Exotic Crossed Products

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For $p \in [2, \infty)$, (π, H) is an *L^p-representation* if the set

$$\{(\eta, \xi) \in H \oplus H \mid g \mapsto \langle\ \eta\ , \pi(g)\xi\ \rangle \text{ is in } L^p(G)\}$$

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- The trivial representation on $\mathbb{C}$ is an L^p-representation only for $p = \infty$.
- If $p > q$, then any L^q-representation is an L^p-representation.

1. L^p-representations

2. Crossed product functors

3. Exactness of the Baum-Connes conjecture

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Lemma (Dixmier?)

If $\phi \in L^p(G)$ is positive type, then π_ϕ is an L^p-representation.

Proof: The dense subset $C_c(G) \cdot \xi$ gives rise to L^p matrix coefficients. $\square$

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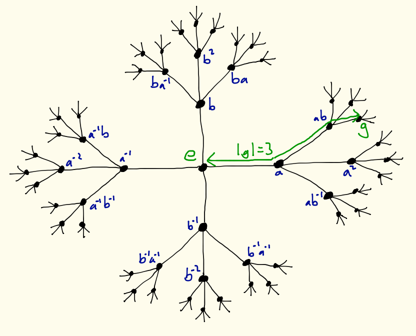
Example: $G = F_2$.



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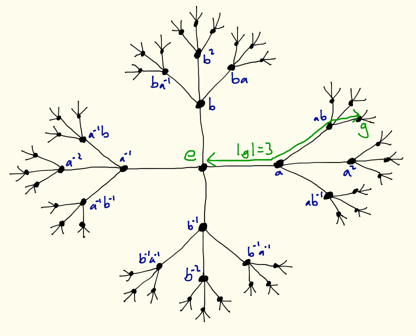

Theorem (Haagerup?)

For each fixed $t \in [0, \infty]$, the function

$$\phi_t : F_2 \to \mathbb{C}, \quad g \mapsto e^{-t|g|}$$

is positive type.

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Similar example: $G = SL(2, \mathbb{R})$: 'the analogue' of $\phi(g) = e^{-|g|}$ is

$$\phi\left(k \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k'\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{2}{|(a^2 + a^{-2}) + (a^2 - a^{-2}) \cos(\theta)|} d\theta.$$

Definition (Brown-Guentner)

$C_p^*(G)$ is the completion of $C_c(G)$ for the norm

$$\|f\| := \sup\{\|\pi(f)\|_{\mathcal{B}(\mathcal{H})} \mid (\mathcal{H}, \pi) \text{ an } L^p\text{-representation}\}.$$

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Remarks:

- For any G, $C^*_\infty(G) = C^*_{\max}(G)$ and $C^*_2(G) = C^*_{\mathrm{red}}(G)$.

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Remarks:

- For any G, $C_\infty^*(G) = C_{\max}^*(G)$ and $C_2^*(G) = C_{\mathrm{red}}^*(G)$.
- The duals $\widehat{G}_p := \widehat{C_p^*(G)}$ are nested, i.e.

$$2 \leqslant p \leqslant q \leqslant \infty \quad \Rightarrow \quad \widehat{G}_2 \subseteq \widehat{G}_p \subseteq \widehat{G}_q \subseteq \widehat{G}$$

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- If G is amenable, $C^*_p(G) = C^*_q(G)$ for all p, q (and so $\widehat{G}_p = \widehat{G}_q$).

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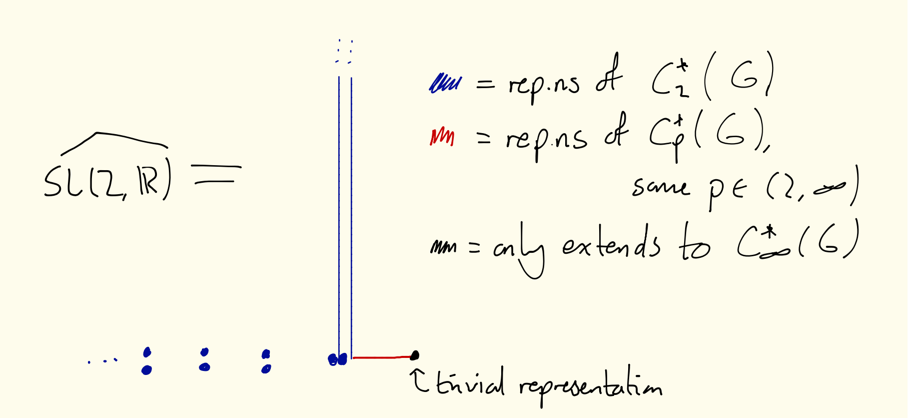

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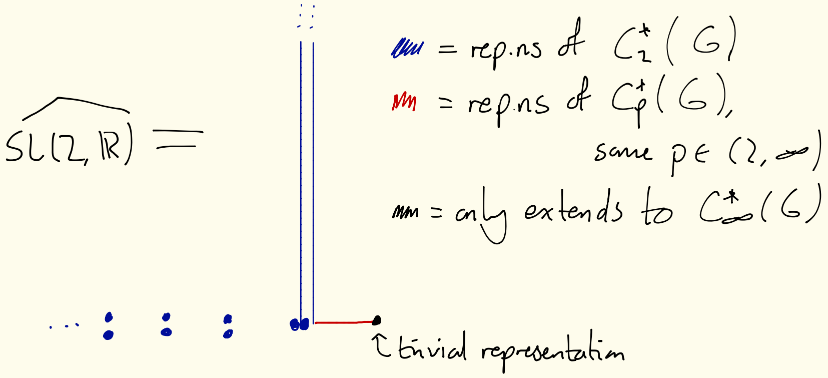

Theorem (Wiersma, Stein?)

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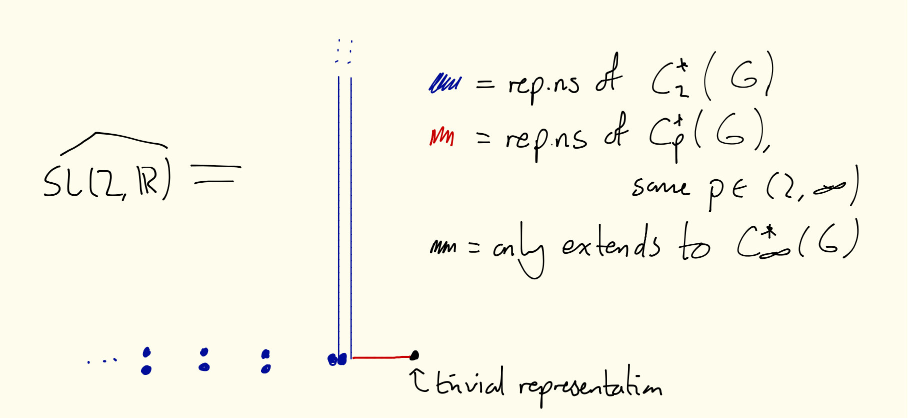

Theorem (Wiersma, Stein?)

$C_p^*(G) \neq C_q^*(G)$ for all $\infty \geqslant p > q \geqslant 2$.
On the other hand, $K(C_p^*(G)) = K_*(C_q^*(G))$ for all p, q.

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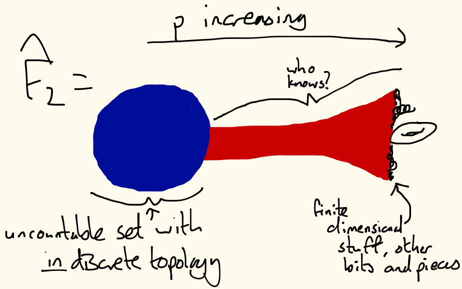

(blue) = rep.ns of $C^*_2(G)$

(red) = rep.ns of $C^*_p(G)$, some $p \in (2, \infty)$

(black) = only extends to $C^*_\infty(G)$

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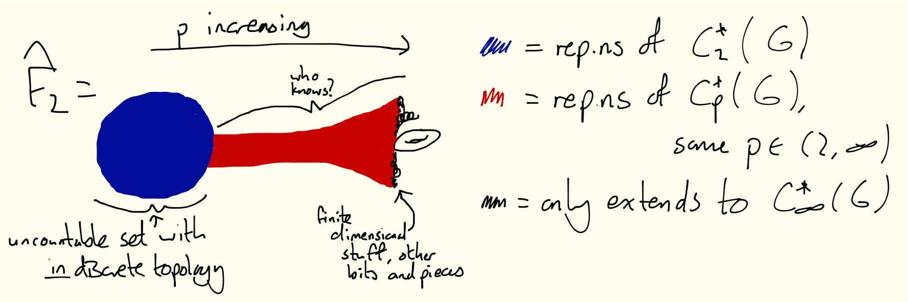

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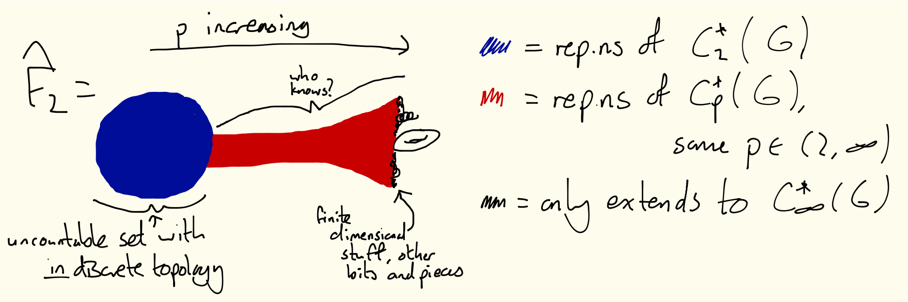

Theorem (Okayasu, Cuntz, Pimsner-Voiculescu, Buss-Echterhoff-W.)

$C^*_p(G) \neq C^*_q(G)$ for all $\infty \geqslant p > q \geqslant 2$. Moreover, for all p:

$$K_i(C^*_p(G)) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \end{cases}$$

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Strategy:

1. Define an *exotic crossed product functor*

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3. Use that in KK^G, $\mathbb{C}$ is equivalent to an amenable G-C^*-algebra A (Cuntz), and here all crossed products are the same.

Definition (Baum-Guentner-W. ?)

A *crossed product* is a functor

$$\rtimes : \{G\text{-}C^*\text{-algebras}\} \to \{C^*\text{-algebras}\}, \quad A \mapsto A \rtimes G$$

such that there are surjective natural transformations

$$A \rtimes_{\mathrm{max}} G \twoheadrightarrow A \rtimes G \twoheadrightarrow A \rtimes_{\mathrm{red}} G$$

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Examples: $\rtimes_{\mathrm{max}}$, $\rtimes_{\mathrm{red}}$, but many others due to Brown-Guentner, Kaliszewski-Landstad-Quigg,...

Theorem (Buss-Echterhoff-W.)

Let $\rtimes$ be a crossed product functor. The following are equivalent.

1. *If pAp is a G-invariant corner of A, then*

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If $\rtimes$ satisfies these conditions, we call it a *correspondence functor*.

Theorem (Buss-Echterhoff-W.)

Any correspondence functor induces a descent morphism

$$\rtimes : KK^G \to KK.$$

With a little more work, if G *is* K*-amenable (e.g.* $G = F_2,\ SL(2,\mathbb{R})$*), then all the crossed products* C^**-algebras*

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Theorem (Kaliszewski-Landstad-Quigg, Buss-Echterhoff-W.)

Define a completion of $C_c(G,A)$ by taking the covariant pair

$$(A, G) \to (A \rtimes_{\max} G) \otimes C^*_p(G), \quad a \mapsto a, \quad g \mapsto g \otimes g,$$

integrating to $C_c(G,A)$, and completing to $A \rtimes_p G$.

Then $\rtimes_p$ is a correspondence functor, and $\mathbb{C} \rtimes_p G = C^*_p(G)$.

For simplicity: G discrete.
The Baum-Connes conjecture predicts that a certain *assembly map*

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Lemma

If

$$0 \to I \to A \to B \to 0$$

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Theorem (Higson-Lafforgue-Skandalis)

For certain non-exact groups, this fails.

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Theorem (Baum-Guentner-W., Buss-Echterhoff-W.)

Define a crossed product $\rtimes_e$ *by completing* $C_c(G, A)$ *in the representation*

$$(A, G) \to (A \otimes l^\infty(G)) \rtimes_{\max} G, \quad a \mapsto a \otimes 1, \quad g \mapsto g$$

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5. *None of the previous Baum-Connes counterexamples apply to the reformulated conjecture using* $\rtimes_e$*.*
6. *Some previous counterexamples become confirming examples.*