A NATURAL EXTENSION OF THE CONTINUOUS FUNCTIONS WITH COMPACT SUPPORT

R. N. BALL, A. W. HAGER, D. G. JOHNSON, AND A. KIZANIS

Abstract. Let Y be a locally compact space, $C_K(Y)$ the collection of real-valued continuous functions with compact support, and $B_L(Y)$ the set of all Baire functions with Lindelöf cozero-set. We show that the embedding $C_K(Y) \leq B_L(Y)$ of archimedean ℓ-groups (or vector lattices) has this universal mapping property: any homomorphism $C_K(Y) \xrightarrow{\varphi} A$, where A has the property

(*) is divisible and both conditionally and laterally σ-complete,

has a unique extension $B_L(Y) \xrightarrow{\overline{\varphi}} A$; also, $B_L(Y)$ has property (*). Our perspective on this is as follows.

In a category **C** an object G is *epicomplete* if the only epic monics out of G are isomorphisms, epic or monic meant in the categorical sense of right or left cancellable. For each of the categories **Arch**, archimedean ℓ-groups with ℓ-homomorphisms, and its companion category **W**, **Arch**-objects with distinguished weak unit and unit-preserving ℓ-homomorphisms, (and for the corresponding categories of vector lattices) epicompleteness has been characterized as divisible and conditionally and laterally σ-complete, and it has been shown to be monoreflective. Denote the reflecting functors by β and $\beta^{\mathbf{W}}$, respectively.

What are they? For **W** the Yosida representation has been used to realize $\beta^{W}A$ as a certain quotient of $B(YA)$, the Baire functions on the Yosida space of A. For **Arch** very little has been known. Here we give a general representation theorem, Theorem A, for βG as a certain subdirect product of W-epicomplete objects derived from G. That result, some W-theory, and the relation between epicity and relative uniform density are then employed to show Theorem B: $\beta C_K(Y) = B_L(Y)$; i.e., the result of the first paragraph.

1. Introduction

Here we set some terminology and context, by way of sketching the proof of Theorem B in the Abstract. This reverses the the order of events in the Abstract and in subsequent sections. We do this because otherwise the point of Theorem A might not be so visible.

Principal references will be: for category theory, [22]; for topology, [13] and [15]; for basics on ℓ-groups (lattice-ordered groups), [1] and [10]; for vector lattices, [25];

Date: 2 October 2003.

1991 *Mathematics Subject Classification.* Primary: 06F20, 18A40, 46A40; Secondary: 54C40.

Key words and phrases. lattice-ordered group, archimedean, epimorphism, epicomplete, monoreflection, relative uniform density, Baire functions, locally compact space, compact support.

for details on epics and epicompletions, various papers in the literature, especially [3]–[7], [24] and [18].

1.1 Let **C** be a category, **R** a full subcategory. A **C**-morphism $A \xrightarrow{\alpha} B$ is **R**-*extendable* if for each $\varphi \in \mathbf{C}(A, \mathbf{R})$ there is $\bar{\varphi}$ with $\bar{\varphi}\alpha = \varphi$. **R** is said to be *monoreflective* if for each object G there is an **R**-extendable $G \xrightarrow{r_G} rG$, which is monic and epic with $rG \in \mathbf{R}$.

When this obtains, since the maps r_G are epic, the extensions $\bar{\varphi}$ are unique. Also, given G, a pair (r_G, rG) is essentially unique, and any pair with its properties is called an **R**-*monoreflection* of G. Also, the situation defines a functor $r : \mathbf{C} \longrightarrow \mathbf{R}$.

We thus have the definition of "**C-ec** is monoreflective in **C**," where **C-ec** denotes the full subcategory of **C**-epicomplete objects. This occurs frequently, and when it does, **C-ec** is the smallest monoreflective subcategory, with the largest reflections, and is thus of especial significance. A familiar example is the category **C** of Tychonoff spaces, where epicomplete means compact and the monoreflection is Stone-Čech compactification. See [17] for further examples and discussion.

We are concerned here with the categories **Arch** and **W**; **Arch-**epicomplete will be denoted just epicomplete, with functor β, and **Arch-ec**-extendable will be shortened to ec-extendable. For **W**, we write **W**-epicomplete with functor $\beta^{\mathbf{W}}$.

1.2 Let us focus, and expand, on the situation $C_K(Y) \leq B_L(Y)$ in the Abstract. Y is a locally compact Hausdorff space, and

$$\begin{aligned}
C(Y) &= \left\{ f \in \mathbb{R}^Y : f \text{ is continuous} \right\}, \\
C_K(Y) &= \left\{ f \in C(Y) : \overline{\operatorname{coz}} f \text{ is compact} \right\}, \\
C_0(Y) &= \left\{ f \in C(Y) : f \text{ vanishes at } \infty \right\}, \\
B(Y) &= \left\{ f \in \mathbb{R}^Y : f \text{ is Baire} \right\}, \\
B_L(Y) &= \left\{ f \in B(Y) : \operatorname{coz} f \text{ is Lindelöf} \right\}.
\end{aligned}$$

Here $\operatorname{coz} f = \{y \in Y : f(y) \neq 0\}$, and f vanishes at ∞ provided that for all $\varepsilon > 0$ there exists a compact set K with $|f(y)| \leq \varepsilon$ for all $y \notin K$. Also, $B(Y)$ is the least subset of $\mathbb{R}^Y$ containing $C(Y)$ and closed under pointwise convergence of sequences, and a space is Lindelöf if each open cover has a countable subcover.

Now any $\mathbb{R}^Y$ is an archimedean ℓ-group in the pointwise operations and order, and so, too, is each of the C's and B's above, as ℓ-subgroups of $\mathbb{R}^Y$.

In $C(Y)$, $B(Y)$, and $\mathbb{R}^Y$, we designate as weak unit the constant function $\mathbf{1}$, and we have the **W**-inclusions $C(Y) \leq B(Y) \leq \mathbb{R}^Y$. In general, the other B's and C's have no weak units, and we have the **Arch**-inclusions (suppressing Y):

$$C_K \leq C_0 \leq B_L.$$

Ultimately we shall show in 4.6 and 5.3 below:

Theorem B. For $G = C_K$ and $G = C_0$, $G \leq B_L$ is an ec-monoreflection of G in **Arch**.

The case $G = C_K$ is the central one, as we shall see. Here, the theorem states exactly:

B1 B_L is epicomplete.
B2 $C_K \leq B_L$ is epic.
B3 $C_K \leq B_L$ is ec-extendable.

We shall prove B1 and B2 more-or-less directly in 4.1 and 4.2; B1 is routine, B2 uses relative uniform density and is more complicated, but most of the work resides in previous knowledge of $C(Y) \leq B(Y)$.

We don't know how to prove B3 directly, but rather employ the Theorem A mentioned in the Abstract. Here, given G, we find a (or several) P which is epicomplete and a product of **W-ec** objects, and an embedding $G \hookrightarrow P$ so that βG is the "epiclosure" of G in P, discussed below in 2.1. For $G = C_K$, we construct P so that visibly B_L embeds in P over C_K. (All of that uses "**W**-decomposition of **Arch**-morphisms," and knowledge of $\beta^{\mathbf{W}}$'s.) Then, using B1 and B2, $B_L = \beta C_K$ follows.

2. Some Preliminaries for Theorem A

Theorem A (3.2 below) results from a combination of the categorical features of (epi, extremal-monic)-factorizations, the method of "reduction by principal perps" from **Arch** to **W**, the properties of epicompleteness, and the idea of a coessential subset. We explain these first. The reader who finds this unnecessary or tedious might go on to Section 3, referring to this section as needed.

2.1 Factorization, etc.

In a category **C**, the morphism $A \xrightarrow{m} B$ is called *extremal monic* if m is monic, and if $m = ge$, with e epic, implies e is an isomorphism. In this circumstance A is called an *extremal subobject of* B . By identifying A with $m(A)$, one thinks of A as a subobject which admits no epic enlargement within B.

If **C** is "sufficiently complete" then:

(a) Any morphism f has an essentially unique (epi, extremal monic)-factorization $f = me$, with e epic and m extremal monic. When f is monic, we call the codomain of e, which is the domain of m, the *epiclosure* of the domain of f.
(b) Any square $fe = mg$ with e epic and m extemal monic "diagonalizes:" there is a unique d with $de = g$ and $md = f$.
(c) The full subcategory **R** is epireflective in **C** if and only if **R** is closed under formation of products and extremal subobjects in **C**.

For the meaning of "sufficiently complete," and details of the results above, see [22]. We don't need to go into this here.

2.2 Arch and W.

(a) **Arch** and **W** are "sufficiently complete," so 2.1 (a), (b), and (c) are valid in **Arch** and **W**.

(b) Let G be a **W**-object (resp., **Arch**-object). G is **W**-epicomplete (resp., **Arch**-epicomplete) if and only if G is divisible and both conditionally and laterally σ-complete. We denote the corresponding categories **W-ec** (resp., **Arch-ec**.)

(c) The class **W-ec** (resp., **Arch-ec**) is monoreflective in **W** (resp., **Arch**).

(d) Let $G \xrightarrow{\varphi} H$ be a surjection of **W** (resp., **Arch**). If G is **W**-epicomplete (resp., **Arch**-epicomplete), then so is H.

(a) is discussed in [4] and [16]. (b) is from [4]. (c) for **W** was first shown in [26], using the theory of σ-frames. Moments later, (c) was derived from (b) and 2.1(c) in [4]. [14] and [27] show that a surjection preserves the algebraic properties in b), so d) follows. Note that (d) implies this stronger form of epicompleteness: each epic out of G is a surjection.

We remark now, and then no more, that all of the considerations of this paper apply to the (less general) corresponding categories of vector lattices because: (1) any **Arch**- (or **W**-) object with property (*) "is" a vector lattice, and (2) it is a result of Bleier and Conrad (see the discussion in [3]) that the category of archimedean vector lattices can be identified with a monoreflective subcategory of **Arch** by forgetting scalar multiplication, and likewise for **W**. In this general vein, we note also that certain aspects of the property (*) have been studued by Fremlin in [14], where (*) is called "universally σ-complete."

2.3 Reduction of Arch to **W**.

(a) Let $G \in$ **Arch** and $u \in G^{+}$. Then $u^{\perp} = \{g \in G : |g| \wedge u = 0\}$ is an ideal (convex sub-ℓ-group); also, $G/u^{\perp} \in$ **Arch** - see [10] - and the quotient map $G \longrightarrow G/u^{\perp}$ is an **Arch**-morphism.

(b) Let $\varphi \in \mathbf{Arch}(G,H)$, and let $u \in G^{+}$. Then $\varphi\left(u^{\perp}\right) \subseteq \varphi(u)^{\perp}$, so

$$\varphi_u\left(g + u^{\perp}\right) \equiv \varphi(g) + \varphi(u)^{\perp}$$

defines a homomorphism $\varphi_u : G/u^{\perp} \longrightarrow H/\varphi(u)^{\perp}$ for which the **Arch**-square below commutes.

$$\begin{array}{ccc} G & \longrightarrow & G/u^{\perp} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_u} \\ H & \longrightarrow & H/\varphi(u)^{\perp} \end{array}$$

(c) Now $u \in G^{+}$ is called a *weak unit* in G when $u^{\perp} = \{0\}$. In any event, the coset $u + u^{\perp}$ is a weak unit in $G/u^{\perp}$. Any element larger than a weak unit is a weak unit. So if $u_1 \geq u$ in G then $u_1 + u^{\perp}$ is a weak unit in $G/u^{\perp}$, and for $\varphi \in \mathbf{Arch}(G,H)$, $\varphi(u_1) + \varphi(u)^{\perp}$ is a weak unit in $H/\varphi(u)^{\perp}$. This makes the φ_u of (b) into a **W**-morphism indicated by

$$\left(G/u^{\perp}, u_1 + u^{\perp}\right) \xrightarrow{\varphi_u} \left(H/\varphi(u)^{\perp}, \varphi(u_1) + \varphi(u)^{\perp}\right) \in \mathbf{W}.$$

(d) Now let $U \subseteq G^+$ and $U^\perp = \bigcap_U u^\perp$. Again we have an ideal, and $G/U^\perp \in$ **Arch**. Let $\varphi \in$ **Arch** (G, H). From the basic properties of the product in **Arch** (which is just the ℓ-group product; i.e., the cartesian product with $+$ and $\leq$ defined coordinate-wise), we have the commuting **Arch**-square below,

$$\begin{array}{ccc} G & \xrightarrow{\ \gamma\ } & \prod G/u^\perp \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\prod \varphi_u} \\ H & \xrightarrow[\ \rho\ ]{} & \prod H/\varphi(u)^\perp \end{array}$$

where the products are indexed by U, $\gamma(g)_u = (g + u^\perp)$ for all $u \in U$, ρ is defined likewise, $\ker \gamma = U^\perp$, and $\ker \rho = \varphi(U)^\perp$. So in the event (to materialize shortly) that $U^\perp = \{0\}$ in G and $\varphi(U)^\perp = \{0\}$ in H, γ and ρ will be embeddings.

2.4 Coessential subsets.

(a) **Definition.** Let $G \in |\mathbf{Arch}|$. An *archimedean kernel* in G is an ideal I of G for which G/I is archimedean.

For $S \subseteq G$, $\mathrm{ak}_G S$ denotes the least archimedean kernel of G containing S; this kernel exists because **Arch** is an SP-class in abelian ℓ-groups. See Section 4 for more on archimedean kernels.

(b) **Definition**. Let $G \in |\mathbf{Arch}|$ and $U \subseteq G^+$. We say that U is *coessential* in G if the only morphism $\alpha \in$ **Arch** (G, H) such that $\alpha(u) = 0$ for all $u \in U$ is the zero map. $\varphi \in$ **Arch** (G, H) is called coessential if $\varphi(G^+)$ is coessential in H.

We record some simple features of coessentiality.

(c) U is coessential in G if and only if the embedding of the generated ℓ-group ideal $\langle U \rangle \leq G$ is a coessential morphism.

(d) An epic is coessential.

(e) The composition of coessential morphisms is coessential.

(f) If U is coessential in G and $G \xrightarrow{\varphi} H$ is coessential then $\varphi(U)$ is coessential in H.

(g) For $U \subseteq G^+$, these are equivalent.

(g1) U is coessential in G.

(g2) $\mathrm{ak}_G U = G$.

(g3) Whenever $\varphi \in$ **Arch** (G, H) is coessential (or is epic, or is surjective), then $\varphi(U)^\perp = \{0\}$.

Proofs. (c), (d), (e), (f), and (g1)$\Longleftrightarrow$(g2) are routine. (g2)$\Longrightarrow$(g3). Suppose (g2) holds, and hence also (g1). Let $\varphi \in$ **Arch** (G, H) be coessential. By (g1) and (f), $\varphi(U)$ is coessential in H. This means that $\mathrm{ak}\,\varphi(U) = H$. Since for any V, $V^{\perp\perp}$ is always an archimedean kernel ([10]), we have $H = \mathrm{ak}\,\varphi(U) \subseteq \varphi(U)^{\perp\perp}$. But $V^{\perp\perp} = H$ iff $V^\perp = \{0\}$. (g3)$\Longrightarrow$(g2). If $\mathrm{ak}_G U \neq G$ then the quotient map $G \xrightarrow{\varphi} G/\mathrm{ak}_G U \equiv H$ is surjective and has $\varphi(U) = \{0\}$, so $\varphi(U)^\perp = H \neq \{0\}$.

We quote the following just by way of context; we won't explicitly use it here. This generalizes a crucial result from [3], and the proof there works here.

(h) Let U be coessential in G, and $\varphi \in \mathbf{Arch}(G, H)$. Then φ is **Arch**-epic if and only if φ is coessential and for each $u \in U$, the **W**-morphism

$$\varphi_u : \left(G/u^{\perp}, u + u^{\perp}\right) \longrightarrow \left(H/\varphi(u)^{\perp}, \varphi(u) + \varphi(u)^{\perp}\right)$$

is **W**-epic.

3. Theorem A

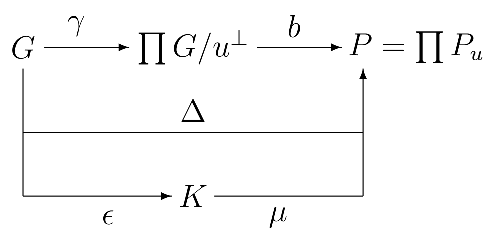
The following situation and notation will obtain throughout this section, and will be referred to later.

3.1 Let $G \in \mathbf{Arch}$ and let $U \subseteq G^{+}$ with $U^{\perp} = \{0\}$. For each $u \in U$ fix $u_1 \in G$ with $u_1 \geq u$. We have, for each $u \in U$, the quotient $G \longrightarrow G/u^{\perp}$ in **Arch**, the **W**-object $\left(G/u^{\perp}, u_1 + u^{\perp}\right)$, and the **W-ec** monoreflection

$$b_u : \left(G/u^{\perp}, u_1 + u^{\perp}\right) \to \beta^{\mathbf{W}}\left(G/u^{\perp}, u_1 + u^{\perp}\right) \equiv P_u.$$

We construe this last in **Arch**, and just write $b_u : G/u^{\perp} \leq P_u$.

Thus we have



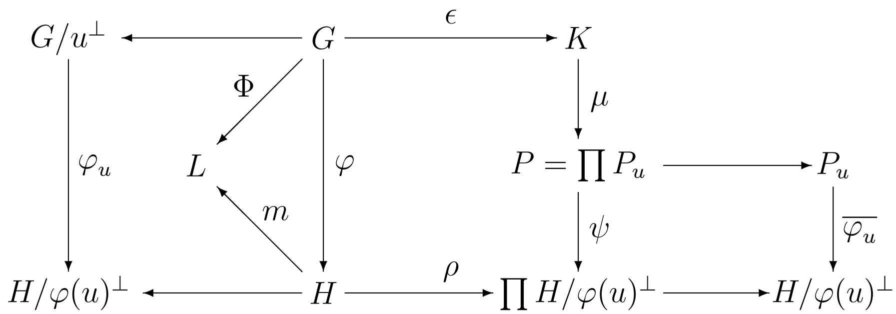
where the notation is as in 2.3(d), $b = \prod_U b_u$, $\Delta \equiv b\gamma$, and $\Delta = \mu\varepsilon$ is the (epi, extremal monic)-factorization of Δ.

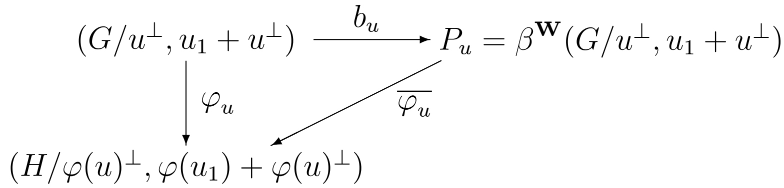
By 2.1 and 2.2, each P_u is **Arch**-epicomplete, and so then are P and K. Also ε is monic as a first factor of the monic Δ. So $G \overset{\varepsilon}{\longrightarrow} K$ is an epic embedding into an epicomplete object, an "epicompletion" of G, and this will be an ec-monoreflection of G if and only if ε is ec-extendable. Relatively simple examples show that, at this level of generality, ec-extendability can fail; see 3.3(b) below.

3.2 Theorem A. Assume the notation of 3.1. If U is coessential in G then $G \overset{\varepsilon}{\longrightarrow} K$ is ec-extendable, and thus is an ec-monoreflection of G.

Proof. Let $\Phi \in \mathbf{Arch}(G, L)$ with L epicomplete, and let $G \overset{\varphi}{\longrightarrow} H \overset{m}{\longrightarrow} L$ be the (epi, extremal monic)-factorization of Φ. We have the following diagram in **Arch**.
Note the following.

(1) H is an extremal subobject of the epicomplete object L, so is epicomplete by 2.1 and 2.2. By 2.2, $H/\varphi(u)^{\perp}$ is also epicomplete.
(2) We may view φ_u as a **W**-morphism, obtaining the following diagram in **W**. $\overline{\varphi_u}$ exists with $\overline{\varphi_u} b_u = \varphi_u$ since b_u is **W-ec**-extendable.
(3) Now view $\overline{\varphi_u}$ as an **Arch**-morphism, and construct $\psi = \prod \overline{\varphi_u}$.





(4) ρ is the standard map into the product, defined by $\rho(h) = (\rho(h)_u)$, where $\rho(h)_u = h + \varphi(u)^\perp$. Because U is coessential in G and φ is epic, $\varphi(U)$ is coessential in H, from which it follows from 2.4 that $\varphi(U)^\perp = \bigcap \varphi(u)^\perp = \{0\}$. Hence ρ is one-to-one. Since H is epicomplete, ρ is extremal monic.

(5) That $(\psi\mu)\varepsilon = \rho\varphi$ is easily checked. So by 2.1 there is $K \xrightarrow{\delta} H$ with $\delta\varepsilon = \varphi$ and $\rho\delta = \psi\mu$.

(6) $m\delta\varepsilon = \Phi$ follows.

□

3.3 Comments.

(a) The point of 3.2 is that for certain G the u's and u_1's can be chosen so that one knows what the P_u's are, and can compute K. This is done for $G = C_K(Y)$ in the next sections. It is to be noted that, in general, the calculation of K is not so straightforward; see 6.1.

(b) The following is shown in [24]. Let $u \in G^+$ be a weak unit, so $(G, u) \in |\mathbf{W}|$. Then $\beta G = \beta^{\mathbf{W}}(G, u)$ iff $\{u\}$ is coessential, in which case u is called a "near unit." 3.2 is a generalization of the "if" portion of this result: take $U = \{u\}$ with $u_1 = u$. A corresponding converse of 3.2 can be formulated; see comment (c) below.

Examples are presented in [24] of $(G, u) \in |\mathbf{W}|$ with $\{u\}$ not coessential, and these are then examples alluded to at the end of 3.1, where $G \xrightarrow{\varepsilon} K$ fails to be ec-extendable.

(c) While we have no use for it, it is interesting and not hard to prove that for any $u \in G^+$, $\beta^{\mathbf{W}}(G/u^\perp, u + u^\perp)$ is naturally $\mathbf{W}$-isomorphic to $(\beta G/u^\perp, u + u^\perp)$. Thus, when $U^\perp = \{0\}$ in βG, which follows from coessentiality, βG embeds in $P = \prod P_u$. Note that this is exactly the use of coessentiality in the proof of 3.2, to make ρ an embedding.

(d) The following comparison of 3.2 with a standard construction is curious. "Everybody knows" that epireflections are constructed by embedding in products via 2.1. In a suitable category $\mathbf{C}$, with subcategory $\mathbf{A}$, given $\mathbf{R}(\mathbf{A})$, the epireflective hull of $\mathbf{A}$, given G, one takes S to be a "skeleton-set" of epis in $\mathbf{C}(G, \mathbf{A})$. Then

$$\begin{array}{ccc} G & \xrightarrow{\quad\Delta\quad} & P = \prod_S A_s \\ & \searrow_{\epsilon} \quad K \quad \nearrow_{\mu} & \end{array}$$

where A_s is the codomain of s, $\pi_s \circ \Delta = s$ for each $s \in S$, and $\Delta = \mu\epsilon$ is the (epi-extremal monic)-factorization. Here, Δ is constructed so that $G \xrightarrow{\Delta} P$ is $\mathbf{A}$-extendable, and it follows that $G \xrightarrow{\epsilon} K$ is also $\mathbf{A}$-extendable. (See 37.1 of [22].)

In our present situation, **Arch-ec** is $\mathbf{R}$(**W-ec**), meaning the epireflective hull in **Arch** generated by **W-ec**, by [4]. But we see no reason why, for the construct $G \xrightarrow{\Delta} P$ in 3.2, Δ should be **W-ec**-extendable (in **Arch**).

The point may be "not all is one."

(e) We note a cardinal generalization of 3.2. For α a regular cardinal or the symbol ∞, let $\mathbf{Arch}_\alpha$ (resp., $\mathbf{W}_\alpha$) have object class $|\mathbf{Arch}|$ (resp., $|\mathbf{W}|$), but the morphisms are α-complete **Arch**- (resp., **W**-) morphisms. Then (see [16]) **Arch-ec** (resp., **W-ec**) is monoreflective in $\mathbf{Arch}_\alpha$ (resp., $\mathbf{W}_\alpha$). Denote the reflecting functors by β_α and $\beta_\alpha^{\mathbf{W}}$. ("$\alpha$-complete" is defined as "$<\alpha$-complete," so $\mathbf{Arch}_\omega = \mathbf{Arch}$ and $\beta_\omega = \beta$; likewise for **W**.) 3.2 generalizes to show that, for U coessential, etc., $\beta_\alpha G$ is to be found in $\prod \beta_\alpha^{\mathbf{W}} \left(G/u^\perp, u_1 + u^\perp\right)$. Note that [5] describes $\beta_{\omega_1}^{\mathbf{W}}$ and $\beta_\infty^{\mathbf{W}}$, but for other α, $\beta_\alpha^{\mathbf{W}}$ is a mystery.

(f) Of course, 3.2 is describing a connection between **W-ec** as a reflective subcategory of **W** and **Arch-ec** as a reflective subcategory of **Arch**. Perhaps in that vein a generalization is possible, which might, in the most general terms, go as follows. Let **S** be a mono-, or perhaps merely epi-, reflective subcategory of **W**, and let **R** be the reflective subcategory of **Arch** generated in some way from **S**. Then the **R**-reflection of G is to be found somehow inside some product of **S**-reflections of G/u's. But, the details escape us; too many crucial features of epicomplete objects are used in 3.2.

(g) To recognize a βG from 3.2, in some specific cases there may be a virtue in minimizing the cardinal number $|U|$. The case $|U| = 1$ is mentioned in (b) above and there is information about the case $|U| = \omega$ in [9]. For $G = C_K(Y)$, which is discussed in the next section, it is not hard to see that the minimum $|U|$ is the so-called Lindelöf degree $L(Y)$ (see [13]), but we see no value in this observation.

Of course, we have here a cardinal invariant of archimedean ℓ-groups which one might study, the "coessentiality character," $cG = \min\{|U| : U \text{ is coessential in } G\}$.

(h) Likewise, there may be a virtue in having a *pairwise disjoint* U, but frequently such does not exist: it's easy to see from 4.10 below that $C_K(Y)$ has a disjoint coessential subset if and only if Y is a topological sum $\sum_I Y_i$ of compact Y_i. Then $U = \{\psi_i : i \in I\}$, where ψ_i is the characteristic function of Y_i. We will return to this topic in [8].

4. $B_L(Y) = \beta C_K(Y)$

Let Y be a fixed locally compact space, $C_K = C_K(Y)$ as in the first section, "Y" usually being suppressed. The proof of "$B_L = \beta C_K$" was outlined in Section 1, and we proceed to the details. The terms epi, epicomplete, etc., refer to **Arch**.

4.1 Proposition. B_L is epicomplete.

Proof. A σ-*ideal* in G is an ideal I in G with this property.

$$(\sigma) \qquad \{g_n\} \subseteq I^+, \bigvee^G g_n = g \Longrightarrow g \in I.$$

The result follows from (1), (2), and (3).

(1) If G is epicomplete then so is each of its σ-ideals.
(2) $B(Y)$ is epicomplete.
(3) B_L is a σ-ideal in $B(Y)$.

Recall that an **Arch** object is epicomplete if and only if it is divisible and both conditionally and laterally σ-complete (2.2). Using that, (1) is routine, and (2) follows because existing countable suprema in $B(Y)$ are pointwise. See [5] for further discussion.

For (3), note first that for $U \subseteq V \subseteq Y$, if U is Baire in Y and V is Lindelöf then U must also be Lindelöf. For by covering V with open sets having compact closure and then taking a countable subcover, we get $U \subseteq \bigcup K_n$ for countably many compact $K_n \subseteq Y$. Since U is Baire in Y, each $U \cap K_n$ is Baire in K_n, and a Baire set in a compact space is Lindelöf. Thus $U = \bigcup (U \cap K_n)$ is Lindelöf as well.

To show that B_L is a sub-ℓ-group of $B(Y)$, observe that, since $\operatorname{coz}(f) = \operatorname{coz}(-f) = \operatorname{coz}|f|$, we have for $f_i \in B_L$ that

$$\begin{aligned}
\operatorname{coz}(f_1 + f_2) &= \operatorname{coz}|f_1 + f_2| \subseteq \operatorname{coz}(|f_1| + |f_2|) = \operatorname{coz} f_1 \cup \operatorname{coz} f_2, \\
\operatorname{coz}(f_1 \vee f_2) &= \operatorname{coz}|f_1 \vee f_2| \subseteq \operatorname{coz}(|f_1| \vee |f_2|) = \operatorname{coz} f_1 \cup \operatorname{coz} f_2, \\
\operatorname{coz}(f_1 \wedge f_2) &= \operatorname{coz}|f_1 \wedge f_2| \subseteq \operatorname{coz}(|f_1| \vee |f_2|) = \operatorname{coz} f_1 \cup \operatorname{coz} f_2.
\end{aligned}$$

Since these are Baire subsets of the Lindelöf set $\operatorname{coz} f_1 \cup \operatorname{coz} f_2$, all are Lindelöf and B_L is closed under the ℓ-group operations. Furthermore, $f \geq g \geq 0$ for $f \in B_L$ and $g \in B(Y)$ implies $\operatorname{coz} f \supseteq \operatorname{coz} g$ and $\operatorname{coz} g$ Lindelöf, hence $g \in B_L$. Finally, B_L has property (σ) in $B(Y)$ because, for non-negative functions f_n,

$$f = \bigvee^B f_n \Longrightarrow f(x) = \bigvee f_n(x) \;\; \forall x \in Y \Longrightarrow \operatorname{coz} f = \bigcup \operatorname{coz} f_n.$$

If all $\operatorname{coz} f_n$ are Lindelőf, so is $\operatorname{coz} f$. □

4.2 Proposition. $C_K \leq B_L$ is epic.

We prove this by showing relative uniform density, which we explain briefly. ([25] is the best general reference; see also [6].) In an archimedean ℓ-group B, choose $0 \leq u \in B$. The sequence $\{a_n\}$ converges to b relatively uniformly regulated by u if

$$\forall k \in \mathbb{N}\ \exists n(k) \in \mathbb{N}\ \langle\ n \geq n(k) \Longrightarrow k\,|a_n - b| \leq u\rangle\,;$$

we write $a_n \longrightarrow b(u)$. For B divisible, this is the more familiar

$$\forall \epsilon = \frac{1}{k}\ \exists n(\epsilon) \in \mathbb{N}\ \langle n \geq n(\epsilon) \Longrightarrow |a_n - b| \leq \epsilon u\rangle\,.$$

Ordinary uniform convergence of real-valued functions is this with u the constant function **1**.

For $A \subseteq B$, the iterated pseudo-closures are

$$\begin{aligned} r_0(A,B) &= A, \\ r_1(A,B) &= \left\{b \in B : a_n \longrightarrow b(u) \text{ for some } (a_n) \subseteq A \text{ and some } u \in B^+\right\}, \\ r_\alpha(A,B) &= \begin{cases} r_1(r_\gamma(A,B), B) & \text{if } \alpha = \gamma + 1 \\ \bigcup_{\gamma<\alpha} r_\alpha(A,B) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases} \end{aligned}$$

Then, $r_{\omega_1+1}(A,B) = r_{\omega_1}(A,B)$, and this is denoted $r(A,B)$ and called the relative uniform closure of A in B.

4.3 Theorem ([6])**.**(a) If $A \leq B$ is relatively uniformly dense then it is epic.
(b) For any Tychonoff space X, $C(X) \leq B(X)$ is relatively uniformly dense.
(c) Suppose B is epicomplete. If $A \leq B$ is epic then it is relatively uniformly dense.

We don't need (c) for the proof of 4.2, but it's worth noting, as is the fact that the hypothesis "B is epicomplete" cannot be dropped.

We also need the following.

4.4 Proposition. Suppose S is locally compact and σ-compact. Then $C_K(S)$ is relatively uniformly dense in $C(S)$. In fact, $r_1(C_K(S), C(S)) = C(S)$.

Proof. Write $S = \bigcup_n K_n$, where each K_n is compact and each $K_n \subseteq \operatorname{int} K_{n+1}$. For each n choose $u_n \in C(S)$ with $0 \leq u_n \leq 1$ such that u_n is 1 on K_n and 0 on $S \setminus K_{n+1}$. This is possible since S is normal.

For any $f \in C_0(S)$ (recall that these are the functions vanishing at infinity), $u_n f \longrightarrow f(1)$. This means that $u_n f \longrightarrow f$ uniformly on S. We explain. For $\epsilon > 0$, there is compact $K \subseteq S$ with $|f(x)| \leq \epsilon$ for $x \notin K$. Since $K \subseteq S = \bigcup \operatorname{int} K_n$, there is $n = n(\epsilon)$ with $K \subseteq K_n$. If $m \geq n$, then $|u_m f - f|$ is 0 on K_m and at most ϵ off K_m.

Now let $g \in C(S)$. Since each g is bounded on K_n, it is easy to build a function $u \in C_0(S)$ with $u(x) > 0$ for each $x \in S$ and $ug \in C_0(S)$. As in the previous paragraph, $u_n(ug) \longrightarrow ug(1)$. Now multiply by $\frac{1}{u} \in C(S)$:

$$u_n g = \frac{1}{u}(u_n(ug)) \longrightarrow \frac{1}{u}(ug)\left(\frac{1}{u}\right),$$

so $u_n g \longrightarrow g\left(\frac{1}{u}\right)$. (Checking simple inequalities justifies the notational sleight-of-hand.) Finally, note that each $u_n g \in C_K(S)$. □

4.5 Proof of 4.2.

By 4.3, it suffices to show that $B_L = r(C_K, B_L)$. Suppose $f \in B_L$; since Y is locally compact and Tychonoff, we can cover $\operatorname{coz} f$ by cozero sets U of Y which have compact closure. Each U is cozero in $\overline{U}$, so is locally compact and σ-compact. Since $\operatorname{coz} f$ is Lindelöf, countably many U's cover, too, and their union S is locally compact and σ-compact. Note that S is a cozero set in Y and $f|S \in B(S)$.

We embed $B(S) \hookrightarrow B_L(Y)$ by $g \hookrightarrow g'$, where g' is g on S and 0 off S. Here $g' \in B(Y)$ because S is zero-set embedded (because S is a cozero set, or because S is Lindelöf ([11])), and thus Baire-set embedded. In fact $g' \in B_L$ since $\operatorname{coz} g' = \operatorname{coz} g$, and $\operatorname{coz} g$ is Lindelöf since it is a Baire (in Y) subset of the Lindelöf set S. This is an embedding as an ℓ-group, so that $a_n \longrightarrow b(u)$ in $B(S)$ implies $a'_n \longrightarrow b'(u')$. Thus for any $A \subseteq B(S)$,

$$r(A, B(S))' \subseteq r\left(A', B(S)'\right) \subseteq r(A', B_L).$$

Finally, observe that $C_K(S)' \subseteq C_K(Y)$.

Recall that $f \in B_L$, so that $f|S \in B(S)$; note that $f = (f|S)'$. By 4.3(b), $C(S)$ is relatively uniformly dense in $B(S)$; by 4.4, $C_K(S)$ is relatively uniformly dense in $C(S)$; since the composition of relatively uniformly dense embeddings is relatively uniformly dense (see 1.1 of [6] if needed), $C_K(S)$ is relatively uniformly dense in $B(S)$. So

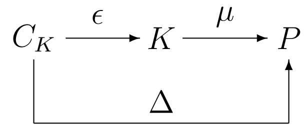
$$f = (f|S)' \in r(C_K(S), B(S))' \subseteq r(C_K, B_L).$$

This completes the proof of 4.2.

We now use 4.1, 4.2, and Theorem A to prove the result in the title of this section.

4.6 Theorem. For any locally compact Y, $C_K(Y) \leq B_L(Y)$ is an ec-monoreflection of $C_K(Y)$.

4.7 Outline of the proof. We shall select coessential $U \subseteq C_K^+$, and correspondingly V, so that the map Δ, as in 3.1 and 3.2, has $C_K \xrightarrow{\epsilon} K$ as ec-monoreflection,



and so that there is an embedding $B_L \xrightarrow{\nu} P$ over C_K with $\nu\alpha = \Delta$, where $C_K \overset{\alpha}{\leq} B_L$. Since B_L is epicomplete (4.1), ν is extremal monic. But then we have two expressions for the (essentially unique) (epi,extremal monic)-factorization of Δ: $\nu\alpha = \Delta = \mu\epsilon$.

Hence there is an isomorphism $K \xrightarrow{d} B_L$ with $\epsilon d = d\nu$. Thus $C_K \leq B_L$ is an ec-monoreflection.

The proof of 4.6 will be completed —- by selecting appropriate U, U_1 and ν — in 4.13, after the necessary facts and tools are assembled. First, we need to know what it means that U be coessential in C_K.

4.8 Proposition. Let $G \in \mathbf{Arch}$.

(a) If $I \subseteq G$ then I is an archimedean kernel of G if and only if I is an ideal which is relatively uniformly closed.

(b) If J is an ideal in G, then so is $r(J, G)$.

(c) For $S \subseteq G$, $\operatorname{ak}_G S = r(\langle S \rangle, G)$ where $\langle S \rangle$ denotes the ideal in G generated by S:

$$\langle S \rangle = \left\{ g \in G : |g| \leq \sum_{1 \leq i \leq n} |s_i|, \; s_i \in S, n \in \mathbb{N} \right\}.$$

(d) If $U \subseteq G^+$ then U is coessential in G if and only if $r(\langle U \rangle, G) = G$.

Proof. For (a) see [25]. (b) is easy, and (c) follows. Then (d) follows using 2.4(e). □

We use the usual notation from [15]: for $s \in \mathbb{R}^X$, $Z(s) = \{x \in X : s(x) = 0\}$ and $\operatorname{coz} s = X \setminus Z(s)$, and for $S \subseteq \mathbb{R}^X$, $Z(S) = \bigcap_S Z(s)$ and $\operatorname{coz} S = \bigcup_S \operatorname{coz} s$.

4.9 Proposition. In the ℓ-group $\mathbb{R}^X$:

(a) If $f_n \longrightarrow f(u)$ then $f_n \longrightarrow f$ pointwise.

(b) Suppose $G \leq \mathbb{R}^X$ and $S \subseteq G$. Then $Z(S) = Z(\operatorname{ak}_G S)$.

(c) Suppose $G \leq \mathbb{R}^X$, $U \subseteq G^+$, and $\operatorname{coz} G = X$. If U is coessential in G then $\operatorname{coz} U = X$.

Proof. (a) is easy and is Lemma 2 of [6]. For (b), $Z(S) \supseteq Z(\operatorname{ak}_G S)$ just because $S \subseteq \operatorname{ak}_G S$. For the reverse inclusion, suppose $x \in Z(S)$. Then $x \in Z(\langle S \rangle)$ from the description of $\langle S \rangle$ in 4.8. By (a), $f(x) = 0$ for every $f \in r_1(\langle S \rangle, G)$; by induction, $f(x) = 0$ for every $f \in r(\langle S \rangle, G)$, which is $\operatorname{ak}_G S$ by 4.8. (c) follows by 2.4(g). □

4.10 Proposition. Consider $G = C_K(Y)$, and let $U \subseteq C_K^+$. U is coessential in C_K if and only if $\operatorname{coz} U = Y$.

Proof. Suppose $\operatorname{coz} U = Y$. We show $\langle U \rangle = C_K$; *a fortiori*, U is coessential. If $f \in C_K$, then $\overline{\operatorname{coz} f}$ is compact, so there are finitely many $u_i \in U$ with $\overline{\operatorname{coz} f} \subseteq \bigcup \operatorname{coz} u_i$. Let $u = \sum u_i \in \langle U \rangle$, so $\operatorname{coz} u = \bigcup \operatorname{coz} u_i$. Since $\overline{\operatorname{coz} f}$ is compact, there is $r > 0$ with $u \geq r$ on $\overline{\operatorname{coz} f}$. Since f is bounded, there is $n \in \mathbb{N}$ with $nu \geq |f|$, hence $f \in \langle U \rangle$.

The converse is 4.9(c). □

4.11 Theorem. ([5] and [26]) Let K be compact. Then, $\beta^{\mathbf{W}}(C(K), 1) = (B(K), 1)$.

4.12 Proposition. Let $G \leq \mathbb{R}^X$ and $u \in G^+$.

(a) $u^{\perp} = \{g \in G : \operatorname{coz} g \cap \operatorname{coz} u = \varnothing\}$.
(b) If $u_1 \in G^+$ has $\operatorname{coz} u_1 \supseteq \overline{\operatorname{coz} u}$ then $u_1 + u^{\perp}$ is a weak unit in $G \diagup u^{\perp}$, and the restriction map effects a **W**-isomorphism from $\left(G \diagup u^{\perp}, u_1 + u^{\perp}\right)$ onto $(G|\overline{\operatorname{coz} u}, u_1|\operatorname{coz} u_1)$, where $G|\overline{\operatorname{coz} u}$ denotes the ℓ-group of restrictions

$$\{g|\overline{\operatorname{coz} u} : g \in G\}\,.$$

(c) If u_1 is 1 on $\overline{\operatorname{coz} u}$ then $\left(G \diagup u^{\perp}, u_1 + u^{\perp}\right)$ is **W**-isomorphic to $(G|\overline{\operatorname{coz} u}\ , 1)$.

(Here (a) is well-known and easily proved, and (b) and (c) follow easily.)

4.13 Proof of 4.6. We keep in mind the outline in 4.7.

Take any $U \subseteq C_K^+$ with $\operatorname{coz} U = Y$ and $0 \leq u \leq 1$ for each $u \in U$. By 4.10, U is coessential.

For each $u \in U$, $\overline{\operatorname{coz} u}$ is compact, so there is $u_1 \in C_K^+$ with $u_1(x) = 1$ for each $x \in \overline{\operatorname{coz} u}$. We elaborate. For each $x \in \overline{\operatorname{coz} u}$ there is open L_x with $x \in L_x$ and $\overline{L_x}$ compact. Finitely many of the L_x's cover $\overline{\operatorname{coz} u}$; their union L is open with $\overline{L} \equiv K$ compact, so $\overline{\operatorname{coz} u} \subseteq \operatorname{int} K \subseteq K$. There is $u_1 \in C(Y)^+$ with u_1 being 1 on $\overline{\operatorname{coz} u}$ and 0 off $\operatorname{int} K$, because a compact set and a disjoint closed set are always completely separated (3.11 of [15]). But $u_1 \in C_K$.

Now $C_K|\overline{\operatorname{coz} u} = C(\overline{\operatorname{coz} u})$. (Any compact set is C-embedded (3.11 of [15]), so any $f \in C(\overline{\operatorname{coz} u})$ extends to $f' \in C(Y)$. Then $u_1 f \in C_K$ is an extension of f. The other inclusion is clear.) Apply 4.12(c): $\left(C_K \diagup u^{\perp}, u_1 + u^{\perp}\right)$ is **W**-isomorphic to $(C(\overline{\operatorname{coz} u}), 1)$. By 4.11, $P_u \equiv \beta^{\mathbf{W}}\left(C_K \diagup u^{\perp}, u_1 + u^{\perp}\right)$ "is" $(B(\overline{\operatorname{coz} u}), 1)$.

Consequently, the $C_K \xrightarrow{\Delta} P = \prod_U P_u$ in 4.7 is (represented as) $P = \prod_U B(\overline{\operatorname{coz} u})$, with $\Delta(g) = (\Delta(g)_u) = (g|\overline{\operatorname{coz} u})$. Then Δ is lifted over $C_K \overset{\alpha}{\leq} B_L$ to $\nu : B_L \longrightarrow P$ defined, obviously, by $\nu(f) = (\nu(f)_u) = (f|\overline{\operatorname{coz} u})$. (The restriction of a Baire function to a subspace is a Baire function on the subspace.)

According to 4.7, the proof of 4.6 is now complete.

4.14 Comments. (a) Referring to the discussion in Section 1 and 3.3(d), 4.6 is equivalent to (4.1, 4.2, and)

$$(*) \qquad C_K(Y) \leq B_L(Y) \text{ is } \mathbf{W\text{-}ec}\text{-extendable in } \mathbf{Arch}.$$

And 4.11 says, among other things, that $C(K) \leq B(K)$ is **W**-**ec**-extendable in **W**, and also in **Arch** by 3.3(b). On the other hand, in [2] we find a locally compact Y for which $C(Y) \leq B(Y)$ fails to be **W**-**ec**-extendable in **W**; i.e., is *not* a **W**-**ec** reflection. This, in light of $(*)$, is somewhat surprising.

(b) The process used here for showing $\beta C_K = B_L$ can surely be applied to any G which is presented as a group of functions in an understandable way, e.g., as in [23], because Theorem A is general and 4.11 and 4.12 are specializations of much more general facts; see [5].

5. $B_L(Y) = \beta C_0(Y)$

We derive this Theorem as a corollary of 4.6 and two simple facts, whose proofs we defer for the moment.

5.1 Proposition. Let $\mathbf{C}$ be a category, $\mathbf{R}$ a full subcategory, and suppose that $G \overset{\alpha}{\longrightarrow} B$ is an epireflection of G into $\mathbf{R}$. If $\alpha = se$,with e epi, as $G \overset{e}{\longrightarrow} A \overset{s}{\longrightarrow} B$, then $A \overset{s}{\longrightarrow} B$ is a reflection of A into $\mathbf{R}$.

5.2 Proposition. Let Y be locally compact. If $f \in C_0(Y)^+$, then $\sqrt{f} \in C_0(Y)$, and there is a sequence $\{g_n\} \subseteq C_K(Y)$ with $g_n \longrightarrow f\left(\sqrt{f}\right)$. So $C_K(Y)$ is relatively uniformly dense and thus epically embedded in $C_0(Y)$.

5.3 Corollary. $B_L(Y)$ is an ec-monoreflection of $C_0(Y)$.

Proof. By 4.6, $C_K \overset{\alpha}{\leq} B_L$ is an ec-reflection, which factors as $C_K \overset{e}{\leq} C_0 \overset{s}{\leq} B_L$, with e epi, by 5.2. The result follows by 5.1. □

5.4 Proof of 5.1. It suffices to show that

$$\varphi \in \mathbf{C}(A, \mathbf{R}) \Longrightarrow \exists \gamma \text{ with } \varphi = \gamma s.$$

Since α is a reflection, there is γ with $\gamma\alpha = \varphi e$. Then $\varphi e = \gamma\alpha = \gamma(se)$, whence $\gamma s = \varphi$, since e is epi.

5.5 Proof of 5.2. (This is somewhat similar to the argument in 4.4.) First, if $h \in C_0^+$, there is a sequence $\{h_n\} \subseteq C_K$ with $\overline{\operatorname{coz} h_n} \subseteq \operatorname{coz} h$ such that $h_n \longrightarrow h^2(h)$. We elaborate. Let $K_n = \left\{x \in Y : h(x) \geq \frac{1}{n}\right\}$; these are compact, since $h \in C_0$. Then choose $u_n \in C(Y)$, $0 \leq u_n \leq 1$, such that u_n is 1 on K_n and 0 off K_{n+1} (by 3.11 of [15]). Then u_n and $u_n h \in C_K$, and $u_n h \longrightarrow h(1)$. Set $h_n = u_n h^2$.

To prove 5.2 let $f \in C_0^+$, and note that $\sqrt{f} \in C_0^+$ also. Now apply the result in the paragraph above, using $h = \sqrt{f}$, to find a sequence $\{h_n\}$ in C_K with $h_n \longrightarrow (\sqrt{f})^2(\sqrt{f})$.

5.6 Remark. Let $C_L(Y) = \{f \in C(Y) : \operatorname{coz} f \text{ is Lindelöf}\}$. It seems likely that $C_K \leq C_L$ is relatively uniformly dense by some argument like 5.5 or 4.4 and thus that $\beta C_L = B_L$, just as 5.3 is proved. But we don't see how to prove that.

6. Concluding Remarks

6.1 Factorization. In view of 4.3 and the central role here of the (epi, extremal monic)-factorization, one might wonder if that factorization of an embedding $A \leq B$, with A divisible and B epicomplete, is

$$A \overset{e}{\leq} r(A, B) \overset{m}{\leq} B.$$

"A divisible" can't be dropped: consider $\mathbb{Z} \leq \mathbb{R}$. "$B$ epicomplete" can't be dropped: there are examples at the end of [6] and in [20]. Here $r(A, B)$ is epicomplete (proof below), so m is extremal monic. So by 4.3, e is epi if and only if $A \leq r(A, B)$ is relatively uniformly dense. But there seems to be no obvious reason for that, because of the loss of regulators upon descent from B to $r(A,B)$.

Proof that $r(A,B)$ is epicomplete. This follows from the more general statement: if $A \leq B$, with A divisible, B epicomplete, and A relatively uniformly closed in B, then A is epicomplete. By 2.2(b), we could show that A is conditionally and laterally σ-complete; it seems easier to use [18] and show that A is relatively uniformly complete and laterally σ-complete. (See [18], also [7] and [25], for discussions of relative uniform completeness.) Since B is epicomplete it is relatively uniformly complete; then, since A is relatively uniformly closed, it is relatively uniformly complete. Now let $\{a_n\}$ be a countable disjoint family in A^+. Then $b = \bigvee a_n$ exists in B. Let $b_n = \bigvee_{i \leq n} a_i$ and $u = \bigvee n a_n$ (in B). 4.3 of [4] says $b_n \longrightarrow b(u)$. So $b \in r(A,B) = A$ since $\{b_n\} \subseteq A$.

6.2 Absolutely relatively uniformly closed. We repeat a question from [6], p. 125. Call A absolutely relatively uniformly closed if

$$A \leq B \text{ in } \mathbf{Arch} \text{ implies } r(A,B) = A.$$

This implies that A is epicomplete, since $A \leq \beta A$ is relatively uniformly dense (4.3). The question is the converse.

6.3 Representation of epicomplete objects It is shown in [5] that (A, e) is **W**-epicomplete if and only if it is **W**-isomorphic to a $(D(X), 1)$, for X compact and basically disconnected. One asks if there is a similar representation of epicomplete objects in **Arch**. (This is, of course closely related to, and perhaps more basic than, the problem treated here of representing $G \leq \beta G$.) More specifically, consider first $B_L(Y)$. Now $B_L(Y) \leq B(Y)$ and $B(Y) \approx D(X)$ as above. Viewing $B_L(Y) \leq D(X)$, $P = Z(B_L(Y))$ is what is called a P-set in X, meaning that the intersection of countably many neighborhoods of P is again a neighborhood, and if we form a quotient of X by collapsing P to a point p, we find that the resulting space X' is still basically disconnected, p is a P-point in X', and

$$B_L(Y) = \{f \in D(X') : f(p) = 0\} \equiv D(X', p).$$

It's fairly easy to see that if X is a compact basically disconnected space with P-point p, then $D(X,p)$ is epicomplete, more-or-less generalizing 4.1. The question is the converse: does G epicomplete imply $G \approx D(X,p)$ for some compact basically disconnected X with P-point p?

We don't know the answer. A sequel to this paper, [8], will consider the issue.

References

[1] M. Anderson, T. Feil, *Lattice-Ordered Groups* (Reidel, Dordrecht, 1988).

[2] R. N. Ball, W. W. Comfort, S. Garcia-Ferreira, A. W. Hager, J. van Mill, L. C. Robertson, ϵ-Spaces, *Rocky Mountain J. Math.* 25 (1995), 867-886.

[3] R. N. Ball, A. W. Hager, Characterization of epimorphisms in Archimedean ℓ-groups and vector lattices, in: A. Glass, W. C. Holland (Eds), *Lattice-Ordered Groups, Advances and Techniques,* (Kluwer Academic Publishers, Dordrecht, 1989), Chapter 8.

[4] R. N. Ball, A. W. Hager, Epicomplete Archimedean ℓ-groups, *Trans. Amer. Math. Soc.* 32 (1990), 459-478.

[5] R. N. Ball, A. W. Hager, Epicompletion of Archimedean ℓ-groups and vector lattices with weak unit,*J. Austral. Math Soc.* 48 (1990), 25-56.

[6] R. N. Ball, A. W. Hager, Relative uniform density of the continuous functions in the Baire functions and of a divisible Archimedean ℓ-group in any epicompletion, *Topology Appl.* 97 (1999), 109-126.

[7] R. N. Ball, A. W. Hager, Algebraic extensions of an archimedean lattice-ordered group, II, *J. Pure and Appl. Alg.* 138 (1999), 197-204.

[8] R. N. Ball, A. W. Hager, D. G. Johnson, A. Kizanis, On the representation of epicomplete archimedean ℓ-groups, work in progress.

[9] R. N. Ball, A. W. Hager, A. Kizanis, Epimorphic adjunction of a weak order unit to an archimedean lattice-ordered group, *Proc. Amer. Math Soc.* 116 (1992) 297-303.

[10] A. Bigard, K. Keimel, S. Wolfenstein, *Groupes et Anneaux Réticulés*, Lecture Notes in Mathematics, Vol. 608 (Springer, Berlin, 1977).

[11] R. L. Blair, A. W. Hager, Extensions of zero-sets and of real-valued functions, *Math. Z.* 136 (1974), 41-58.

[12] W. W. Comfort, S. Negrepontis, *Continuous Pseudometrics,* (Dekker, New York, 1975).

[13] R. Engelking, *General Topology,* (Heldermann, Berlin, 1989).

[14] D. H. Fremlin, Inextensible, Riesz spaces, *Math. Proc. Cambridge Phil. Soc.* 77(1975), 71-89.

[15] L. Gillman, M. Jerison, *Rings of Continuous Functions* (Van Nostrand, New York, 1960, reprinted as: Graduate Texts, Vol. 43, Springer, Berlin, 1976).

[16] A. W. Hager, A. Kizanis, Certain extensions of α-complete homomorphisms in archimedean lattice-ordered groups, *J. Austral. Math. Soc.* 62 (1997), 239-258.

[17] A. W. Hager, J. Martinez, Maximum Monoreflections, *Appl. Cat. Str.* 2 (1994), 315-329.

[18] A. W. Hager, J. Martinez, α-projectable and laterally α-complete archimedean lattice-ordered groups, *Proc. Conf. in Memory of T. Retta, Temple, PA/Addis-Ababa, 1995;* S. Berhanu, Ed.; Ethiopian Jour. Sci. (1996), 73-84.

[19] A. W. Hager, J. Martinez, Hulls for various kinds of α-completeness in archimedean lattice-ordered groups, *Order* 16 (1999), 89-103.

[20] A. W. Hager, J. Martinez, **C**-epic compactifications, *Topology Appl.* 117 (2002), 113-138.

[21] M. Henriksen, J. R. Isbell, D. G. Johnson, Residue class fields of lattice-ordered algebras, *Fund. Math.* L(1961), 107-117.

[22] H. Herrlich, G. Strecker, *Category Theory* (Allyn and Bacon, Boston, 1973).

[23] D. G. Johnson, On a representation theory for a class of lattice-ordered rings, *Proc. London Math Soc.* (3) 12(1962), 207-226.

[24] A. Kizanis, Weak units in epicompletions of lattice-ordered groups, *Rocky Mountain J. Math. 30 (2000), 1323-1341.*

[25] W. Luxemburg, A. Zaanen, *Riesz Spaces, Vol. 1* (North-Holland, Amsterdam, 1971).

[26] J. J. Madden, J. Vermeer, Epicomplete Archimedean ℓ-groups via a localic Yosida theorem, *J. Pure Appl. Alg.* 68 (1990), 243-252.

[27] Z. Tzeng, Extended real-valued functions and the projective resolution of a compact Hausdorff space, Thesis, Wesleyan Univ., 1970.

(Ball) Department of Mathematics, University of Denver, Denver, CO 80208, USA
E-mail address: rball@du.edu

(Hager) Department of Mathematics, Wesleyan University, Middletown, CT 06457, USA
E-mail address: ahager@wesleyan.edu

(Johnson) 15 Barclay Ct., Somerset, NJ 08873, USA
E-mail address: dgjohnson@member.ams.org

(Kizanis) Department of Mathematics, Western New England College, 1215 Wilbraham Road, Springfield, MA 01119, USA
E-mail address: akizanis@wnec.edu