Algebra Preliminary Examination

Department of Mathematics, University of Denver

Fall 2012 (September 6, 2012)

NAME:

INSTRUCTIONS:

- The duration of the exam is 4 hours.
- The exam has three parts, each part consisting of four problems.
- Each problem is worth 10 points.
- All problems from part 1 and the best 6 problems from parts 2 and 3 (combined) will determine your score.
- A score of 70% guarantees a pass.

POINTS:

Problem 1.1	....	/10
Problem 1.2	....	/10
Problem 1.3	....	/10
Problem 1.4	....	/10
Problem 2.1	....	/10
Problem 2.2	....	/10
Problem 2.3	....	/10
Problem 2.4	....	/10
Problem 3.1	....	/10
Problem 3.2	....	/10
Problem 3.3	....	/10
Problem 3.4	....	/10

TOTAL POINTS: PERCENTAGE: PASSED: Yes No

Part 1: Introduction to Abstract Algebra

Problem 1.1: [2, 4, 4 points] Every permutation $\sigma \in S_n$ can be written uniquely as a product of disjoint cycles. We say that $\sigma \in S_n$ has *cycle structure* $c(\sigma) = (m_1, m_2, \ldots, m_n)$ if, when written as a product of disjoint cycles, it contains precisely m_i cycles of length i, for every $1 \le i \le n$.

(i) Describe the set $C = \{c(\sigma);\ \sigma \in S_n\} \subseteq \mathbb{N}^n$.
(ii) Show that whether $\sigma \in S_n$ belongs to A_n depends only on $c(\sigma)$.
(iii) Construct an explicit function $f : C \to \{-1, 1\}$ such that $f(c(\sigma)) = 1$ iff $\sigma \in A_n$.

Problem 1.2: [each problem 2.5 points] For the following pairs of groups, determine if they are isomorphic or not. Justify your answers.

(i) The multiplicative group of positive rationals and $(\mathbb{Z}, +)$.
(ii) $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$.
(iii) $(\mathbb{R}, +) \times (\mathbb{R}, +)$ and $(\mathbb{C}, +)$.
(iv) $(\mathbb{Q}, +)$ and the multiplicative group of positive rationals.

Problem 1.3: [10 points] Let F be a field, let n be a positive integer, and let $R = M_n(F)$ be the ring of all $n \times n$ matrices with entries in F with the usual operations of matrix addition and multiplication.

(i) Prove that the only two-sided ideals in R are $\{0\}$ and R.
(ii) When is R a field?

Problem 1.4: [4, 3, 3 points]

(i) Construct a field F of order 9.
(ii) Describe all subfields of F.
(iii) Describe all subgroups of the multiplicative group $F^* = F \setminus \{0\}$.

Part 2: Group Theory

Problem 2.1: [10 points] A group G is said to be an *internal semidirect product* of its normal subgroup N and its subgroup H if $G = NH$ and $N \cap H = 1$. A group G is an *external semidirect product* of two groups N and H if there is a homomorphism $\varphi : H \to \mathrm{Aut}(N)$, $x \mapsto \varphi_x$, such that $G = (N \times H, *)$, where

$$(a, x) * (b, y) = (a\varphi_x(b), xy).$$

Show that every internal semidirect product of N and H is isomorphic to an external semidirect product of N and H, and vice versa. (You do not need to show that $(N \times H, *)$ is a group.)

Problem 2.2: [10 points] Let G be a simple group with $|G| > 60$. Show that G has no subgroups of index less or equal to 5.

Problem 2.3: [6, 4 points] A group G is called *metabelian* if it has a normal subgroup N such that both N and G/N are abelian.

(i) Show that metabelian groups are closed under homomorphic images, subgroups and direct products. (For the direct products, give the proof for products of arbitrarily many metabelian groups, not just finitely many.)

(ii) Show that every group of order 2012 is metabelian.

Problem 2.4: [10 points] Let n be a positive integer and let $A = \mathbb{Z}^n$. Prove that if B is any subgroup of A that is generated by fewer than n elements, then the index $[A : B]$ is infinite. (Hint: The group $\mathbb{Z}^n$ is the free abelian group on n generators.)

Part 3: Rings and Fields

Problem 3.1: [2, 2, 6 points] An integral domain R is *Euclidean* if there exists $f : R\setminus\{0\} \to \mathbb{N}$ such that for every a, $b \in R$, $b \neq 0$ there are q, $r \in R$ such that $a = bq + r$ AND either $r = 0$ or $f(r) < f(b)$.

(i) Show that every field is a Euclidean domain.
(ii) Give an example of a Euclidean domain that is not a field. (Give explicitly a suitable function f.)
(iii) Prove that every Euclidean domain is a PID.

Problem 3.2: [4, 3, 3 points]

(i) State Eisenstein's criterion for irreducibility of $f \in \mathbb{Z}[x]$ over $\mathbb{Q}$.
(ii) Is $f(x) = x^3 - 9x^2 + 6x - 12$ irreducible over $\mathbb{Q}$?
(iii) Is $f(x)$ from (ii) irreducible over $\mathbb{R}$?

Problem 3.3: [10 points] Let R be a commutative ring with unity. Show that the set of nilpotent elements in R forms an ideal of R.

Problem 3.4: [5, 5 points]

(i) Prove that every finite field has p^n elements for some prime p and some positive integer n.
(ii) Prove that any two finite fields with the same number of elements are isomorphic.