Noncommutative Geometry and Conformal Geometry: Local Index Formula and Conformal Invariants (joint work with Hang Wang)

Raphaël Ponge

Seoul National University & UC Berkeley

November 1, 2014

Main Results

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Main Results (RP+HW)

- Local index formula in conformal-diffeomorphism invariant geometry.

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- Local index formula in conformal-diffeomorphism invariant geometry.
- Construction of a new class of conformal invariants.

References

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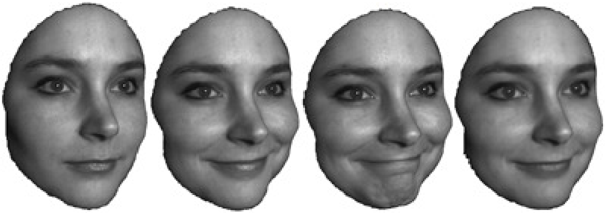
- RP+HW: *Noncommutative geometry and conformal geometry. III. Poincaré duality and Vafa-Witten inequality.* arXiv:1310.6138.

Conformal Geometry



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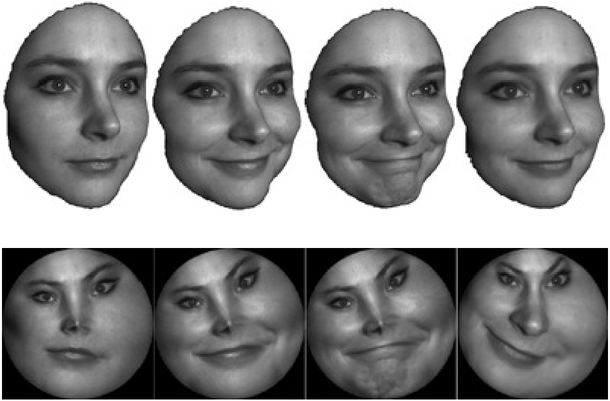

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Solution Provided by NCG

Trade the space M/G for the crossed product algebra,

$$C_c^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C_c^\infty(M) \right\},$$

$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

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Proposition (Green)

If G acts freely and properly, then $C_c^\infty(M/G)$ is Morita equivalent to $C_c^\infty(M) \rtimes G$.

The Noncommutative Torus

Example

Given $\theta \in \mathbb{R}$, let $\mathbb{Z}$ act on S^1 by

$$k.z := e^{2ik\pi\theta} z \qquad \forall z \in S^1 \ \forall k \in \mathbb{Z}.$$

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The crossed-product algebra $\mathcal{A}_\theta := C^\infty(S^1) \rtimes_\theta \mathbb{Z}$ is generated by two operators U and V such that

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Remark

The algebra $\mathcal{A}_\theta$ is called the *noncommutative torus*.

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Overview of Noncommutative Geometry

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Remark

We also get spectral triples by taking

- $\mathcal{H} = L^2(M, \Lambda^\bullet T^*M)$ and $D = d + d^*$.
- $\mathcal{H} = L^2(M, \Lambda^{0,\bullet} T^*_{\mathbb{C}}M)$ and $D = \overline{\partial} + \overline{\partial}^*$ (when M is a complex manifold).

Diffeomorphism-Invariant Geometry

Setup

- M smooth manifold.
- $G = \text{Diff}(M)$ full group diffeomorphism group of M.

Fact

The only G-invariant geometric structure of M is its manifold structure.

Theorem (Connes-Moscovici '95)

There is a spectral triple $\left(C_c^\infty(P) \rtimes G, L^2(P, \Lambda^\bullet T^*P), D\right)$, where

- *$P = \left\{g_{ij}dx^i \otimes dx^j; (g_{ij}) > 0\right\}$ is the metric bundle of M.*
- *D is a "mixed-degree" signature operator, so that*

$$D|D| = d_H + d_H^* + d_V d_V^* - d_V^* d_V.$$

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Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ *is a twisted spectral triple.*

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Example (RP+HW)

Connes-Tretkoff's twisted spectral triples over NC tori associated to conformal weights.

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- Twisted spectral triples over NC tori associated to conformal weights (Connes-Tretkoff).
- Twisted spectral triples associated to some quantum statistical systems (e.g., Connes-Bost systems, supersymmetric Riemann gas) (Greenfield-Marcolli-Teh '13).

Setup

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Connections over a Spectral Triple

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Definition

A *σ-translate* of $\mathcal{E}$ is a finitely generated projective module $\mathcal{E}^\sigma$ together with a linear isomorphism $\sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma$ such that

$$\sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \qquad \forall \xi \in \mathcal{E}\ \forall a \in \mathcal{A}.$$

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The datum of σ-connection on $\mathcal{E}$ defines a coupled operator,

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2. The corresponding coupled operator agrees with $\not{D}_{\nabla^E}$.

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When $\sigma = \mathrm{id}$, and in all the main examples with $\sigma \neq \mathrm{id}$, we have

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Lemma (RP+HW)

Suppose that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ *is p-summable, i.e.,* $\operatorname{Tr}|D|^{-p} < \infty$ *for some* $p \geq 1$.

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$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M),\ k_\phi > 0.$$
4. $C^\infty(M) \rtimes G$ is the crossed-product algebra, i.e.,
$$C^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C_c^\infty(M) \right\},$$
$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

Conformal Dirac Spectral Triple

Lemma (Connes-Moscovici)

For $\phi \in G$ *define* $U_\phi : L^2_g(M, \$) \to L^2_g(M, \$)$ *by*

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Lemma (Connes-Moscovici)

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The datum of any metric $g \in \mathcal{C}$ *defines a twisted spectral triple* $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g \right)_{\sigma_g}$ *given by*

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3. *The automorphism $\sigma_g(fu_\phi) := k_\phi^{-1} fu_\phi$.*

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1. *The Connes-Chern character* $\mathrm{Ch}(\not{D}_g)_{\sigma_g} \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ *is an invariant of the conformal class* $\mathcal{C}$.
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$$\langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta \rangle,$$
is a scalar conformal invariant.

Definition

The conformal Connes-Chern character $\mathrm{Ch}(\mathcal{C}) \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ is the Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ for any metric $g \in \mathcal{C}$.

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Consequence

When $\mathcal{C}$ is non-flat, we are reduced to the computation of the Connes-Chern character of $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)$, where G is a group of isometries.

Remark

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4. This approach was various other applications (equivariant JLO cocycle, equivariant eta cochain, Yong Wang's papers).

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- Over M^ϕ, with respect to $TM_{|M^\phi} = TM^\phi \oplus \mathcal{N}^\phi$, there are decompositions,

$$\phi' = \begin{pmatrix} 1 & 0 \\ 0 & \phi'_{|\mathcal{N}^\phi} \end{pmatrix}, \qquad \nabla^{TM} = \nabla^{TM^\phi} \oplus \nabla^{\mathcal{N}^\phi}.$$

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$$\hat{A}\left(R^{TM^\phi}\right) := \det{}^{\frac{1}{2}} \left[\frac{R^{TM^\phi}/2}{\sinh\left(R^{TM^\phi}/2\right)} \right],$$

$$\nu_\phi\left(R^{\mathcal{N}^\phi}\right) := \det{}^{-\frac{1}{2}} \left[1 - \phi'_{|\mathcal{N}^\phi} e^{-R^{\mathcal{N}^\phi}} \right].$$

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This represents Connes' transverse fundamental class of M/G.

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where $\tilde{f}^j$ is a G^ϕ-invariant smooth extension of f^j to M.

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Remark

The above invariants are not the type of conformal invariants appearing in the Deser-Schwimmer conjecture solved by Spyros Alexakis in 2007 (about 600 pages).