CONFIGURATIONS IN COPRODUCTS OF PRIESTLEY SPACES

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Abstract. Let P be a configuration, i.e., a finite poset with top element. Let $\mathsf{Forb}(P)$ be the class of bounded distributive lattices L whose Priestley space $\mathcal{P}(L)$ contains no copy of P. We show that the following are equivalent: $\mathsf{Forb}(P)$ is first-order definable, i.e., there is a set of first-order sentences in the language of bounded lattice theory whose satisfaction characterizes membership in $\mathsf{Forb}(P)$; P is coproductive, i.e., P embeds in a coproduct of Priestley spaces iff it embeds in one of the summands; P is a tree. In the restricted context of Heyting algebras, these conditions are also equivalent to $\mathsf{Forb}_H(P)$ being a variety, or even a quasivariety.

Introduction

Some properties of a distributive lattice L can be expressed in terms of configurations forbidden in the order structure of the corresponding Priestley dual $\mathcal{P}(L)$, i.e., configurations forbidden in the poset of prime ideals on L. Thus for instance, L is Boolean if and only if $\mathcal{P}(L)$ contains no non-trivial chain, which is to say that the order of $\mathcal{P}(L)$ is trivial. Among several more involved, but also well-known, results of this type, we mention the characterization of relative normality in [7], or the treatment of the question of n-chains in [1].

In [3] we have shown that if P is a tree then the prohibition of a copy of P in $\mathcal{P}(L)$ can be characterized by a first-order condition on L. Furthermore, we have presented some formulas, containing an unspecified number of variables and therefore not first-order, for some other connected configurations, for instance, for the diamond. The question naturally arises as to whether one can replace these formulas

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by first-order ones, and if not, to determine which configurations have the feature that their prohibition in $\mathcal{P}(L)$ can be characterized by first-order formulas on L. In this article we prove that, among the finite posets with top, these configurations are precisely the trees.

This turns out to be closely connected with the problem of configurations in coproducts. Coproducts in Priestley spaces are not quite transparent compactifications of the disjoint sums. Can a configuration embed in a coproduct without embedding in a summand? A part of the result above is that a tree cannot, while any other configuration, i.e., any configuration with a cycle, can.

Although we have restricted ourselves to the configurations with top, the results are in fact more general. It is only that the general situation is tied up with several other issues (to which we intend to devote a special study) so that an adequate treatment would multiply the size of this article.

1. Preliminaries

1.1. If M is a subset of a poset $(X, \leqslant)$ we write $\downarrow M$ for $\{x \mid x \leqslant m \in M\}$ and $\uparrow M$ for $\{x \mid x \geqslant m \in M\}$; we abbreviate $\downarrow\{x\}$ (resp. $\uparrow\{x\}$) to $\downarrow x$ (resp. $\uparrow x$).

If $\downarrow M = M$ (resp. $\uparrow M = M$) we say that M is a *down-set* (resp. *up-set*).

The fact that Q is isomorphic to a subposet of P will be indicated by $Q \hookrightarrow P$. For the negative we use the symbol $Q \not\hookrightarrow P$.

1.2. Recall that a *Priestley space* X is a compact ordered space such that whenever $x \nleqslant y$ there is a clopen down-set $U \subseteq X$ such that $x \notin U \ni y$. The category of Priestley spaces and monotone continuous maps (Priestley maps) will be denoted by **PSp**.

Further recall (see, e.g., [4], [8], [9]) the Priestley duality between **PSp** and the category **DLat** of bounded distributive lattices and 01-lattice homomorphisms; the equivalence functors are usually given as

$$\begin{aligned} &\mathcal{P}(L) = \{x \mid L \neq x \subseteq L \text{ prime ideal}\}, \quad \mathcal{P}(h)(x) = h^{-1}[x], \\ &\mathcal{D}(X) = \{U \mid U = \downarrow U \subseteq X \text{ clopen }\}, \quad \mathcal{D}(f)(U) = f^{-1}[U], \end{aligned} \tag{1.2}$$

$\mathcal{P}(L)$ is endowed with a suitable topology and ordered by inclusion.

If lattices L, M are Heyting algebras, the *Heyting* homomorphisms between them correspond to the Priestley maps h such that, moreover,

$$h(\downarrow x) = \downarrow h(x).$$

Such maps will be referred to as *h-maps*.

A one-point space $\{\cdot\}$ will be indicated as $\bullet$ and the unique map $X \to \bullet$ will be denoted by σ_X.

1.3. For a given family of Priestley spaces X_i, $i \in J$, we view the coproduct $X \equiv \coprod_J X_i$ as $\mathcal{P}(\prod_J \mathcal{D}(X_i))$. Denoting $\mathcal{D}(X_i)$ by A_i and $\prod_J A_i$ by A, an element $x \in X$ is then a proper prime ideal of A. For $i \in J$, the insertion map $\iota_i : X_i \to X$, dual to the projection map $A \to A_i$, is given by the rule

$$x_i \mapsto \{a \in A \mid x_i \notin a(i)\}, \quad x_i \in X_i.$$

A particular coproduct will play an important role in our analysis. A one-point space will be denoted $\bullet$, for which $\mathcal{P}(\bullet)$ is the two-element lattice $\mathbf{2} \equiv \{0 < 1\}$. The coproduct of a family of one-point spaces, indexed by J, consists therefore of all proper prime ideals of the lattice $\mathbf{2}^J$; we denote it ${}^J\bullet$.

Of course, the proper prime ideals of $\mathbf{2}^J$ are all maximal and so trivially ordered. Therefore $\mathbf{2}^J$ is **PSp**-isomorphic to βJ, the Stone-Čech compactification of the discrete space J. For our purposes, the points of βJ are the ultrafilters on J, and this set of points is regarded as a Priestley space under its usual compact topology and trivial order. More concretely, the map $\alpha : {}^J\bullet \to \beta J$ defined by

$$x \mapsto \{J \smallsetminus K \mid \chi_K \in x\}, \quad x \in {}^J\bullet,$$

provides the **PSp**-isomorphism. Here χ_K is the characteristic function of the subset $K \subseteq J$.

1.4. Now consider a family of Priestley spaces X_i, $i \in J$, and assume the notation of 1.3. For each $i \in J$ let $\sigma_i : X_i \to \bullet$ be the unique **PSp**-map, with dual injection $\delta_i : \mathbf{2} \to A_i$. If we let $\delta \equiv \prod_J \delta_i : \mathbf{2}^J \to A$, then the coproduct map $\sigma \equiv \coprod_J \sigma_i : X \to {}^J\bullet$ is given simply by the rule $\sigma(x) = \delta^{-1}(x)$, $x \in X$.

Crucial for our purposes is the map $\varepsilon \equiv \alpha\sigma : X \to \beta J$. If we abbreviate $\delta(\chi_K)$ to a_K for $K \subseteq J$, so that

$$a_K(i) \equiv \begin{cases} 1_{A_i} = X_i & \text{if} \quad i \in K \\ 0_{A_i} = \emptyset & \text{if} \quad i \notin K \end{cases},$$

we get a simple formula for ε:

$$\varepsilon(x) = \{J \smallsetminus K \mid \chi_K \in \delta^{-1}(x)\} = \{J \smallsetminus K \mid a_K \in x\}.$$

Now we have the coproduct X represented as the disjoint union $\bigcup_{u \in \beta J} X_u$, where

$$X_u \equiv \varepsilon^{-1}(u) = \{x \mid \{J \smallsetminus K \mid a_K \in x\} \subseteq u\}.$$

This viewpoint brings to the surface several important facts.

- Because the order in βJ is trivial, all the X_u's are order independent.
- Hence an embedding of a connected poset P into the coproduct has to embed P into one of the X_u's.
- For each $i \in J$, ι_i provides a **PSp**-isomorphism from X_i onto $X_{\widetilde{i}}$, where $\widetilde{i} \equiv \{Z \mid i \in Z\} \in \beta J$, and $X_{\widetilde{i}}$ is clopen in X.
- $\bigcup_{i\in J} X_{\widetilde{i}}$ is a dense open subset of X.

1.5. By a result from [5],

the Priestley duals $\mathcal{D}(X_u)$ of X_u are the ultraproducts $\prod_u \mathcal{D}(X_i)$.

1.6. A *configuration* is a finite poset (finite Priestley space) with a top element. For a configuration P denote by

$$\mathsf{Forb}(P) \quad \text{resp.} \quad \mathsf{Forb}_H(P)$$

the class of distributive lattices L (resp. Heyting algebras L) such that $P \not\hookrightarrow L$.

A configuration P is said to be *coproductive* if it cannot be embedded into a coproduct $\coprod_{i\in J} X_i$ unless it is embeddable into one of the X_i's.

1.7. By Łoś's Theorem (see, e.g., [6]), a system that can be characterized by first order formulas in a first order theory is closed under ultraproducts. In consequence we have

Proposition. *If the class $\mathsf{Forb}(P)$ is characterized by first order formulas in the theory of distributive lattices then the configuration P is coproductive.*

Proof. Suppose we have $P \hookrightarrow \coprod_{i\in J} X_i$. Since P is connected, we have $P \hookrightarrow X_u$ for an ultrafilter $u \in \beta J$. Now by 1.5 the ultraproduct $\prod_u \mathcal{D}(X_i)$ is not in $\mathsf{Forb}(P)$ and hence there is an $i \in J$ such that $\mathcal{D}(X_i) \notin \mathsf{Forb}(P)$, that is, $P \hookrightarrow X_i$. $\square$

2. A special coproduct

2.1. Conventions. Immediate precedence in a poset will be indicated by $\prec$.

A *generalized diamond* is a poset that is obtained from a tree that is not a chain by augmenting it with an extra bottom point.

A point x in finite poset P (not necessarily with top) is said to be *confluent* if there are incomparable y, z such that $x \leqslant y, z$. The set of all confluent points of P, resp. maximal confluent points of P, will be denoted by

$$\mathsf{Cf}(P) \quad \text{resp.} \quad \max \mathsf{Cf}(P).$$

We have the obvious

2.1.1. Observations. *1. If $f : P \to Q$ is an embedding then*

$$f[\mathsf{Cf}(P)] \subseteq \mathsf{Cf}(Q), \quad \textit{and} \tag{2.1.a}$$

$$f[\mathsf{Cf}(P) \setminus \max \mathsf{Cf}(P)] \subseteq \mathsf{Cf}(Q) \setminus \max \mathsf{Cf}(Q) \tag{2.1.b}$$

2. If p is in $\max \mathsf{Cf}(P)$ then $\uparrow p$ is a generalized diamond.

2.2. If P is a configuration that is not a tree then obviously $\mathsf{Cf}(P) \neq \emptyset$. Choose an $r \in \max \mathsf{Cf}(P)$ such that

(2.2.1) the number k of the immediate successors of r is smallest possible.

The generalized diamond $\uparrow r$ will be denoted by D, and the notation

$$P,\ r,\ k,\ D$$

will be observed in the sequel; we will write $j : D \to P$ for the embedding.

2.3. The posets D_n and P_n. *The posets D_n*: In the set M of the elements minimal above r choose a fixed s. This symbol will also be retained in the sequel. Set

$$D_n = D \times \{1, 2, \ldots, n\}$$

(we will write pi for (p, i)) and order it by

$$\begin{aligned} (\mathbf{!!}) \qquad & ri \leq sj \text{ iff } i \neq j, \\ & ri \leq pj \text{ iff } i = j \text{ for } p \in M,\ p \neq s, \text{ and} \\ & pi \leq qj \text{ iff } p \leq q \text{ and } i = j \text{ for } p \neq r. \end{aligned}$$

The posets P_n are obtained from P by replacing the subset D by D_n and leaving the other points alone; the order among the elements of $P \setminus D$ is kept as in P, and the p and qi are put in the same order as p and q. Note that P_n in general does not have a top. The natural maps defined by $p \longmapsto p$ for $p \in P \smallsetminus D$ and $pi \longmapsto p$ for $p \in D$ will be denoted by

$$f_n : P_n \to P.$$

Note that

$$f_n(\downarrow z) = \downarrow f_n(z) \quad \text{for any } z. \tag{2.3.1.}$$

2.4. Proposition. *There is no embedding $P \hookrightarrow P_n$.*

Proof. We leave the demonstration for $n = 1$ to the reader, and assume $n \geq 2$. Suppose φ is such an embedding. Considering a longest path $p_0 < p_1 < \cdots < \top$ and its image $\varphi(p_0) < \varphi(p_1) < \cdots < \varphi(\top)$ we see that $\varphi(\top) = \top i_0$ for some i_0. Consequently we have

$$\varphi[P] \subseteq Q = \downarrow\!\top i_0.$$

Since $n \geq 2$,

$$Q = (P \setminus D) \cup ((D \setminus \{r\}) \times \{i_0\}) \cup (\{r\} \times \{1, 2, \ldots, n\})$$

with the relation in $P \setminus D$ as in P, $pi_0 \leqslant qi_0$ for $p, q \in D \setminus \{r\}$ iff $p \leqslant q$, $p \leqslant qi$ iff $p \leqslant q$, and $ri < qi_0$ for any i and any $q \neq s$ such that $r < q$.

Thus

$$\mathsf{Cf}(Q) = (\mathsf{Cf}(P) \setminus \{r\}) \cup \{ri \mid i = 1, \ldots, n\}$$

and the ri are maximal and have only $k-1$ immediate successors in Q. By the choice (2.2.1), $ri \neq \varphi(p)$ for $p \in \mathsf{Cf}(P)$. By (2.1.b) they cannot be $\varphi(p)$ for any other $p \in \mathsf{Cf}(P)$ either, which yields, using also (2.1.a), $\varphi[\mathsf{Cf}(P)] \subseteq \mathsf{Cf}(P) \setminus \{r\}$. This cannot be, since φ is one-one. □

3. The fibers in the coproduct of the P_n's

3.1. We apply the notation and formula from 1.4 to the family P_n, $n \in \mathbb{N}$. To reiterate, $X = \mathcal{P}(A)$, where $A = \prod_{\mathbb{N}} A_n$ and $A_n = \mathcal{D}(P_n)$, $n \in \mathbb{N}$. We then have for the function $\varepsilon : X \to \beta\mathbb{N}$ the formula

$$\varepsilon(x) = \{\mathbb{N} \smallsetminus K \mid a_K \in x\}$$

where

$$a_K(n) = \begin{cases} 1_{A_n} = P_n & \text{if} \quad n \in K \\ 0_{A_n} = \emptyset & \text{if} \quad n \notin K \end{cases} .$$

In consequence we have

$$P_u = \varepsilon^{-1}(u) = \{x \in X \mid \{a_{\mathbb{N} \smallsetminus K} \mid K \in u\} \subseteq x\}. \tag{3.2}$$

3.3. Recall the maps $f_n : P_n \to P$ from 2.3. Define $f : X \to P$ by $f \cdot \iota_n = f_n$, $n \in \mathbb{N}$. In order to express f concretely, let $A_P \equiv \mathcal{D}(P)$, $h_n \equiv \mathcal{D}(f_n) : A_P \to A_n$, and $h \equiv \prod_{n \in \mathbb{N}} h_n : A_P \to A$. Then

$$f = \lambda^{-1} \cdot \mathcal{D}(h) = \lambda^{-1} h^{-1},$$

where $\lambda : P \cong \mathcal{P}(\mathcal{D}(P))$ is the duality equivalence given by

$$\lambda(p) = \{a \in A_P \mid p \notin a\}, \quad p \in P.$$

Because P is finite, the ideal $\lambda(p)$ contains the largest element $a_p = \{q \mid q \not\geqslant p\}$, and we have

$$\lambda(p) = \{a \in A_P \mid a \leqslant a_p\}.$$

Similarly, the filter F_p complementary to $\lambda(p)$ contains the smallest element $b_p = \downarrow p$, and we have $F_p = \{a \in A_P \mid a \geqslant b_p\}$. Set

$$\widetilde{a}_p = h(a_p) = (f_n^{-1}(a_p))_n,$$
$$\widetilde{b}_p = h(b_p) = (f_n^{-1}(b_p))_n.$$

Take a prime ideal x, and let y be the complementary filter. Then $h^{-1}(x) = \lambda(p)$ iff $h^{-1}(y) = F_p$. Thus

$$f(x) = p \quad \text{iff} \quad h^{-1}(x) = \lambda(p) \quad \text{iff} \quad \widetilde{b}_p = h(b_p) \notin x \ni h(a_p) = \widetilde{a}_p.$$

We have the formula

$$f^{-1}(p) = \{x \mid \widetilde{b}_p \notin x \ni \widetilde{a}_p\}, \quad p \in P. \tag{3.3}$$

3.4. Proposition. *For each $u \in \beta\mathbb{N}$ and $p \in P$,*

$$P_u \cap f^{-1}(p) = \{x \mid \widetilde{b}_p \notin x \supseteq \{a \mid \{n \mid f_n^{-1}(p) \cap a(n) = \emptyset\} \in u\}\}.$$

Proof. Let Z label the set displayed on the right. Consider $x \in P_u \cap f^{-1}(p)$. To show that $x \in Z$, first note that $\widetilde{b}_n \notin x$ by (3.3). Let $a \in A$ be such that $K \equiv \{n \mid f_n^{-1}(p) \cap a(n) = \emptyset\} \in u$. Since $x \in P_u$ then by (3.2) $a_{\mathbb{N}\smallsetminus K} \in x$, and since $x \in f^{-1}(p)$ then by (3.3) $\widetilde{a}_p \in x$; consequently $a_{\mathbb{N}\smallsetminus K} \vee \widetilde{a}_p \in x$. To conclude $x \in Z$ we need only establish the claim that $a \leqslant a_{\mathbb{N}\smallsetminus K} \vee \widetilde{a}_p$. If $n \in K$ then $f_n^{-1}(p) \cap a(n) = \emptyset$ and hence $a(n) \leqslant f_n^{-1}(a_p) = \widetilde{a}_p(n)$, for otherwise there would be a $z' \in a(n)$ such that $f_n(z) \geqslant p$ and by 2.3.1 we would have $z \leqslant z'$ such that $f_n(z) = p$. On the other hand, if $n \notin K$ then $a(n) \leqslant 1_{A_n} = a_{\mathbb{N}\smallsetminus K}(n)$. This proves the claim.

Now let $x \in Z$. Trivially, for any $K \subseteq \mathbb{N}$, $\{n \mid f_n^{-1}(p) \cap a_{\mathbb{N}\smallsetminus K}(n) = \emptyset\} = K$ and hence by (3.2) our x is in P_u. Finally, $f_n^{-1}(p) \cap \widetilde{a}_p(n) = f_n^{-1}(\{p\} \cap a_p) = \emptyset$ for all n and hence $\widetilde{a}_p \in x$ and $x \in f^{-1}(p)$ by (3.3). $\square$

3.5. Proposition. *For each $u \in \beta\mathbb{N}$ and $p \in P \smallsetminus D$, $P_u \cap f^{-1}(p)$ contains exactly one element, namely*

$$\{a \mid \{n \mid p \notin a(n)\} \in u\}.$$

Proof. For $p \notin D$, $f_n^{-1}(p) = \{p\}$ and hence

$$\{a \mid \{n \mid f_n^{-1}(p) \cap a(n) = \emptyset\} \in u\} = \{a \mid \{n \mid p \notin a(n)\} \in u\}.$$

Denote this set by x. It is easy to check that x is a prime ideal; trivially it does not contain $\widetilde{b}_p$. Now let $x' \supsetneq x$ be a prime ideal. Choose $a \in x' \smallsetminus x$. Then $K \equiv \{n \mid p \notin a(n)\} \notin u$, hence $\mathbb{N} \smallsetminus K \in u$, and by (3.2), $a_K \in x$, and hence $a \vee a_k \in x'$. The proof will be complete if we can establish the claim that $\widetilde{b}_p \leqslant a_K \vee a$. Certainly for $n \in K$ we have $\widetilde{b}_p(n) = \downarrow p \leqslant P_n = 1_{A_n} = a_K(n)$. But also for $n \notin K$ we have $p \in a(n)$, so that $\widetilde{b}_p(n) = \downarrow p \leqslant a(n)$. This proves the claim and the proposition. $\square$

4. Trees, coproductivity, and first order

4.1. Recall from [3] that

> *if T is a tree then L is in* $\mathsf{Forb}(T)$ *iff every $a : T \to L$ has a T-complement in L*

where a T-complement of a map $a : T \to L$ is a map $c : T \to L$ such that

- for $t \in \min T$, $a(t) \wedge c(t) \leq a'(t)$ where $a'(t) = \bigvee_{t \nleq \tau} a(\tau)$,
- $a(t) \wedge c(t) \leq \bigvee_{\tau \prec t} c(\tau) \vee a'(t)$ otherwise,
- and $a(\top_T) = 1$,

and that

> $\mathsf{Forb}_H(P)$ *is a variety of Heyting algebras iff* $\mathsf{Forb}_H(P)$ *is a quasivariety of Heyting algebras iff P is a tree.*

4.2. We return to the notation of Section 3. Fix $u \in \beta\mathbb{N} \smallsetminus \mathbb{N}$. Our objective is to define a function $g_u : P \to P_u$ which we will show is an embedding in Proposition 4.3. For that purpose we need an auxiliary index set $I = \{(n, j) \mid 1 \leq j \leq n\}$, and an (arbitrary) ultrafilter v on I containing the proper filter base

$$F \equiv \{J \subseteq I \mid \exists m \ (\{n \mid |\{j \mid (n, j) \notin J\}| \leq m\} \in u)\}.$$

For each $p \in D$ let

$$g_u(p) = \{a \mid \{(n, j) \mid pj \notin a(n)\} \in v\}.$$

Clearly $g_u(p)$ is a prime ideal on $A = \prod A_n$. For $p \in P \smallsetminus D$ define

$$g_u(p) = \{a \mid \{n \mid p \notin a(n\} \in u\}.$$

4.2.1. Observation. *For any p, $\widetilde{b}_p \notin g_u(p) \ni \widetilde{a}_p$.*

Indeed,

$$\{(n,j) \mid pj \notin f_n^{-1}(b_p)\} = \{(n,j) \mid p \nleqslant p\} = \emptyset \quad \text{and}$$
$$\{(n,j) \mid pj \notin f_n^{-1}(a_p)\} = \{(n,j) \mid p \geqslant p\} = I.$$

4.2.2. Lemma. *If $q \nleqslant p$ then $\widetilde{a}_q \notin g_u(p)$.*

Proof. $p \notin \widetilde{a}_q(n)$ resp. $pj \notin \widetilde{a}_q(n)$ implies $q \leqslant p$. Thus, in both the cases, the sets that should be in the ultrafilters are void. □

4.3. Proposition. *For each $u \in \beta N \smallsetminus N$ there is an embedding $g_u : P \to P_u$ such that*

$$f \cdot g_u = id_P.$$

Proof. We will show that the formula g_u defined in 4.2 constitutes an embedding. By 4.2.2, $q \nleqslant p$ implies that $g_u(q) \nsubseteq g_u(p)$, so that it suffices to prove that g_u is monotone. Thus, let $p \leqslant q$.

If $p, q \in P \smallsetminus D$, $g_u(p) \subseteq g_u(q)$ since $\{n \mid p \notin a(n)\} \subseteq \{n \mid q \notin a(n)\}$.

Suppose $p \in P \smallsetminus D$ and $q \in D$, and consider $a \in g_u(p)$, so that $J \equiv \{n \mid p \notin a(n)\} \in u$. Then $L \equiv \{(n,j) \mid n \in J,\ 1 \leqslant j \leqslant n\} \in F$ by definition. Since $p \leqslant qj$ for all j in any P_n,

$$\{(n,j) \mid qj \notin a(n)\} \supseteq L \in F \subseteq v.$$

It follows that $a \in g_u(q)$ and that $g_u(p) \subseteq g_u(q)$.

The case $p, q \in D$, $(p,q) \neq (r,s)$ (recall 2.2, 2.3) is much the same.

It remains to prove that $g_u(r) \subseteq g_u(s)$. Thus, let $a \in g_u(r)$ so that $J \equiv \{(n,j) \mid rj \notin a(n)\} \in v$; we seek to show that $K \equiv \{(n,j) \mid sj \notin a(n)\} \in v$. First, observe that

$$L \equiv \{n \mid |\{j \mid (n,j) \in J\}| \geqslant 2\} \in u.$$

Indeed, if not then $\{n \mid |\{j \mid (n,j) \notin I \smallsetminus J\}| \leqslant 1\} \in u$ and $I \smallsetminus J \in F \subseteq v$. Now note that $(n,j) \in K$ for all $n \in L$ and $1 \leq j \leq n$, for every point of $a(n) \cap f_n^{-1}(s)$ lies above a point of $a(n) \cap f_n^{-1}(r)$ as long as there are at least two points of the latter type. We have

$$K \supseteq \{(n,j) \mid n \in L,\ 1 \leq j \leq n\} \in F \subseteq v.$$

This completes the proof. □

4.4. Theorem. *The following statements on a configuration P are equivalent*

1. *P is a tree,*
2. *P is coproductive,*
3. $\mathsf{Forb}(P)$ *can be characterized by first order formulas.*

Proof. (1)⇒(3) by 4.1, (3)⇒(2) by 1.7, and (2)⇒(1) by 2.4 and 4.3. □

4.5. Recalling the second statement in 4.1, and using the fact that the products in the category of Heyting algebras are as in the category of distributive lattices, we obtain, in the Heyting context,

Theorem. *The following statements on a configuration P are equivalent*

(1) *P is a tree,*
(2) *P is coproductive,*
(3) $\mathsf{Forb}_H(P)$ *can be characterized by first order formulas,*
(4) $\mathsf{Forb}_H(P)$ *is a quasivariety,*
(5) $\mathsf{Forb}_H(P)$ *is a variety.*

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