Duality for C*-algebras and applications

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Poincaré duality for C*-algebras

Definition

Two C*-algebras A and B are *Poincaré dual* if there exist classes

$$\Delta \in \mathrm{KK}(A \otimes B, \mathbb{C}) \text{ (the `unit'), } \quad \widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, A \otimes B) \text{ (`co-unit')}$$

such that the map

$$\mathrm{KK}(D_1, B \otimes D_2) \to \mathrm{KK}(A \otimes D_1, A \otimes B \otimes D_2) \xrightarrow{\cdot \otimes_{A \otimes B} \Delta} \mathrm{KK}(A \otimes D_1, D_2),$$

is an isomorphism for all D_1, D_2.

Remark

It is enough to check that there is some class $\widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, B \otimes A)$ which maps to 1_A, in the case $D_1 = \mathbb{C}$, $D_2 = A$, and so that the map $\mathrm{KK}(A \otimes B, \mathbb{C}) \to \mathrm{KK}(B, B)$ defined analogously using $\widehat{\Delta}$, maps Δ to 1_B. (The zig-zag equations).

Self-duality for K-oriented manifolds

Example

If X is a compact, K-oriented manifold, there is a distinguished elliptic operator on X called the *Dirac operator*. It determines a class $[D] \in \mathrm{KK}(C(X), \mathbb{C})$. Let $\delta\colon X \to X \times X$ be the diagonal map. Set $\Delta := \delta_*([D]) \in \mathrm{KK}(C(X) \otimes C(X), \mathbb{C})$. For $\widehat{\Delta}$, let

- ν be the normal bundle to the embedding $\delta\colon X \to X \times X$
- ξ_ν be the Thom class in $\mathrm{KK}(\mathbb{C}, C_0(\nu))$ of the vector bundle ν over X
- $\widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, C(X \times X))$ be the image of ξ_ν under the map $\mathrm{KK}(\mathbb{C}, C_0(\nu)) \to \mathrm{KK}(\mathbb{C}, C(X \times X))$ induced from tubular neighbourhood embedding of ν in $X \times X$.

Then Δ and $\widehat{\Delta}$ induce a Poincaré duality between $C(X)$ and itself.

Remark

$\Delta \cap [E]$ is the class of the Dirac operator 'twisted' by E.

The Fock space extension of a Cuntz-Krieger algebra

A an n-by-n symmetric matrix of 0's and 1's, the adjacency matrix of a graph.

O_A the Cuntz-Krieger algebra, generated by partial isometries $s_1, \ldots, s_n$ with $s_i^* s_i = \sum_j A_{ij} s_j s_j^*$.

F_A the 'Fock space' Hilbert space $\sum_{n=0}^{\infty} F_A^n$ where F_A^n is the span of the elementary tensors $\xi_{r_1} \otimes \cdots \otimes \xi_{r_n}$ where $A_{r_k, r_{k+1}} = 1$. Then for each i, j the left and right 'creation operators'

$$S_i(\xi_{r_1} \otimes \cdots \otimes \xi_{r_n}) := \xi_i \otimes \xi_{r_1} \otimes \cdots \otimes \xi_{r_n}$$

,

$$R_j(\xi_{r_1} \otimes \cdots \otimes \xi_{r_n}) := \xi_{r_1} \otimes \cdots \otimes \xi_{r_n} \otimes \xi_j,$$

commute with each other mod compacts and the S_i's and R_i's each respectively satisfy the Cuntz-Krieger relations mod compacts.

Duality of Kaminker and Putnam

Using the Fock space construction we obtain an extension

$$0 \to \mathcal{K}(F_A) \to C^*(\{S_i, R_j\}_{i,j}) \to O_A \otimes O_A \to 0. \qquad (0.1)$$

of $O_A \otimes O_A$ by the compact operators.

Theorem (Kaminker, Putnam)

The class $\Delta \in \mathrm{KK}^1(O_A \otimes O_A, \mathbb{C})$ *of the Fock Space extension* (0.1) *induces a self-duality of* O_A.

Remark

The world of *spaces* contains very few self-dual examples which are not manifolds!

Hyperbolic groups

If G is a Gromov hyperbolic group, it has a boundary ∂G which carries a continuous action of G.

Theorem (E)

The crossed-product $C(\partial G) \rtimes G$ is self-dual for any hyperbolic group G.

The duality class $\Delta \in \mathrm{KK}^1(C(\partial G) \rtimes G \otimes C(\partial G) \rtimes G, \mathbb{C})$ is built by constructing a certain extension

$$0 \to \mathcal{K}(l^2 G) \to \mathcal{E} \to C(\partial G) \rtimes G \otimes C(\partial G) \rtimes G$$

that only depends on the fact that ∂G is part of a compactification $G \subset \overline{G}$ of G in the topological sense, and the geometric property that if $g_i \to \xi$ is a sequence of group elements converging to a boundary point $\xi \in \partial G$, then for any $g \in G$, $g_i g \to \xi$ as well.

What Poincaré duality is good for

To describe the K-homology of a C*-algebra in some geometric fashion.

Example

Poincaré self-duality for K-oriented manifolds implies that every K-homology class for $C(X)$ is represented by a first order elliptic operator on X – and hence a $d := \dim(X)$-dimensional spectral triple (H, π, D) over $C^\infty(X)$ in the sense that the principal values of D grow like $n^{\frac{1}{d}}$ – important for noncommutative geometry.

This is because Poincaré duality for manifolds is geometrically *computable*, and because of the nice form of the fundamental class, *i.e.* the Dirac operator.

Theorem (E, Nica)

If G is a Gromov hyperbolic group, d_ϵ a visibility metric on the boundary, then every K-homology class for $C(\partial G) \rtimes G$ is represented by a p-summable Fredholm module over $\mathrm{Lip}(\partial G, d_\epsilon) \rtimes_{\mathrm{alg}} G$ for $p > \mathrm{hdim}(\partial G, d_\epsilon)$.

This follows from some ergodic properties of the boundary extension

$$0 \to \mathcal{K}(l^2 G) \cong C_0(G) \rtimes G \to C(\overline{G}) \rtimes G \to C(\partial G) \rtimes G \to 0,$$

and Poincaré duality. The theorem follows from a description of the K-homology entirely in terms of a kind of pseudodifferential calculus for hyperbolic groups.

Other examples

- Any compact manifold X is dual to TX.
- The irrational rotation algebra A_θ is self-dual.
- $C_0(X) \rtimes G$, X smooth K-oriented manifold, G discrete acting smoothly and properly, is self-dual.
- For a torsion-free K-amenable group, Baum-Connes says that $C_r^* G$ is dual to $C(BG)$.
- Certain quantum groups...

The formal Lefschetz trace theorem

Another thing duality is good for rests on the following general calculation using the Künneth and Universal Coefficient theorems.

Theorem

If A and B are Poincaré dual with unit $\Delta \in \mathrm{KK}(A \otimes B, \mathbb{C})$ and co-unit $\widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, A \otimes B)$, then $\mathrm{K}_*(A)$ (and $\mathrm{K}_*(B)$) have finite rank and if $f \in \mathrm{KK}(A, A)$, then

$$(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta = \mathrm{Tr}_s(f_*)$$

where Tr_s is the graded trace of f acting on $\mathrm{K}_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If one twists the class $\widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, A \otimes B)$ by f, pairs this with $\Delta \in \mathrm{KK}(A \otimes B, \mathbb{C})$, one gets the Lefschetz number of f.

The proof

The proof is essentially an exercise in the Künneth and UC Theorems. Duality gives a non degenerate bilinear form

$$\mathrm{K}_*(A) \otimes_{\mathbb{C}} \mathrm{K}_*(B) \to \mathbb{Z}$$

and so if (x_i) is a basis for $\mathrm{K}_*(A)$ there is a corresponding dual basis (y_i) for the K-theory of B.
Now verify that

$$\widehat{\Delta} = \sum_i x_i \otimes y_i$$

because of duality, and compute.

Example

The integer $\widehat{\Delta} \otimes_{A \otimes B} \Delta$ obtained by pairing $\widehat{\Delta}$ and Δ, is the 'Euler characteristic' $\mathrm{rank}(\mathrm{K}_0(A)) - \mathrm{rank}(\mathrm{K}_1(A))$. (The case $f = 1_A \in \mathrm{KK}(A, A)$.)

Deducing the Lefschetz fixed-point formula

Idea: whereas $\mathrm{tr}_s(f_*)$ is a global homological invariant of f, $(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta$ is a Kasparov product that, when all the data is given geometrically, can be computed geometrically.

Exercise If X is a smooth K-oriented manifold and $f\colon X \to X$ is a smooth map whose graph $X \to X \times X$ is transverse to the diagonal, Δ and $\widehat{\Delta}$ the duality classes explained above, then the integer $([f^*] \otimes 1_{C(X)})_*(\widehat{\Delta}) \otimes_{C(X \times X)} \Delta$ is the algebraic fixed-point set

$$\sum_{x \in \mathrm{Fix}(f)} \det(1 - D_x f) \in \mathbb{Z}$$

of f. (This is because both duality classes are supported near the diagonal in $X \times X$.)

From the Exercises and the formal Lefschetz Theorem we deduce the traditional Lefschetz fixed-point theorem

$$\mathrm{tr}_s(f^*) = \sum_{x \in \mathrm{Fix}(f)} \det(1 - D_x f) \in \mathbb{Z}.$$

Idea

Are there noncommutative analogues of the Lefschetz fixed-point theorem deducible from the general formal Lefschetz theorem

$$(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta = \mathrm{tr}_s(f_*)?$$

Example – a Lefschetz fixed-point theorem for orbifolds

Let G be a discrete group acting

- Properly
- Isometrically
- Co-compactly

on a smooth Riemannian manifold X.

Example

- The group $\mathbb{Z}/2$ acting on the circle by complex conjugation.
- The infinite dihedral group G, generated by $x \mapsto x + 1$, $x \mapsto -x$, acting on $\mathbb{R}$.

A duality between $C_0(X) \rtimes G$ and $C_0(TX) \rtimes G$ was constructed by [Echterhoff, E, Kim] using differential topology.

Automorphisms $f \in \mathrm{KK}(C_0(X) \rtimes G, C_0(X) \rtimes G)$

(Smooth) automorphisms of $C_0(X) \rtimes G$: covariant pairs (ϕ, ζ), $\phi\colon X \to X$ diffeomorphism, $\zeta \in \mathrm{Aut}(G)$ a group automorphism, such that $\phi\big(\zeta(g)x\big) = g\phi(x)$ $\forall x \in X$.

The transversality assumption: *If* $x \in X$, $g \in G$ *such that* $\phi(gx) = x$, *then the map*

$$\mathrm{Id} - d(\phi \circ g)(x)\colon T_xX \to T_xX \tag{0.2}$$

is non-singular.

This implies that the fixed-point set of the induced map on the space $G\backslash X$ of orbits is finite.

Theorem

(Echterhoff-Emerson-Kim) Choose a point p from each fixed orbit of the induced map $\dot{\phi}\colon G\backslash X \to G\backslash X$.
For each p, let $L_p := \{g \in G \mid \phi(gp) = p\}$ (it is finite); then the isotropy subgroup $\mathrm{Stab}_G(p)$ acts on L_p by twisted conjugation $h \cdot g := \zeta(h)gh^{-1}$. Let the orbits of this action be represented by elements $g_1, \ldots, g_m$. For each i, let $H_{p,i} \subset \mathrm{Stab}_G(p)$ be the stabilizer of g_i under this action – $H_{p,i}$ commutes with $\phi \circ g_i$.
Then

$$\mathrm{tr}_s\big((\phi, \zeta)_*\big) = \sum_{\dot{p}\in \mathrm{Fix}(\dot{\phi})} \sum_i \frac{1}{|H_{p,i}|} \sum_{h\in H_{p,i}} \mathrm{sign}\,\det(\mathrm{id} - D_{p_i}(\phi\circ g_i)|_{\mathrm{Fix}(h)})$$

If X is a point and G is a finite group, $\zeta \in \mathrm{Aut}(H)$, the theorem gives the following. Say that two elements g_1, g_2 of G are *twisted conjugate* if $g_1 = \zeta(h) g_2 h^{-1}$ for some $h \in G$.

The automorphism ζ induces a map on the representation ring $\mathrm{Rep}(G)$ of G – a free abelian group.

Theorem

The trace of $\zeta_* \colon \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ *equals the number of* ζ*-twisted conjugacy classes in* G*.*

Can the Lefschetz theorem be made equivariant?

Let G be a compact group, A and B G-C*-algebras which are G-equivariantly dual with unit and co-units Δ and $\widehat{\Delta}$. For example A and B could be $C(X)$ for a compact, smooth G-equivariantly K-oriented manifold, by results of Kasparov.
As before we consider the invariant

$$(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C}).$$

We call it the *geometric trace* of f. The ring $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is canonically isomorphic to the representation ring $\mathrm{Rep}(G)$ of G.

Example

If $G = \mathbb{T}$ then $\mathrm{Rep}(G)$ is the ring $\mathbb{Z}[X, X^{-1}]$ of integer-coefficient Laurent polynomials in one variable. The G-equivariant K-theory of any A is a module over $\mathbb{Z}[X, X^{-1}]$.

Problem

Can the geometric trace be expressed in purely homological terms in the equivariant situation, *e.g.* in terms of $\mathrm{Rep}(G)$-module traces?

Various problems

- Since not all $\mathrm{Rep}(G)$-modules are free, or even projective, there is no well-defined notion of 'trace' (nor even of 'rank').
- The UCT and Künneth Theorem do not hold equivariantly.
- (Worse) There is finite group G and two elements f with different geometric traces but which induce the *same map* on equivariant K-theory!

Hodgkin groups

Definition

A *Hodgkin group* is a compact group which is connected with torsion-free fundamental group.

Lemma

If G is a Hodgkin group then $\mathrm{Rep}(G)$ is an integral domain.

In this case $\mathrm{Rep}(G)$ embeds in its field of fractions F_G and any $\mathrm{Rep}(G)$-module (*i.e.* $\mathrm{KK}^G(A, B)$ for any A, B), can be made into an F_G-vector space by replacing it by

$$\mathrm{KK}^G(A, B) \otimes_{\mathrm{Rep}(G)} F_G.$$

If $f \in \mathrm{KK}^G(A, A)$ then f induces a canonical vector space map on $\mathrm{K}_*^G(A) \otimes_{\mathrm{Rep}(G)} F_G$ and we can define $\mathrm{tr}_s(f_*)$ to be the graded trace of f acting on $\mathrm{K}_*^G(A) \otimes_{\mathrm{Rep}(G)} F_G$. *A priori* it lies in F_G.

The formal equivariant Lefschetz formula for Hodgkin groups

Theorem

(Emerson, Meyer, Dell'Ambrogio) If G is a Hodgkin group and A a dualizable object of KK^G, then the homological trace $\mathrm{tr}_s(f_*)$ of f defined above actually lies in the image of $\mathrm{Rep}(G) \to F_G$ and agrees with the geometric trace $(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta$ for any duality between A and B.

Geometric consequence

Theorem

Let X be a smooth compact G-manifold, $\Lambda \in \mathrm{KK}^G(C(X), C(X))$ the class of a smooth G-equivariant correspondence $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$ from X to X. Assume that the map $(b, f)\colon M \to X \times X$ is transverse to the diagonal $X \to X \times X$. Then the intersection space

$$Q_{b,f} := \{m \in M \mid b(m) = f(m)\}$$

admits a canonical G-invariant smooth structure and equiviariant K-orientation, and the graded trace $\mathrm{tr}_s(\Lambda_*)$ of Λ acting on $\mathrm{K}^*_G(X)$ is equal to the Atiyah-Singer G-index of the Dirac operator on $Q_{b,f}$ twisted by ξ.

Example - the case of (equivariant) maps

A smooth G-equivariant map $b\colon X \to X$ is encoded by the correspondence

$$X \xleftarrow{b} X \xrightarrow{\mathrm{id}} X$$

and the transversality assumption that (b, id) is transverse to the diagonal is the traditional general position assumption of the Lefschetz fixed-point theorem. Moreover,

$$Q_{b,\mathrm{id}} = \{x \in X \mid b(x) = x\}$$

is the fixed-point set of b, with a suitable G-equivariant K-orientation – *i.e.* a suitable G-equivariant $\mathbb{Z}/2$-graded complex line bundle L on Q (next slide).

The equivariant K-orientation on $Q_{\mathrm{id},b}$

(continuing the case of maps...)

$Q_{\mathrm{id},b}$ is a finite, G-invariant set of points of X.

Choose $q \in Q_{\mathrm{id},b}$, let $H := \mathrm{Stab}_G(q)$. The function

$$\chi_q \colon H \to \{\pm 1\}, \quad \chi_q(h) := \mathrm{sign}\,\det(\mathrm{id} - D_q b|_{\mathrm{Fixed}(h)})$$

is $\pm$ a character of H, corresponding to $\pm$ a one-dimensional representation V_q of H, and

$$L|_{Gq} = \mathrm{ind}_H^G(V_q) := G \times_H V_q \in \mathrm{Rep}(G)$$

describes the K-orientation L along the orbit Gq.

An example – Euler characteristics

Fact For any smooth G-manifold X, the G-equivariant correspondence

$$X \xleftarrow{\mathrm{id}} (X, \xi) \xrightarrow{\mathrm{id}} X$$

from X to X acts on $\mathrm{K}^*_G(X)$ by the map λ_ξ of *ring multiplication by* ξ (thus using the ring structure on $\mathrm{K}^*_G(X)$.) The Equivariant Lefschetz Theorem (and some more work) describes the graded module trace of this map geometrically. We state the result only in a special case.

Theorem

Assume the compact group $\mathbb{T}$ acts smoothly on X with a finite set of stationary points in X. Then $\forall \xi \in \mathrm{K}^0_{\mathbb{T}}(X)$,

$$\mathrm{tr}_s(\lambda_\xi) = \sum_P \xi_{|_P} \in \mathrm{Rep}(\mathbb{T}).$$

An example

Let $X = \mathbb{CP}^1$ with the $\mathbb{T}$-action induced by the embedding $\mathbb{T} \to \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^2)$, $z \mapsto \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$. There are two stationary points, with homogeneous coordinates $[1, 0]$ and $[0, 1]$ respectively. Let H be the dual of the canonical line bundle on $\mathbb{CP}^1$ with its natural $\mathbb{T}$-action. Restricting H to the stationary points $[1, 0]$ and $[0, 1]$ yields respectively the characters X and X^{-1}, whence by the Lefschetz theorem

$$\mathrm{tr}_s(\Lambda_{[H]}) = X + X^{-1} \in \mathbb{Z}[X, X^{-1}].$$

To show this directly requires computing $\mathrm{K}^*_{\mathbb{T}}(\mathbb{CP}^1)$ as a ring and as a $\mathbb{Z}[X, X^{-1}]$-module: it is a free $\mathbb{Z}[X, X^{-1}]$-module supported in dimension zero, with basis $1, [H]$. One can check that $[H]^2 = (X + X^{-1})[H] + 1$, so that the matrix of ring multiplication by $[H]$ is $\begin{bmatrix} 0 & 1 \\ 1 & X + X^{-1} \end{bmatrix}$ (the trace is as claimed).