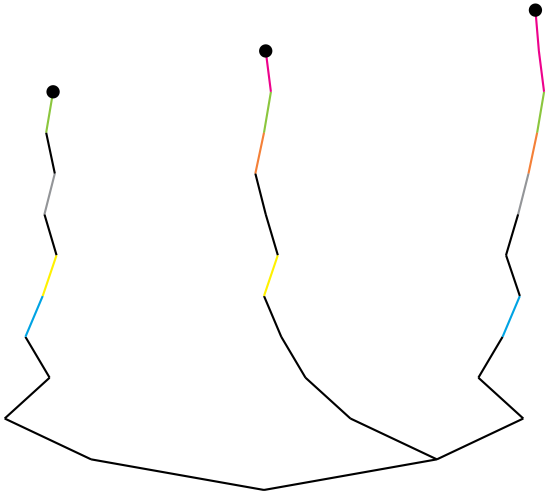
D is not strictly similar to either A or C.



The tree E codes G and is not strictly similar to A - D



The tree F codes G and is strictly similar to E



Envelopes and Witnessing Coding Nodes

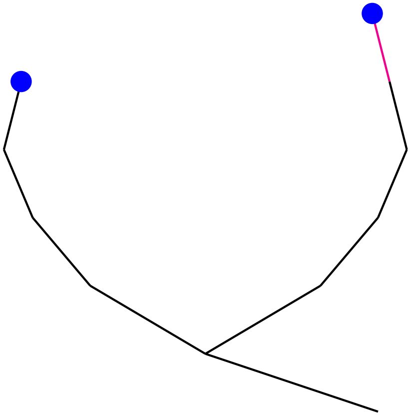
Envelopes add some neutral coding nodes to a finite tree to make it satisfy the Strict Witnessing Property.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

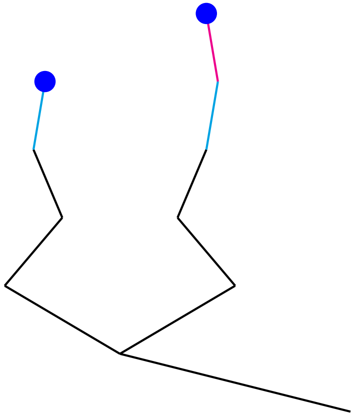
We now give some examples of envelopes.

H codes a non-edge



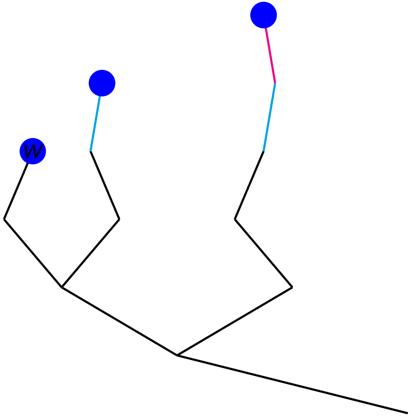
H is its own envelope.

I codes a non-edge



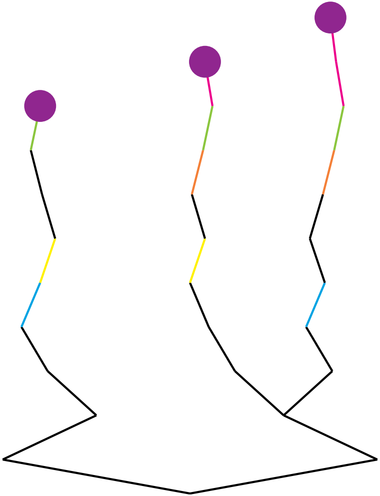
I is not its own envelope.

An Envelope $\mathbf{E}(I)$

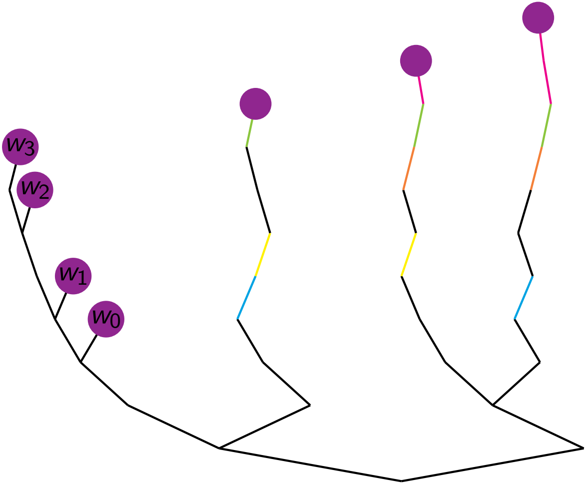


An envelope of I.

The antichain E from before

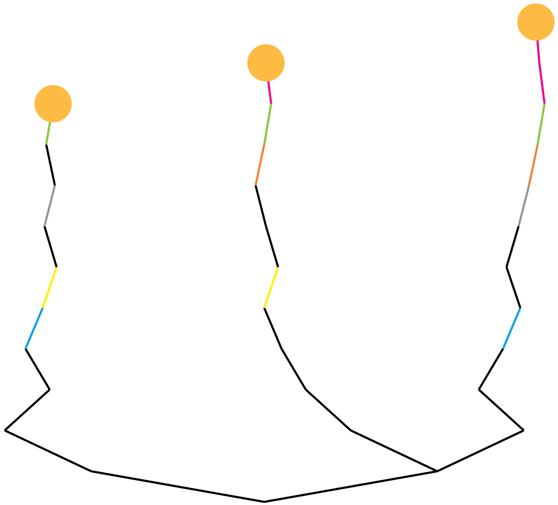


An envelope $\mathbf{E}(E)$

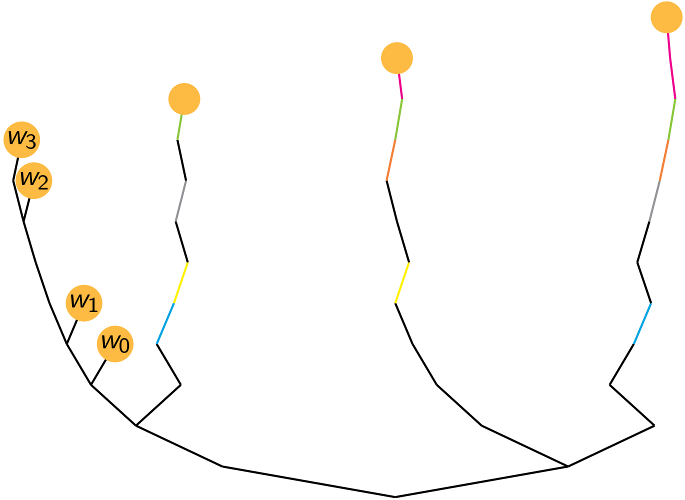


The coding nodes $w_0, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E



$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



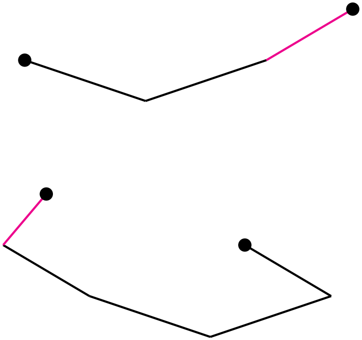
The coding nodes $w_0, \ldots, w_3$ make an envelope of F.

Part III: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for Big Ramsey Degrees $T(G, \mathcal{H}_k)$

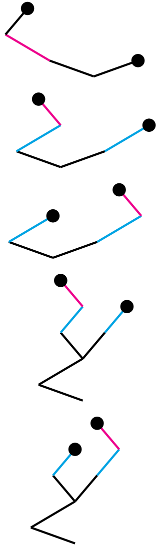
1. Let G be a finite K_k-free graph, and let f color the copies of G in $\mathcal{H}_k$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.
4. Apply the Ramsey Theorem from Part III, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take an antichain of coding nodes, $\mathbb{A}$ in S, which codes $\mathcal{H}_k$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_k$ coded by $\mathbb{A}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of strict similarity types of antichains coding G.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



These are their own envelopes.
$T(Edge, \mathcal{G}_3) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types in $\mathcal{H}_3$ (D.)



Part II: Ramsey Theorem for Finite Trees with the Strict Witnessing Property.

Goal: Find a Ramsey theorem of the form, “Given a finitary coloring of all copies of a finite k-clique-free graph A inside the k-clique-free Henson graph, as coded by a stron coding tree T, find a subtree S, which is again a strong coding tree, in which all copies of A of a given strict similarity type have the same color.

Ideas:

(a) Use forcing to find Halpern-Läuchli style theorems for colorings of level sets. This builds on ideas from Harrington's ‘forcing proof’ of the Halpern-Läuchli Theorem.

(b) Then weave together to obtain an analogue of Milliken's Theorem.

Set-up for level set colorings

Let $T \in \mathcal{T}_k$ and $A \subseteq B \subseteq T$ finite subtrees of T with $\max(A) \subseteq \max(B)$, and both have the Witnessing Property.

Let A^+ be the set of immediate extensions in $\widehat{T}$ of $\max(A)$.

Let $A_e \subseteq A^+$ contain $0^{(l_A+1)}$ and have at least two members.

Suppose that $\tilde{X}$ is a level set of nodes in T extending A_e and $A \cup \tilde{X}$ is a finite valid subtree of T satisfying WP, and assume $0^{(l_{\tilde{X}})} \in \tilde{X}$.

Case (a). $\tilde{X}$ contains a splitting node.

Case (b). $\tilde{X}$ contains a coding node.

$$\mathrm{Ext}_T(A, \tilde{X}) = \{X \subseteq T : X \sqsupseteq \tilde{X} \text{ is a level set, } A \cup X \cong A \cup \tilde{X}, \text{ and } A \cup X \text{ is valid in } T\}.$$

Ramsey Theorem for Level Sets with a Splitting Node

Thm. (D.) Assume Case (a) in the previous set-up.
Given any coloring $h : \mathrm{Ext}_T(A, \tilde{X}) \to 2$, there is a strong coding tree $S \le T$ such that $B \sqsubset S$ and h is monochromatic on $\mathrm{Ext}_S(A, \tilde{X})$.

‘Forcing Proof’

Case (i): level set $\tilde{X}$ contains a splitting node.

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in $\tilde{X}$ extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \le d$.

Fix κ large enough so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The Forcing

$\mathbb{P}$ is the set of functions p such that

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) $\text{ran}(p)$ has no pre-determined new pre-cliques in T.

$q \leq p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \geq l_p$, and

(i) $q(d) \supset p(d)$, and $q(i, \delta) \supset p(i, \delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) $\text{ran}(q \restriction \vec{\delta}_p)$ has no new pre-cliques above $\text{ran}(p)$.

Set-up for applying Erdős-Rado

For $i < d$, $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ denote the α-th generic branch in T_i, and $\dot{b}_d$ the generic branch in T_d.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}$, a name for the levels in $\dot{b}_d$.

For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \rangle$.

- For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that
  1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l";
  2. $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

The Countable Coloring

Let $\mathcal{I}$ be the collection of functions $\iota : 2d \to 2d$ such that

$$\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d-2), \iota(2d-1)\}.$$

For $\vec{\theta} \in [\kappa]^{2d}$, $\iota \in \mathcal{I}$ determines two sequences of ordinals in $[\kappa]^d$:

$$\iota_e(\vec{\theta}) := (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \iota_o(\vec{\theta}) := (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p(d), \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \\ \langle \langle i, j \rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i \rangle, \\ \langle \langle j, k \rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle, \end{aligned} \tag{2}$$

where $\vec{\alpha} = \iota_e(\vec{\theta})$, $\vec{\beta} = \iota_o(\vec{\theta})$, $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}} \rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$ in increasing order. For $\vec{\theta} \in [\kappa]^{2d}$, define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$.

f provides a large homogeneous set of conditions

Note: $\text{dom}(f) = [\kappa]^{2d}$ and $\text{ran}(f)$ is a countable set.

Since $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$, take $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [K]^{\aleph_0}$ so that $K_0 < \cdots < K_{d-1}$ and $K' := \bigcup_{i<d} K_i$ thin in K.

Main Lemma. There are $\varepsilon^* \in 2$ and $t_i^* \in T_i$ such that for all $\vec{\alpha} \in \prod_{i<d} K_i$, $\varepsilon_{\vec{\alpha}} = \varepsilon^*$ and $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$. Furthermore, $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

The $t_0^*, \ldots, t_d^*$ provide good starting nodes for constructing the tree homogeneous for the coloring on $\text{Ext}_T(A, \tilde{X})$.

Building a tree homogeneous for level set coloring

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

(2) These forcings are not simply Cohen forcings; the partial orderings are stronger in order to guarantee that the new levels we obtain by forcing are extendible inside T to another strong coding tree.

(3) The assumption that $A \cup \tilde{X}$ satisfies the Witnessing Property is necessary.

Case (b): Coloring level sets with a coding node

This case is harder, because the forcing proof only produces an end-homogeneous strong coding tree.

Then there is a third forcing argument needed to homogenize over monochromatic cones.

Much induction produces the Milliken analogue: The Ramsey Theorem for trees with the Strict Witnessing Property.

Envelopes are then used to obtain the final Ramsey Theorem for Strict Similarity Types.

Strict Witnessing Property

A subtree A of $\mathbb{T}_k$ satisfies the Strict Witnessing Property (SWP) if A satisfies the Witnessing Property and for each interval $(|d^A_m|, |d^A_{m+1}|]$:

1. If d^A_{m+1} is a splitting node, A has no new pre-cliques in the interval.
2. If d^A_{m+1} is a coding node, A has at most one new pre-clique in this interval.
3. If Y is a new pre-clique in this interval, then each proper subset of Y has a new pre-clique in some interval $(|d^A_j|, |d^A_{j+1}|]$, where $j < m$.

Lem. (D.) If $A \subseteq \mathbb{T}_k$ has the Strict Witnessing Property and $B \cong A$, then B also has the Strict Witnessing Property.

Any B stably isomorphic to A is a copy of A.

Ramsey Theorem for Finite Trees with SWP

Thm. (D.) Let $T \in \mathcal{T}_k$ and A be a finite subtree of T with the Strict Witnessing Property. Let c be a coloring of all copies of A in T. Then there is a strong $\mathcal{H}_k$-coding tree $S \leq T$ in which all copies of A in S have the same color.

This is an analogue of Milliken's Theorem for strong coding trees.

Future Directions

1. Extend methods to other infinite structures with or without forbidden configurations.
2. Trees with coding nodes and forcing arguments have allowed the development of infinite dimensional Ramsey theory on copies of the Rado graph: analogues of the Galvin-Prikry Theorem. Extend these methods to other structures with finite big Ramsey degrees.
3. Milliken was used to determine Ramsey theory of the profinite graph (Huber-Geshke-Kojman, and Zheng). Extend to other uncountable structures.
4. Prove lower bounds cohere so that Zucker's work may be applied to obtain new examples of minimal completion flows.

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