Free monotone transport without a trace

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November 2, 2014

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Joint laws

Let $X_1, \ldots, X_N$ be self-adjoint elements in a von Neumann algebra with a state φ and denote $X = (X_1, \ldots, X_N)$. The *joint law of $X_1, \ldots, X_N$ with respect to φ*, denoted φ_X, is the linear functional on $\mathscr{P}$ defined by

$$\varphi_X(P(t_1, \ldots, t_N)) := \varphi(P(X_1, \ldots, X_N)) \qquad P \in \mathscr{P}$$

Classical Transport

- Transport from (X, μ) to (Z, ν) is a map $T: X \to Z$ such that $T_*\mu = \nu$
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Free transport

Let $X = (X_1, \ldots, X_N) \subset M$ and $Z = (Z_1, \ldots, Z_N) \subset L$, where M and L are von Neumann algebras with faithful states φ and ψ, respectively. Then *transport from* φ_X *to* ψ_Z is an N-tuple $Y = (Y_1, \ldots, Y_N) \subset W^*(X_1, \ldots, X_N) \subset M$ so that $\varphi_Y = \psi_Z$.

Note that the existence of transport implies $Z_j \mapsto Y_j$ is a state-preserving embedding of $W^*(Z_1, \ldots, Z_N)$ into $W^*(X_1, \ldots, X_N)$.

Shlyakhtenko 1997:

- Given $\{U_t\}_{t\in\mathbb{R}}$ a strongly continuous one-parameter group of orthogonal transformations on a real Hilbert space $\mathcal{H}_\mathbb{R}$
- Extend each U_t to unitary on $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$ and let $A > 0$ be generator: $U_t = A^{it}$

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- Examples:
  1. $U_t = I_N \in M_N(\mathbb{R})$ for all $t \Rightarrow A = I_N$
  2. For $\lambda > 0$

$$U_t = \begin{pmatrix} \cos(t\log\lambda) & -\sin(t\log\lambda) \\ \sin(t\log\lambda) & \cos(t\log\lambda) \end{pmatrix}$$

$$A = \frac{1}{2}\begin{pmatrix} \lambda+\lambda^{-1} & -i(\lambda-\lambda^{-1}) \\ i(\lambda-\lambda^{-1}) & \lambda+\lambda^{-1} \end{pmatrix}$$

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- Define new inner product:

$$\langle x, y\rangle_U = \left\langle \frac{2}{1+A^{-1}}x, y \right\rangle \qquad x, y \in \mathcal{H}_\mathbb{C}$$

and let $\mathcal{H}$ be the closure of $\mathcal{H}_\mathbb{C}$ with respect to this inner product

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- Define $s(x) = \ell(x) + \ell(x)^* \in \mathcal{B}(\mathcal{F}(\mathcal{H}))$ for $x \in \mathcal{H}$ where

$$\ell(x) f_1 \otimes \cdots \otimes f_n = x \otimes f_1 \otimes \cdots \otimes f_n \qquad \ell(x)^* f_1 \otimes \cdots \otimes f_n = \langle x, f_1 \rangle_U f_2 \otimes \cdots \otimes f_n$$

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- The *free Araki-Woods factor* is $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'' := W^*(s(x) \colon x \in \mathcal{H}_{\mathbb{R}}) \subset \mathcal{B}(\mathcal{F}(\mathcal{H}))$
- The *free quasi-free state* φ is the vector state on $\mathcal{B}(\mathcal{F}(\mathcal{H}))$ corresponding to Ω, is normal and faithful on $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$

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- $\Gamma(\mathbb{R}^N, I_N)'' = L\mathbb{F}_N$
- In fact, if $G \subset \mathbb{R}_+^\times$ is the closed subgroup generated by the spectrum of A then

$$\Gamma(\mathcal{H}_\mathbb{R}, U_t)'' \text{ is a factor of type } \begin{cases} \mathrm{III}_1 & \text{if } G = \mathbb{R}_+^\times \\ \mathrm{III}_\lambda & \text{if } G = \lambda^\mathbb{Z},\ 0 < \lambda < 1 \\ \mathrm{II}_1 & \text{if } G = \{1\} \end{cases}$$

We let $M = \Gamma(\mathbb{R}^N, U_t)''$ and $X_j = s(e_j)$, where $\{e_1, \ldots, e_N\} \subset \mathbb{R}^N$ is the standard basis. Thus

$$M = W^*(X_1, \ldots, X_N).$$

Our goal is to construct transport from φ_X (with $X = (X_1, \ldots, X_N)$).

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Voiculescu's free difference quotients

- For each $j = 1, \ldots, N$ the *jth free difference quotient* $\delta_j \colon \mathscr{P}(X) \to \mathscr{P}(X) \otimes \mathscr{P}(X)^{op}$ is defined by

$$\delta_j(X_k) = \delta_{j=k} 1 \otimes 1^\circ$$
$$\delta_j(PQ) = \delta_j(P) \cdot Q + P \cdot \delta_j(Q).$$

- $\delta_1(X_1X_2X_1X_2) = 1 \otimes (X_2X_1X_2)^\circ + X_1X_2 \otimes X_2^\circ$

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We then define the *jth σ-difference quotient* $\partial_j \colon \mathscr{P}(X) \to \mathscr{P}(X) \otimes \mathscr{P}(X)^{op}$ as

$$\partial_j := \sum_{k=1}^{N} \left[\frac{2}{1+A}\right]_{kj} \delta_k.$$

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Proposition

Let M, $X_1, \ldots, X_N$, and φ be as above. Then for each $j = 1, \ldots, N$ and every $P \in \mathscr{P}(X)$

$$\varphi(X_j P) = \varphi \otimes \varphi^{op}(\partial_j P)$$

For each $j = 1, \ldots, N$ we define the *jth σ-cyclic derivative* $\mathscr{D}_j \colon \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathscr{D}_j P = \sum_{k=1}^{N} \left[\frac{2}{1+A}\right]_{jk} \sum_{P=AX_kB} \sigma^{\varphi}_{-i}(B)A,$$

where we note

$$\sigma^{\varphi}_z(X_j) = \sum_{k=1}^{N} [A^{iz}]_{jk} X_k = [A^{iz}X]_j.$$

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A special potential

We first note

$$\mathscr{D}_\ell (X_k X_j) = \left[\frac{2}{1+A}\right]_{\ell k} \sigma_{-i}^{\varphi}(X_j) + \left[\frac{2}{1+A}\right]_{\ell j} X_k$$

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&= \left[\frac{2}{1+A}\right]_{k\ell} X_j + \left[\frac{2}{1+A}\right]_{\ell j} X_k
\end{aligned}$$

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Thus

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and consequently if

$$V_0 := \frac{1}{2}\sum_{j,k=1}^{N} \left[\frac{1+A}{2}\right]_{jk} X_k X_j,$$

then its easy to see that

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Proposition

Let M, $X_1,\ldots,X_N$, φ, and V_0 be as above. Then for each $j=1,\ldots,N$ and every $P \in \mathscr{P}(X)$

$$\varphi(\mathscr{D}_j(V_0)P) = \varphi \otimes \varphi^{op}(\partial_j P)$$

Observe that provided we fix A we can make sense of ∂_j and$\mathscr{D}_j$ on $\mathscr{P}$.

Free Gibbs state

Fix $V \in \mathscr{P}$. A linear functional ψ on $\mathscr{P}$ is called a *free Gibbs state with potential V* if for each $j = 1, \ldots, N$ and every $P \in \mathscr{P}$

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- φ_X is the free Gibbs state with potential V_0.
- For $R > 0$, we consider particular Banach norms $\|\cdot\|_{R,\sigma}$ on $\mathscr{P}$ such that $\overline{\mathscr{P}}^{\|\cdot\|_{R,\sigma}}$ can be thought of as convergent power series with radius of convergence at least R.

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- In the definition of a free Gibbs state, V can be taken from $\overline{\mathscr{P}}^{\| \cdot \|_{R,\sigma}}$.
- If $R \geq \|X_1\|, \ldots, \|X_N\|$ then $P(X) \in M$ whenever $P \in \overline{\mathscr{P}}^{\| \cdot \|_{R,\sigma}}$ and $\|\sigma_{ik}^{\varphi}(P(X))\|_{R,\sigma} = \|P(X)\|_{R,\sigma}$ for all $k \in \mathbb{Z}$.

Theorem (N. 2013)

- Let M, $X_1, \ldots, X_N$, φ, and V_0 as above.
- Let $Z = (Z_1, \ldots, Z_N)$ be elements in a von Neumann algebra L with faithful state ψ.
- Assume ψ_Z is a free Gibbs state with potential $V \in \overline{\mathscr{P}}^{\|\cdot\|_{R,\sigma}}$, with $R \geq \|X_1\|, \ldots, \|X_N\|$.

There exists $\epsilon > 0$ (depending on N, A, and R) such that if $\|V - V_0\|_{R,\sigma} < \epsilon$ then transport from φ_X to ψ_Z exists.

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In particular, the transport elements $Y_1, \ldots, Y_N$ are convergent power series in $X_1, \ldots, X_N$. By shrinking ϵ if necessary, we can write $X_1, \ldots, X_N$ as power series in $Y_1, \ldots, Y_N$ and therefore $Z_j \mapsto Y_j$ gives a state-preserving isomorphism

$$C^*(Z_1, \ldots, Z_N) \cong C^*(X_1, \ldots, X_N)$$
$$W^*(Z_1, \ldots, Z_N) \cong M.$$

Theorem (Guionnet, Maurel-Segala 2006)

Let $R > 0$. There exists $\epsilon > 0$ (depending on N, A, and R) so that the free Gibbs state with potential V is unique whenever $\|V - V_0\|_{R,\sigma} < \epsilon$.

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Proof of non-tracial transport theorem.

- Suffices to construct $Y_1, \ldots, Y_N \in M$ whose joint law with respect to φ is the free Gibbs state with potential V

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- Joint law of Y_j's being free Gibbs state with potential V is equivalent to G being the fixed point of a mapping F_V on the unit ball of $\overline{\mathscr{P}(X)}^{\|\cdot\|_{R,\sigma}}$
- Can show F_V is locally Lipschitz

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Hiai 2003:

- Given $q \in (-1,1)$, the *q-Fock space* $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{U,q} = \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{k=1}^{n} \langle f_k, g_{\pi(k)} \rangle_U$$

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  Note $\mathcal{F}_0(\mathcal{H}) = \mathcal{F}(\mathcal{H})$

- $\ell_q(x)$ for $x \in \mathcal{H}$ is defined as before but now

$$\ell_q(x)^* f_1 \otimes \cdots \otimes f_n = \sum_{k=1}^{n} q^{k-1} \langle x, f_k \rangle_U f_1 \otimes \cdots \otimes \hat{f}_k \otimes \cdots \otimes f_n$$

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- Given $q \in (-1,1)$, the *q-Fock space* $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{U,q} = \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{k=1}^{n} \langle f_k, g_{\pi(k)} \rangle_U$$

  Note $\mathcal{F}_0(\mathcal{H}) = \mathcal{F}(\mathcal{H})$

- $\ell_q(x)$ for $x \in \mathcal{H}$ is defined as before but now

$$\ell_q(x)^* f_1 \otimes \cdots \otimes f_n = \sum_{k=1}^{n} q^{k-1} \langle x, f_k \rangle_U f_1 \otimes \cdots \otimes \hat{f}_k \otimes \cdots \otimes f_n$$

- Letting $s_q(x) = \ell_q(x) + \ell_q(x)^*$, the *$q$-deformed Araki-Woods algebra* is $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' = W^*(s_q(x) \colon x \in \mathcal{H}_\mathbb{R})$
- The *q-quasi-free state* φ_q is the vector state corresponding to Ω, is normal and faithful on $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$

Hiai 2003:

- When A has infinitely many mutually orthogonal eigenvectors $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is a factor and the type classification is the same as the one obtained by Shlyakhtenko (the $q = 0$ case)
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Theorem (N. 2013)

There exists $\epsilon > 0$ (depending on N and A) such that whenever $|q| < \epsilon$ we have

$$\Gamma_q(\mathbb{R}^N, U_t)'' \cong \Gamma(\mathbb{R}^N, U_t)'' \text{ and}$$
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and these isomorphisms are state-preserving.

Proof.

- Fix $q \in (-1, 1)$ and denote $M = \Gamma_q(\mathbb{R}^N, U_t)''$ and $X_j = s_q(e_j)$ for $j = 1, \ldots, N$.

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- We show the q-quasi-free state is the free Gibbs state with a potential V_q and that $V_q \to V_0$ as $|q| \to 0$.
- If we can find for each $j = 1, \ldots, N$ a $\xi_j^{(q)} \in M$ satisfying

$$\varphi_q(\xi_j^{(q)} P) = \varphi_q \otimes \varphi_q^{op}(\partial_j P)$$

for all $P \in \mathscr{P}(X)$, then φ_q is the free Gibbs state with potential

$$V_q = \Sigma\left(\sum_{j,k=1}^{N} \left[\frac{1+A}{2}\right]_{jk} \xi_k^{(q)} X_j \right),$$

and

$$\|V_q - V_0\|_{R,\sigma} \leq C(N,A,R) \max_{1 \leq k \leq N} \|\xi_k^{(q)} - X_k\|_R$$

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where $\Xi_q \in L^2(M \otimes M^{op}, \varphi_q \otimes \varphi_q^{op})$ is the element identified with $\sum_{n\geq 0} q^n P_n \in HS(F_q(\mathcal{H}))$ via the identification

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- If we define $\partial_j^{(q)}(P) := \partial_j(P)\Xi_q$, then our desired elements are

$$\xi_j^{(q)} = \partial_j^{(q)*} \circ (\sigma_{-i}^{\varphi_q} \otimes \sigma_i^{\varphi_q}) \left([\Xi_q^{-1}]^*\right),$$

estimates similar to those of Dabrowski imply these are well-defined

Proof.

- Adapting a well known identity of Voiculescu to the non-tracial case yields

$$\begin{aligned}\xi_j^{(q)} =& (\sigma_{-i}^{\varphi_q} \otimes 1)\left([\Xi_q^{-1}]^*\right) \# X_j \\ &- m \circ (1 \otimes \varphi_q \otimes 1) \circ \left(1 \otimes \partial_j^{(q)} + \bar{\partial}_j^{(q)} \otimes 1\right) \circ (\sigma_{-i}^{\varphi_q} \otimes 1)\left([\Xi_q^{-1}]^*\right),\end{aligned}$$

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where $(a \otimes b^\circ)\#c = acb$

- Using methods of Dabrowski one can show

$$(\sigma_{-i}^{\varphi_q} \otimes 1)\left([\Xi_q^{-1}]^*\right) = 1 \otimes 1^\circ + o(|q|)$$

- Hence

$$\xi_j^{(q)} = X_j + o(|q|)$$

□

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Theorem 3.1 (N. 2014)

Using non-tracial transport it is possible to perturb the above embeddings without altering the tower of non-commutative probability spaces $(M_k)_{k\geq 0}$. Furthermore, given another state τ_k on $Gr_k\mathcal{P}$ one obtains criterion for when the von Neumann algebra associated to the GNS representation of τ_k is isomorphic to M_k.