THE VARIETIES OF QUASIGROUPS OF BOL-MOUFANG TYPE: AN EQUATIONAL REASONING APPROACH

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Abstract. A quasigroup identity is of Bol-Moufang type if two of its three variables occur once on each side, the third variable occurs twice on each side, the order in which the variables appear on both sides is the same, and the only binary operation used is the multiplication, viz. $((xy)x)z = x(y(xz))$. Many well-known varieties of quasigroups are of Bol-Moufang type. We show that there are exactly 26 such varieties, determine all inclusions between them, and provide all necessary counterexamples. We also determine which of these varieties consist of loops or one-sided loops, and fully describe the varieties of commutative quasigroups of Bol-Moufang type. Some of the proofs are computer-generated.

1. Introduction

The purpose of this paper is twofold: to provide the classification of varieties of quasigroups of Bol-Moufang type, and to demonstrate that the equational reasoning and finite model builder software currently available is powerful enough to answer questions of interest in mathematics.

Since we hope to attract the attention of both mathematicians and computer scientists, we give the necessary background for both groups.

Recall that a set X of universal algebras of the same type is a *variety* if it is closed under products, subalgebras, and homomorphic images. Equivalently, X is a variety if and only if it consists of all universal algebras of the same type satisfying some identities.

Generally speaking, to establish inclusions between varieties (or sets), it suffices to use only two types of arguments:

(i) Given varieties $\mathcal{A}$, $\mathcal{B}$, show that $\mathcal{A} \subseteq \mathcal{B}$.
(ii) Given varieties $\mathcal{A}$, $\mathcal{B}$, decide if there is $C \in \mathcal{B} \setminus \mathcal{A}$.

Throughout this paper, the varieties will be varieties of quasigroups defined by a single Bol-Moufang identity. Thus, if $i_{\mathcal{A}}$ is the identity defining $\mathcal{A}$, and $i_{\mathcal{B}}$ is the identity defining $\mathcal{B}$, then to establish (i) it suffices to show that $i_{\mathcal{A}}$ implies $i_{\mathcal{B}}$. We use the equational reasoning tool `Otter` [7] to assist with some of these proofs. As for (ii), we use the finite model builder `Mace4` [7] to find algebras C in $\mathcal{B} \setminus \mathcal{A}$. We call such algebras C *distinguishing examples.*

The computer-generated `Otter` proofs are cumbersome and difficult to read; we do not include them in this paper. However, all have been carefully organized and are available electronically at [10]. To be able to read and understand [10], see Section 3.

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2. Quasigroups of Bol-Moufang type

A set Q with binary operation $\cdot$ is a *quasigroup* if the equation $a \cdot b = c$ has a unique solution in Q whenever two of the three elements a, b, $c \in Q$ are given. Note that multiplication tables of finite quasigroups are exactly Latin squares.

An element $e \in Q$ is called the *left* (*right*) *neutral element* of Q if $e \cdot a = a$ ($a \cdot e = a$) holds for every $a \in Q$. An element $e \in Q$ is the *neutral element* if $e \cdot a = a \cdot e = a$ holds for every $a \in Q$. In this paper, we use the term *left* (*right*) *loop* for a quasigroup with a left (right) neutral element. A *loop* is a quasigroup with a neutral element. Hence loops are precisely 'not necessarily associative groups', as can also be seen from the lattice of varieties depicted in Figure 1. (Recall that a *semigroup* is an associative groupoid, and a *monoid* is a semigroup with a neutral element.)

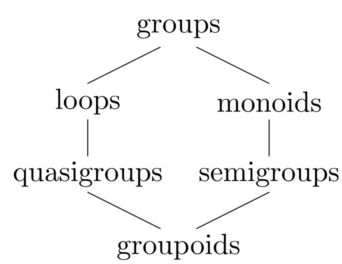

Figure 1. From groupoids to groups

The above definition of a quasigroup cannot be written in terms of identities, as it involves existential quantifiers. Fortunately, as is the tradition in the field, it is possible to introduce a certain kind of universal algebra with 3 binary operations axiomatized by identities (hence forming a variety) that describes the same objects.

The *variety of quasigroups* consists of universal algebras $(Q, \cdot, \backslash, /)$ whose binary operations $\cdot$, $/$, $\backslash$ satisfy

$$a \cdot (a \backslash b) = b, \quad (b/a) \cdot a = b, \quad a \backslash (a \cdot b) = b, \quad (b \cdot a)/a = b.$$

It is customary to think of $a \backslash b$ as division of b by a on the left, and of a/b as division of a by b on the right. Note that $a \backslash b$ is the unique solution to the equation $ax = b$, and similarly for b/a.

This latter description of quasigroups is necessary if one wants to work with equational reasoning software, such as `Otter`.

We are coming to the title definition of this paper. An identity $\alpha = \beta$ is of *Bol-Moufang type* if (i) the only operation in α, β is $\cdot$, (ii) the same 3 variables appear on both sides, in the same order, (iii) one of the variables appears twice on both sides, (iv) the remaining two variables appear once on both sides. For instance, $(x \cdot y) \cdot (x \cdot z) = x \cdot ((y \cdot x) \cdot z)$ is an identity of Bol-Moufang type. A systematic notation for identities of Bol Moufang type was introduced in [13], and will be reviewed in Section 4.

A variety of quasigroups (loops) is said to be of *Bol-Moufang* type if it is defined by one additional identity of Bol-Moufang type.

We say that two identities (of Bol-Moufang type) are *equivalent* if they define the same variety. This definition must be understood relative to some underlying variety, since two identities can be equivalent for loops but not for quasigroups, as we shall see.

Several well-known varieties of loops are of Bol-Moufang type. Their classification was initiated by Fenyves in [2], [3], and completed by the authors in [13]. None of these three papers required computer calculations. However, shortly after this current project was undertaken, it became obvious that the situation for varieties of quasigroups of Bol-Moufang type is more intricate and complex. That is why we opted for presenting the results in this format, with the lengthier computer-generated proofs omitted.

3. Otter and Mace4

Our investigations were aided by the automated reasoning tool `Otter` and by the finite model builder `Mace4` [7], both developed by William McCune (Argonne). `Otter` implements the Knuth-Bendix algorithm, and has proven to be effective at equational reasoning [6]. See [8] for more about `Otter`'s technical specifications, as well as links to results assisted by `Otter` and `Mace4`. A self-contained, elementary introduction to `Otter` can be found in [12].

Many authors simply use the `Otter` output file as the proof of a theorem; it is common practice to publish untranslated `Otter` proofs [8]. This is mathematically sound since the program can be made to output a simple *proof object*, which can be independently verified by a short `lisp` program. We have posted all proofs omitted in this paper at [10]. In the proofs at [10], "para_into" and "para_from" are short for "paramodulation into" and "paramodulation from", and they are the key steps in any `Otter` proof. Very crudely, paramodulation is an inference rule that combines variable instantiation and equality substitution into one step [8].

The proofs generated by `Otter` contain all information necessary for their translation into human language; nevertheless, they are not easy to read. This is because `Otter` often performs several nontrivial substitutions at once. Many of the proofs can be made significantly shorter, especially with some knowledge of the subject available, however, some proofs appear to be rather clever even after being translated. In other words, one often obtains no insight into the problem upon seing the `Otter` proof. We would be happy to see more intuitive proofs, but we did not feel that they are necessary for our programme. If the reader wants to come up with such proofs, he/she should be aware of the standard techniques of the field, such as autotopisms, pseudo-automorphisms, and their calculus [1], [11].

`Mace4` is a typical finite model builder. Thus, given a finite set of identities (or their negations), it attempts to construct a universal algebra satisfying all of the identities. Given the huge number of nonisomorphic (or nonisotopic) quasigroups of even small orders (cf. [11, p. 61]), it is not easy to construct such examples by hand, without some theory. `Mace4` was therefore invaluable for the purposes of this work, specifically Section 10.

4. Systematic notation

The following notational conventions will be used throughout the paper. We omit $\cdot$ while multiplying two elements (eg $x \cdot y = xy$), and reserve $\cdot$ to indicate priority of multiplication (eg $x \cdot yz = x(yz)$). Also, we declare $\backslash$ and $/$ to be less binding than the omitted multiplication (eg $x/yz = x/(yz)$), and if $\cdot$ is used, we consider it to be less binding than any other operation (eg $x \cdot yz \backslash y = x((yz) \backslash y)$).

Let x, y, z be all the variables appearing in an identity of Bol-Moufang type. Without loss of generality, we can assume that they appear in the terms in alphabetical order. Then there are exactly 6 ways in which the 3 variables can form a word of length 4, and there are exactly 5 ways in which a word of length 4 can be bracketed, namely:

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	$o(o(oo))$
2	$o((oo)o)$
3	$(oo)(oo)$
4	$(o(oo))o$
5	$((oo)o)o$

Let Xij with $X \in \{A, \dots, F\}$, $1 \le i < j \le 5$ be the identity whose variables are ordered according to X, whose left-hand side is bracketed according to i, and whose right-hand side is bracketed according to j. For instance, $C25$ is the identity $x((yy)z) = ((xy)y)z$.

It is now clear that any identity of Bol-Moufang type can be transformed into some identity Xij by renaming the variables and interchanging the left-hand side with the right-hand side. There are therefore $6 \cdot (4+3+2+1) = 60$ "different" identities of Bol-Moufang type, as noted already in [3], [4], [13].

The *dual* of an identity I is the identity obtained from I by reading it backwards, i.e., from right to left. For instance, the dual of $(xy)(xz) = ((xy)x)z$ is the identity $z(x(yx)) = (zx)(yx)$. With the above conventions in mind, we can rewrite the latter identity as $x(y(zy)) = (xy)(zy)$. One can therefore identify the dual of any identity Xij with some identity $(Xij)' = X'j'i'$. The name $X'j'i'$ of the dual of Xij is easily calculated with the help of the following rules:

$$A' = F, \quad B' = E, \quad C' = C, \quad D' = D, \quad 1' = 5, \quad 2' = 4, \quad 3' = 3.$$

5. Canonical definitions of some varieties of quasigroups

Table 1 defines 26 varieties of quasigroups. As we shall see, these varieties form the complete list of quasigroup varieties of Bol-Moufang type.

We have carefully chosen the defining identities in such a way that they are either self-dual (GR, EQ, CQ, FQ, MNQ) or coupled into dual pairs (Lx′=Rx). The only exception to this rule is the Moufang identity $D34$. We will often appeal to this duality.

The reasoning behind the names of the new varieties in Table 1 is as follows: If any of the quasigroups LGi, RGi is a loop, it becomes a group; if any of the quasigroups LCi, RCi is a loop, it becomes an LC-loop, RC-loop, respectively. All of this will be clarified in the next section.

Although we will use the new names and abbreviations of Table 1 for the rest of the paper, the reader is warned that other names exist in the literature. Fenyves [3] assigned numbers to the 60 identities of Bol-Moufang type in a somewhat random way, and Kunen [4] developed a different systematic notation that does not reflect inclusions between varieties. Our notation is an extension of [13].

6. Loops of Bol-Moufang type

The varieties of loops of Bol-Moufang type were fully described in [13]. The situation is summarized in Figure 2—a Hasse diagram of varieties of loops of Bol-Moufang type. For every variety in the diagram we give: the name of the variety, abbreviation of the name, defining identity, and all equivalent Bol-Moufang identities defining the variety. Inclusions among varieties are indicated by their relative position and connecting lines, as is usual in a Hasse diagram. The higher a variety is in Figure 2, the smaller it is.

We use Figure 2 as the starting point for our investigation of the quasigroup case.

TABLE 1. Definitions of varieties of quasigroups

variety	abbrev.	defining identity	its name	reference
groups	GR	$x(yz) = (xy)z$		folklore
RG1-quasigroups	RG1	$x((xy)z) = ((xx)y)z$	$A25$	new
LG1-quasigroups	LG1	$x(y(zz)) = (x(yz))z$	$F14$	new
RG2-quasigroups	RG2	$x(x(yz)) = (xx)(yz)$	$A23$	new
LG2-quasigroups	LG2	$(xy)(zz) = (x(yz))z$	$F34$	new
RG3-quasigroups	RG3	$x((yx)z) = ((xy)x)z$	$B25$	new
LG3-quasigroups	LG3	$x(y(zy)) = (x(yz))y$	$E14$	new
extra q.	EQ	$x(y(zx)) = ((xy)z)x$	$D15$	[2]
Moufang q.	MQ	$(xy)(zx) = (x(yz))x$	$D34$	[9]
left Bol q.	LBQ	$x(y(xz)) = (x(yx))z$	$B14$	[14]
right Bol q.	RBQ	$x((yz)y) = ((xy)z)y$	$E25$	[14]
C-quasigroups	CQ	$x(y(yz)) = ((xy)y)z$	$C15$	[3]
LC1-quasigroups	LC1	$(xx)(yz) = (x(xy))z$	$A34$	[3]
LC2-quasigroups	LC2	$x(x(yz)) = (x(xy))z$	$A14$	new
LC3-quasigroups	LC3	$x(x(yz)) = ((xx)y)z$	$A15$	new
LC4-quasigroups	LC4	$x(y(yz)) = (x(yy))z$	$C14$	new
RC1-quasigroups	RC1	$x((yz)z) = (xy)(zz)$	$F23$	[3]
RC2-quasigroups	RC2	$x((yz)z) = ((xy)z)z$	$F25$	new
RC3-quasigroups	RC3	$x(y(zz)) = ((xy)z)z$	$F15$	new
RC4-quasigroups	RC4	$x((yy)z) = ((xy)y)z$	$C25$	new
left alternative q.	LAQ	$x(xy) = (xx)y$		folklore
right alternative q.	RAQ	$x(yy) = (xy)y$		folklore
flexible q.	FQ	$x(yx) = (xy)x$		[11]
left nuclear square q.	LNQ	$(xx)(yz) = ((xx)y)z$	$A35$	new
middle nuclear square q.	MNQ	$x((yy)z) = (x(yy))z$	$C24$	new
right nuclear square q.	RNQ	$x(y(zz)) = (xy)(zz)$	$F13$	new

7. Equivalences

Our first task is to determine which quasigroup varieties of Bol-Moufang type are equivalent. Clearly, if two identities are equivalent for quasigroups, they must also be equivalent for loops, but not *vice versa*.

The results can be found in Figure 3. (Note that the right half of Figure 3 is dual to the left side, with the exception of CQ. The 5 varieties in the middle and CQ are self-dual.) Actually, all sets of equivalent identities in Figure 3 are maximal equivalent sets, but we will not verify the maximality yet. (See Section 8.) Let us go through diagram 3 and comment on it.

We note that there is nothing to show if we list a single identity for a given variety in Figure 3, since we merely need to show that all identities in a given box of Figure 3 are equivalent. This takes care of 18 varieties.

Kunen [5] showed that the identities $B15$, $D23$, $D34$, $E15$ are equivalent for quasigroups, hence settling MQ. He also showed that the three EQ identities $B23$, $D15$, $E34$ are equivalent [4]. We will talk more about these results in Section 8.

Lemma 7.1. *All identities listed under* GR *in Figure* 3 *are equivalent, and imply associativity.*

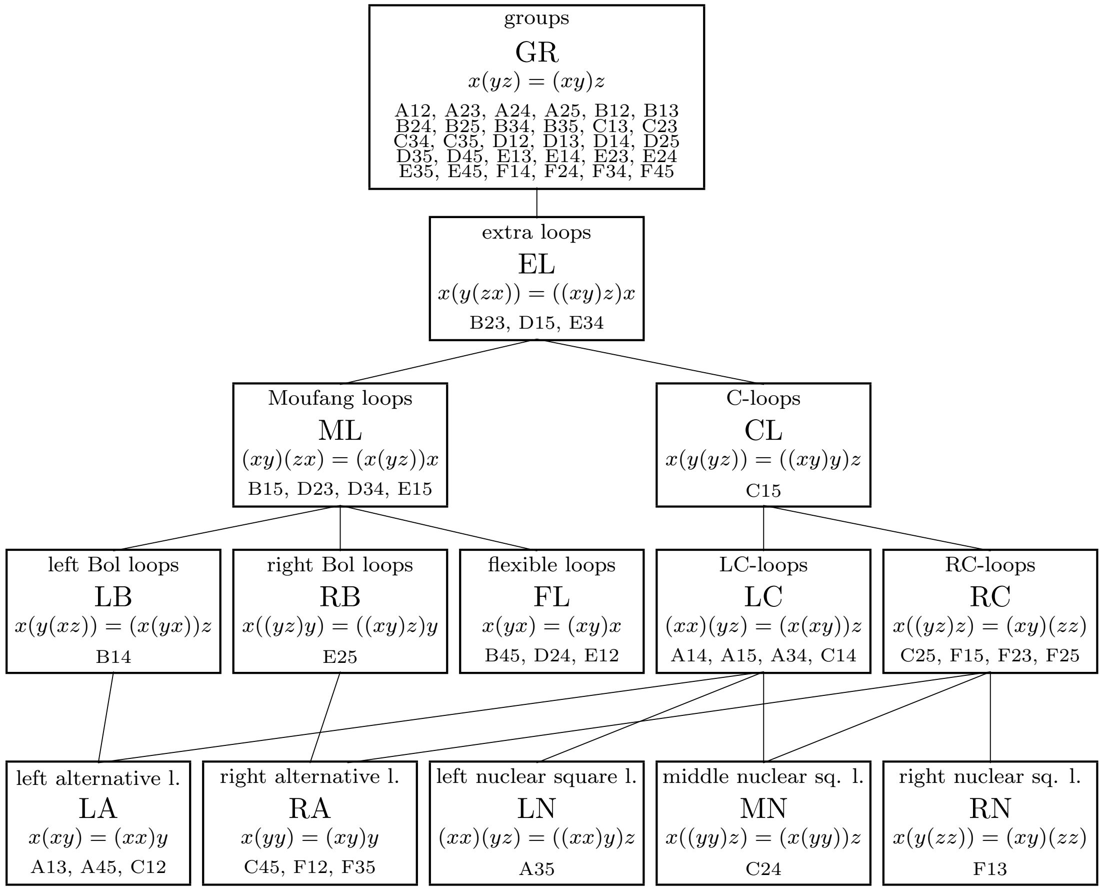

FIGURE 2. Varieties of loops of Bol-Moufang type

Proof. It is very easy to show, just as we did in [13, Proposition 4.1], that all the identities with the exception of $C23$ and $(C23)' = C34$ imply associativity. (To show that $C23$ implies associativity, we took advantage of a neutral element in [13].)

Assume that a quasigroup Q satisfies $C23$. We will first show that Q possesses a right neutral element.

Substitute $(yy) \setminus z$ for z in $C23$ to obtain

$$xz = x(yy \cdot yy \setminus z) = (xy)(y \cdot yy \setminus z). \tag{1}$$

With x/y instead of x and y instead of z we obtain

$$x = (x/y)y = (x/y \cdot y)(y \cdot yy \setminus y) = x(y \cdot yy \setminus y).$$

This means that $y(yy \setminus y) = e$ is independent of y, and is the right neutral element of Q. Since $x(x \setminus x) = x$, too, the right neutral element can be written as $e = x \setminus x = y \setminus y$.

Now, (1) implies

$$xy \setminus xz = y \cdot yy \setminus z, \tag{2}$$

which, with $x \setminus y$ instead of y, implies

$$y \setminus xz = x \setminus y \cdot (x \setminus y)(x \setminus y) \setminus z. \tag{3}$$

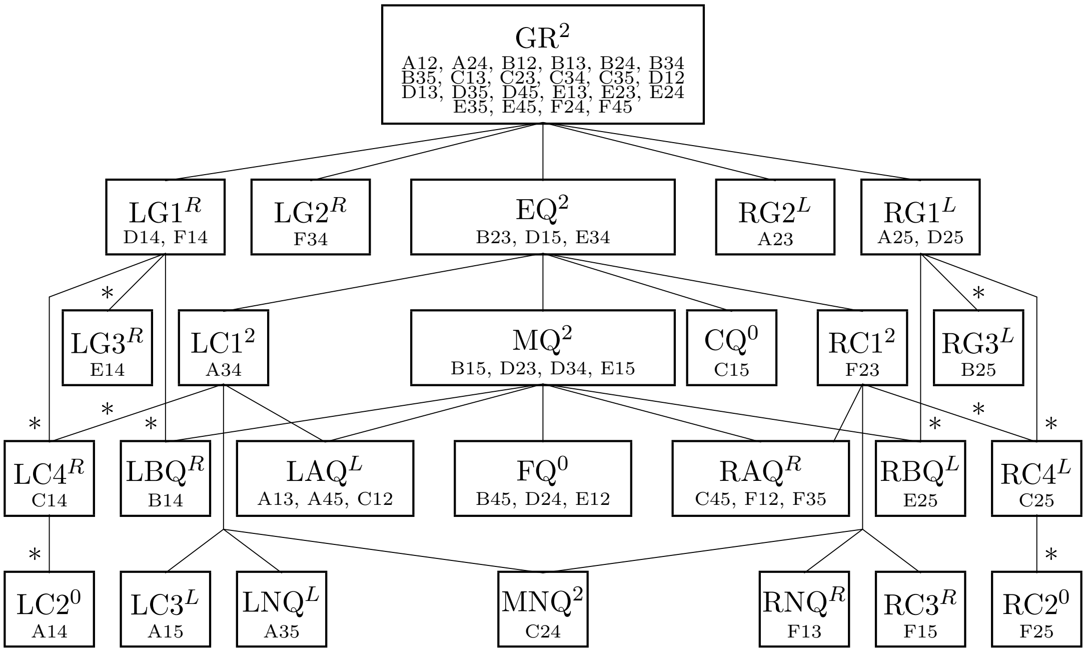

Figure 3. Varieties of quasigroups of Bol-Moufang type

Since $xe = x$, we can use e instead of z in (2) to get

$$xy \setminus x = y(yy \setminus e). \tag{4}$$

Finally, from $e = y(yy \setminus y)$, we have

$$yy \setminus y = y \setminus e. \tag{5}$$

We now show that $e \setminus y = y$, which is equivalent to saying that $y = ey$, or, that e is the neutral element of Q. The associativity of the loop Q then easily follows, just as in [13].

We have $e \setminus y = y(y \setminus e) \setminus y$, by one of the quasigroup axioms. Using (4) with y instead of x and $y \setminus e$ instead of y, we can write

$$e \setminus y = y(y \setminus e) \setminus y = y \setminus e \cdot (y \setminus e)(y \setminus e) \setminus e.$$

By (5), the right-hand side can be written as

$$yy \setminus y \cdot (yy \setminus y)(yy \setminus y) \setminus e.$$

By (3) with yy instead of x and e instead of z, this is equal to $y \setminus (yy)e = y \setminus yy = y$, and we are through. □

Lemma 7.2. *Each of the three* FQ *quasigroup identities $B45$, $D24$, $E12$ is equivalent to flexibility.*

Proof. By duality, it suffices to show that $B45$ is equivalent to flexibility, and that $D24$ is equivalent to flexibility. Cancel z on the right in $B45$ to obtain flexibility. Substituting u for yz in $D24$ yields flexibility. Flexibility clearly implies both $B45$ and $D24$. □

Lemma 7.3. *The* LG1 *identities $D14$ and $F14$ are equivalent for quasigroups.*

Proof. By `Otter`, [10, Thms. 1, 2]. □

Lemma 7.4. *Each of the* LAQ *identities* $A13$, $A45$ *and* $C12$ *is equivalent to the left alternative law.*

Proof. It is easy to see that any of these three identities implies the left alternative law (e.g. let $u = yz$ in $A13$, cancel z on the right in $A45$, cancel x on the left in $C12$). With the left alternative law available, all three identities clearly hold. □

8. Neutral elements in quasigroups of Bol-Moufang type

In 1996, Kunen [5] discovered the surprising fact that a Moufang quasigroup is necessarily a (Moufang) loop. In other words, if a quasigroup Q satisfies any of the Moufang identities (cf. MQ in Figure 3), it possesses a neutral element. He then went on to investigate this property for many quasigroups of Bol-Moufang type [4].

Since we wish to translate the diagram 2 into the quasigroup case, we need to know which of the quasigroup varieties defined in Table 1 are in fact loop varieties. It will also be useful later to know if the varieties consist of one-sided loops.

The results are summarized in Figure 3. The superscript immediately following the abbreviation of the name of the variety $\mathcal{A}$ indicates if:

(2) every quasigroup in $\mathcal{A}$ is a loop,
(L) every quasigroup in $\mathcal{A}$ is a left loop, and there is $Q \in \mathcal{A}$ that is not a loop,
(R) every quasigroup in $\mathcal{A}$ is a right loop, and there is $Q \in \mathcal{A}$ that is not a loop,
(0) there is $Q_L \in \mathcal{A}$ that is not a left loop, and there is $Q_R \in \mathcal{A}$ that is not a right loop.

The justification follows.

8.1. **Neutral elements.** GR consists of groups, hence loops. MQ consists of loops by the above-mentioned result of Kunen [5]. EQ consists of loops by [4]. Every MNQ-quasigroup and every LC1-quasigroup is a loop by [4, Thm. 3.1].

8.2. **One-sided neutral elements.** If one wants to show that a quasigroup Q is a right loop, it suffices to prove that $x \setminus x = y \setminus y$ for every x, $y \in Q$, since then we can set $e = x \setminus x$ for some fixed x, and we get $y \cdot e = y \cdot x \setminus x = y \cdot y \setminus y = y$ for every y. Similarly, Q will be a left loop if $x/x = y/y$ for every x, $y \in Q$.

Lemma 8.1. *Every* LG3*-quasigroup is a right loop. Every* LNQ*-quasigroup is a left loop. Every* LC4*-quasigroup is a right loop. Every* LBQ*-quasigroup is a right loop. Every* LAQ*-quasigroup is a left loop.*

Proof. Let Q be an LG3-quasigroup. Fix x, $y \in Q$ and choose $z \in Q$ so that $y(zy) = y$. Then $xy = x(y(zy)) = x(yz) \cdot y$ by $E14$, and thus $x = x(yz)$, or $x \setminus x = yz$. Since z depends only on y, we see that $x \setminus x$ is independent of x. In other words, $x \setminus x = y \setminus y$ for every x, $y \in Q$.

Let Q be an LNQ-quasigroup. Fix $x \in Q$ and choose $y \in Q$ such that $xx \cdot y = xx$. Then $(xx)(yz) = ((xx)y)z = xx \cdot z$ by $A35$, and thus $yz = z$, or $z/z = y$. Since y does not depend on z, we are done.

Let Q be an LC4-quasigroup. We have $x = (x/yy)(yy) = (x/yy)(y \cdot y(y \setminus y))$, which is by $C14$ equal to $(x/yy \cdot yy)(y \setminus y) = x(y \setminus y)$. Thus $x \setminus x = y \setminus y$ and Q is a right loop.

Let Q be an LBQ-quasigroup. Then $xy = x(y/x \cdot x(x \setminus x))$, which is by $B14$ equal to $(x(y/x \cdot x))(x \setminus x) = (xy)(x \setminus x)$. With $u = xy$, we see that $u = u(x \setminus x)$, or $u \setminus u = x \setminus x$.

Let Q be a quasigroup satisfying the left alternative law $x(xy) = (xx)y$. With $xx \setminus y$ instead of y we obtain $x(x \cdot xx \setminus y) = y$, or

$$xx \setminus y = x \setminus (x \setminus y). \tag{6}$$

The left alternative law also yields $x(xy)/y = xx$, which, upon substituting x/x for x and x for y, becomes $x/x = x/x \cdot x/x$. Then (6) can be used to conclude that $(x/x) \setminus y = (x/x \cdot x/x) \setminus y = (x/x) \setminus ((x/x) \setminus y)$. Upon multiplying this equation on the left by x/x, we get $y = (x/x) \setminus y$, or $x/x = y/y$. □

Lemma 8.2. *Every* LG2*-quasigroup is a right loop. Every* LC3*-quasigroup is a left loop.*

Proof. See [10, Thm. 3] and [10, Thm. 4]. □

8.3. **Missing one-sided neutral elements.** All examples of order n below have elements 0, 1, ..., $n-1$, and their multiplication tables have columns and rows labelled by 0, ..., $n-1$, in this order. Most of the Examples 1–5 will also be used in Section 9.

Example 1. LNQ-quasigroup that is not a right loop:

0	1	2	3
2	0	3	1
1	3	0	2
3	2	1	0

For another example, see $I(2,1,3)$ of [4].

Example 2. FQ- and LC2-quasigroup that is neither a left loop nor a right loop:

0	2	1
2	1	0
1	0	2

Example 3. LG1-, LG2-, LG3- and LBQ-quasigroup that is not a left loop:

0	2	1
1	0	2
2	1	0

Example 4. LC3-quasigroup that is not a right loop:

0	1	2	3	4
2	3	1	4	0
3	0	4	2	1
4	2	0	1	3
1	4	3	0	2

Example 5. A left alternative quasigroup that is not a a right loop.

0	1	2	3	4	5
2	3	4	0	5	1
1	4	3	5	0	2
4	0	5	2	1	3
3	5	0	1	2	4
5	2	1	4	3	0

8.4. **Summary for neutral elements.** We have recalled that LC1, GR, EQ, MQ, MNQ and RC1 consist of loops. Moreover, if the inclusions of Figure 3 are correct (and they are, as we will see in Section 9), we have just established that all varieties in the left half of diagram 3 behave as indicated with respect to neutral element. The right half of the diagram then follows by duality.

9. Implications

Many of the implications in diagram 3 can be established as follows: Let $\mathcal{A}$ be a quasigroup variety consisting of loops, let $\mathcal{B} \supseteq \mathcal{A}$ be a loop variety, and let $\mathcal{C}$ be a quasigroup variety defined by any of the (equivalent defining) identities of $\mathcal{B}$. Then $\mathcal{C} \supseteq \mathcal{B}$, and thus $\mathcal{C} \supseteq \mathcal{A}$. For example: LC1 is contained in LNQ because LC1 is a quasigroup variety consisting of loops by Section 8, and because LC1=LC⊆LN as loops, by [13].

The four inclusions GR⊆LG1, GR⊆LG2, GR⊆RG2, GR⊆RG1 are trivial, as GR implies everything (associativity).

Lemma 9.1. *Every* LC1-*quasigroup is an* LC3-*quasigroup.*

Proof. We know that an LC1-quasigroup is left alternative. Then $x(x \cdot yz) = xx \cdot yz = (x \cdot xy)z = (xx \cdot y)z$, where the middle equality is just $A34$. □

The remaining 10 inclusions are labelled by asterisk (*) in Figure 3, and can be verified by `Otter` as follows:

An LG1-quasigroup is an LG3-quasigroup by [10, Thm. 5], an LG1-quasigroup is an LC4-quasigroup by [10, Thm. 6], an LC4-quasigroup is an LC2-quasigroup by [10, Thm. 7], an LG1-quasigroup is an LBQ-quasigroup by [10, Thm. 8], and an LC1-quasigroup is an LC4-quasigroup by [10, Thm. 9].

The remaining 5 inclusions follow by duality.

At this point, we have justified all inclusions in Figure 3. We have not yet shown that the inclusions are proper (i.e., that the sets of equivalent identities are maximal), and that no inclusions are missing. All of this will be done next.

10. Distinguishing examples

Since there are 26 varieties to be distinguished, we will proceed systematically. Our strategy is as follows:

There are 17 minimal elements (maximal with respect to inclusion) in Figure 3, namely LC2, LG3, LBQ, LC3, LNQ, LG2, LAQ, FQ, MNQ, CQ, RAQ, RG2, RNQ, RBQ, RC3, RG3 and RC2. Assume we want to find $C \in \mathcal{A} \setminus \mathcal{B}$, for some varieties $\mathcal{A}$, $\mathcal{B}$ of Figure 3. It then suffices to find $C \in \mathcal{A} \setminus \mathcal{C}$, where $\mathcal{C}$ is any of the minimal elements below $\mathcal{B}$ (but, of course, not below $\mathcal{A}$). For example, in order to distinguish $LG1$ from the varieties not below LG1 in Figure 3, we only need $14 = 17 - 3$ distinguishing examples.

Assume that given $\mathcal{A}$, all distinguishing examples $C \in \mathcal{A} \backslash \mathcal{B}$ were found. Then our task is much simpler for any $\mathcal{C}$ below $\mathcal{A}$, since we only must look at minimal elements below $\mathcal{A}$ that are not below $\mathcal{C}$. For instance, when all distinguishing examples for LG1 are found, we only need 2 additional distinguishing examples for LG3, namely $C \in LG3 \setminus LC2$ and $D \in LG3 \setminus LBQ$.

We get many distinguishing examples for free by taking advantage of the existence of (one-sided) neutral elements in the varieties under consideration. For instance, there must be a quasigroup in LG1\LC3, as it suffices to take any LG1-quasigroup that is not a left loop. Such examples will be called of *type* 1.

Some of the remaining distinguishing examples can be obtained from the results of [13]. For instance, there must be a quasigroup in LC4\LBQ, because, by [13], there is a loop in LC\LB. Such examples will be called of *type* 2.

Finally, about half of the examples follows by duality. Hence, it suffices to find all distinguishing examples of the form $C \in \mathcal{A} \setminus \mathcal{B}$, where $\mathcal{A}$ is in the left half of Figure

3, including EQ, MQ, FQ, MNQ and CQ (these five varieties are self-dual, as we have already noticed).

It turns out that only 9 additional examples are needed for a complete discussion. Here they are:

Example 6. An LG1-quasigroup with additional properties (see below).

0	2	1	3	5	4
1	4	0	5	3	2
2	0	5	4	1	3
3	5	4	0	2	1
4	1	3	2	0	5
5	3	2	1	4	0

Example 7. An LG1-quasigroup with additional properties (see below).

0	3	4	1	2	5
1	2	5	0	3	4
2	1	0	5	4	3
3	0	1	4	5	2
4	5	2	3	0	1
5	4	3	2	1	0

Example 8. An LC4-quasigroup with additional properties (see below).

0	1	3	2	5	4
1	5	0	4	2	3
2	0	4	5	3	1
3	4	5	0	1	2
4	2	1	3	0	5
5	3	2	1	4	0

Example 9. An LG3-quasigroup with additional properties (see below).

0	2	3	1	4	5
1	0	4	5	2	3
2	4	5	0	3	1
3	5	0	4	1	2
4	3	1	2	5	0
5	1	2	3	0	4

Example 10. An LG2-quasigroup with additional properties (see below).

0	2	3	1
1	3	2	0
2	0	1	3
3	1	0	2

Example 11. An LG2-quasigroup with additional properties (see below).

0	3	1	2
1	2	0	3
2	1	3	0
3	0	2	1

Example 12. An LC3-quasigroup with additional properties (see below).

0	1	2	3	4
3	2	4	1	0
4	3	1	0	2
2	0	3	4	1
1	4	0	2	3

Example 13. An MNQ-quasigroup with additional properties (see below).

0	1	2	3	4
1	0	3	4	2
2	4	0	1	3
3	2	4	0	1
4	3	1	2	0

Example 14. A CQ-quasigroup with additional properties (see below).

1	3	0	2	7	6	4	5
4	2	6	0	5	3	7	1
0	6	2	4	3	5	1	7
7	5	4	6	1	2	0	3
2	0	3	1	6	7	5	4
6	4	5	7	2	1	3	0
3	7	1	5	0	4	2	6
5	1	7	3	4	0	6	2

Here is the systematic search for all distinguishing examples, as outlined in the strategy above.

$\mathcal{A}$ =LG1: Type 1 examples are LG1$\backslash\mathcal{B}$, where $\mathcal{B}\in$ {LC3, LNQ, LAQ, MNQ, RG2, RBQ, RG3}. The remaining examples are: $\mathcal{B}$=LG2 by Example 6 (check that LG2 fails with $x=0$, $y=0$, $z=1$), $\mathcal{B}$=FQ by Example 3 (with $x=1$, $y=0$), $\mathcal{B}$=CQ by Example 3 (with $x=0$, $y=1$, $z=0$), $\mathcal{B}$=RAQ by Example 3 (with $x=0$, $y=1$), $\mathcal{B}$=RNQ by Example 7 (with $x=0$, $y=0$, $z=1$), $\mathcal{B}$=RC3 by Example 3 (with $x=0$, $y=0$, $z=1$), and $\mathcal{B}$=RC2 by Example 3 (with $x=0$, $y=0$, $z=1$).

$\mathcal{A}$ =LC4: Type 2 example is $\mathcal{B}$=LBQ. The remaining example is $\mathcal{B}$=LG3 by Example 8 (with $x=0$, $y=1$, $c=0$).

$\mathcal{A}$ =LC2: With $\mathcal{B}$=LC4 by Example 2 (with $x=1$, $y=0$, $z=0$).

$\mathcal{A}$ =LG3: With $\mathcal{B}\in$ {LC2, LBQ} by Example 9 (both with $x=0$, $y=0$, $z=1$).

$\mathcal{A}$ =LBQ: Type 2 examples are $\mathcal{B}\in$ {LC2, LG3}.

$\mathcal{A}$ =LG2: Type 1 examples are $\mathcal{B}\in$ {LC3, LNQ, LAQ, MNQ, RG2, RBQ, RG3}. The remaining examples are: $\mathcal{B}\in$ {LC2, LBQ, CQ} by Example 10 (all with $x=0$, $y=0$, $z=1$), $\mathcal{B}$=LG3 by the same example (with $x=0$, $y=1$, $z=0$), $\mathcal{B}$=FQ by Example 3 (with $x=1$, $y=0$), $\mathcal{B}$=RAQ by the same example (with $x=0$, $y=1$), $\mathcal{B}\in$ {RC3, RC2} by the same example (both with $x=0$, $y=0$, $z=1$), and $\mathcal{B}$=RNQ by Example 11 (with $x=0$, $y=0$, $z=1$).

$\mathcal{A}$ =LC1: Type 2 examples are $\mathcal{B}\in$ {LG3, LBQ, LG2, FQ, CQ, RAQ, RG2, RNQ, RBQ, RC3, RG3, RC2}.

$\mathcal{A}$ =LC3: Type 1 example is $\mathcal{B}$=MNQ. The remaining examples are: $\mathcal{B}$=LC2 by Example 4 (with $x=1$, $y=0$, $z=0$), $\mathcal{B}$=LAQ by the same example (with $x=1$, $y=0$), and $\mathcal{B}$=LNQ by Example 12 (with $x=1$, $y=0$, $z=0$).

$\mathcal{A}$ =LNQ: Type 1 example is $\mathcal{B}$=MNQ. Type 2 examples are $\mathcal{B}\in$ {LC2, LC3, LAQ}.

$\mathcal{A}$ =LAQ: Type 1 example is $\mathcal{B}$=MNQ. Type 2 examples are $\mathcal{B}\in$ {LC2, LC3, LNQ}.

$\mathcal{A}$ =EQ: Type 2 examples are $\mathcal{B} \in$ {LG3, LG2}.

$\mathcal{A}$ =MQ: Type 2 examples are $\mathcal{B} \in$ {LC2, LC3, LNQ, MNQ, CQ}.

$\mathcal{A}$ =FQ: Type 1 examples are $\mathcal{B} \in$ {LBQ, LAQ}.

$\mathcal{A}$ =MNQ: Type 2 examples are $\mathcal{B} \in$ {LNQ, LAQ}. The remaining examples are: $\mathcal{B} \in$ {LC2, LC3} by Example 13 (both with $x = 1$, $y = 0$, $z = 2$).

$\mathcal{A}$ =CQ: Type 1 examples are $\mathcal{B} \in$ {LC3, LAQ, LNQ, MNQ}. Type 2 example is $\mathcal{B}$=FQ. The remaining example is $\mathcal{B}$=LC2 by Example 14 (with $x = 0$, $y = 1$, $z = 2$).

11. Main result

Theorem 11.1. *There are* 26 *varieties of quasigroups of Bol-Moufang type. Their names and defining identities can be found in Table* 1*. The maximal sets of equivalent identities for these varieties can be found in the Hasse diagram* 3*, together with all inclusions between the varieties. For every variety, we also indicate in Figure* 3 *if it consists of loops, left loops or right loops.*

12. Commutative quasigroups of Bol-Moufang type

We conclude the paper with the classification of varieties of commutative loops of Bol-Moufang type and commutative quasigroups of Bol-Moufang type. With the exception of the comments below, we omit all proofs and distinguishing examples.

Figure 4 depicts the Hasse diagram of commutative loops of Bol-Moufang type. We indicate the equivalent identities defining a given variety by pointing to the noncommutative case. For instance, commutative groups can be defined by any defining identity of GR or EL of Figure 2. Note that flexibility follows from commutativity, as $x(yx) = x(xy) = (xy)x$.

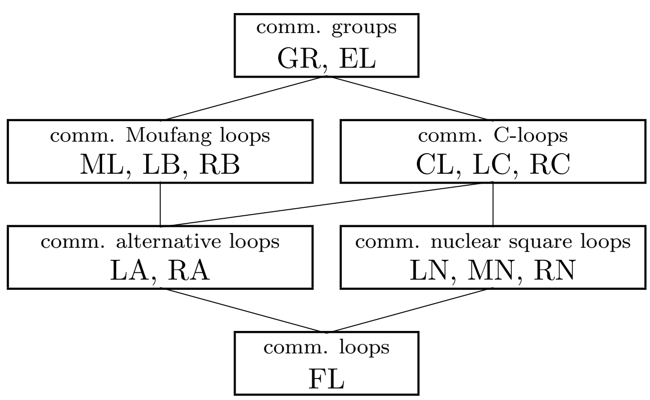

Figure 4. Varieties of commutative loops of Bol-Moufang type

Figure 5 depicts the Hasse diagram of commutative quasigroups of Bol-Moufang type, using conventions analogous to those of Figure 4. Note that if a quasigroup variety possesses a one-sided neutral element, its commutative version consists of loops. Hence only the noncommutative varieties LC2, FQ, CQ and RC2 behave differently than in the commutative loop case. Example 2 gives a commutative C-quasigroup that is not a loop.

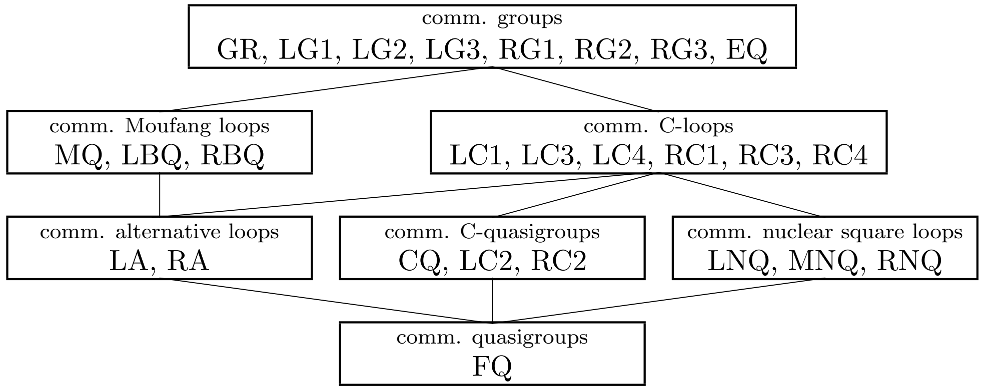

FIGURE 5. Varieties of commutative quasigroups of Bol-Moufang type

13. Acknowledgement

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