On simplicity and uniqueness of trace for reduced twisted group C^*-algebras

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Introduction

G will always denote a discrete group.

The reduced group C^*-algebra $C^*_r(G)$ is simple with a unique trace in many cases, for example when G is a Powers group, including e.g. free nonabelian groups $\mathbb{F}_n$, (nontrivial) free products $G * H$, and others.

Recently it was shown (by Breuillard, Kalantar, Kennedy, and Ozawa) that $C^*_r(G)$ has a unique trace if and only if the amenable radical of G is trivial. In particular, this means that simplicity of $C^*_r(G)$ is stronger than uniqueness of trace.

What is different in the twisted case vs. the ordinary case?

Many interesting examples of simple twisted group C^*-algebras with unique trace come from amenable groups.

Why twisted group C^*-algebras?

$C_r^*(G)$ is nuclear $\iff$ G is amenable $\iff$ $C^*(G) \cong C_r^*(G)$.
The trivial representation $G \to \mathbb{C}$, $g \mapsto 1$ for all $g \in G$ gives an ideal of codimension 1 of the full group C^*-algebra $C^*(G)$.
So if $C_r^*(G)$ is simple and nuclear, then $G = \{e\}$.

A reduced twisted group C^*-algebra can be both nuclear and simple, e.g. the irrational rotation algebras are isomorphic to $C_r^*(\mathbb{Z}^2, \sigma)$.

In general, $C_r^*(G, \sigma)$ is nuclear $\iff$ G is amenable.

Question

1. $C_r^*(G)$ is simple $\implies$ $C_r^*(G, \sigma)$ is simple for all σ?
2. $C_r^*(G)$ has unique trace $\implies$ $C_r^*(G, \sigma)$ has unique trace for all σ?

(both 1. and 2. hold for weak Powers groups)

Two-cocycles

Definition

A *(normalized circle-valued) two-cocycle on* G is a function $\sigma : G \times G \to \mathbb{T}$ such that

$$\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk)$$
$$\sigma(g,e) = \sigma(e,g) = 1$$

for all $g, h, k \in G$. Such functions are sometimes called *multipliers on* G.

Definition

A *σ-projective unitary representation* of G on a Hilbert space $\mathcal{H}$ is a map $U : G \to U(\mathcal{H})$ such that

$$U(g)U(h) = \sigma(g,h)U(gh)$$

for all $g, h \in G$.

The twisted ℓ^1-algebra

Define the Banach *-algebra $\ell^1(G,\sigma)$ as the set $\ell^1(G)$ with twisted convolution and involution

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(h)\sigma(h, h^{-1}g)f_2(h^{-1}g)$$
$$f^*(g) = \overline{\sigma(g, g^{-1})f(g^{-1})}$$

together with the usual $\|\cdot\|_1$-norm. Define the left regular σ-projective unitary representation $\lambda = \lambda_\sigma$ of G on $B(\ell^2(G))$ by

$$(\lambda(g)\xi)(h) = \sigma(g, g^{-1}h)\xi(g^{-1}h)$$

and its integrated form on $\ell^1(G,\sigma)$ by

$$\lambda(f) = \sum_{g\in G} f(g)\lambda(g).$$

Twisted group C^*-algebras

Definition

- $C^*_r(G,\sigma)$ is the C^*-algebra generated by $\lambda(\ell^1(G,\sigma))$.
- $W^*(G,\sigma)$ is the von Neumann algebra generated by $\lambda(\ell^1(G,\sigma))$.
- $C^*(G,\sigma)$ is the enveloping C^*-algebra of $\ell^1(G,\sigma)$.

Representations of $C^*(G,\sigma)$ are in 1-1-correspondence with σ-projective unitary representations of G.

If G is amenable, then λ is faithful, so $C^*_r(G,\sigma) \cong C^*(G,\sigma)$.

Question

1. $C^*(G,\sigma) \cong C^*_r(G,\sigma) \implies G$ amenable? (holds if σ is trivial)
2. $C^*(G,\sigma)$ is simple $\implies G$ amenable?

Canonical trace on $C_r^*(G, \sigma)$

Definition

Let τ be the vector state on $C_r^*(G, \sigma)$ given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$.
Then τ is a faithful trace on $C_r^*(G, \sigma)$ and $\tau(\lambda(g)) = 0$ if $g \neq e$.

Question (open in both directions; when σ is trivial $\Longrightarrow$ holds)

$C_r^*(G, \sigma)$ simple $\Longleftrightarrow$ $C_r^*(G, \sigma)$ has unique trace?

Kleppner's condition

- $g \in G$ is called *σ-regular* if $\sigma(g,h) = \sigma(h,g)$ whenever $gh = hg$.
- If g is σ-regular then hgh^{-1} is also σ-regular for all h.

Theorem (Kleppner, Murphy, O)

The following are equivalent:

(i) Every nontrivial σ-regular conjugacy class in G is infinite.

(ii) $W^*(G,\sigma)$ is a factor.

(iii) $C_r^*(G,\sigma)$ is prime (i.e. nonzero ideals have nonzero intersection).

(iv) $C_r^*(G,\sigma)$ has trivial center.

Definition

We say that (G,σ) satisfies *Kleppner's condition* if (i) holds.

Remark

Kleppner's condition is necessary for both simplicity and unique trace of $C_r^*(G,\sigma)$, but is in general far from being sufficient for any of them.

Examples

Example 1 (icc)

If G is icc (i.e. its nontrival conjugacy classes are infinite), then (G, σ) satisfies Kleppner's condition for all σ. E.g. $C_r^*(\mathbb{F}_n)$ is simple for all $n \geq 2$, but $C_r^*(G)$ is nonsimple when G is amenable and icc.

Example 2 (abelian)

If G is abelian, then (G, σ) satisfies Kleppner's condition if there are no nontrivial σ-regular points,
i.e. for all $g \neq e$ there exists h such that $\sigma(g, h) \neq \sigma(h, g)$.
E.g. $C_r^*(\mathbb{Z}_n \times \mathbb{Z}_n, \sigma) \cong M_n(\mathbb{C}) \iff (\mathbb{Z}_n \times \mathbb{Z}_n, \sigma)$ satisfies Kleppner's cond.
The noncommutative n-tori are isomorphic to $C_r^*(\mathbb{Z}^n, \sigma_\theta)$, where the two-cocycles are parametrized by $\theta \in \mathbb{T}^{n(n-1)/2}$.

Example 3 (nonamenable, non-icc)

$G = \mathbb{F}_2 \times \mathbb{Z}$, then $H^2(G, \mathbb{T}) \cong \mathbb{T}^2$. Then $(G, \sigma_{\mu,\nu})$ satisfies Kleppner's condition $\iff$ at least one of μ and ν is nontorsion.

Summary of what is previously known

- For a large class of nonamenable icc groups, $C_r^*(G)$ is simple with a unique trace.
- If G is finite or abelian or more generally nilpotent [Packer '89], and σ is a cocycle of G, then the following are equivalent:

(i) (G, σ) satisfies Kleppner's condition
(ii) $C_r^*(G, \sigma)$ is simple
(iii) $C_r^*(G, \sigma)$ has unique trace

First Goal

Let $\mathcal{K}$ the class of groups such that for any σ, the conditions (i)-(iii) are equivalent. Describe the subclass $\mathcal{K}^{\mathrm{am}}$ of all amenable groups in $\mathcal{K}$.

$\mathcal{K}^{\mathrm{am}}$ does not contain any amenable icc group (except $\{e\}$).
$\mathcal{K}^{\mathrm{am}}$ does not contain *all* amenable groups that admit a reduced twisted group C^*-algebra which is simple with unique trace.

FC-hypercentral groups

The *FC-center of G* is given by

$$FC(G) = \{g \in G \mid \text{the conjugacy class of } g \text{ is finite}\}.$$

The FC-center of G is a normal subgroup of G.
The *upper FC-central series* $\{F_\alpha\}_\alpha$ of G is a normal series of subgroups of G indexed by the ordinal numbers. It is defined as follows:

We set $F_0 = \{e\}$, $F_\alpha/F_\beta = FC(G/F_\beta)$ if $\alpha = \beta + 1$, and $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ when α is a limit ordinal. This series eventually stabilizes and

$$FCH(G) = \lim_\alpha F_\alpha = \bigcup_\alpha F_\alpha$$

is called the *FC-hypercenter of G*, and is a normal subgroup of G.
If $G = FCH(G)$ then G is called *FC-hypercentral*.
$FCH(G)$ is trivial if and only if $FC(G)$ is trivial if and only if G is icc.

FC-hypercentral groups

Proposition

The quotient group $G/FCH(G)$ is icc. Moreover, if N is a normal subgroup of G such that G/N is icc, then $FCH(G) \subset N$.

The FC-hypercenter of a group G is the smallest normal subgroup of G that produces an icc quotient group.
We define $ICC(G) = G/FCH(G)$.
The class of FC-hypercentral groups is closed under subgroups, direct products, and FC-hypercentral extensions. Moreover:

$$\text{virtually nilpotent} \implies \text{FC-hypercentral} \implies \text{polynomial growth}$$

If we restrict to finitely generated groups, these classes coincide by Gromov's theorem.

Main theorem

Theorem (B-O)

Every FC-hypercentral group G belongs to $\mathcal{K}^{\text{am}}$, that is, for any two-cocycle σ of G, the following are equivalent:

(i) (G, σ) satisfies Kleppner's condition

(ii) $C_r^*(G, \sigma)$ is simple

(iii) $C_r^*(G, \sigma)$ has unique trace

The QTS property

Definition

A C^*-algebra is said to have the QTS property if each (nontrivial) quotient A/J admits a trace.

Theorem (Murphy '00)

Let A be a unital C^*-algebra having the QTS property.
Then A is simple if and only if all its traces are faithful.

Corollary

If $C_r^*(G, \sigma)$ has the QTS property and a unique trace, then $C_r^*(G, \sigma)$ is simple.

The QTS property

$C_r^*(G,\sigma)$ has QTS property + unique trace $\Longrightarrow$ $C_r^*(G,\sigma)$ is simple.

Proposition (Murphy, Bédos)

If G is amenable or if G is exact and $C_r^*(G,\sigma)$ has stable rank 1, then $C_r^*(G,\sigma)$ has the QTS property.

Question

Is there any relationship between $C_r^*(G,\sigma)$ being simple and having stable rank 1? In all cases where $C_r^*(G,\sigma)$ is known to be simple and the stable rank of $(C_r^*(G,\sigma))$ has been computed, it is 1.

Main theorem

Theorem (B-O)

Every FC-hypercentral group G belongs to $\mathcal{K}^{\mathrm{am}}$, that is, for any two-cocycle σ of G, the following are equivalent:

(i) (G, σ) satisfies Kleppner's condition.

(ii) $C_r^*(G, \sigma)$ is simple.

(iii) $C_r^*(G, \sigma)$ has unique trace.

Remark that we always have (ii) $\Longrightarrow$ (i), and since FC-hypercentral groups are amenable, $C_r^*(G, \sigma)$ has the QTS property, so (iii) $\Longrightarrow$ (ii). Hence, it is suffices to show that (i) $\Longrightarrow$ (iii).

(The proof of (i) $\Longrightarrow$ (iii) is inspired by Packer's for nilpotent groups; and uses some techniques by Carey and Moran)

Sketch of proof

Let φ be a trace on $C_r^*(G,\sigma)$.
Suppose (G,σ) satisfies Kleppner's condition. For each $h \in FC(G) \setminus \{e\}$, there exists $g \in G$ s.t. $hg = gh$ and $\sigma(h,g) \neq \sigma(g,h)$, and then

$$\varphi(\lambda(h)) = \varphi(\lambda(g)\lambda(h)\lambda(g)^*) = \varphi(\sigma(g,h)\overline{\sigma(ghg^{-1},g)}\lambda(ghg^{-1})) = \sigma(g,h)\overline{\sigma(h,g)}\varphi(\lambda(h)) = z\varphi(\lambda(h))$$

for some $z \neq 1$, and thus $\lambda(h) = 0$.
That is, φ agrees with τ on $C^*\{\lambda(h) : h \in FC(G)\}$.
The rest of the (rather technical) proof is to show the following lemma:
If φ agrees with τ on $C^*\{\lambda(h) : h \in FC(G)\}$, then φ agrees with τ on $C^*\{\lambda(h) : h \in FCH(G)\}$.
This is done by (transfinite) induction on the upper FC-central series $\{F_\alpha\}_\alpha$, i.e.: we show that when α is an ordinal and $\varphi(\lambda(h)) = 0$ for all $h \in F_\beta \setminus \{e\}$ and $\beta < \alpha$, then $\varphi(\lambda(h)) = 0$ for all $h \in F_\alpha \setminus \{e\}$.

More results

Theorem (B-O)

Assume that $K = ICC(G)$ is a (weak) Powers group. Then we have:

a) (G, σ) satisfies Kleppner's condition if and only if (G, σ) has the unique trace property.

b) Set $H = FCH(G)$ and let σ_H denote the restriction of σ to $H \times H$. If (H, σ_H) satisfies Kleppner's condition, then $C_r^*(G, \sigma)$ is simple and has the unique trace property.

For part a): if, in addition, G is exact and $C_r^*(G, \sigma)$ has stable rank one, then $C_r^*(G, \sigma)$ is simple.

Proposition (B-O)

Suppose (G, σ) satisfies Kleppner's conditon.
If the action of $ICC(G)$ on G is freely acting on $W^*(G, \sigma)$ (i.e. $\alpha(S)T = TS$ for all $S \Rightarrow T = 0$), then $C_r^*(G, \sigma)$ has a unique trace.
If, in addition, G is exact and $C_r^*(G, \sigma)$ has stable rank one, then $C_r^*(G, \sigma)$ is simple.

Examples I

Example 3 cont.

$G = \mathbb{F}_2 \times \mathbb{Z}$, then $H^2(G, \mathbb{T}) \cong \mathbb{T}^2$. Then $(G, \sigma_{\mu,\nu})$ satisfies Kleppner's condition $\iff$ at least one of μ and ν is nontorsion.
Here $FCH(G) = \mathbb{Z}$ and $ICC(G) = \mathbb{F}_2$ is Powers, so by the theorem:
Kleppner's condition $\iff$ $C_r^*(G, \sigma_{\mu,\nu})$ has unique trace.
By a different technique, we can show that Kleppner's condition is equivalent to simplicity, so $G \in \mathcal{K}$.

Example 4

Let $n \in \mathbb{N}$, $n \geq 2$ and set $G = \langle a, b \mid ab^n = b^n a \rangle$. Then G is the so-called Baumslag-Solitar group often denoted by $BS(n, n)$. We have

$$FCH(G) = FC(G) = Z(G) = \langle b^n \rangle \simeq \mathbb{Z}$$

and $ICC(G) \simeq \mathbb{Z} * \mathbb{Z}_n$ is Powers.

Examples II

Example 4 cont.

Let f denote the surjective homomorphism $f\colon G \to \mathbb{Z}^2$ satisfying $f(a) = (1,0)$ and $f(b) = (0,1)$. For $\theta \in \mathbb{R}$, let $\omega_\theta \in Z^2(\mathbb{Z}^2, \mathbb{T})$ be given by

$$\omega_\theta(m,n) = e^{2\pi i \theta m_2 n_1},$$

and define $\sigma_\theta \in Z^2(G, \mathbb{T})$ by

$$\sigma_\theta(x,y) = \omega_\theta(f(x), f(y)).$$

It can be shown that every two-cocycle on G is cohomologous to one of this form.
Then one checks that (G, σ_θ) satisfies Kleppner's condition if and only if θ is irrational. Hence, by the theorem, $C_r^*(G, \sigma_\theta)$ has a unique trace if and only if θ is irrational.
Using a different technique one can conclude that $C_r^*(G, \sigma_\theta)$ is simple if and only if θ is irrational, and that $G = BS(n,n)$ belongs to $\mathcal{K}$.