Classification of C^*-algebras of minimal diffeomorphisms on odd dimensional spheres

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Introduction

Let X be an infinite compact Hausdorff space and $\alpha : X \to X$ a homeomorphism.
Then α induces an action of $\mathbb{Z}$ on the C^*-algebra $C(X)$:

$$n.f \longmapsto f \circ \alpha^{-n}.$$

What can we say about the crossed product C^*-algebra?

$$\mathrm{C(X)} \rtimes_\alpha \mathbb{Z} \text{ `` } = \text{ '' } \mathrm{C}^*(C(X), u),$$

where u is a unitary implementing the action, i.e,

$$ufu^* = f \circ \alpha^{-1}.$$

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- C(X) $\rtimes_\alpha \mathbb{Z}$ is nuclear.
- Tracial states are in one-to-one correspondence with invariant Borel probability measures.
- When X is metrisable, C(X) $\rtimes_\alpha \mathbb{Z}$ is separable.
- (X, α) is a minimal dynamical system (ie. there are no proper closed α-invariant subsets of X) if and only if C(X) $\rtimes_\alpha \mathbb{Z}$ is simple.

Classification of C^*-algebras

To every (simple separable unital nuclear) C^*-algebra A, one may assign the Elliott invariant,

$$\begin{aligned} \mathrm{Ell}(A) &= (K_0(A), K_0(A)_+, [1_A]_+, K_1(A), \\ &\quad T(A), \rho : T(A) \to SK_0(A)). \end{aligned}$$

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Given a class $\mathcal{C}$ of simple separable unital nuclear C^*-algebras, we want to show that if $A, B \in \mathcal{C}$ then

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Example: Cantor minimal systems

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and in fact

$$\iff (X_A, \alpha_A) \text{ and } (X_B, \alpha_B) \text{ strong topological orbit equivalent.}$$

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Example [Fathi-Herman]: Let $m \neq n \geq 3$, odd. Then there are uniquely ergodic minimal dynamical systems $(S^m, \beta_1), (S^n, \beta_2)$ such that $C(S^m) \rtimes_{\beta_1} \mathbb{Z} \cong C(S^n) \rtimes_{\beta_2} \mathbb{Z}$.

Minimal diffeomorphisms of odd dimensional spheres

Let $n = 2k + 1, k \geq 1$.

There are minimal diffeomorphisms β of S^n having any predefined number of β-invariant measures [Windsor, 2003].

In particular there exist nonuniquely ergodic minimal dynamical systems.

Fast approximation

Definition

Let Y be infinite, compact, metrisable and $\beta : Y \to Y$ a homeomorphism. Say that β is a fast approximation by periodic homeomorphisms if $\beta : Y \to Y$ that can be written as the limit of a sequence $(T_i)_{i\in\mathbb{N}}$ of homeomorphisms such that $T_i : Y \to Y$ has period m_i, each m_i divides m_{i+1}, and

$$\sup_{\substack{t\in Y\\ j=1,\ldots,m_i}} |\beta^j(t) - T_i^j(t)| \to 0 \text{ as } i \to \infty.$$

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Windsor's diffeomorphisms above are all fast approximations by periodic homeomorphisms.

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Problem: Having no nontrivial projections and many tracial states means there is less information in the Elliott invariant. This makes classification more difficult!

Classification

Theorem (S, 2014)

Suppose $\beta_1 : S^n \to S^n, \beta_2 : S^m \to S^m$ *are fast approximations by periodic homeomorphisms. Then*

$$C(S^n) \rtimes_{\beta_1} \mathbb{Z} \cong C(S^m) \rtimes_{\beta_2} \mathbb{Z} \iff T(C(S^n) \rtimes_{\beta_1} \mathbb{Z}) \cong T(C(S^m) \rtimes_{\beta_2} \mathbb{Z}).$$

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A look back at previous classification techniques

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When $x \in X$, then $A_{\{x\}}$ retains a lot of information from A, but has an easier structure to deal with.

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$\implies$ classification, up to $\mathcal{Z}$-stability when the C^*-algebras have projections separating tracial states via results of Winter and Lin.
$\implies$ classification when $\dim(X) < \infty$ and the C^*-algebras have projections separating tracial states. [Toms–Winter, 2009].

Classification by embedding

Theorem (Winter, 2013)

Let A and B be separable, simple, unital C^*-algebras. Suppose that $\dim_{nuc} A < \infty$ *and that A has only finitely many extremal tracial states. Let B be TAI and suppose there is a unital embedding*

$$\iota : A \to B$$

such that

$$T(\iota) : T(B) \xrightarrow{\cong} T(A)$$

and such that

$$\tau_* = \tau'_* \in S(K0(B))$$

for $\tau, \tau' \in T(B)$. Then $A \otimes \mathcal{Q}$ is TAI.

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It is easier to show that A is classifiable!

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Proposition

Let $\beta : S^n \to S^n$ be a minimal homeomorphism. Then there is a uniquely ergodic minimal homeomorphism $\alpha : X \to X$ such that the homeomorphism $\alpha \times \beta : X \times S^n \to X \times S^n$ is minimal.

Proof.

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Let $(\varprojlim \mathbb{Z}/m_j, \alpha)$ be an odometer system (so $\alpha(x) = x + 1$). We can show that any $(x_0, y_0) \in X \times S^n$ has dense orbit in $X \times S^n$. Let $(x, y) \in X \times S^n$. Take j sufficiently large to find elements $x_0', x' \in \mathbb{Z}/m_j$ lying very close to x_0 and x, respectively.

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Since $\mathbb{Z}/m_j$ is finite, there is a $k \in \mathbb{Z}$ such that $\alpha^k(x_0') = x'$. Since β^{m_j} is minimal, there is $l \in \mathbb{N}$ such that $\beta^{lm_j}(\beta^k(y_0))$ is very close to y.

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Then we have $(\alpha \times \beta)^{lm_j + k}(x_0, y_0)$ close to (x, y). □

Proposition

Let $\alpha \times \beta : X \times S^n \to X \times S^n$ be a minimal homeomorphism where (X, α) is an odometer system. Then every tracial state $\tau \in T(A)$ comes from the product of the unique tracial state τ_1 on $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ and a tracial state $\tau_2 \in T(\mathcal{C}(S^n) \rtimes_\beta \mathbb{Z})$.

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Proof.

Consider the measure-preserving dynamical systems $(X, \mu \circ \pi_X)$, $(S^n, \mu \circ \pi_{S^n})$ and $(X \times S^n, \mu)$, where π_X, π_{S^n} are projections from $X \times S^n$ onto X and S^n. Since β is completely minimal and (X, α) is an odometer, it follows that (X, μ_0) and (S^n, μ_1) are disjoint as measurable dynamical systems, that is, $\mu = \mu_0 \times \mu_2$ [Downarowicz]. □

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Proof.

Follows from the fact every $\tau \in T(A)$ is of the form $\tau_X \otimes \tau_{S^n}$ where τ_X is the unique α-invariant tracial state on $\mathcal{C}(X)$ and τ_{S^n}, that $\tau, \tau' \in T(S^n)$, we have $\tau_* = \tau'_*$ has range $\mathbb{Z}$ [Phillips, 2007].

$H^1(X \times S^n, \mathbb{Z}) = 0 \implies$ range of a state τ_* induced by any tracial state $\tau \in T(A)$ is determined by range of τ_* on $K_0(\mathcal{C}(X \times S^n))$. $\square$

Breaking the orbit at a fibre

Instead of breaking the orbit at a point in $X \times S^n$, we take $x \in X$ and break the orbit at a fibre $\{x\} \times S^n$:

$$A_{\{x\}\times S^n} := C^*(C(X \times S^n), uC_0((X \setminus \{x\}) \times S^n)).$$

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Remarks: This is similar to what Lin and Matui did for minimal dynamical systems of $X \times \mathbb{T}$ and what Sun did for minimal dynamical systems on $X \times \mathbb{T}^2$.

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Can show (1) in the same way as one shows $\mathrm{C}^*(C(Y), vC_0(Y \setminus \{y\}))$ is simple for arbitrary minimal dynamical system (Y, γ) [Lin–Phillips, 2010], or by considering the C^*-algebra associated to a subgroupoid

Definition

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$$\sup_{\substack{t \in Y \\ j=1,\ldots,m_i}} |\beta^j(t) - T_i^j(t)| \to 0 \text{ as } i \to \infty.$$

Finding the projection p

Lemma

There is an odometer system (X, α) *such that the following holds: For any* $y \in X$, *any* $\epsilon > 0$, *any* $N_0 \in \mathbb{R}_+$, *and any pair of finite sets* $\mathcal{F}_X \subset \mathcal{C}(X)$, $\mathcal{F}_{S^n} \subset \mathcal{C}(S^n)$ *there are* $M > N_0 \in \mathbb{N}$ *and* $y \in U \subset X$ *a clopen subset and a partial isometry* $w \in A_{\{y\}}$ *such that*

1. $\alpha^{-M}(U), \ldots, \alpha^{-1}(U), U, \alpha(U), \ldots, \alpha^{M}(U)$ *are pairwise disjoint*
2. $w^*w = 1_{U \times S^n}$ *and* $ww^* = 1_{\alpha^M(U) \times S^n}$,
3. $\|wa - aw\| < \epsilon$ *for every* $a \in \{f \otimes 1_{S^n} \mid f \in \mathcal{F}_X\} \cup \{1_X \otimes f \mid f \in \mathcal{F}_{S^n}\}$.

Classification for $C(S^n) \rtimes_\beta \mathbb{Z}$.

Theorem

Let $\mathcal{A}$ be the simple unital C^*-algebras associated to minimal diffeomorphisms $\beta : S^n \to S^n$ as constructed by Fathi and Herman or Windsor. Then for any $A, B \in \mathcal{A}$,

$$A \cong B$$

if and only if

$$T(A) \cong T(B).$$

Proof

By the above, there is a minimal Cantor system (X, α) such that $C(S^n) \rtimes_\beta \mathbb{Z}$ embeds in a trace-preserving way into $C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$. Moreover, every tracial state on $C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$ induces the same state on K_0.

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We have that $C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$ is TAI, thus by Winter's theorem $(C(S^n) \rtimes_\beta \mathbb{Z}) \otimes \mathcal{Q}$ is TAI. Since these C^*-algebras all satisfy the UCT, classification up to $\mathcal{Z}$-stability by Elliott invariants follows from a result of Lin (the class of simple unital nuclear UCT C^*-algebras that are "rationally" TAI are classifiable by Elliott Invariants).

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Since $\dim(S^n) < \infty$, all C^*-algebras are $\mathcal{Z}$-stable [Toms–Winter, 2009].
Finally, Phillips via Connes showed that every such C^*-algebra has isomorphic K-theory. Thus the Elliott invariant collapses to the tracial state space.

Remarks

Huaxin Lin was able to remove that β be a fast approximation by periodic homeomorphisms by showing that
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This is used to produce the partial isometry w.