On positive definiteness over locally compact quantum groups

Ami Viselter

(Jointly with Volker Runde)

University of Haifa, Israel

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Outline

- Introduction
  - Positive-definite functions
  - Locally compact quantum groups (LCQGs)
  - Positive-definite functions over LCQGs
- Topologies on the positive-definite functions
  - Results of Granirer and Leinert
  - Over LCQGs
- Square-integrable positive-definite functions
  - Results of Godement and Phillips
  - Over LCQGs
- Amenability
  - Quick intro & Results of Godement and Valette
  - Over LCQGs
- The separation property
  - Basic results for groups
  - Over LCQGs

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Positive-definite functions

G – a locally compact group

Definition (Godement, 1948)

A continuous function $x : G \to \mathbb{C}$ is positive definite if

$$\left(x(s_i^{-1}s_j)\right)_{1\leq i,j\leq n} \text{ is positive in } M_n \text{ whenever } s_1, \ldots, s_n \in G.$$

Such x is always bounded. In fact, $\|x\| = x(e)$.

Examples

1. Any character of G is positive definite.

For $g : G \to \mathbb{C}$, let $\tilde{g}(s) := \overline{g(s^{-1})}$.

2. If $g \in L^2(G)$ then $g * \tilde{g}$ is positive definite.
3. x is positive definite $\iff$ $\overline{x}$ is positive definite.

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Positive-definite functions

Theorem (Bochner, Weil, Godement, De Cannière–Haagerup)

Let $x : G \to \mathbb{C}$ be continuous and bounded. Then TFAE:

1. *x is positive definite;*
2. $\langle x, f^* * f \rangle \geq 0$ *for each* $f \in L^1(G)$, *where* $f^*(s) := \overline{f(s^{-1})}\Delta(s^{-1})$;
3. *There is a continuous unitary rep π of G on $\mathcal{H}_\pi$ and $\xi \in \mathcal{H}_\pi$ s.t.*

$$x(g) = \langle \pi(g)\xi, \xi \rangle \qquad (\forall g \in G);$$

   Equivalently, it is (identified with) a positive element of $B(G)$;
4. *x is a completely positive multiplier of $A(G)$.*

Legend

- $A(G) := \mathrm{VN}(G)_*$ (the Fourier algebra), realized in $C_0(G)$ as

$$\{f * \tilde{g} : f, g \in L^2(G)\}.$$

- $B(G) := C^*(G)^*$ (the Fourier–Stieltjes algebra), realized in $C_b(G)$.

A group as a quantum group

G – a locally compact group.

1. A von Neumann algebra: $L^\infty(G)$
2. Co-multiplication: the $*$-homomorphism $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \cong L^\infty(G \times G)$ defined by

$$(\Delta(f))(t,s) := f(ts) \qquad (f \in L^\infty(G)).$$

   By associativity, we have $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

3. Left and right Haar measures. View them as n.s.f. weights $\varphi, \psi : L^\infty(G)_+ \to [0,\infty]$ by $\varphi(f) := \int_G f(t)\,\mathrm{d}t_\ell$, $\psi(f) := \int_G f(t)\,\mathrm{d}t_r$.

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Locally compact quantum groups

Motivation

Lack of Pontryagin duality for non-Abelian l.c. groups.

Definition (Kustermans & Vaes, 2000)

A locally compact quantum group is a pair $\mathbb{G} = (M, \Delta)$ such that:

1. M is a von Neumann algebra
2. $\Delta : M \to M \otimes M$ is a co-multiplication: a normal, faithful, unital $*$-homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

3. There are two n.s.f. weights φ, ψ on M (the Haar weights) with:
   - $\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\varphi(x) < \infty$
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Locally compact quantum groups

- Rich structure theory, including an unbounded antipode and duality $\mathbb{G} \mapsto \hat{\mathbb{G}}$ within the category satisfying $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.
- $L^1(\mathbb{G})$ is a Banach algebra with convolution $\omega * \theta := (\omega \otimes \theta) \circ \Delta$.
- $L^1(\mathbb{G})$ has a dense involutive subalgebra $L^1_*(\mathbb{G})$.

Example (commutative LCQGs: $\mathbb{G} = G$)

$L^\infty(\mathbb{G}) = L^\infty(G), \quad (L^1(\mathbb{G}), *) = (L^1(G), \text{convolution})$

Example (co-commutative LCQGs: $\mathbb{G} = \hat{G}$)

The dual $\hat{G}$ of G (as a LCQG) has

- $L^\infty(\mathbb{G}) = \mathrm{VN}(G), \quad (L^1(\mathbb{G}), *) = (A(G), \text{pointwise product})$
- $\Delta : \mathrm{VN}(G) \to \mathrm{VN}(G) \otimes \mathrm{VN}(G)$ given by $\Delta(\lambda_g) := \lambda_g \otimes \lambda_g$
- $\varphi = \psi =$ the Plancherel weight on $\mathrm{VN}(G)$.

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Every LCQG $\mathbb{G}$ has 3 equivalent "faces":

1. von Neumann-algebraic: vN alg $L^\infty(\mathbb{G})$
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- The left regular representation for groups: $\lambda : L^1(G) \to C_r^*(G)$ generalizes to

$$\lambda : L^1(\mathbb{G}) \to C_0(\hat{\mathbb{G}}).$$

- It extends to $C_0^u(\mathbb{G})^*$ as

$$\lambda^u : C_0^u(\mathbb{G})^* \to M(C_0(\hat{\mathbb{G}})).$$

- The GNS constructions of $(L^\infty(\mathbb{G}), \varphi)$ and $(L^\infty(\hat{\mathbb{G}}), \hat{\varphi})$ yield the same Hilbert space, $L^2(\mathbb{G})$. When $\mathbb{G} = G$, $L^2(\mathbb{G}) = L^2(G)$. Let $\Lambda : \mathcal{N}_\varphi \to L^2(\mathbb{G})$ be the canonical injection.

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Positive-definite functions over LCQGs

Definitions (Daws, 2012; Daws & Salmi, 2013)

Let $\mathbb{G}$ be a LCQG. Say that $x \in L^\infty(\mathbb{G})$ is...

1. a completely positive-definite function if...
2. a positive-definite function if $\langle x^*, \omega^* * \omega \rangle \geq 0$ for all $\omega \in L^1_*(\mathbb{G})$
3. a Fourier–Stieltjes transform of a positive measure if

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4. a completely positive multiplier if there exists a completely positive multiplier of $L^1(\hat{\mathbb{G}})$ associated to x.

Theorem (Daws, 2012; Daws & Salmi, 2013)

(1) $\iff$ (3) $\iff$ (4) $\implies$ (2) . If $\mathbb{G}$ is co-amenable, all are equivalent.

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Comparison of topologies

Let G be a l.c. group.

Theorem (Raĭkov, 1947; Yoshizawa, 1949)

On the set of positive-definite functions of norm 1*, the w*-topology* $\sigma(L^\infty(G), L^1(G))$ *coincides with the topology of uniform convergence on compact subsets.*

Actually, much more can be said.

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Comparison of topologies

The $A(G)$-strict topology on $B(G)$ is induced by the semi-norms $u \mapsto \|uf\|_{A(G)}$, $f \in A(G)$.

Theorem A (Granirer & Leinert, 1981)

On the unit sphere of $B(G)$, the w^*-topology coincides with the $A(G)$-strict topology.

Taking positive elements in $B(G)$, we get Raĭkov, Yoshizawa.
Generalizes results of Derighetti (1970) and McKennon (1971).

The $L^1(G)$-strict topology on $M(G) := C_0(G)^*$ is induced by the semi-norms $\mu \mapsto \|\mu * f\|_{L^1(G)}$, $f \in L^1(G)$.

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On the unit sphere of $M(G)$, the w^*-topology coincides with the $L^p(G)$-strict topology.

Comparison of topologies

Hu, Neufang & Ruan asked (2012) whether Theorems A,B generalize to LCQGs. We answer this affirmatively.

Let $\mathbb{G}$ be a LCQG. The $L^1(\mathbb{G})$-strict topology on $C_0^{\mathrm{u}}(\mathbb{G})^*$ is induced by the semi-norms $\mu \mapsto \|\mu * \omega\|_{L^1(\mathbb{G})}$, $\omega \in L^1(\mathbb{G})$.

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Relying on the theory of the non-commutative L^p spaces, we have:

Theorem (Caspers, 2013)

For $1 \leq p \leq 2$, consider $L^p(\mathbb{G}) := L^p(L^\infty(\mathbb{G}))$. One can give a proper definition to convolution of elements of $L^1(\mathbb{G})$ with elements of $L^p(\mathbb{G})$.

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Apparently it is not possible to do that for $p > 2$.
Even classically, convolutions are not "well behaved" outside $L^1(G)$.

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Theorem (Godement, 1948)

Let G be a l.c. group. Every positive-definite, square-integrable function over G has a square root: it is of the form $g * g$, $\tilde{g} = g \in L^2(G)$.

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Let $\mathcal{A}$ be a full left Hilbert algebra w/ completion $\mathcal{H}$. Consider the cone

$$\mathcal{P}^\flat := \left\{ \eta \in \mathcal{H} : \langle \eta, \zeta^\# \zeta \rangle \geq 0 \quad (\forall \zeta \in \mathcal{A}) \right\}$$

(appearing in works of Araki, Connes, Haagerup, Perdrizet, Takesaki).

Say that $\eta \in \mathcal{P}^\flat$ is integrable if

$$\sup \left\{ \langle \eta, \xi \rangle : \xi \in \mathcal{A} \text{ and } \pi(\xi) \text{ is a projection} \right\} < \infty.$$

Example

- Let $\mathcal{A}$ be the (full) left Hilbert algebra associated with the Plancherel weight on $\mathrm{VN}(G)$ ($\mathcal{A}_0 := C_c(G)$ with the inner product of $L^2(G)$, product = convolution, $\zeta^\# = \zeta^*$, $\mathcal{A} := \mathcal{A}_0''$).
- Then $\mathcal{H} = L^2(G)$, and if $f \in L^\infty(G) \cap L^2(G)$, then

$$f \in \mathcal{P}^\flat \qquad \Longleftrightarrow \qquad f \text{ is positive definite.}$$

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Theorem (J. Phillips, 1973)

Let $\eta \in \mathcal{P}^\flat$. Then η is integrable $\iff$ it has a square root $\zeta \in \mathcal{P}^\flat$, namely $\langle \xi, \eta \rangle = \langle \pi(\xi)\zeta, \zeta \rangle$ for every $\xi \in \mathcal{A}$.

Going back to Godement's Theorem:

- Take $\mathcal{A}$ from the last example.
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Proof (sketch)

We follow similar lines:

1. We let $\mathcal{A}_{\hat{\varphi}}$ be the (full) left Hilbert algebra associated with the left Haar weight $\hat{\varphi}$ of $\hat{\mathbb{G}}$.
2. If $x \in \mathcal{N}_\varphi$ (in $L^\infty(\mathbb{G})$ and "square integrable"), then
$$x \text{ is positive definite} \iff \Lambda(x) \in \mathcal{P}^\flat_{\hat{\varphi}}.$$
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Amenability

Definition

A locally compact group G is amenable if it admits a left-invariant mean: a state of $L^\infty(G)$ that is invariant under left translations.

Examples

Compact groups, abelian (even solvable) groups, locally-finite groups.
Non-example: $\mathbb{F}_n$, $n \geq 2$.

Definitions

A LCQG $\mathbb{G}$ is amenable if it admits a left-invariant mean: a state m of $L^\infty(\mathbb{G})$ with $m((\omega \otimes \mathrm{id})\Delta(x)) = m(x)\omega(\mathbb{1})$ for all $x \in L^\infty(\mathbb{G})$, $\omega \in L^1(\mathbb{G})$.

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$\mathbb{G}$ is amenable $\underset{?}{\overset{\Longleftarrow}{\Longrightarrow}}_{?}$ $\hat{\mathbb{G}}$ is co-amenable.

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Positive-definite measures

Let G be a locally compact group.

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Say that $\mu \in M(G)$ is positive-definite if $\int f * \tilde{f} \, \mathrm{d}\mu \geq 0$ for all $f \in C_c(G)$.
Equivalently: $\lambda(\mu) \geq 0$ (recall: it belongs to $M(C_r^*(G))$).

Theorem (Godement, 1948; extended by Valette, 1998)

TFAE:

1. *G is amenable;*
2. *$\int f \, \mathrm{d}\mu \geq 0$ for each pos-def $\mu \in M(G)$ and pos-def function f;*
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If $\mathbb{G}$ is co-amenable, then TFAE:

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The separation property

Definition (Lau & Losert, 1986; Kaniuth & Lau, 2000)

Let G be a locally compact group and H be a closed subgroup of G. Say that G has the *H-separation property* if for every $g \in G \setminus H$ there exists a positive-definite function φ on G with $\varphi|_H \equiv 1$ but $\varphi(g) \neq 1$.

Theorem

G has the H-separation property in these cases:

- *H is normal (easy)*
- *H is compact (Eymard, 1964)*
- *H is open (Hewitt & Ross)*
- *G is [SIN] (Kaniuth & Lau, based on Forrest, 1992).*

Example

Let $G =$ the "$ax + b$ group" and $H \leq G$. Then:

G has the H-separation property $\iff$ H is either compact or normal.

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Definition (Daws, Kasprzak, Skalski & Sołtan, 2012)

Let $\mathbb{G}, \mathbb{H}$ be LCQGs. Say that $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$ if there exists a surjective $*$-homomorphism $\Phi : C_0^{\mathrm{u}}(\mathbb{G}) \to C_0^{\mathrm{u}}(\mathbb{H})$ intertwining the co-multiplications.

Such a map has a dual, $\hat{\Phi} : C_0^{\mathrm{u}}(\hat{\mathbb{H}}) \to M(C_0^{\mathrm{u}}(\hat{\mathbb{G}}))$ (Meyer, Roy & Woronowicz, 2012).

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If $H \leq G$, then Φ is the restriction map $C_0(G) \to C_0(H)$, $f \mapsto f|_H$, and the dual $\hat{\Phi}$ is the natural embedding $C^*(H) \hookrightarrow M(C^*(G))$.

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Say that $\mathbb{G}$ has the $\mathbb{H}$-separation property if whenever μ is a state of $C_0^{\mathrm{u}}(\mathbb{G})$ such that $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \notin \hat{\Phi}(M(C_0^{\mathrm{u}}(\hat{\mathbb{H}})))$, there is a state $\hat{\omega}$ of $C_0^{\mathrm{u}}(\hat{\mathbb{G}})$ so that $\Phi((\mathrm{id} \otimes \hat{\omega})(\mathbb{W}_{\mathbb{G}})) = \mathbb{1}$ but $\mu((\mathrm{id} \otimes \hat{\omega})(\mathbb{W}_{\mathbb{G}})) \neq 1$.

In English:

if μ is a state of $C_0^{\mathrm{u}}(\mathbb{G})$ that is "not supported by $\mathbb{H}$", then there exists a positive-definite function that "restricts to $\mathbb{1}$ on $\mathbb{H}$" but is "not 1 w.r.t. μ".

The separation property

Example

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Definitions (Woronowicz)

A LCQG $\mathbb{H}$ is compact if $\mathbb{1} \in C_0(\mathbb{H})$.

The dual $\hat{\mathbb{H}}$ is then called discrete. It admits a (bounded) co-unit $\hat{\epsilon} \in C_0(\hat{\mathbb{H}})^*$ and a central minimal projection $\hat{p} \in L^\infty(\hat{\mathbb{H}})$ so that $\hat{a}\hat{p} = \hat{\epsilon}(\hat{a})\hat{p} = \hat{p}\hat{a}$ for every $\hat{a} \in L^\infty(\hat{\mathbb{H}})$.

Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ a compact quantum subgroup of $\mathbb{G}$. Then there is an embedding $\gamma : L^\infty(\hat{\mathbb{H}}) \hookrightarrow L^\infty(\hat{\mathbb{G}})$ "interacting well with $\hat{\Phi}$".

Theorem (Runde–V, 2014)

If

$$(\forall \hat{z} \in M(C_0(\hat{\mathbb{G}}))) \qquad \hat{\Delta}_{\mathbb{G}}(\hat{z})(\gamma(\hat{p}) \otimes \mathbb{1}) = \gamma(\hat{p}) \otimes \hat{z} \qquad \Longrightarrow \qquad \hat{z} \in \operatorname{Im}\gamma,$$

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This condition holds, even for every $\hat{z} \in L^\infty(\hat{\mathbb{G}})$, in these cases:

- The commutative case: $H \leq G$
- The co-commutative case:
  $\mathbb{G} = \hat{G}$ and $\mathbb{H} = \widehat{G/A}$ where $A \trianglelefteq G$ is open
- Cocycle bicrossed products
  - $\mathbb{H}$ is a compact normal quantum subgroup of $\mathbb{G}$ whose ambient extension is cleft
  - the construction involves two LCQGs, $\mathbb{G}_1, \mathbb{G}_2$
  - it is a (von Neumann algebraic) generalization of the Packer–Raeburn twisted crossed products.

Thank you for your attention!