Algebra Preliminary Examination

Department of Mathematics, University of Denver

Fall 2011 (September 8, 2011)

NAME:

INSTRUCTIONS:

- The exam's duration is 4 hours.
- The exam has three parts, each part consisting of four problems.
- Each problem is worth 10 points.
- All problems from part 1 and the best 6 problems from parts 2 and 3 (combined) will determine your score.
- A score of 70% guarantees a pass.

POINTS:

Problem 1.1	....	/10
Problem 1.2	....	/10
Problem 1.3	....	/10
Problem 1.4	....	/10
Problem 2.1	....	/10
Problem 2.2	....	/10
Problem 2.3	....	/10
Problem 2.4	....	/10
Problem 3.1	....	/10
Problem 3.2	....	/10
Problem 3.3	....	/10
Problem 3.4	....	/10

TOTAL POINTS: PERCENTAGE: PASSED: Yes No

Part 1: Introduction to Abstract Algebra

Problem 1.1: Let X be a finite set of size n and let $\mathcal{P}(X)$ be the power set of X. Let $G = (\mathcal{P}(X), *)$, where $*$ is the symmetric difference on X, that is, for A, $B \subseteq X$ we have $A * B = \{x \in A \cup B;\ x \notin A \cap B\}$.

(a) Show that G is a group of order 2^n. (You can use Venn diagrams.)
(b) Show that every non-identity element of G is of order 2.
(c) Construct an explicit isomorphism $f : G \to (\mathbb{Z}_2)^n$.

Problem 1.2: Let R be a commutative ring and I, J ideals of R such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \times R/J$.

Problem 1.3: Let $R = \mathbb{Z}[\sqrt{-7}] \leq \mathbb{C}$.

(a) Find all units in R.
(b) Show that R possesses an element that is irreducible but not prime.

Problem 1.4: Let $GL(2,3)$ be the group of 2×2 invertible matrices with entries in $GF(3)$, and let $SL(2,3) = \{A \in GL(2,3);\ \det(A) = 1\}$.

(a) Find $|GL(2,3)|$.
(b) Show that $SL(2,3) \leq GL(2,3)$ and find $|SL(2,3)|$.
(c) Determine all primes p such that $GL(2,3)$ contains an element of order p.

Part 2: Group Theory

Problem 2.1: Let p and $q = 2p - 3$ be distinct primes such that $p \neq 7$. Show that every group of order p^3q^3 has a normal subgroup of order p^3. (Hint: The smallest situation occurs with $p = 5$, $q = 7$. You might want to look at this case first to see what's going on.)

Problem 2.2: Consider the abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}$.

(a) Find $m = \{\max |x|;\ x \in G\}$.
(b) Let $y \in G$ be an element of order $m = \{\max |x|;\ x \in G\}$, and let $H = G/\langle y \rangle$. Write H as a direct product of cyclic groups. Explain!

Problem 2.3: Let N be a group such that $\mathrm{Aut}(N) = \mathrm{Inn}(N)$. Suppose that $N \trianglelefteq G$. Show that for every $g \in G$ there is $a \in C_G(N)$ such that $g \in aN$.

Problem 2.4: Find presentations for the groups $\mathbb{Z}$, $\mathbb{Z}^2$ and D_{2n}, where D_{2n} is the dihedral group of order $2n$. Do you know the groups D_4, D_6 under other names?

Part 3: Rings and Fields

Problem 3.1: Define *maximal ideal* in a ring with 1. Show that every ring with 1 has a maximal ideal.

Problem 3.2 Recall that a commutative ring is *Noetherian* if it contains no infinite strictly ascending chain of ideals, and it is *Artinian* if it contains no strictly descending chain of ideals.

(a) Is $\mathbb{Z}$ Noetherian, Artinian?
(b) For a field F, is $F[x]$ Noetherian, Artinian?

Problem 3.3

(a) Show that in an integral domain R of characteristic p the mapping $\varphi : R \to R$, $x \mapsto x^p$ is a ring homomorphism.
(b) Let F be the p-element field, E an extension of F, and $\alpha \in E$ such that $f(\alpha) = 0$ for some $f \in F[x]$. Show that α^p is a root of f.

Problem 3.4:

(a) Show that a finite extension of fields is algebraic.
(b) Show that a simple algebraic extension of fields is finite.