MONOMORPHISMS IN SPACES WITH LINDELÖF FILTERS

RICHARD N. BALL AND ANTHONY W. HAGER

Abstract. **SpFi** is the category of spaces with filters: an object is a pair $(X,\mathcal{F})$, X a compact Hausdorff space and $\mathcal{F}$ a filter of dense open subsets of X. A morphism $f:(Y,\mathcal{G})\to(X,\mathcal{F})$ is a continuous function $f:Y\to X$ for which $f^{-1}(F)\in\mathcal{G}$ whenever $F\in\mathcal{F}$. This category arises naturally from considerations in ordered algebra, e.g., Boolean algebra, lattice-ordered groups and rings, and from considerations in general topology, e.g., the theory of the absolute and other covers, locales, and frames, though we shall specifically address only one of these connections here in an appendix. Now we study the categorical monomorphisms in **SpFi**. Of course, these monomorphisms need not be one-to-one. For general **SpFi** we derive a criterion for monicity which is rather inconclusive, but still permits some applications. For the category **LSpFi** of spaces with Lindelöf filters, meaning filters with a base of Lindelöf, or cozero, sets, the criterion becomes a real characterization with several foci ($C(X)$, Baire sets, etc.), and yielding a full description of the monofine coreflection and a classification of all the subobjects of a given $(X,\mathcal{F})\in$ **LSpFi**. Considerable attempt is made to keep the discussion "topological," i.e., within **SpFi**, and to not get involved with, e.g., frames. On the other hand, we do not try to avoid Stone duality. An appendix discusses epimorphisms in archimedean ℓ-groups with unit, roughly dual to monics in **LSpFi**.

Contents

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Part I. General spaces with filters

0. Introduction and Preliminaries

The category **SpFi** was defined in the abstract. We first defined it in [3, p. 183], in a slightly and inconsequentially different way, in connection with a roughly dual problem in ℓ-group theory. Further account of the meager literature specifically on, or closely related to, **SpFi** requires some terminology.

0.1. Main references for topology are [**?**] and [13]. All topological spaces are completely regular Hausdorff, usually compact. **Comp** is the category of compact (Hausdorff) spaces with continuous maps. For X a space, $C(X)$ is the set (or ℓ-group, vector lattice, ring, ℓ-ring,...) of continuous real-valued functions on X. The *cozero set of* $f \in C(X)$ is $\operatorname{coz} f \equiv \{x : f(x) \neq 0\}$, and $\operatorname{coz} X \equiv \{\operatorname{coz} f : f \in C(X)\}$. Each cozero set is an F_σ, hence Lindelöf when $X \in |\mathbf{Comp}|$.

Let α be a regular cardinal or the symbol ∞, thought of as larger than every cardinal. In a space X, an *α-cozero set* is the union of strictly fewer than α cozero sets, so an ω_1-cozero set is a cozero set, and an ∞-cozero set is simply an open set. When $X \in |\mathbf{Comp}|$ each α-cozero set is *α-Lindelöf*, meaning that each open cover has a subcover consisting of strictly fewer than α sets. So ω_1-Lindelöf means Lindelöf.

For $(X, \mathcal{F}) \in |\mathbf{SpFi}|$, we say that the filter $\mathcal{F}$ is *α-Lindelöf* if $\mathcal{F}$ has a base of α-cozero sets, and $\alpha\mathbf{SpFi}$ is the full subcategory of **SpFi** whose objects have α-Lindelöf filters. Thus

$$\mathbf{SpFi} = \infty\mathbf{SpFi} = \bigcup_{\alpha<\infty} \alpha\mathbf{SpFi}.$$

We use the alternate notation **LSpFi** for $\omega_1\mathbf{SpFi}$; this category is the primary domain of this paper.

For $X \in |\mathbf{Comp}|$, $\mathcal{G}_\alpha(X)$ is the filter of dense open sets generated by the dense α-cozero sets. For the resulting $(X, \mathcal{G}_\alpha(X)) \in |\mathbf{SpFi}|$ we usually write $(X, \mathcal{G}_\alpha)$, and for $(X, \mathcal{G}_{\omega_1})$ we write $(X, \mathcal{C})$. Fixing $X \in |\mathbf{Comp}|$, $(X, \mathcal{G}_\alpha) \in |\alpha\mathbf{SpFi}|$ and $\mathcal{G}_\alpha$ is the finest such filter. α**-SpFi** stands for the full subcategory of **SpFi**, or of $\alpha\mathbf{SpFi}$, whose objects are the $(X, \mathcal{G}_\alpha)$'s. Thus ∞**-SpFi** is the category of compact spaces with skeletal maps ([17], [27], [35]).

In a general category, a monomorphism, or by abusing language, monic, is a morphism m which is left-cancellable: $mf = mg$ implies $f = g$. A one-to-one map is monic in **SpFi** since it is monic in **Comp** since it is monic in **Sets**, but hardly conversely. (Thus the present article.) And it is easy to see that m is **SpFi**-monic iff it is $\alpha\mathbf{SpFi}$-monic for some $\alpha < \infty$. More precisely, let $m \in \mathbf{SpFi}$. Then

$m \in \alpha\mathbf{SpFi}$ for some $\alpha < \infty$, and with respect to any such α, m is **SpFi**-monic iff m is $\alpha\mathbf{SpFi}$-monic.

0.2. We indicate connections of **SpFi** and **LSpFi** with some other categories. This isn't intended to be a primer on the other categories, nor on these connections, but just to suggest the topic of **SpFi**-monics is of wider significance, and that our results here have various other interesting interpretations.

First there are functors

$$\mathbf{SpFi} \underset{\beta}{\overset{\cap}{\rightleftarrows}} \mathbf{Loc} \overset{\text{op}}{\approx} \mathbf{Frm}$$

detailed in [7], in which **Loc** is the category of completely regular locales, **Frm** is its opposite category of completely regular frames, and $[\beta, \cap]$ is an adjunction. By virtue of this,

0.2.1. m is **SpFi**-monic iff $\cap m$ is **Loc**-monic iff $(\cap m)^{\text{op}}$ is **Frm**-epic. Then, letting $\alpha\mathbf{Frm}$ be the full subcategory of **Frm** whose objects are α-Lindelöf, with $\alpha\mathbf{Loc} \equiv (\alpha\mathbf{Frm})^{\text{op}}$, the pair $[\beta, \cap]$ "respects α" and, abusing notation, we have

$$\alpha\mathbf{SpFi} \underset{\beta}{\overset{\cap}{\rightleftarrows}} \alpha\mathbf{Loc} \overset{\text{op}}{\approx} \alpha\mathbf{Frm},$$

$[\beta, \cap]$ still being an adjunction. So

0.2.2. m is $\alpha\mathbf{SpFi}$-monic iff $\cap m$ is $\alpha\mathbf{Loc}$-monic iff $(\cap m)^{\text{op}}$ is $\alpha\mathbf{Frm}$-epic. (The process of this paragraph was described in the lecture [14].) Then there is the categorical equivalence

$$\alpha\mathbf{Frm} \underset{F}{\overset{E}{\rightleftarrows}} \alpha\text{-}\mathbf{Frm},$$

α-**Frm** being the category of completely regular α-frames described in 4.3 of [24], so

0.2.3. ϕ is $\alpha\mathbf{Frm}$-epic iff $E(\phi)$ is α-**Frm**-epic.

0.2.4. Now for $\alpha < \infty$, α-**Frm**-epics are shown in 5.2 of [24] to be the morphisms which become surjective when lifted over the Boolean reflection in α-**Frm**. This has a formal topological, i.e. **SpFi**, equivalent which we state in §5 below. We painstakingly explain what that means for $\alpha = \omega_1$ in §6–9 below, independent of the various apparati of frame theory alluded to above. This is possible because we can write down what the Boolean reflection in ω_1-**Frm** is and, in effect, we do below. But for $\alpha > \omega_1$ the situation is opaque.

Finally, in this vein, [25] presents a characterization of **Frm**-epics (and complete regularity isn't needed) which, from our point of view, seems to be an amalgam of the version in **Frm** of 0.1, 0.2, and 0.3 above. This is, in some sense, fairly simple *quā* frames, etc., but what it means topologically, i.e., in **SpFi**, is unclear.

We note that Molitor's thesis [28] presents an elegant and readable account of the functor $\cap : \mathbf{SpFi} \overset{\cap}{\to} \mathbf{Loc}$ in Chapter 4, and of the situation with monomorphisms in Chapter 5, which we have outlined above.

0.3. Finally, we recall our original motivation for defining and studying **SpFi**, the application to problems in ℓ-groups via the "**SpFi** Yosida functor" $\mathbf{W} \overset{SY}{\to} \mathbf{LSpFi}$, $\mathbf{W}$ being the category of archimedean ℓ-groups with distinguished weak unit. Here, for $G \in |\mathbf{W}|$, one begins with the usual Yosida representation of G on $YG \in \mathbf{Comp}$,

$$G \approx \widehat{G} \subseteq D\left(YG\right) \equiv \left\{ f \in C\left(YG, R \cup \{\pm\infty\}\right) : f^{-1}R \text{ dense in } YG \right\}.$$

Thus $\left\{ \widehat{g}^{-1}R : g \in G \right\}$ generates a Lindelöf filter $\mathcal{F}_G$ on YG, and we set $(SY)(G) \equiv (YG, \mathcal{F}_G)$. It devolves that φ is $\mathbf{W}$-epic iff $(SY)(\phi)$ is **LSpFi**-monic, though SY is not half of an appropriate adjunction. We shall return to this point later.

A number of the results in the present paper were announced without proof in [2], and actually, most of the present proofs are vast improvements over those there envisioned. Also, the reader familiar with our paper [5] about **W**-epicompletions will notice an overlap of technicalities there and here, mostly regarding Baire sets. We apologize for the repetition, but it seems necessary to our present goal of a somewhat self-contained and readable topological treatment. (We admit that we have not avoided Stone duality, and Stone duality can be viewed as a starting point for all the functors in §0.2.)

0.4. **Acknowledgement.** During roughly 1987–1992, there was considerable dialogue between and among the present authors and the second author's students, Anthony Macula and Andrew Molitor, about **SpFi**, locales, and ℓ-groups.

1. A basic construction: subspaces

We "reduce" a continuous map f into a **SpFi**-object to a **SpFi**-morphism. The case of "subspaces" occurs when f is a topological inclusion.

Let $(X, \mathcal{F}) \in |\mathbf{SpFi}|$, and let $X \overset{f}{\leftarrow} Y$ be continuous. Here and in what follows we use $\mathcal{F}_\delta$ to denote the filter on X generated by countable intersections of sets from $\mathcal{F}$. Let $Y_0 \equiv Y$, let

$$Y_{\alpha+1} \equiv \bigcap_{E \in \mathcal{F}_\delta} \overline{f^{-1}(E) \cap Y_\alpha},$$

and for a limit ordinal β let $Y_\beta \equiv \bigcap_{\alpha<\beta} Y_\alpha$. Let $f_\alpha \equiv f|Y_\alpha$ for each α. The process must terminate, perhaps in $\emptyset$, for there are ordinals α for which $Y_{\alpha+1} = Y_\alpha$. For any, or the first, such ordinal α, let $Y_\infty \equiv Y_\alpha$ and $f_\infty \equiv f_\alpha$.

Proposition 1.1. (1) *$Y_\alpha = Y_{\alpha+1}$ $(= Y_\infty)$ iff $f^{-1}(F) \cap Y_\alpha$ is dense in Y_α for each $F \in \mathcal{F}$.*

(2) *$\left\{ f_\infty^{-1}F : F \in \mathcal{F} \right\}$ is the base for a filter of dense open sets in Y_∞, denoted $f_\infty^{-1}\mathcal{F}$, and $(X, \mathcal{F}) \overset{f_\infty}{\leftarrow} \left(Y_\infty, f_\infty^{-1}\mathcal{F}\right) \in \mathbf{SpFi}$.*

(3) *Y_∞ is the largest among topological subspaces W of Y with the property: $f^{-1}(F) \cap W$ is dense in W for each $F \in \mathcal{F}$.*

Proof. (1). Clearly $Y_\alpha = Y_{\alpha+1}$ iff $f^{-1}E \cap Y_\alpha$ is dense in Y_α for all $E \in \mathcal{F}_\delta$, which of course implies $f^{-1}F \cap Y_\alpha$ dense in Y_α for all $F \in \mathcal{F}$. The converse holds by the Baire Category Theorem.

(2) follows from (1).

(3). If $W = \overline{W} \subseteq Y$, define W_α for all α with respect to $X \overset{f|W}{\leftarrow} W$. Clearly $W_\alpha \subseteq Y_\alpha$ for all α. Let

$$\mathcal{W} \equiv \left\{ W \subseteq Y : f^{-1}F \cap W \text{ is dense in } W \text{ for all } F \in \mathcal{F} \right\}.$$

Then $W \in \mathcal{W}$ implies $\overline{W} \in \mathcal{W}$, and $\overline{W} \in \mathcal{W}$ means $\overline{W} = \left(\overline{W}\right)_\alpha$ for all α, whence

$$W \subseteq \overline{W} = \bigcap_\alpha \left(\overline{W}\right)_\alpha \subseteq \bigcap_\alpha Y_\alpha = Y_\infty.$$

This completes the proof. □

The crux of the problem of understanding monics in **SpFi** is in the process

$$Y \supseteq \cdots Y_\alpha \supseteq \cdots Y_\infty.$$

We focus on "subspaces" immediately below, but the additional generality of 1.1 is regained in §8.

1.2. **Subspaces.** Let $(X, F) \in |\mathbf{SpFi}|$. To say that $S \in \operatorname{sub}(X, \mathcal{F})$ is to say that S is a closed subspace of X with the property that $S \cap F$ is dense in S for all $F \in \mathcal{F}$. Then $\mathcal{F} \cap S \equiv \{F \cap S : F \in \mathcal{F}\}$ is a filter of dense open subsets of S, and the inclusion $X \hookleftarrow S$ is a **SpFi**-morphism as $(X, \mathcal{F}) \hookleftarrow (S, \mathcal{F} \cap S)$.

For general closed T in X, we apply the process in 1.1 to the inclusion $X \hookleftarrow T$, and relabel T_∞ as T'. From 1.1 we have

1.2.1. $T = T'$ iff $T = T_1$ iff $T \cap E$ is dense in T for all $E \in \mathcal{F}_\delta$.

1.2.2. T' is the largest member of $\operatorname{sub}(X, \mathcal{F})$ contained in T.

Remark 1.3. (1) *The development in* [7] *employs the slower descent to T' using $T_{\alpha+1} = \bigcap_{\mathcal{F}} \overline{T_\alpha \cap F}$. That is, the intersection is over all sets in $\mathcal{F}$ rather than $\mathcal{F}_\delta$.*

(2) $\operatorname{sub}(X, \{X\}) = \{S : S$ *is closed in* $X\}$. *If S is regular closed then $S \in \operatorname{sub}(X, \mathcal{F})$, no matter what $\mathcal{F}$.*

(3) *A space is called* α-disconnected *if each α-cozero set has open closure. A closed set P in a space X is called a* P_α-set in X, *and we write $P \in \mathcal{P}_\alpha(X)$, if the intersection of strictly fewer than α neighborhoods of P is again a neighborhood. A P_{ω_1}-set is referred to simply as a P-set.*

(4) *Theorem 2.6 of* [6] *states that for X compact and α-disconnected, $S \in \operatorname{sub}(X, \mathcal{G}_\alpha)$ iff $S \in \mathcal{P}_\alpha(X)$, and in this case S is α-disconnected and $\mathcal{G}_\alpha(X) \cap S = \mathcal{G}_\alpha(S)$. The ω_1 case of this will be used later.*

(5) *Let $X \in |\mathbf{Comp}|$ have no isolated points, and let* $\operatorname{cof} X \equiv \{F : |X \smallsetminus F| < \omega\}$. *Then T' is the familiar perfect kernel of T, and it is well-known that transfinite descent is frequently needed to achieve T'.*

(6) *The following is 1.5(1) of* [7], *and will be used later. If $f : (Y, \mathcal{G}) \to (X, \mathcal{F}) \in \mathbf{SpFi}$ and $S \in \operatorname{sub}(Y, \mathcal{G})$ then $f(S) \in \operatorname{sub}(X, \mathcal{F})$.*

(7) *If $(X, F) \in \alpha\mathbf{SpFi}$ and $S \in \operatorname{sub}(X, \mathcal{F})$ then $(S, \mathcal{F} \cap S) \in \alpha\mathbf{SpFi}$; that is, $\alpha\mathbf{SpFi}$ is "closed under subspace formation."*

2. Products and pullbacks in **SpFi**

We assume the reader is familiar with the definitions of products and pullbacks in a general category and their construction in **Comp**.

Proposition 2.1. *Let $\{(X_i\mathcal{F}_i) : i \in I\}$ be a set in $|\mathbf{SpFi}|$. Let $(X, \{\pi_i\}_I)$ be the* **Comp** *product, i.e., $X = \prod_I X_i$ is the topological product, and $\pi_i : X \to X_i$, $i \in I$, are the projection maps. Let $\mathcal{F}$ be the filter of dense opens sets in X generated by*

$$\left\{\bigcap_J \pi_i^{-1}(F_i) : \text{finite } J \subseteq I,\ i \in J,\ F_i \in \mathcal{F}_i\right\}.$$

(1) *$\mathcal{F}$ is the smallest filter on X for which all $\pi_i : (X, \mathcal{F}) \to (X_i, \mathcal{F}_i)$ are* **SpFi***-morphisms.*
(2) *$((X, \mathcal{F}), \{\pi_i\}_I)$ is the* **SpFi***-product of $\{(X_i, F_i) : i \in I\}$.*

We write $(X, \mathcal{F}) = \prod_I (X_i, \mathcal{F}_i)$, and sometimes $\mathcal{F} = \prod_I \mathcal{F}_i$.

Proof. First, for any D_i dense in X_i, $\pi_i^{-1}D$ is dense in X. So (1) is clear. For (2), given $f_i : (Y, \mathcal{G}) \to (X_i, \mathcal{F}_i) \in$ **SpFi**, there is unique $f : Y \to X \in$ **Comp** with $\pi_i f = f_i$, $i \in I$, since $(X, \{\pi_i\}_I)$ is the **Comp**-product. One checks that $f : (Y, \mathcal{G}) \to (X, \mathcal{F}) \in$ **SpFi**. □

Proposition 2.2. *Let $f_i : (X_i, \mathcal{F}_i) \to (Y, \mathcal{G})$, $i \in I$, be a set of* **SpFi***-morphisms. Let $(T, \{t_i\})$ be their pullback in* **Comp***, i.e., with $X = \prod X_i$,*

$$T = \{t \in X : f_i\pi_i t = f_j\pi_j t \text{ for all } i, j\},\ t_i = \pi_i|T.$$

Let $P \equiv T'$ (with respect to $\prod \mathcal{F}_i$ on X), $\mathcal{P} \equiv (\prod \mathcal{F}_i)|P$, and $p_i \equiv t_i|p$. Then $((P, \mathcal{P}), \{p_i\}_I)$ is the pullback in **SpFi**.

Proof. If $h_i : (Z, \mathcal{G}) \to (X_i, \mathcal{F}_i)$ has $f_ih_i = f_jh_j$ for all i, j, there is a unique $g : Z \to T \in$ **Comp** with $t_ig = f_i$ for each i, since $(T, \{t_i\})$ is the **Comp**-pullback. By 6, $g(Z) \subseteq T' = P$. The desired $h : (Z, \mathcal{H}) \to (P, \mathcal{P})$ is just the range restriction of g. The details are routine. □

Proposition 2.3. α**SpFi** *is closed under product and pullback constructions.*

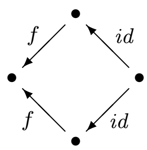
Proof. For products, observe that the preimage of an α-cozero set is an α-cozero set, and the intersection of finitely many α-cozero sets is an α-cozero set. Closure of α**SpFi** under pullbacks follows from its closure under products and from 1.3(7). □

2.1 and 2.2 appear in [28]; 2.2 was probably noticed first by A. J. Macula.

3. Monomorphisms in SpFi

The following ([18, 21.12]) comes immediately from the definitions.

Lemma 3.1. *In any category, f is a monic iff this is a pullback square.*



We interpret this for $(Y, \mathcal{G}) \xleftarrow{f} (X, \mathcal{F}) \in$ **SpFi**. Let $\Delta \equiv \{(x, x) : x \in X\} \subseteq X \times X$.

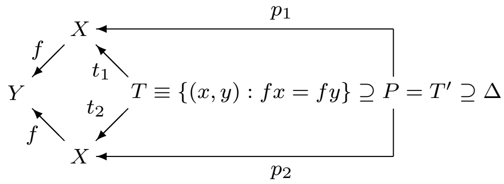
Lemma 3.2. (1) $\Delta \in \text{sub}(X \times X, \mathcal{F} \times \mathcal{F})$
(2) *$\delta(x) \equiv (x, x)$, $x \in X$, defines a* **SpFi***-isomorphism $\delta : (X, \mathcal{F}) \to (\Delta, (\mathcal{F} \times \mathcal{F}) \cap \Delta)$.*

Proof. (1) $(F_1 \times F_2) \cap \Delta = \{(x,x) : x \in F_1 \cap F_2\}$, and D dense in X implies that $\{(x,x) : x \in D\}$ is dense in Δ. (2) Of course $\delta : X \to \Delta$ is a homeomorphism. And

$$\delta^{-1}((F_1 \times F_2) \cap \Delta) = F_1 \cap F_2,\ \delta(F) = (F \times F) \cap \Delta,$$

showing δ and δ^{-1} are **SpFi**-morphisms. □

3.3. Consequently we have this diagram for $(Y, \mathcal{G}) \overset{f}{\leftarrow} (X, \mathcal{F}) \in$ **SpFi**, using the notation of 2.2.



The square "from Y to P" is a pullback. By 3.2, $P \supseteq \Delta$.

Proposition 3.4. *These* **SpFi** *squares are isomorphic via δ, i.e.,* $(\pi_i|\Delta)\,\delta = \mathrm{id}_i$.

So one is a pullback square iff the other is.

Corollary 3.5. *f is* **SpFi***-monic iff $T' = \Delta$ in 3.3.*

Proof. Combine 3.1, 3.4, and the uniqueness of pullbacks. □

This observation focuses sharply only when we understand T', which as we shall see, we do completely when the filters are Lindelöf—the content of §6 and following—but otherwise only in some special but instructive circumstances, which are the content of the next section.

4. Some examples

Consider $(Y, \mathcal{G}) \overset{f}{\leftarrow} (X, \mathcal{F}) \in$ **SpFi**, and the situation in 3.3-3.5. Recall

$$T_1 = \bigcap_{E \in \mathcal{F}_\delta} \overline{T \cap (E \times E)} \supseteq \cdots T' \supseteq \Delta.$$

Proposition 4.1. *$T_1 = \Delta$ (perforce $T' = \Delta$ and f is monic) iff for all $x_1 \neq x_2$ in X there are $E \in \mathcal{F}_\delta$ and neighborhoods U_i of x_i with $f(U_1 \cap E) \cap f(U_2 \cap E) = \emptyset$.*

Proof. $(x_1, x_2) \notin T_1$ iff there exist $E \in \mathcal{F}_\delta$ with $(x_1, x_2) \notin \overline{T \cap (E \times E)}$ iff there are neighborhoods U_i of x_i with $U_1 \times U_2 \cap (T \cap (E \times E)) = \emptyset$ iff $f(U_1 \cap E) \cap f(U_2 \cap E) = \emptyset$. The assertion follows. □

Corollary 4.2. *Each implies the next.*

(1) *$x_1 \neq x_2$ in X implies there are $F \in \mathcal{F}$ and neighborhoods U_i of x_i with $f(U_1 \cap F) \cap f(U_2 \cap F) = \emptyset$.*
(2) *$x_1 \neq x_2$ in X implies there are $E \in \mathcal{F}_\delta$ and neighborhoods U_i of x_i with $f(U_1 \cap E) \cap f(U_2 \cap E) = \emptyset$.*

(3) *f is monic.*

Corollary 4.3. *If there is E in $\mathcal{F}$ or in $\mathcal{F}_\delta$ such that f is one-to-one on E then f is monic.*

Proof. The condition in 4.2 (1) or (2) obtains. □

Corollary 4.4. *Let $Y \in |\mathbf{Comp}|$, let X be dense in Y, let X_d be discrete X, and let $Y \overset{f}{\leftarrow} \beta X_d$ be the Stone-Čech extension of the inclusion $Y \hookleftarrow X_d$. Then $(Y, \{Y\}) \overset{f}{\leftarrow} (\beta X_d, \mathcal{G}_\infty)$ is monic.*

Proof. By 4.3, with $E = X_d$. □

Remark 4.5. (1) *The implication 4.2(1)⟹(3) is given an easy direct proof in* [6, 3.1].

(2) [6, 5.6, 5.9] *says these are equivalent: $(Y, \mathcal{G}_\infty) \overset{f}{\leftarrow} (X, \mathcal{G}_\infty)$ is monic; $f(X) \overset{f}{\leftarrow} X$ is irreducible; 4.2(1) holds (with $\mathcal{F} = \mathcal{G}_\infty$ of course).*

(3) *Note $(Y, \mathcal{G}_\infty)$ in (2). 4.4 shows that, replacing $\mathcal{G}_\infty$ by $\{X\}$, monic does* not *imply irreducible.*

(4) *The bulk of this paper, §6 and following, is based on the fact that in* **LSpFi**, *4.2 (3)⟹(2), i.e., $T_1 = T'$.*

(5) *One can very well ask for simpler examples of monics (not in* **LSpFi**, *of course) not satisfying 4.2 (2). That seems not so easy, but in §6 we at least point out an indirect route to many such examples.*

Finally, we describe a class of monics which have arisen naturally in various situations, usually motivated by algebraic considerations, whose **SpFi**-monicity seems to be a central feature. Given $(X, \mathcal{F}) \in \mathbf{SpFi}$, let

$$L \equiv \varprojlim \{\beta F : F \in \mathcal{F}\} \text{ and } M \equiv \varprojlim \{\beta E : E \in \mathcal{F}_\delta\}.$$

For L, the bonding maps are: when $A, B \in \mathcal{F}$ and $A \supseteq B$, $\beta A \overset{b_A^B}{\leftarrow} \beta B$ is the Stone-Čech extension of the inclusion. For $A \in \mathcal{F}$, there is the projection $\beta A \overset{l_A}{\leftarrow} L$, so there is $X \overset{l_X}{\leftarrow} L$. This l_X is irreducible, so inversely preserves dense sets. Let $l_X^{-1}(\mathcal{F})$ be the filter of dense open sets generated by $\{l_X^{-1}(F) : F \in \mathcal{F}\}$, so $(X, \mathcal{F}) \overset{l_X}{\leftarrow} (L, l_X^{-1}(\mathcal{F})) \in \mathbf{SpFi}$. Likewise for M: with $X \overset{m_X}{\leftarrow} M$ the projection, $(X, \mathcal{F}) \overset{m_X}{\leftarrow} (M, m_X^{-1}(\mathcal{F})) \in \mathbf{SpFi}$.

Proposition 4.6. *l_X satisfies 4.2(1), and m_X satisfies 4.2(2), so both are monic.*

Proof. If, in $L \subseteq \prod_{\mathcal{F}} \beta F$, $(x_F) \neq (y_F)$ then for some $F \in \mathcal{F}$ we have $x_F \neq y_F$ in βF. So there are disjoint neighborhoods U_x and U_y in βF, and

$$l_X\left(l_F^{-1}(U_x) \cap l_F^{-1}(F)\right) \cap l_X\left(l_F^{-1}(U_y) \cap l_F^{-1}(F)\right) = \emptyset.$$

And similarly for M. □

There is also the projection $\prod_{\mathcal{F}} \beta F \leftarrow \prod_{\mathcal{F}_\delta} \beta E$, which restricts to a projection $L \overset{\pi:}{\leftarrow} M$ for which $l_X \pi = m_X$, so π is also monic. Frequently, but not always, π is one-to-one, and we write $L = M$.

Instances of the situation are the $(X, \mathcal{G}_\alpha) \leftarrow L \equiv L_\alpha$ and $(X, \mathcal{G}_\alpha) \leftarrow M \equiv M_\alpha$ for any $X \in |\mathbf{Comp}|$. Here M_α is the quasi-F_α cover of X; for $\alpha = \infty$ this is the absolute of X. For $\alpha = \omega_1$ or $\alpha = \infty$, $L_\alpha = M_\alpha$; for other α it is not known. See

[8]. See also [22], [29], [28], and [24], where the point of view is closer to the present one.

For general $(X,\mathcal{F}) \in$ **LSpFi**, the associated M is described as the "maximum subspace preserving preimage" in [1, 4.12]; as discussed there, this was motivated by issues in lattice-ordered groups. See also 3.2 and §4 of [7], where the situation in general **SpFi** is discussed.

5. Monofine in αSpFi

The discussion here is presented as a point of reference to the sequel, which is all about **LSpFi**.

In a category, one might call an object A *monofine* if the only morphisms into A which are epic and monic are isomorphisms. Thus, in α**SpFi**, $(X,\mathcal{F})$ is monofine iff $(X,\mathcal{F}) \xleftarrow{f} (Y,\mathcal{G})$ surjective and monic implies f is one-to-one (a homeomorphism) and $f^{-1}\mathcal{F} = \mathcal{G}$, $f^{-1}\mathcal{F}$ being the filter generated by $\left\{f^{-1}F : F \in \mathcal{F}\right\}$. Note that we are permitting $\alpha = \infty$ here, and $\mathbf{SpFi} = \infty\mathbf{SpFi} = \bigcup_{\alpha<\infty} \alpha\mathbf{SpFi}$.

A space is called α-disconnected if each α-cozero set has open closure. (The compact α-disconnected spaces are exactly the Stone spaces of α-complete Boolean algebras [31].) We adapt the term to **SpFi**. Call the **SpFi**-object $(X,\mathcal{F})$ *α-disconnected in* **SpFi** if X is α-disconnected as a space and $\mathcal{F} = \mathcal{G}_\alpha(X)$, the filter generated by all dense α-cozero sets.

Theorem 5.1 (3.3 and 3.4 of [6])**.** *For α**SpFi**, these conditions on $(X,\mathcal{F})$ are equivalent.*

1. *$(X,\mathcal{F})$ is monofine.*
2. *$(X,\mathcal{F}) \xleftarrow{f} (Y,\mathcal{G})$ monic implies f one-to-one and $f^{-1}(f(Y)\cap\mathcal{F}) = \mathcal{G}$. (This condition might be put: the only subobjects of $(X,\mathcal{F})$—i.e., monics into $(X,\mathcal{F})$—are subspaces.)*
3. *$(X,\mathcal{F})$ is α-disconnected.*

In [6], α**-SpFi** stands for the full subcategory of the present α**SpFi** whose objects are of the form $(Y,\mathcal{G}_\alpha(Y))$, and the results there are stated for these objects. But since, for $(X,\mathcal{F}) \in \alpha\mathbf{SpFi}$, the identity function $(X,\mathcal{F}) \leftarrow (X,\mathcal{G}_\alpha(X))$ is in α**SpFi** and is monic, 5.1 follows.

Let $\mathcal{A}$ be a category with full subcategory $\mathcal{C}$. For $A \in |\mathcal{A}|$, a (the) coreflection of A into $\mathcal{C}$ is a morphism $A \xleftarrow{c_A} cA$ with $cA \in |\mathcal{C}|$, such that for each $A \xleftarrow{f} C$ with $C \in |\mathcal{C}|$ there is unique $C \xrightarrow{\overline{f}} cA$ with $c_A \circ \overline{f} = f$; c_A is called the coreflection morphism, cA the coreflection. When every $A \in |\mathcal{A}|$ has a coreflection in $\mathcal{C}$, $\mathcal{C}$ is called coreflective. A functor $\mathcal{A} \xrightarrow{C} \mathcal{C}$ is defined (for $A \xleftarrow{f} B$, $C(f) \equiv \overline{f \circ c_B}$), and this is adjoint to the inclusion $\mathcal{A} \hookleftarrow \mathcal{C}$. See [18] for many instances of such situations.

Theorem 5.2. *In α**SpFi** with $\alpha < \infty$, the α-disconnected **SpFi**-objects form a coreflective subcategory. For each $(X,\mathcal{F}) \in \alpha$**SpFi**, the coreflection morphism $(X,\mathcal{F}) \xleftarrow{\alpha_X} (X^\alpha, \mathcal{G}_\alpha(X^\alpha))$ is surjective and monic.*

The theorem can, with some work, be derived from the result in [22] asserting that each $(X,\{X\})$ has such a coreflection. Or, for those sufficiently conversant with the situation sketched in 0.2,

$$\alpha\mathbf{SpFi} \underset{\beta}{\overset{\cap}{\rightleftarrows}} \mathbf{L}_\alpha\mathbf{Loc} \overset{\text{op}}{\approx} \mathbf{L}_\alpha\mathbf{Frm} \approx \alpha\mathbf{Frm} \approx \alpha\text{-}\mathbf{Frm}, \tag{5.3}$$

5.2 is recognizable in and around [24, 5.3], which also has the algebraic version of 5.1, and of the following, built into it.

Corollary 5.3. *In $\alpha\mathbf{SpFi}$ with $\alpha < \infty$, f is monic iff $\alpha(f)$ is monic, i.e., one-to-one, by 5.1.*

Here the forward implication is true for any coreflection α with the α_X's monic, and the backward implication is true for any coreflection α with the α_X's monic and epic. The details are routine.

The reader will note the caveat $\alpha < \infty$ in 5.2 and 5.3. For $\alpha = \infty$ the statements are not true because, with reference to (5.3), $\infty\mathbf{Frm} = \mathbf{Frm}$ is not co-well-powered—see [19]—and thus $\infty\mathbf{SpFi} = \mathbf{SpFi}$ is not well-powered.

Part II. Spaces with Lindelöf filters

6. Subspaces and more, and monics, in LSpFi

As in §1, let $(X,\mathcal{F}) \in \mathbf{SpFi}$ and let $X \overset{f}{\longleftarrow} Y$ be continuous. The first step in the (generally transfinite) descent from Y to Y_∞ is $Y_1 = \bigcap_{E\in\mathcal{F}_\delta} \overline{f^{-1}E}$.

Theorem 6.1. *If $(X,\mathcal{F}) \in \mathbf{LSpFi}$ then $Y_1 = Y_\infty$.*

Proof. We are to show that $Y_1 \cap f^{-1}E$ is dense in Y_1 for $E \in \mathcal{F}_\delta$. It suffices to take E to be a basic Lindelöf $F \in \mathcal{F}$; these are cozero sets and so are the $f^{-1}F$ since f is continuous.

So take such an F and suppose $y_0 \notin \overline{Y_1 \cap f^{-1}F}$. Then y_0 has a cozero neighborhood U with $U \cap Y_1 \cap f^{-1}F = \emptyset$, so that $\left(U \cap f^{-1}F\right) \cap Y_1 = \emptyset$. For each $y \in U \cap f^{-1}F$, $y \notin Y_1 = \bigcap_{\mathcal{F}_\delta} \overline{f^{-1}E}$, so there is some $E(y)$ with $y \notin \overline{f^{-1}E(y)}$, so y has a neighborhood $V(y)$ with $V(y) \cap f^{-1}E(y) = \emptyset$. Since $U \cap f^{-1}F$ is a cozero set, hence Lindelöf, it is contained in a set of the form $\bigcup_N V(y_n)$, and $\bigcup V(y_n) \cap \bigcap f^{-1}E(y_n) = \emptyset$. Thus

$$\emptyset = \left(U \cap f^{-1}F\right) \cap \bigcap f^{-1}E(y_n) = U \cap f^{-1}\left(F \cap \bigcap E(y_n)\right).$$

With $E = F \cap \bigcap E(y_n) \in \mathcal{F}_\delta$, we have $U \cap f^{-1}E = \emptyset$, so that $y_0 \notin \overline{f^{-1}E}$, whence $y_0 \notin Y_1$. □

Applying 6.1 to an inclusion $T \hookrightarrow X$, T closed, we get the following as in 1.2.

Corollary 6.2. *Let $(X,\mathcal{F}) \in \mathbf{LSpFi}$, and let T be a closed subset of X. Then $T_1 \in \operatorname{sub}(X,\mathcal{F})$, so $T' = T_1$.*

Then applying 4.1 and 4.2 to $(Y,\mathcal{G}) \overset{f}{\longleftarrow} (X,\mathcal{F})$ and T the topological pullback $\{(x_1,x_2) \in X \times X : fx_1 = fx_2\}$ with $T_1 = \bigcap_{E\in\mathcal{F}_\delta} \overline{T \cap (E \times E)}$, we get this.

Corollary 6.3. *Let $(Y,\mathcal{G}) \overset{f}{\longleftarrow} (X,\mathcal{F}) \in \mathbf{LSpFi}$. Then f is monic iff for all $x_1 \neq x_2$ in X there are $E \in \mathcal{F}_\delta$ and neighborhoods U_i of x_i with $f(U_1 \cap E) \cap f(U_2 \cap E) = \emptyset$.*

This result was announced in [2] without proof but with a proof suggested which was much different and less clear. The sequel shows how 6.3 permits a thorough analysis of monics in **LSpFi**.

7. Monicity via $C(X)$

We present two theorems describing monicity of $(Y,\mathcal{G}) \overset{\tau}{\leftarrow} (X,\mathcal{F})$ in **LSpFi** in terms of $C(X)$. The first is a fairly straightforward translation of 6.2, and the second is an interesting condition of "pointwise density modulo the filter $\mathcal{F}$."

For $X \in |\mathbf{Comp}|$, $C(X)$ is the usual vector lattice (or ℓ-group or ring or f-ring) of continuous real-valued functions on X. For many standard facts, one may see [13] and [30].

Theorem 7.1. *Let $(Y,\mathcal{G}) \overset{\tau}{\leftarrow} (X,\mathcal{F}) \in$ **LSpFi**. The following are equivalent (to τ monic, by 6.2).*

1. *If $x_1 \neq x_2$ in X then there are $E \in \mathcal{F}_\delta$ and neighborhoods U_i of x_i for which $\tau(U_1 \cap E) \cap \tau(U_2 \cap E) = \emptyset$.*
2. *If K_1 and K_2 are disjoint compact sets in X then there are $E \in \mathcal{F}_\delta$ and neighborhoods U_i of K_i for which $\tau(U_1 \cap E) \cap \tau(U_2 \cap E) = \emptyset$.*
3. *For each $b \in C(X)$ there is $E \in \mathcal{F}_\delta$ for which $x_1, x_2 \in E$ and $\tau x_1 = \tau x_2$ imply $bx_1 = bx_2$.*

Proof. (3)$\Longrightarrow$(1). If $x_1 \neq x_2$, choose $b \in C(X)$ with $bx_1 \neq bx_2$, then choose $E \in \mathcal{F}_\delta$ by (3), and neighborhoods U_i of x_i for which $x_i' \in U_i$ implies $bx_1' \neq bx_2'$. If $y \in \tau(U_1 \cap E) \cap \tau(U_2 \cap E)$ then there would be $x_i' \in U_i \cap E$ with $y = \tau x_i'$, so $bx_1' = bx_2'$ by (3) while $bx_1' \neq bx_2'$ since $x_i' \in U_i$. We conclude there is no such y.

(1)$\Longrightarrow$(2). Fix $x_1 \in K_1$. By (1) there are, for each $x_2 \in K_2$, $J \in F_\delta$ and neighborhoods V_i of x_i such that $\tau(V_1 \cap J) \cap \tau(V_2 \cap J) = \emptyset$. These V_2's cover K_2, so finitely many do; call their finite union W_2, and let W_1 be the intersection of the corresponding V_1's, and let $L \in \mathcal{F}_\delta$ be the intersection of the corresponding J's. Now we have neighborhoods W_1 of x_1 and W_2 of K with $\tau(W_1 \cap L) \cap \tau(W_2 \cap L) = \emptyset$. Again, finitely many W_1's cover K_1, so let U_1 be their union, and let U_2 be the intersection of the corresponding W_2's, and let $E \in \mathcal{F}_\delta$ be the intersection of the corresponding L's. Then E, U_1, and U_2 satisfy (2).

(2)$\Longrightarrow$(3). Let $b \in C(X)$. For rationals $\alpha < \beta$, $\{x : bx \leq \alpha\}$ and $\{x : bx \geq \beta\}$ are disjoint compact sets, so by (2) they have neighborhoods $U_1(\alpha,\beta)$ and $U_2(\alpha,\beta)$, and there is $E(\alpha.\beta) \in \mathcal{F}_\delta$ fulfilling the condition. Let $E \equiv \bigcap \{E(\alpha,\beta) : \alpha < \beta \text{ rational}\}$, so $E \in \mathcal{F}_\delta$. We have $\tau(U_1(\alpha,\beta) \cap E) \cap \tau(U_2(\alpha,\beta) \cap E) = \emptyset$ for each $\alpha < \beta$. Suppose $x_1, x_2 \in E$ with $\tau x_1 = \tau x_2$. Were $bx_1 \neq bx_2$, there would be $\alpha < \beta$ with $x_i \in U_i(\alpha,\beta)$, which would be a contradiction. We conclude that $bx_1 = bx_2$. $\square$

7.2. We give a simple, but convenient, reformulation of 7.1. Whenever $Y \overset{\tau}{\leftarrow} X \in$ **Comp**, a homomorphism of vector lattices (and other algebraic structures) $C(Y) \overset{\widetilde{\tau}}{\rightarrow} C(X)$ is defined by $\widetilde{\tau}(g) = g \circ \tau$. Then $A \equiv \widetilde{\tau}(C(Y))$ is a sub-vector-lattice of $C(X)$ containing all constant functions, and for which $\tau(x_1) = \tau(x_2)$ iff $a(x_1) = a(x_2)$ for all $a \in A$ (since $y_1 = y_2$ in Y iff $g(y_1) = g(y_2)$ for all $g \in C(Y)$).

Given a set A of functions defined on X, and $x_1, x_2 \in X$, we say that A *separates x_1 and x_2* if there is $a \in A$ with $a(x_1) \neq a(x_2)$.

Corollary 7.3. *Let $(Y,\mathcal{G}) \overset{\tau}{\leftarrow} (X,\mathcal{F}) \in$ **LSpFi**, and let $A \equiv \widetilde{\tau}(C(Y))$. Then τ is monic iff for each $b \in C(X)$ there is $E \in \mathcal{F}_\delta$ such that A separates every pair from E which b separates.*

We now recast monicity in terms of "pointwise density mod the filter."

7.4. Note the following for any $C(Y)$: if $y_1, \ldots, y_n$ are distinct points and $r_1, \ldots, r_n$ are real numbers, there is $g \in C(Y)$ with $gy_i = r_i$ for $i = 1, \ldots, n$.

Theorem 7.5. *Let* $(Y, \mathcal{G}) \overset{\tau}{\longleftarrow} (X, \mathcal{F}) \in$ **LSpFi**, *and let* $A \equiv \widetilde{\tau}(C(Y))$. *Then the following are equivalent.*

1. τ *is monic in* **LSpFi**.
2. *For every* $b \in C(X)$ *there is an* $E \in \mathcal{F}_\delta$ *such that for every* $x_1, x_2 \in E$ *there is* $a \in A$ *with* $ax_1 = bx_1$ *and* $ax_2 = bx_2$.
3. *For every* $b \in C(X)$ *there is* $E \in \mathcal{F}_\delta$ *such that for every finite* $F \subseteq E$ *there is* $a \in A$ *with* $a|F = b|F$.

Proof. We actually show this: for fixed b and E, the conditions in 7.1 (3) and 7.5 (2) are equivalent, and imply the stronger condition in 7.5 (3). So fix b and E.

Suppose the condition in 7.1 (3). Let $x_1, x_2 \in E$. If $bx_1 = bx_2$, just choose $a \in A$ which is constantly bx_1. If $bx_1 \neq bx_2$, then $\tau x_1 \neq \tau x_2$ by 7.1 (3). Choose $g \in C(X)$ with $g\tau x_i = bx_i$ by 7.4, and let $a \equiv g\tau$. Conversely, suppose the condition in 7.5 (2), and let $x_1, x_2 \in E$ with $bx_1 \neq bx_2$. Choosing $a \in A$ per 7.5 (2) shows $\tau x_1 \neq \tau x_2$ by 7.2.

Now we show by induction that the two-point condition in 7.5 (2) implies the condition in 7.5 (3), i.e., the n-point condition for any n. The assertion is: for any $n \in N$, $F \subseteq E$, $|F| \leq n$ implies there is $a \in A$ with $a|F = b|F$. By hypothesis this is true for $n = 2$. Suppose it is so for n, and let $|F| = n+1$, as $F = \{x_1, \ldots, x_{n+1}\}$. If the τx_i's are all distinct, just let $a \equiv g \circ \tau$ for $g \in C(Y)$ with $g\tau x_i = bx_i$, using 7.4. Otherwise there is j_0 such that for some i_0, $ax_{i_0} = ax_{j_0}$ for all $a \in A$ by 7.2. Then $bx_{i_0} = bx_{j_0}$ by the two-point condition. Now, by the n-point condition, there is $a \in A$ with $ax_i = bx_i$ for $i \neq j_0$, which also makes $ax_{j_0} = b_{j_0}$. This completes the induction step, and by induction we are done. □

Remark 7.6.

1. *The procedure in 7.5 can be described as follows. Given* $(X, \mathcal{F})$, *a monic* $(Y, \mathcal{G}) \overset{\tau}{\longleftarrow} (X, \mathcal{F})$ *generates a sub-vector-lattice* $A \equiv \widetilde{\tau}(C(Y))$ *of* $C(X)$ *containing constants which satisfies 7.5 (3), let us say is "pointwise dense modulo* $\mathcal{F}$*." Conversely, given such* A, *define an equivalence relation by the rule* $x_1 \sim x_2$ *iff* $ax_1 = ax_2$ *for all* $a \in A$, *and let* $X/\sim \; \equiv Y \overset{\tau}{\longleftarrow} X$ *be the quotient in* **Comp**. *Then each* $a \in A$ *factors through* τ *to* $a' \in C(Y)$, *and* $\{a' : a \in A\}$ *separates points of* Y, *and so is uniformly dense in* $C(Y)$ *by the Stone-Weierstrass Theorem. Since* A *was "pointwise dense mod* $\mathcal{F}$*," so is* $\widetilde{\tau}C(Y)$, *so* $(Y, \{Y\}) \overset{\tau}{\longleftarrow} (X, \mathcal{F})$ *is monic.*
2. *The filter* $\{Y\}$ *appears in (1), but* $(Y, \mathcal{G}) \overset{\tau}{\longleftarrow} (X, \mathcal{F})$ *is monic iff* $(Y, \{Y\}) \overset{\tau}{\longleftarrow} (X, \mathcal{F})$ *is. Also, the* τ *created from* A *is a surjection, but general* $(Y, \mathcal{G}) \overset{\tau}{\longleftarrow} (X, \mathcal{F})$ *is monic iff* $(\tau(X), \mathcal{G} \cap \tau(X)) \overset{\tau}{\longleftarrow} (X, \mathcal{F})$ *is. (Note 7.1(1) here.)*
3. *Another view of this subsection, which we shall examine in detail in a later paper, is this. Given* $(X, \mathcal{F})$, *there is on* $C(X)$ *the "topology of pointwise convergence mod* $\mathcal{F}$*." Basic neighborhoods of* b *are indexed by* $E \in \mathcal{F}_\delta$ *and* $\varepsilon > 0$:

$$\eta(b, E, \varepsilon) \equiv \{f : \exists \text{ finite } F \subseteq E \; \forall x \in F \; (|f - b|(x) \leq \varepsilon)\}.$$

For A *a sub-vector-lattice of* $C(X)$ *containing constants, the condition in 7.5 (3) is density in this topology.*

We now explain how the developments of this section show indirectly that there are many monics in **SpFi** which fail the criterion under consideration, i.e., 7.5 (3) = 4.2 (2). The essential observation is that scrutiny of the proofs of 7.1 and 7.5 reveals a certain non-dependence on **LSpFi**.

Proposition 7.7. *For* $(Y, \mathcal{G}) \overset{\tau}{\leftarrow} (X, \mathcal{F}) \in$ **SpFi***, the following are equivalent (and imply* τ *monic by 4.2).*

(1) *If* $x_1 \neq x_2$ *in* X*, then there are* $E \in \mathcal{F}_\delta$ *and neighborhoods* U_i *of* x_i *with* $\tau(U_1 \cap E) \cap \tau(U_2 \cap E) = \emptyset$.

(2) *If* $b \in C(X)$ *then there is* $E \in \mathcal{F}_\delta$ *for which* $x_1, x_2 \in E$ *and* $\tau x_1 = \tau x_2$ *imply* $bx_1 = bx_2$.

For a cardinal α, $\exp \alpha$ denotes 2^α; for a space X, $\operatorname{wt} X$ denotes the weight of X, i.e., the minimum cardinality of a base.

Corollary 7.8. *If* τ *satisfies 7.7 then* $|X| \leq \exp\exp|Y|$.

Proof. We have $\operatorname{wt} X \leq (\operatorname{wt} X)^\omega = |C(X)|$ by a theorem of Smirnov for compact X; see §7 of [10]. And $|X| \leq \exp(\operatorname{wt} X)$ since X is Hausdorff; see [11]. So it suffices to show that $|C(X)| \leq \exp|Y|$.

Let b and $E \equiv E(b)$ satisfy 7.7 (2). Define $b' : \tau(E(b)) \to R$ by $b'(\tau x) \equiv bx$. This is well-defined. Now $\mathcal{H} \equiv \{\tau(E) : E \in \mathcal{F}_\delta\}$ is a filter base of dense sets in Y, $H_1 \supseteq H_2$ implies a restriction map $R^{H_1} \to R^{H_2}$, and with these as bondings, we have a direct limit $L \equiv \lim_{\longrightarrow} \{R^H : H \in \mathcal{H}\}$, and for each H there is a map $R^H \to L$. It devolves that, given b and $E(b)$, with $H = \tau(E(b))$, $C(X) \ni b \longmapsto b' \in R^H \to L$ is a well-defined one-to-one map. (Actually, it's a homomorphism for, say, vector lattices, of $C(X)$ into L, and L is even archimedean, but we needn't pursue that now.) It remains to note $|L| \leq |\mathcal{H}| \cdot \exp|Y| = \exp|Y|$. ☐

7.8 implies there are $(Y, \mathcal{G})$ with monic preimages not satisfying 7.7. This is simply because **SpFi** is not well-powered: the category of locales is not (see [19]), and there is the adjunction **SpFi** $\leftrightarrows$ **Loc** mentioned in §0.

8. The Stone space of the Baire field

See [31] or [16] or [20] for details of this sketch of Stone duality. This is the contravariant equivalence **BA** $\leftrightarrows$ **BS**, where **BA** is the category of Boolean algebras, with their homomorphisms, and **BS** is the category of Boolean spaces, meaning zero-dimensional compact Hausdorff spaces, with continuous maps. The functor clop assigns to a space X the Boolean algebra $\operatorname{clop} X$ of clopen (closed and open) subsets of X, and assigns to a map $X \overset{f}{\to} Y$ the Boolean homomorphism $\operatorname{clop} X \overset{f^{-1}}{\leftarrow} \operatorname{clop} Y$. For $\mathcal{A} \in |\mathbf{BA}|$, $S\mathcal{A}$ is the set of Boolean ultrafilters $\mathcal{U}$ in $\mathcal{A}$, with basic sets $\xi A = \{\mathcal{U} : A \in \mathcal{U}\}$, $A \in \mathcal{A}$. The map $\mathcal{A} \ni A \longmapsto \xi A \in \operatorname{clop} S\mathcal{A}$ is an isomorphism, the Stone representation of $\mathcal{A}$. For $\mathcal{A} \overset{\phi}{\to} \mathcal{B} \in \mathbf{BA}$, $S\mathcal{A} \overset{S\phi}{\leftarrow} S\mathcal{B}$ is $(S\phi)(\mathcal{V}) = \phi^{-1}(\mathcal{V})$. The induced $\operatorname{clop} S\mathcal{A} \overset{(S\phi)^{-1}}{\to} \operatorname{clop} S\mathcal{B}$ is the Stone representation of ϕ.

Now let X be a space. The Baire field on X, $\mathcal{B}(X)$, is the least σ-field on X, i.e., the least sub-σ-algebra of the power set of X, containing $\operatorname{coz} X = \{\operatorname{coz} f : f \in C(X)\}$.

The only case of present interest in the next theorem is $\mathcal{A} = \mathcal{B}(X)$, but the generalization costs nothing and will be useful for later reference.

Let $X \in |\mathbf{Comp}|$ and let $\mathcal{A}$ be a sub-Boolean-algebra of the power set $\mathcal{P}(X)$ which interacts with the topology as:

(*) $\quad \forall\, x \in X\ \forall$ neighborhood H of $x\ \exists\ A \in \mathcal{A}\ \left(x \in \operatorname{int} A \subseteq \overline{A} \subseteq H\right).$

($\mathcal{B}(X)$ satisfies this; indeed, $\operatorname{coz} X$ does.) Let $X_{\mathcal{A}}$ be the set X with the topology which has $\mathcal{A}$ as basis, and let $X \overset{\iota}{\leftarrow} X_{\mathcal{A}}$ be the identity function, which is continuous by $(*)$.

Theorem 8.1. (1) *For each $x \in X$, $\mathcal{U}_x = \{M \in \mathcal{A} : x \in M\}$ is an ultrafilter in $\mathcal{A}$. Define $p : X_{\mathcal{A}} \to S\mathcal{A}$ by $p(x) = \mathcal{U}_x$. Then p is a homeomorphism of $X_{\mathcal{A}}$ onto a dense set in $S\mathcal{A}$.*

(2) *For each $\mathcal{U} \in S\mathcal{A}$, $\bigcap_{\mathcal{U}} \overline{U}$ is a singleton, which we denote $\mu(\mathcal{U})$. Thus a function $\mu : S\mathcal{A} \to X$ is defined, and this is a continuous surjection.*

(3) *$\mu p = \iota$.*

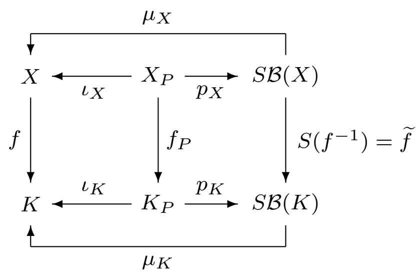
(4) *For each $A \in \mathcal{A}$, $\xi A = \overline{p(A)}$, $p^{-1}\xi A = \iota^{-1} A$, and $\mu(\xi A) = \overline{A}$.*

8.1 is familiar to many, and the proof is routine, so we omit it. Regarding μ, one may note that condition $(*)$ implies $\operatorname{clop} X \subseteq \mathcal{A}$ by an easy covering argument. Suppose $X \in |\mathbf{BS}|$. Then $S \operatorname{clop} X = X$ up to homeomorphism, and the Stone dual of the Boolean inclusion $\operatorname{clop} X \hookrightarrow \mathcal{A}$ is $\mu : S\mathcal{A} \to X$. This requires an observation augmenting 8.1 (4): $\mu^{-1} A = \xi A$ when A is open.

We turn to the relation of 8.1 to measurable functions. One may do the analysis in the generality of 8.1, but it doesn't focus very sharply, so we consider only situations $\mathcal{A} = \mathcal{B}(X)$. A function $X \overset{f}{\to} Y$ of spaces is Baire-measurable, or just Baire, if $f^{-1}(M) \in \mathcal{B}(X)$ whenever $M \in \mathcal{B}(Y)$, and $B(X, Y)$ denotes the set of all these. $B(X, R)$ is denoted $B(X)$, and this is a vector lattice and ring which is sequentially uniformly closed, i.e., for f_n's in $B(X)$, if $f_n \to f$ uniformly on X then $f \in B(X)$. The substructure of bounded functions is $B^*(X)$.

Let $X, K \in |\mathbf{Comp}|$. For each of X and K, we have the maps μ, p, and ι of 8.1; we subscript these as μ_X and μ_K, etc. We also have $X_{\mathcal{B}(X)}$, which we denote X_P. ($X \overset{\iota_X}{\leftarrow} X_P$ is sometimes called the P-space coreflection of X.) Likewise for K_P. Let $f : X \to K$ be a Baire function. Then f is continuous for the P-topologies. We denote this continuous map $f_P : X_P \to K_P$; we have $f \iota_X = \iota_K f_P$. Now $\mathcal{B}(X) \overset{f^{-1}}{\leftarrow} \mathcal{B}(K)$ is a Boolean homomorphism, and even a σ-homomorphism, a fact which will be important later. So there is continuous $S\mathcal{B}(X) \overset{S(f^{-1})}{\to} S\mathcal{B}(K)$ for which $\operatorname{clop} S\mathcal{B}(X) \overset{S(f^{-1})^{-1}}{\to} \operatorname{clop} S\mathcal{B}(K)$ is the Stone representation of f^{-1}. For the sake of the typography we put $\widetilde{f} \equiv S\left(f^{-1}\right)$.

8.2. It may be helpful to display the situation:



Theorem 8.3. (1) *$\widetilde{f}p_X = p_K f_P$, and $\widetilde{f}$ is the unique continuous function satisfying that.*

(2) *$(\mu_K \widetilde{f})p_X = f\iota_X$, and $\mu_K \widetilde{f}$ is the unique continuous function satisfying that.*

(3) *$\mu_K \widetilde{f} = f\mu_X$.*

Proof. For the uniqueness in (1) and (2), note that for continuous g_i, $g_1 p_X = g_2 p_X$ implies $g_1 = g_2$ because p_X has dense image by 8.1(1), i.e., p_X is an epimorphism in Hausdorff spaces. For the equation in (1), note that $\widetilde{f}$ is $S\left(f^{-1}\right)$, and the action of a general $S(\phi)$ is $S(\phi)(\mathcal{U}) = \phi^{-1}(\mathcal{U}) = \{L : \phi(L) \subseteq \mathcal{U}\}$. Using $\phi = f^{-1}$ and $\mathcal{U} = \mathcal{U}_x$, we have

$$\begin{aligned}\widetilde{f}p_X(x) &= S\left(f^{-1}\right)(\mathcal{U}_x) = \left\{L : f^{-1}L \in \mathcal{U}_x\right\} = \left\{L : x \in f^{-1}L\right\}\\ &= \{L : f(x) \in L\} = \mathcal{U}_{f(x)} = p_K f_P(x).\end{aligned}$$

For (2), apply μ_K on the left to the equation from (1), as $\mu_K \widetilde{f} p_X = \mu_K p_K f_P = \iota_K f_P = f\iota_X$. For (3), $(f\mu_X)p_X = f\iota_X = \left(\mu_K \widetilde{f}\right)p_X$ from (1), and since f is continuous so is $f\mu_X$, and then the p_X can be canceled since it is an epimorphism in the category of Hausdorff spaces with continuous maps, as we remarked above. □

In (3), actually, $\widetilde{f}$ is the unique continuous map satisfying the equation. This seems not so obvious, and is really a statement about **SpFi**-monicity. We discuss this in the next section.

8.3 (1) and/or (2) can be paraphrased: $(S\mathcal{B}(X), \mu_X)$ has the universal mapping property that each $f \in \mathcal{B}(X, K)$, K compact, has the unique continuous "extension" $\mu_K \widetilde{f}$. This was first brought to our attention by A. J. Macula, with a different proof. There is also this converse.

Proposition 8.4. *If $g \in C(S\mathcal{B}(X), K)$, K compact, then $f \equiv g \circ p_X \circ \iota_X^{-1} \in \mathcal{B}(X, K)$, and $\mu_K \widetilde{f} = g$.*

Proof. To show f is Baire, it is enough to show that $f^{-1}U \in \mathcal{B}(X)$ when $U \in \operatorname{coz} K$. Now $f^{-1}U = \left(gp_X\iota_X^{-1}\right)^{-1}(U) = \iota_X\left(p_X^{-1}\left(g^{-1}U\right)\right)$. Hence $g^{-1}U \in \operatorname{coz} S\mathcal{B}(X)$, so $g^{-1}U$ is a countable union of clopen sets as $g^{-1}U = \bigcup_n \xi M_n$. So $f^{-1}U = \iota_X\left(p_X^{-1}\bigcup_n \xi M_n\right) = \bigcup_n \iota_X\left(p_X^{-1}\xi M_n\right)$ since ι_X is one-to-one, and this is $\bigcup_n M_n \in \mathcal{B}(X)$, using 8.1(4). $\mu_K \widetilde{f} = g$ follows from the uniqueness in 8.3(2). □

Thus for $f \in B^*(X)$, let K be the closed interval $[\inf f(X), \sup f(X)]$, and let $\widehat{f} = \mu_K \widetilde{f}$, construed as an element of $C(S\mathcal{B}(X))$, as opposed to $C(S\mathcal{B}(X), K)$. This is the unique $g \in C(S\mathcal{B}(X))$ which extends f, and every $g \in C(S\mathcal{B}(X))$ arises in this way. Because X_P is dense in $S\mathcal{B}(X)$, $f_n \to f$ uniformly on X within $B^*(X)$ iff $\widehat{f}_n \to \widehat{f}$ uniformly on $S\mathcal{B}(X)$ within $C(S\mathcal{B}(X))$.

Corollary 8.5. *$B^*(X) \ni f \longmapsto \widehat{f} \in C(S\mathcal{B}(X))$ is an isomorphism of vector lattices, unitary f-algebras, etc., which preserves uniform convergence of sequences. For $M \in \mathcal{B}(X)$, $\widehat{\chi}(M) = \chi(\xi M) = \chi\left(\overline{pM}\right)$.*

9. The basically disconnected coreflection in **LSpFi**

In this section and the next, we return to the situation sketched in §5, in the case $\alpha = \omega_1$: ω_1-disconnected is called basically disconnected (BD). We repeat: $(X, \mathcal{F})$

is BD if X is BD as a space (each cozero set has open closure) and $\mathcal{F} = \mathcal{G}_{\omega_1}(X) = \mathcal{C}(X)$, the filter generated by all dense cozero sets. The full subcategory of **LSpFi** whose objects are BD may also be denoted **BD**.

According to what is said in §5, **BD** is coreflective in **LSpFi**. (In spaces without filters **BD** is not coreflective [34].) The reader who studies our references for this for general α, [22] and [24], may have difficulty recognizing a proof at all, and will certainly have difficulty describing in terms of **SpFi** what the BD coreflection *is*, since the constructions referred to are as universal objects.

9.1. We now explicitly construct the BD coreflection of each $(X, \mathcal{F}) \in$ **LSpFi**. Continuing the discussion in §8, we have $\mu_X : S\mathcal{B}(X) \to X$, which has nothing to do with $\mathcal{F}$. Let $I = I(\mathcal{F})$ be the Boolean σ-ideal in $\mathcal{B} = \mathcal{B}(X)$ generated by $\{X \smallsetminus F : F \in \mathcal{F}\}$. We then have the Boolean quotient $\mathcal{B} \to \frac{\mathcal{B}}{I}$, whose Stone dual $S\mathcal{B} \leftarrow S\frac{\mathcal{B}}{I}$ is one-to-one, and so can be viewed as an inclusion $S\mathcal{B} \hookleftarrow S\frac{\mathcal{B}}{I}$, and in this view

$$S\frac{\mathcal{B}}{I} = S\mathcal{B} \smallsetminus \bigcup_{L \in I} \xi L = \bigcap_{X \smallsetminus E \in I} \xi E = \bigcap_{E \in \mathcal{F}_\delta \cap \mathcal{B}} \xi E = \{\mathcal{U} \in S\mathcal{B} : \mathcal{F}_\delta \cap \mathcal{B} \subseteq \mathcal{U}\}.$$

(ξL designates the clopen set corresponding to L, so here $\xi L = \overline{p_X(L)}$.) Note that because $(X, \mathcal{F}) \in$ **LSpFi**, $\mathcal{F}$ is generated by its sets in $\mathcal{B}$, and the same is true of $\mathcal{F}_\delta$. Therefore it is only a slight abuse of notation to write $S\frac{\mathcal{B}}{I}$ as $\{\mathcal{U} \in S\mathcal{B} : \mathcal{F}_\delta \subseteq \mathcal{U}\}$.

Given $(X, \mathcal{F}) \in$ **LSpFi**, let $X^\# \equiv S\frac{\mathcal{B}}{I}$, and let m_X be the composition $\mu_X j$

$$X^\# = S\frac{\mathcal{B}}{I} \xrightarrow{j} S\mathcal{B} \xrightarrow{\mu_X} X,$$

j being the inclusion, so that $m_X = \mu_X | X^\#$.

Theorem 9.2. *Let* $(X, \mathcal{F}) \in$ **LSpFi**.

(1) $X \xleftarrow{m_X} X^\#$ *is a surjection.*
(2) $X^\#$ *is BD.*
(3) $(X, \mathcal{F}) \xleftarrow{m_X} (X^\#, \mathcal{C})$ *is a* **SpFi** *isomorphism iff* $(X, \mathcal{F})$ *is BD as a* **SpFi** *object.*
(4) $(X, \mathcal{F}) \xleftarrow{m_X} (X^\#, \mathcal{C})$ *is in* **LSpFi**, *and is monic there.*
(5) *If* $(X, \mathcal{F}) \xleftarrow{f} (Y, \mathcal{G}) \in$ **LSpFi** *then there is a unique* $(X^\#, \mathcal{C}(X^\#)) \xleftarrow{f^\#} (Y^\#, \mathcal{C}(Y^\#)) \in$ **LSpFi** *for which* $m_X f^\# = f m_Y$.
(6) $(X, \mathcal{F}) \xleftarrow{m_X} (X^\#, \mathcal{C}(X^\#))$ *is the* **BD** *coreflection of* $(X, \mathcal{F})$.

The proof of Theorem 9.2 occupies the rest of the section; it constitutes a considerable mass of information.

Proof of 9.2(1). $I(\mathcal{F}) = \{L \in \mathcal{B} : L \subseteq X \smallsetminus F,\ F \in \mathcal{F}_\delta\}$, and for each $L \in I$, $\operatorname{int}_X L = \emptyset$ by the Baire Category Theorem. Thus the following lemma applies and completes the proof. □

We state and prove the following lemma in the same generality as 8.1, since it requires no more work. Here we have

$$X \xleftarrow{\mu} S\mathcal{A} \xhookleftarrow{j} S\frac{\mathcal{A}}{J} = S\mathcal{A} \smallsetminus \bigcup_{L \in J} \xi L = \bigcap_{A \in \operatorname{co} J} \xi A = \bigcap_{A \in \operatorname{co} J} \overline{pA},$$

J being any ideal in $\mathcal{A}$ and $\operatorname{co} J$ being $\{S : X \smallsetminus S \in J\}$.

Lemma 9.3. *These are equivalent.*

(1) μj *is onto* X.
(2) *For each* $L \in J$, $\operatorname{int}_X L = \emptyset$.
(3) *For each* $A \in \operatorname{co} J$, A *is dense in* X.

Proof. (2)$\Longleftrightarrow$(3) is clear. (1)$\Longrightarrow$(3). For any $A \in \mathcal{A}$, $\mu\left(\overline{pA}\right) \subseteq \overline{A}$, so if $A \in \operatorname{co} J$ has $\overline{A} \neq X$ then certainly μj is not onto. (3)$\Longrightarrow$(1). Assume (3). Then for $A \in \operatorname{co} J$, $\mu\left(\overline{pA}\right) = X$ since A is dense and $A = \mu p A \subseteq \mu\left(\overline{pS}\right)$ and the last is closed. So let $x \in X$. For $A \in \operatorname{co} J$, $x \in \mu\left(\overline{pA}\right)$, so $\mu^{-1}\{x\} \cap \overline{pA} \neq \emptyset$. Since $\operatorname{co} J$ is a filter, the family $\left\{\mu^{-1}\{x\} \cap \overline{pA} : A \in \operatorname{co} J\right\}$ has the finite intersection property. By compactness, the total intersection is nonvoid, so there exists $\mathcal{U} \in \bigcap_{A \in \operatorname{co} J} \overline{pA} = S\frac{\mathcal{A}}{J}$ such that $\mu(\mathcal{U}) = x$. □

Alternative proof of 9.2(1) . For any $x \in X$, the family

$$\{F \cap A : F \in \mathcal{F}_\delta \cap \mathcal{B},\ x \in \operatorname{int} A \subseteq A \in \mathcal{B}\} \subseteq \mathcal{B}$$

has the finite intersection property because each $F \in \mathcal{F}_\delta$ is dense, and is therefore contained in some $\mathcal{U} \in S\mathcal{B}$. Then $\mathcal{U} \in S\frac{\mathcal{B}}{I}$ by 9.1, and $m_X(\mathcal{U}) = \bigcap_{\mathcal{U}} \overline{U} = x$. □

Proof of 9.2(2). Since $\mathcal{B} = \mathcal{B}(X)$ is a σ-complete Boolean algebra and $I = I(\mathcal{F})$ is a σ-ideal, $\frac{\mathcal{B}}{I}$ is also σ-complete ([31, 21.1]), and thus $S\frac{\mathcal{B}}{I} = X^{\#}$ is basically disconnected ([31, 22.4]). Alternatively, since $\mathcal{B}$ is σ-complete, $S\mathcal{B}$ is basically disconnected , and since I is a σ-ideal, $S\frac{\mathcal{B}}{I}$ is a P-set in $S\mathcal{B}$ ([31, 21.6]), and thus basically disconnected by 1.3(4). □

Proof of 9.2(3). If m_X is an isomorphism then it is a homeomorphism, so by (2), $X \in$ **BD**. But any homeomorphism carries dense cozero-sets back and forth, so $\mathcal{F} = \mathcal{C}(X)$, and $(X, \mathcal{F}) \in$ **BD**. Conversely, suppose $(X, \mathcal{F}) \in$ **BD**. We need only show m_X one-to-one, for then it is a homeomorphism by (1), and in fact a **SpFi**-isomorphism since $\mathcal{F} = \mathcal{C}$ by assumption. Now X is BD, and $X \stackrel{\mu_X}{\leftarrow} S\mathcal{B}$ is the Stone dual of the Boolean inclusion $\operatorname{clop} X \stackrel{e}{\hookrightarrow} \mathcal{B}$; see §8. So $m_X : X \stackrel{\mu_X}{\leftarrow} S\mathcal{B} \stackrel{j}{\hookleftarrow} S\frac{\mathcal{B}}{I}$ is the dual of $\operatorname{clop} X \stackrel{e}{\hookrightarrow} \mathcal{B} \stackrel{q}{\to} \frac{\mathcal{B}}{I}$, and since $\mathcal{F} = \mathcal{C}(X)$,

$$I = \left\{ M \in \mathcal{B} : M \subseteq \bigcup_n Z_n \text{ for nowhere dense zero sets } Z_1, Z_2, \ldots \right\}.$$

That m_X is one-to-one just means that qe is onto. This is the consequence of the following version of the Loomis-Sikorski-Stone Theorem given in [3, 3.5]; the usual version uses the ideal of meagre Baire sets instead of I.

Proposition 9.4. *If* X *is compact BD, then for each* $M \in \mathcal{B}$ *there is* $U \in \operatorname{clop} X$ *for which the symmetric difference* $M \Delta U \in I$*; that is,* qe *is onto.*

This completes the proof of 9.2(3). □

Proof of the first part of 9.2(4). We next show $(X, \mathcal{F}) \stackrel{m_X}{\leftarrow} \left(X^{\#}, \mathcal{C}\right) \in$ **SpFi**. (Note that $(X, \mathcal{F}) \stackrel{\mu_X}{\leftarrow} (S\mathcal{B}, \mathcal{C})$ is in **SpFi** iff $\mathcal{F} = \{X\}$.) We shall apply 6.1 to $X \stackrel{\mu_X}{\leftarrow} S\mathcal{B}$; set $\mu \equiv \mu_X$, $m \equiv m_X$, and $S \equiv S\mathcal{B}$. We have $\mu_1 : S_1 \to X$ defined by

$$X \stackrel{\mu}{\leftarrow} S \hookleftarrow S_1 \equiv \bigcap_{E \in \mathcal{F}_\delta} \overline{\mu^{-1} E}$$

from 8.5, with $\mu_1^{-1}F$ dense in S_1 for each $F \in \mathcal{F}$. We are interested in $m : X^{\#} \to X$ defined by

$$X \overset{\mu}{\leftarrow} S \hookleftarrow X^{\#} = \bigcap_{\xi \in \mathcal{F}_\delta} \xi E.$$

(See the description of $X^{\#} = S\frac{\mathcal{B}}{I}$ just before 8.4; the co J there is co $I(\mathcal{F}) = \mathcal{F}_\delta$.) By 7.5, for any $M \in \mathcal{B}$ we have $\xi M = \overline{p^{-1}M} \subseteq \overline{\mu^{-1}M}$. So we have $m : X^{\#} \to X$ provided by $X \overset{\mu_1}{\leftarrow} S_1 \hookleftarrow X^{\#}$. We want $(X, \mathcal{F}) \overset{m}{\leftarrow} \left(X^{\#}, \mathcal{C}\right) \in \mathbf{SpFi}$, which just means $m^{-1}F = \mu^{-1}F \cap X^{\#}$ is dense in $X^{\#}$ for $F \in \mathcal{F}$. For $F \in \mathcal{F}$, $\mu^{-1}F \cap S_1 = \mu_1^{-1}F$ is a dense cozero in S_1.

Lemma 9.5 (Tzeng [33])**.** *Let Y be compact BD, and K be a closed subspace. If $U \in \mathcal{C}(K)$ then there is $V \in \mathcal{C}(X)$ with $V \cap K = U$.*

Proof. Let $U = \bigcup_n U_n$ for $U_n \in \operatorname{clop} K$ (since K is zero-dimensional), and each n there is $V_n \in \operatorname{clop} Y$ with $V_n \cap K = U_n$. Let

$$V \equiv \bigcup_n V_n \smallsetminus \overline{\bigcup_n V_n}.$$

This is clearly dense, and cozero since $\bigcup_n V_n$ is cozero and $Y \smallsetminus \overline{\bigcup_n V_n} \in \operatorname{clop} Y$ since Y is BD. And $V \cap K = U$. □

By 9.3, $\mu^{-1}F \cap S = V \cap S_1$ for some $V \in \mathcal{C}(S)$. Since $X^{\#}$ is a P-set in S, $V \cap X^{\#}$ is dense in $X^{\#}$ (see 1.3(4)). Finally,

$$V \cap X^{\#} = (V \cap S_1) \cap X^{\#} = \left(\mu^{-1}F \cap S_1\right) \cap X^{\#} = \mu^{-1}F \cap X^{\#} = \mu^{-1}F.$$

This completes the proof of the lemma, and of the first part of the proof of 9.2(4). □

Alternative proof of the first part of 9.2(4). To show that $(X, \mathcal{F}) \overset{m_X}{\leftarrow} \left(X^{\#}, \mathcal{C}\right) \in \mathbf{SpFi}$, we need only show that $m_X^{-1}(K)$ is dense in $X^{\#}$ for any $K \in \mathcal{F}$, i.e., that $\mu_X^{-1}(K)$ meets $\xi A \cap X^{\#}$ for any $A \in \mathcal{B}$ such that $\xi A \cap X^{\#} \neq \emptyset$. Now if $\mathcal{U} \in \xi A \cap X^{\#}$ then, since $\mathcal{F}_\delta \cup \{A\} \subseteq \mathcal{U}$ by 9.1, it follows that A meets every $F \in \mathcal{F}_\delta$. Express K as a countable union $\bigcup_N K_n$ of zero sets K_n. We claim that there must be some index n for which $A \cap K_n$ meets every $F \in \mathcal{F}_\delta$. For it not, then for every n there exists $F_n \in \mathcal{F}_\delta$ such that $A \cap K_n \cap F_n = \not\emptyset$, yielding $F \equiv \bigcap_N F_n \in \mathcal{F}_\delta$ such that $A \cap K \cap F = \emptyset$, a contradiction since $K \cap F \in \mathcal{F}_\delta$. The claim establishes that the family

$$\{A \cap K_n \cap F : F \in \mathcal{F}_\delta\} \subseteq \mathcal{B}$$

has the finite intersection property, and is therefore contained in some ultrafilter $\mathcal{U}$. But $\mathcal{U} \in X^{\#}$ because $\mathcal{F}_\delta \subseteq \mathcal{U}$, $\mathcal{U} \in \xi A$ because $A \in \mathcal{U}$, and $m_X(\mathcal{U}) \in F$ because $\mu_X(\mathcal{U}) = \bigcap_{\mathcal{U}} \overline{U} \in K_n \subseteq F$. □

Second part of the proof of 9.2(4). Now we show $(X, \mathcal{F}) \overset{m_X}{\leftarrow} \left(X^{\#}, \mathcal{C}\right)$ is monic. Monicity has nothing to do with the filter in the codomain, so this is the same as $(X, \{X\}) \overset{m_X}{\leftarrow} \left(X^{\#}, \mathcal{C}\right)$ being monic. For this version of m_X, we have the **SpFi**-factorization of m_X as $(X, \{X\}) \overset{\mu_X}{\leftarrow} (S\mathcal{B}, \mathcal{C}) \overset{j_X}{\hookleftarrow} \left(X^{\#}, \mathcal{C}\right)$. As noted in the proof of 9.2(2), $X^{\#}$ is a P-set in $S\mathcal{B}$, so by 1.3(3), $X^{\#} \in \operatorname{sub}(S\mathcal{B}, \mathcal{C}(S\mathcal{B}))$ and $\mathcal{C}(S\mathcal{B}) \cap X^{\#} = \mathcal{C}\left(X^{\#}\right)$. Therefore $j_X \in \mathbf{SpFi}$, and, being one-to-one, is monic. Since the composition of monics is monic, the following theorem completes the proof of 9.2(4). □

Theorem 9.6. $(X, \{X\}) \overset{\mu_X}{\leftarrow} (S\mathcal{B}, \mathcal{C})$ *is monic.*

Proof. Let $\mu \equiv \mu_X$ and $S \equiv S\mathcal{B}$. We shall verify the condition in 7.3 in the following form.

(*)For each $b \in C(S)$ there is $E \in \mathcal{C}_\delta$ such that if $y_1, y_2 \in E$ and $by_1 \neq by_2$, there is $a \in A \equiv \widetilde{\mu}C(X)$ for which $ax_1 \neq ax_2$.

Recall the isomorphism $B^*(X) \ni f \longmapsto \widetilde{f} \in C(S)$ of 8.5, and that $\widehat{\chi}(M) = \chi\left(\overline{pM}\right)$, $M \in \mathcal{B}$, and this alternative mode of generation

$$B^*(X) = \mathrm{ucl}\left(\mathrm{ls}\left\{\chi\left(\overline{pM}\right) : M \in \mathcal{B}\right\}\right),$$

in which ls denotes linear span and ucl denotes uniform closure. (See [26]). Thus by 8.5

$$C(S) = \mathrm{ucl}\left(\mathrm{ls}\left\{\chi\left(\overline{pM}\right) : M \in \mathcal{B}\right\}\right).$$

Then the following easy lemma applies to our situation.

Lemma 9.7. *Let* $(Y, \mathcal{G}) \in$ **SpFi** *and let* $X \overset{\mu}{\leftarrow} Y$ *be continuous, so* $A \equiv \widetilde{\mu}C(X) \subseteq C(Y)$. *Then*

$$A_0 \equiv \{b \in C(Y) : \exists E \in \mathcal{G}_\delta \ \forall y_1, y_2 \in E \ \ (by_1 \neq by_2 \Longrightarrow \exists a \in A \ \ (ay_1 \neq ay_2))\}$$

satisfies $A_0 = \mathrm{ucl}(\mathrm{ls}\, A_0)$*, i.e.,* A_0 *is a uniformly closed sub-vector-lattice of* $C(Y)$.

Using $(Y, \mathcal{G}) = (S, \mathcal{C})$ and $X \overset{\mu_X}{\leftarrow} S$ in 9.7, we see that it is enough to show that $\widehat{\chi}(M) = \chi\left(\overline{pM}\right) \in A_0$ for $M \in \mathcal{B}$. Let $\mathcal{A}_0 \equiv \left\{M \in \mathcal{B} : \chi\left(\overline{pM}\right) \in A_0\right\}$. We show that $\mathcal{A}_0$ contains $\mathrm{coz}\, X$ and is a σ-field; $\mathcal{A}_0 = \mathcal{B}$ follows, which will prove the theorem.

To simplify notation, $(\ )'$ denotes Boolean (set-theoretic) complement, and b_M denotes $\chi\left(\overline{pM}\right)$. Observe that $b_M y_1 \neq b_M y_2$ means $y_1 \in \overline{pM}$ and $y_2 \in \overline{pM}' = \overline{pM'}$, or vice-versa. So clearly, $M \in \mathcal{A}_0$ iff $M' \in \mathcal{A}_0$.

Now let $M \in \mathrm{coz}\, X$, so $M = \mathrm{coz}\, g$ for some $g \in C(X)$. Now $X = M \cup M'$ and $S = \overline{pM} \cup \overline{pM'}$, so $E \equiv \mu^{-1}M \cup \overline{pM'} \in \mathcal{C}$. ($E$ is cozero because M was, and $\overline{pM'}$ is clopen. E is dense because $\mu^{-1}M \supseteq pM$.) Then $a \equiv g\mu$ satisfies

$$\forall y_1, y_2 \in E \ \ (b_M y_1 \neq b_M y_2 \Longrightarrow ay_1 \neq ay_2).$$

For if $y_1, y_2 \in E$ are given different values by b_M then one lies in $\overline{pM'}$ and the other in $\mu^{-1}M$. Since $\mu pM' = \iota M' = M'$ is closed in X and μ is continuous, $\mu\overline{pM'} \subseteq \overline{M'} = M'$. That is, one of the y_i's must be mapped by μ to a point of M and the other to a point of M'. The conclusion is that $\mathcal{A}_0 \supseteq \mathrm{coz}\, X$.

Suppose $M_1, M_2, \ldots \in \mathcal{A}_0$. We want $M \equiv \bigcup_n M_n \in \mathcal{A}_0$. Let $f \equiv \sum 2^{-n} b_{M_n}$. Since $M_n \in \mathcal{A}_0$, $b_{M_n} \in A_0$ and so $f \in A_0$ by 9.7. So there is $E(f) \in \mathcal{C}_\delta$ for which

$$\forall y_1, y_2 \in E(f) \ \ (fy_1 \neq fy_2 \Longrightarrow \exists a \in A \ \ (ay_1 \neq ay_2)).$$

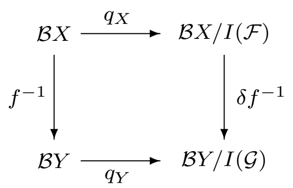
Note that $\bigcup_n \overline{pM_n}$ is dense in $\overline{\bigcup_n pM_n} = \overline{p\bigcup_n M_n} = \overline{pM}$, so that

$$F \equiv \bigcup \overline{pM_n} \cup \left(\overline{pM}\right)' \in \mathcal{C}.$$

Let $E \equiv F \cap E(f)$. Then if $y_1, y_2 \in E$ have, say, $y_1 \in \overline{pM}$ and $y_2 \in \left(\overline{pM}\right)'$ then $y_1 \in \bigcup \overline{pM_n}$ so $fy_1 > 0$, while clearly $fy_2 = 0$. Since $y_1, y_2 \in E(f)$, there is $a \in A$ with $ay_1 \neq ay_2$. Thus $M \in \mathcal{A}_0$. This completes the proof of Theorem 9.6 $\square$

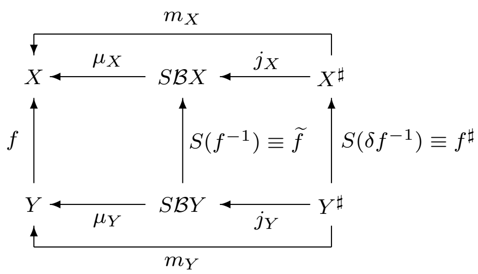
Here is a generalization of 9.6. Given $X \in \mathbf{Comp}$, let $\mathcal{B}_\alpha$ be the α-field in $\mathcal{B}(X)$ generated by coz X (so $\mathcal{B}_{\omega_1}$ is the Baire field). 8.1 provides a continuous surjection $X \overset{\mu}{\leftarrow} S\mathcal{B}_\alpha$. Then it is shown in [29] that $(X, \{X\}) \overset{\mu}{\leftarrow} (S\mathcal{B}_\alpha, \mathcal{G}_\alpha)$ is **SpFi**-monic. Molitor's proof uses locales. Whether this has any relevance to the α-disconnected (monofine) coreflection in α**SpFi** (see §5) is completely unclear.

Proof of 9.2(5). Let $(X, \mathcal{F}) \overset{f}{\leftarrow} (Y, \mathcal{G}) \in \mathbf{SpFi}$. Consider the diagram below. Since



f is continuous it is Baire, and so we have the Boolean homomorphism f^{-1}, which is in fact a σ-homomorphism. Since $f \in \mathbf{SpFi}$, $f^{-1}(F) \in \mathcal{G}$ for $F \in \mathcal{F}$, and thus $f^{-1}(I(\mathcal{F})) \subseteq I(\mathcal{G})$. Thus f^{-1} "drops" over the quotients q_X and q_Y to a unique Boolean homomorphism δf^{-1}. Since $I(\mathcal{F})$ and $I(\mathcal{G})$ are σ-ideals, q_X, q_Y, and δf^{-1} are σ-homomorphisms.

Now consider this diagram. The left square is the outer square in 8.2, the present



X, Y being the K, X of 8.2, and this commutes by 8.3(3). The right square is the Stone dual of the commutative square in the preceding diagram, so it commutes. Consequently, $m_X f^\# = f m_Y$.

Now we note that $\left(X^\#, \mathcal{C}\right) \overset{f^\#}{\leftarrow} \left(Y^\#, \mathcal{C}\right) \in \mathbf{SpFi}$: δf^{-1} is a σ-homomorphism and

Lemma 9.8 ([31])**.** *Let $\mathcal{A}_1 \overset{\phi}{\rightarrow} \mathcal{A}_2$ be a Boolean homomorphism with Stone dual $S\mathcal{A}_1 \overset{S\phi}{\leftarrow} S\mathcal{A}_2$. Then ϕ is a σ-homomorphism iff $(S\mathcal{A}_1, \mathcal{C}) \overset{S\phi}{\leftarrow} (S\mathcal{A}_2, \mathcal{C})$ is* **SpFi**.

Finally, $f^\#$ is unique because m_X is monic by 9.2(4). □

Proof of 9.2(6). Consider (3), (4), and (5). □

10. Monofine in LSpFi

We pick up where we left off at the end of §5, now armed with the quite full knowledge in **LSpFi** of monics and the basically disconnected = monofine coreflection of 9.2.

Corollary 10.1. *In* **LSpFi**, *these condition on $(X, \mathcal{F}) \overset{f}{\leftarrow} (Y, \mathcal{G})$ are equivalent.*

(1) *f is monic.*

(2) *The function* $\left(X^{\#},\mathcal{C}\right)\overset{f^{\#}}{\longleftarrow}\left(Y^{\#},\mathcal{C}\right)$*, in 9.2(5), is monic.*

(3) *The function* $\left(X^{\#},\mathcal{C}\right)\overset{f^{\#}}{\longleftarrow}\left(Y^{\#},\mathcal{C}\right)$*, in 9.2(5), is one-to-one.*

(4) *The function* $\frac{\mathcal{B}(X)}{I(\mathcal{F})}\overset{\delta f^{-1}}{\longrightarrow}\frac{\mathcal{B}(Y)}{I(\mathcal{G})}$*, in the first diagram of the proof of 9.2(5), is onto. This means that for each* $M\in\mathcal{B}(Y)$ *there is* $L\in\mathcal{B}(X)$ *and there is* $E\in\mathcal{G}_\delta$ *for which* $f(M\cap E)\subseteq L$ *and* $f(M'\cap E)\subseteq L'$*. (Here* $(\ )'$ *denotes Boolean, i.e., set-theoretic, complement.)*

Proof. The equivalence of (1), (2), and (3) is the ω_1-case of 5.3.

(2) iff (3). Since $f^{\#}=S\left(\delta f^{-1}\right)$, $f^{\#}$ is one-to-one iff δf^{-1} is onto, by Stone duality. Now to say that δf^{-1} is onto is to say that for each $M\in\mathcal{B}(Y)$ there is $L\in\mathcal{B}(X)$ for which

$$M\bigtriangleup f^{-1}L\equiv\left(M\smallsetminus f^{-1}L\right)\cup\left(f^{-1}L\smallsetminus M\right)\in I(\mathcal{G}),$$

which means $M\bigtriangleup f^{-1}L\subseteq\bigcup_n(Y\smallsetminus G_n)$ for some G_n's in $\mathcal{G}$. Taking complements yields $E\equiv\bigcap_n G_n\in\mathcal{G}_\delta$ for which $M\cap E=f^{-1}L\cap E$, meaning $f(M\cap E)\subseteq L$ and $f(M'\cap E)\subseteq L'$. □

Remark 10.2. *One can show that the condition that* $X\overset{f}{\longleftarrow}Y$ *be monic in 7.1(2) implies that for each* $M\in\mathcal{B}(Y)$ *there is* $E\in\mathcal{G}_\delta$ *for which* $f(M\cap E)\cap f(M'\cap E)=\emptyset$*. (The proof is a relatively straightforward induction on the Baire class of* M*.) It would appear that this implies the condition in 10.1(3) via application of Frolik's generalization of Lusin's First Separation Principle: disjoint analytic sets are separated by Baire sets,* [12, Theorem 3]*. Ignorance prevents further comment here.*

We indicate the classification of subobjects of a given $(X,\mathcal{F})$, i.e., monics into $(X,\mathcal{F})$ [18], which results from 10.1. Let $(X,\mathcal{F})\overset{f}{\longleftarrow}(Y,\mathcal{G})$ be monic, so $\left(X^{\#},\mathcal{C}\right)\overset{f^{\#}}{\longleftarrow}\left(Y^{\#},\mathcal{C}\right)$ is one-to-one. Let $S\equiv f^{\#}\left(X^{\#}\right)$. Then $S\in\operatorname{sub}\left(X^{\#},\mathcal{C}\right)$, so $S\in\mathcal{P}_{\omega_1}\left(X^{\#}\right)$, S is basically disconnected, and $\mathcal{C}(S)=\mathcal{C}\left(X^{\#}\right)\cap S$. Consequently, the "range-restriction of $f^{\#}$" $(S,\mathcal{C})\overset{f_S^{\#}}{\longleftarrow}\left(Y^{\#},\mathcal{C}\right)$ is an isomorphism. Now we have $fm_Y=m_Xf^{\#}$ and $f^{\#}=i_Sf_S^{\#}$, where $\left(X^{\#},\mathcal{C}\right)\overset{i_S}{\hookleftarrow}(S,\mathcal{C})$ is the inclusion. So $fm_Y=m_Xi_Sf^{\#}$. Here $f_S^{\#}$ is an isomorphism, and m_Xi_S is the restriction $m_X|S$ to the P-set S.

We insert an aside. The above shows that if $(X,\mathcal{F})\overset{f}{\longleftarrow}(Y,\mathcal{G})$ is monic, then there is a monic surjection g such that $fg=m_X|S$ for some $S\in\mathcal{P}\left(X^{\#}\right)$, up to isomorphism. The converse fails, though it holds if we insist $g=m_Y$: just consider any infinite compact space X and $(X,\{X\})\overset{f}{\longleftarrow}\left(X^{\#},\left\{X^{\#}\right\}\right)$ with the function $f=m_X$, and $\left(X^{\#},\mathcal{C}\right)\overset{g}{\rightarrow}\left(X^{\#},\left\{X^{\#}\right\}\right)$ with g the identity on $X^{\#}$. Then $fg=m_X$, but f is not monic. Put more abstractly, **LSpFi** is failing the property

$$fg\text{ monic},\ g\text{ monic surjection }\Longrightarrow f\text{ monic},$$

another small complication in the theory of monomorphisms.

Now, in any compact basically disconnected Z, $\mathcal{P}_{\omega_1}(Z)$ is in one-to-one order-reversing correspondence with the set of σ-ideals in $\operatorname{clop}Z$. Since $\operatorname{clop}Z\approx\frac{\mathcal{B}(Z)}{I(\mathcal{C})}$, the family of σ-ideal is in one-to-one correspondence with the set of σ-ideals of $\mathcal{B}(Z)$ which contain $I(\mathcal{C})$.

Suppose $(X,\mathcal{F})\overset{f_i}{\longleftarrow}(Y_i,\mathcal{G}_i)$, $i=1,2$, are monic. Call them *equivalent* if $f_2=f_1\phi$ for an isomorphism ϕ, a general definition [18, §6], and (weaker) $\sharp$-*equivalent* if

$f_1 m_{Y_1}\phi = f_2 m_{Y_2}$. It's easy to see that these are equivalence relations on the class of subobjects of $(X, \mathcal{F})$. Then from the preceding paragraphs, together with 10.1 and its constructions, we have this.

Corollary 10.3. *Let $(X, \mathcal{F}) \in$ **LSpFi**. The following sets are in pairwise one-to-one correspondence.*

1. *$\sharp$-equivalence classes of subobjects of $(X, \mathcal{F})$.*
2. *equivalence classes of subobjects of $(X, \mathcal{F})$ with basically disconnected domains.*
3. *$\mathcal{P}_{\omega_1}\left(X^{\#}\right)$ $\left(= \left\{P \in \mathcal{P}_{\omega_1}(S\mathcal{B}(X)) : P \subseteq X^{\#}\right\}\right)$.*
4. *σ-ideals in $\mathcal{B}(X)/I(\mathcal{F})$.*
5. *σ-ideals in $\mathcal{B}(X)$ which contain $I(\mathcal{F})$.*

Part III. Appendix: Epimorphisms of archimedean ℓ-groups with unit.

$\mathcal{W}$ is the category of archimedean ℓ-groups (lattice-ordered groups) with distinguished weak unit, and unit-preserving ℓ-group homomorphisms. It was our interest in epimorphisms in $\mathcal{W}$ which motivated our invention and investigations of **SpFi**, including the present paper. In this appendix, we derive some high points of that theory from the theory of monomorphisms in **LSpFi**. These "high points" were earlier described in our papers [3], [5], [4], without real reference to **SpFi**. The reader familiar with those papers will recognize the **SpFi** undercurrent there, and the similar technicalities. One notes that the material for **LSpFi** is distinctly more general, and that the results for **W** follow economically, without reproducing the technicalities.

11. The **SpFi**-Yosida representation.

We begin with a discussion of the classical Yosida Theorem. $\overline{R}$ is the two-point compactification of the reals R. For X a space,

$$D(X) \equiv \left\{f \in C\left(X, \overline{R}\right) : f^{-1}R \text{ is dense in } X\right\}.$$

With $f \leq g$ defined pointwise, $D(X)$ is a lattice. For $f, g, h \in D(X)$, "$f + g = h$ in $D(X)$" means $f(x) + g(x) = h(x)$, where all three are real, i.e., lie in the dense set $f^{-1}R \cap g^{-1}R \cap h^{-1}R$. This sometimes/rarely is fully defined in $D(X)$; see 14 below. By a "**W**-object in $D(X)$" we mean a subset $G \subseteq D(X)$ which is a sublattice, which is closed under the partly defined addition, and which contains the constant function 1. It can easily be seen that such a G is an archimedean ℓ-group in which 1 is a weak unit.

Objects in **W** will be denoted as G, then the distinguished weak unit is e_G. YG is the set of ℓ-ideals of G which are maximal for not containing e_G, with the hull-kernel topology.

Here is Yosida's Representation Theorem [36], augmented with recognition of functoriality [15].

Theorem 11.1. *Let $G \in |\mathbf{W}|$.*

1. *YG is compact Hausdorff, and there is a **W**-isomorphism $G \to \widehat{G}$ onto a **W**-object $\widehat{G}$ in $D(YG)$ such that $\widehat{G}$ separates the points of YG.*
2. *For any space X, if $G \to \overline{G}$ is any **W**-isomorphism onto the **W**-object $\overline{G}$ in $D(X)$ then there is a continuous $\tau : X \to YG$ with $\tau(X)$ dense such*

that $\overline{a} = \widehat{a} \circ \tau$ for each $a \in G$. If X is compact Hausdorff and $\overline{G}$ separates the points of X then τ is a homeomorphism.

(3) *If $\phi : G \to H$ is in $\mathbf{W}$, then there is a unique continuous function $Y\phi : YG \leftarrow YH$ for which $\widehat{\phi(a)} = \widehat{a} \circ Y\phi$ for each $a \in G$. Then ϕ is one-to-one iff $Y\phi$ is onto, and if ϕ is onto then $Y\phi$ is one-to-one.*

(4) *A contravariant functor $Y : \mathbf{W} \to \mathbf{Comp}$ is defined by (1) and (3).*

We shall identify $\mathbf{W}$-objects G with the Yosida representations $\widehat{G}$, and the action of morphisms ϕ with the action $\widehat{\phi(a)} = \widehat{a} \circ Y\phi$ above; we suppress all ˆ's. We note further:

Corollary 11.2. (1) *For $G \in |\mathbf{W}|$, let $G^{-1}R \equiv \left\{F \subseteq YG : \exists g \in G \ \left(F \supseteq g^{-1}R\right)\right\}$. Then $\left(YG, G^{-1}R\right) \in \mathbf{LSpFi}$.*

(2) *For $\phi : G \to H \in \mathbf{W}$ with $Y\phi : YG \leftarrow YH$, $(Y\phi)^{-1}\left(g^{-1}R\right) = (\phi g)^{-1} R \in H^{-1}R$ for $g \in G$. Thus*

$$Y\phi : \left(YG, G^{-1}R\right) \leftarrow \left(YH, H^{-1}R\right) \in \mathbf{LSpFi}.$$

(3) *A contravariant functor $S : \mathbf{W} \to \mathbf{LSpFi}$ is defined by (1) and (2). Its action on an object is $SG \equiv \left(YG, G^{-1}R\right)$, and its action on a morphism $G \xrightarrow{\phi} H$ is*

$$S\phi \equiv \left[Y\phi : \left(YG, G^{-1}R\right) \leftarrow \left(YH, H^{-1}R\right)\right].$$

(4) *S is faithful, i.e., one-to-one on Hom-sets.*

The functor $S : \mathbf{W} \to \mathbf{LSpFi}$ in 11.2 is called the **SpFi**-*Yosida functor.*

Corollary 11.3. *Let $\phi \in \mathbf{W}$. If $S\phi$ is* **LSpFi**-*monic, then ϕ is* $\mathbf{W}$-*epic.*

Proof. If $\alpha\phi = \beta\phi$ then $(S\phi)(S\alpha) = S(\alpha\phi) = S(\beta\phi) = (S\phi)(S\beta)$. If $S\phi$ is **LSpFi**-monic then $S\alpha = S\beta$. Then, by 11.2(4), $\alpha = \beta$. □

Naive consideration of the converse of 11.3 (which will turn out to be true) suggests the need for a functor $\mathbf{W} \leftarrow \mathbf{LSpFi}$ in some kind of alliance with S. (An adjoint to S would do the job, but there isn't one. Among other reasons for this, S is not "dense," meaning that it is not the case that each object in **LSpFi** is isomorphic to an object in the range of S; see 11.7 below.) In the next section, we produce a suitable such functor.

We return to a few useful details about the Yosida Representation.

Proposition 11.4 ([17])**.** *Let X be a space. In $D(X)$, addition is fully defined (hence $D(X)$ is an archimedean ℓ-group, and given the weak unit 1, $D(X) \in \mathbf{W}$) iff X is a quasi-F space, meaning that each dense cozero-set is C^*-embedded. Thus, when X is compact and quasi-F, $SD(X) = (X, \mathcal{C})$.*

Now any basically disconnected (BD) spaces is quasi-F; BD implies F-space implies quasi-F-space [13]. A point of considerable present interest is that the monofine objects in **LSpFi** are precisely those of the form $S(D(X))$ for X compact and basically disconnected. Finally for this section, we consider the interesting

Problem 11.5. *What are the objects of $S(\mathbf{W})$? This is a question about filters, not spaces. For compact X, $SC(X) = (X, \{X\})$.*

Here is a fragment of an answer.

Proposition 11.6. *Let $(X, \mathcal{F}) \in \mathbf{LSpFi}$. If $(X, F) \in S(|\mathbf{W}|)$ then $\mathcal{F}$ satisfies: for each $F_1 \neq F_2$ in $\mathcal{F}$, the sets $(X \smallsetminus F_i) \cap F_j$, $i \neq j$, have disjoint closures.*

Corollary 11.7. *For any infinite compact metric space X, $(X, \mathcal{C}) \notin S(|\mathbf{W}|)$.*

Proof of 11.7. Let p be non-isolated, and let (p_n) be a sequence of distinct points such that $p_n \to p$. Let $F_1 \equiv X \smallsetminus \{p_n : n \text{ odd}\}$ and $F_2 \equiv X \smallsetminus \{p_n : n \text{ even}\}$. Then p is in each $\overline{(X \smallsetminus F_i) \cap F_j}$. □

Proof of 11.6. Note that if $f + g = h$ in $D(X)$ for any X then, whenever $f(p) = \infty$ and $g(p) \in R$, $h(p) = \infty$. Then for $G \in |\mathbf{W}|$, $g_1, g_2 \in G$ and $g_i \geq 0$, if $p \in \overline{g_1^{-1}(\infty) \cap g_2^{-1}(R)}$ then $(g_1 - g_2)(p) = +\infty$. Consequently $p \notin \overline{g_2^{-1}(\infty) \cap g_1^{-1}(R)}$, for if it were then $(g_2 - g_1)(p) = +\infty$ by interchanging g_1 and g_2 in the previous argument, and this is a contradiction. □

Proposition 11.8. *For compact X the following are equivalent.*

(1) $(X, \mathcal{C}) \in S(|\mathbf{W}|)$.
(2) $(X, \mathcal{C})$ *satisfies the condition in 11.6.*
(3) X *is quasi-F.*

Proof. The equivalence of the latter two conditions in the considerably greater generality of completely regular frames is Proposition 8.4.10 of [9]. □

12. The functor $E : \mathbf{W} \leftarrow \mathbf{LSpFi}$.

It will be convenient to compress some previous notation. The category **LSpFi** will be denoted **L**. An object of **L** will be denoted X, Z, etc., suppressing the filters unless that is confusing. Hom-sets in **L** or **W** are $\mathbf{L}(X, Z)$ or $\mathbf{W}(G, H)$.

We now define and describe a functor E in the position

$$\mathbf{W} \underset{E}{\overset{S}{\rightleftarrows}} \mathbf{L}$$

which makes clear the association between **W**-epics and **L**-monics.

12.1. The BD coreflection in **L**, from §9 and §10, of $X \in |\mathbf{L}|$ is $m_X : X \leftarrow X^{\#}$; m_X is **L**-monic and surjective, hence **L**-epic. Recall that $X^{\#}$ carries the filter $\mathcal{C}(X^{\#})$ generated by all dense cozeros. Given $f \in \mathbf{L}(X, Z)$, we have unique $f^{\#} \in \mathbf{L}(X^{\#}, Z^{\#})$ with $m_Z f^{\#} = f m_X$. The coreflection functor itself can be denoted $(\ )^{\#}$.

For $X \in |\mathbf{L}|$, set $E(X) \equiv D(X^{\#})$. (Strictly speaking, we mean D of the "space part of $X^{\#}$" here, but let's overlook that.) Since $X^{\#}$ is BD, $D(X^{\#}) \in |\mathbf{W}|$ by 11.4 and the remarks following.

For $f \in \mathbf{L}(X, Z)$, $E(f) \in \mathbf{W}(E(Z), E(X))$ is defined as follows. For $b \in E(Z) = D(Z^{\#})$, $E(f)(b) \equiv b \circ f^{\#}$. Since $f^{\#} \in \mathbf{L}(X^{\#}, Z^{\#})$,

$$F \in \mathcal{C}(Z^{\#}) \Longrightarrow (f^{\#})^{-1} F \in \mathcal{C}(X^{\#}),$$

so $(b \circ f^{\#})^{-1} R = (f^{\#})^{-1} b^{-1} R \in \mathcal{C}(X^{\#})$, and thus $b \circ f^{\#} \in D(X^{\#}) = E(X)$. Clearly $E(f)$ preserves $+$, $\vee$, $\wedge$, and 1, so $E(f) \in \mathbf{W}$.

Proposition 12.2. $\mathbf{W} \overset{E}{\leftarrow} \mathbf{L}$ *is a faithful contravariant functor, and $SE = (\ )^{\#}$.*

Proof. Since $(\ )^{\#}$ is a covariant functor, $(\mathrm{id}_X)^{\#} = \mathrm{id}_{X^{\#}}$ and $(fg)^{\#} = f^{\#}g^{\#}$. These imply $E(\mathrm{id}_X) = \mathrm{id}_{E(X)}$ and $E(fg) = E(g)\,E(f)$.

Suppose $f, g \in \mathbf{L}(X, Z)$ and $E(f) = E(g)$. By 11.1(2) (and noted in 11.4), we have for the Yosida spaces $YE(X) = X^{\#}$ and $YE(Z) = Z^{\#}$, so by the uniqueness 11.1(3), $E(f) = E(g)$ means $f^{\#} = g^{\#}$. Thus $fm_Z = m_Z f^{\#} = m_Z g^{\#} = gm_Z$. Since m_Z is epic, $f = g$. In 11.4 and 11.2 above, we noted $SE(X) = SD\left(X^{\#}\right) = X^{\#}$, filtered by $\mathcal{C}\left(X^{\#}\right)$, and the uniqueness in 11.1(3) shows $SE(f) = f^{\#}$. □

12.3. We consider the composition $ES : \mathbf{W} \to \mathbf{W}$. For $G \in \mathbf{W}$ we have $m \equiv m_{SG} : SG \leftarrow (SG)^{\#}$, which is an **L**-monic surjection. Define $e_G : G \to ESG = D\left((SG)^{\#}\right)$ as $e_G(g) \equiv g \circ m$, $g \in G$. ($g \circ m \in D\left((SG)^{\#}\right)$ since $m^{-1}\left(g^{-1}R\right) \in \mathcal{C}\left((SG)^{\#}\right)$, and e_G clearly preserves $+$, $\vee$, $\wedge$, and 1.) Then $Se_G = m$ by uniqueness in 11.1, so e_G is one-to-one, again by 11.1, and **W**-epic by 11.3. So $G = ESG$ iff e_G is a surjection, thus an isomorphism.

Proposition 12.4. (1) *$G \in E(|\mathbf{L}|)$ iff $G = ESG$ iff YG is BD and $G = D(YG)$.*

(2) *For each $G \in |\mathbf{W}|$, $e_G : G \to ESG$ is the $E(|\mathbf{L}|)$-monoreflection of G.*

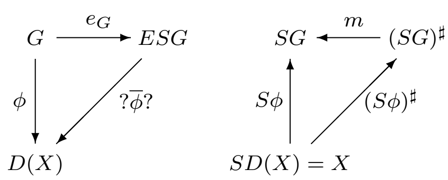
Proof. (1) The implications

$$G = ESG \Longrightarrow G \in E(|\mathbf{L}|) \Longrightarrow [YG \text{ BD},\ G = D(YG)]$$

are clear. Now suppose the last. Then $SG = (YG, \mathcal{C}(YG))$, and $(SG)^{\#} = SG$ because $(\ )^{\#}$ is the BD coreflection in **L**. Thus $ESG = D\left((SG)^{\#}\right) = D(YG) = G$.

(2)In view of (1), the monoreflectivity assertion is that, given X compact BD and $\phi \in \mathbf{W}(G, D(X))$, there is unique $\overline{\phi} \in \mathbf{W}(ESG, D(X))$ with $\overline{\phi}e_G = \phi$; uniqueness is automatic by epicity of e_G.

Consider these diagrams in **W** and **L**, respectively. For the latter, we *do* have



$(S\phi)^{\#}$ with $m(S\phi)^{\#} = S\phi$, since $(SG)^{\#}$ is the BD coreflection. Then, define $\overline{\phi}(b) \equiv b \circ (S\phi)^{\#}$ for $b \in ESG = D\left((SG)^{\#}\right)$. Since $(S\phi)^{\#} \in \mathbf{L}$ and the filter on $(SG)^{\#}$ is that generated by all dense cozeros, each $\overline{\phi}(b) \in D(X)$. As usual, $\overline{\phi} \in \mathbf{W}$ and $S\overline{\phi} = (S\phi)^{\#}$. Since the **L** diagram commutes, so does the **W** diagram. □

13. Epimorphisms in W.

Corollary 13.1. (1) *For $f \in \mathbf{L}$, these are equivalent: f is monic; $E(f)$ is $\mathbf{W}$-epic; SEF is $\mathbf{L}$-monic.*

(2) *For $\phi \in \mathbf{W}$, these are equivalent: ϕ is $\mathbf{W}$-epic; $S(\phi)$ is $\mathbf{L}$-monic; $ES(\phi)$ is $\mathbf{W}$-epic.*

Proof. (1). Since $SEf = f^{\#}$, f is monic iff SEf is monic iff SEf is one-to-one by 10.1. SEf monic implies Ef epic by 11.3. Suppose $E(f)$ is epic and suppose $fg = fh$ in **L**. Then

$$E(g)\,E(f) = E(fg) = E(fh) = E(h)\,E(f),$$

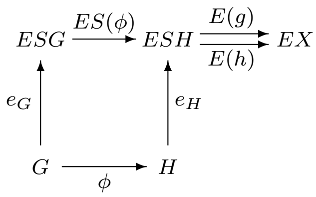
so $E(g) = E(h)$. Since E is faithful, $g = h$.

(2). Using $f = S(\phi)$ in (1) shows $S\phi$ monic iff $ES(\phi)$ epic, and 11.3 says $S(\phi)$ monic implies ϕ epic. Now suppose ϕ is epic, and $S(\phi)\,g = S(\phi)\,h$ in **L**. Then

$$(*) \qquad E(g)\,ES(\phi) = E(h)\,ES(\phi)$$

in **W**.

Displaying helps. Now $e_H\phi = ES(\phi)\,e_G$. This and $(*)$ yield $E(g)\,e_H\phi =$



$E(h)\,e_H\phi$. Since ϕ was supposed epic, and e_H is epic, so is $e_H\phi$. Thus $E(g) = E(h)$, and since E is faithful, $g = h$. □

We now derive the main theorem of [3], a characterization of epimorphisms in **W**. This result was the genesis of our interest in **SpFi**.

Consider $\phi : G \to H$ in **W**, viewed in the Yosida representation. There is the associated $S\phi : SG \leftarrow SH$ in **L**, and the action of ϕ is $\phi(g) = g \circ (S\phi)$. Note that SH, qua **L**-object, carries the filter $H^{-1}R$.

Corollary 13.2. *ϕ is **W**-epic iff for each $h \in H$, there is $E(h) \in \left(H^{-1}R\right)_\delta$ such that $\phi(G)$ separates each pair from $E(h)$ which h separates.*

Proof. By 13.1, ϕ is **W**-epic iff $S\phi$ is **L**-monic, and by 7.3, $S\phi$ is **L**-monic iff for each $b \in C(SH)$ there is $E(b) \in \left(H^{-1}R\right)_\delta$ such that $(\widetilde{S\phi})\,(C(SG))$ separates each pair from $E(b)$ which b separates. ($(\widetilde{S\phi})\,(a) \equiv a \circ S\phi$, the same action as the action of ϕ.) Of course this condition is equivalent to the stated one.

For any element of any **W**-object $k \in K$ we have the "bounded truncates" $k_n \equiv (k \wedge ne_K) \vee (-ne_K)$, $n \in N$, or in the Yosida representation, $k_n = (k \wedge n) \vee (-n) \in C(SK)$. Then, regarding our $\phi : G \to H$,

(1) $\phi(g)$ separates x_1, x_2 iff some $\phi(g_n)$ does, and $\phi(g_n) = (\widetilde{Sg})\,(g_n)$. (This is obvious.)
(2) If each $b \in C(SH)$ has an $E(b)$, then each $h \in H$ has an $E(h)$. (For each n, $h_n \in C(SH)$, so there is $E(h_n)$. Then $E(h) = \bigcap_n E(h_n) \in \left(H^{-1}R\right)_\delta$, and works for h.)

The proof of 13.2 is complete. □

Of course, 7.5 also can be translated into a criterion for **W**-epicity. We omit, or perhaps defer, discussion.

14. Epicompleteness in W.

Of course, so far we have ignored §10, which is this crucial feature of E.

Theorem 14.1. *f is* **L***-monic iff $SE(f)$ $(= f^{\#})$ is one-to-one.*

14.1 will follow from

Theorem 14.2. *ϕ is* **W***-epic iff $ES(\phi)$ is onto.*

Proof. By 13.1 and 10.1, ϕ is epic iff $S\phi$ is monic iff $(S\phi)^{\#}$ is one-to-one. $ES(\phi)$ onto implies this last, by 11.1(3) and the definition of $ES(\phi)$. Conversely, suppose that $(S\phi)^{\#} : (SG)^{\#} \leftarrow (SH)^{\#}$ is one-to-one. In effect then, $(SH)^{\#}$ is a closed subspace of the BD space $(SG)^{\#}$, and for $b \in ES(G)$, $ES(\phi)(b)$ is the restriction $b|(SH)^{\#}$. The assertion "$ES(\phi)$ is onto $ES(H) = D\left((SH)^{\#}\right)$" becomes exactly the following. □

Lemma 14.3 (Tzeng [33])**.** *Let Y be a closed subspace of the compact BD space X. For each $a \in D(Y)$ there is $b \in D(X)$ with $b|Y = a$.*

Proof. Let $a \in D(Y)$. First suppose $a \geq 1$. Then $\frac{1}{a} \in C(Y)$, so there is $g \in C(X)$ with $g|Y = \frac{1}{a}$ and $0 \leq g \leq 1$, by the Tietze-Urysohn Theorem. Define $b : X \to [-\infty, \infty]$ by

$$b(x) \equiv \begin{cases} \frac{1}{g(x)} & \text{if} \quad x \in \operatorname{coz} g \\ 1 & \text{if} \quad x \notin \overline{\operatorname{coz} g} \\ \infty & \quad\text{otherwise} \end{cases}.$$

Since X is BD, $\overline{\operatorname{coz} g}$ is open, and $b \in D(X)$. For general a, let $a_1 \equiv (a \vee 0) + 1$, $a_2 \equiv ((-a) \vee 0) + 1$, so $a = a_1 - a_2$, and extend a_i to $b_i \in D(X)$ by the above. Since $D(X)$ is a group because BD implies quasi-F and Proposition 11.4 applies, $b = b_1 - b_2 \in D(X)$; and $b|Y = a$. □

We now derive some of the main results from [4] and [5].

Call $G \in \mathbf{W}$ *epicomplete* if $G \xrightarrow{\phi} H$ monic and epic in **W** implies ϕ is an isomorphism. (It's easy to see that **W**-monic means one-to-one.)

Corollary 14.4. *For $G \in \mathbf{W}$, these are equivalent.*

(1) *G is epicomplete.*
(2) *$G = ES(G)$.*
(3) *YG is BD and $\widehat{G} = D(YG)$.*
(4) *Each* **W***-epic out of G is a surjection.*

Thus, for any $G \in \mathbf{W}$, $e_G : G \to ES(G)$ is an epicomplete monoreflection.

Proof. (4)$\Longrightarrow$(1) is clear, and (2)$\Longleftrightarrow$(3) by 12.4(1). If (1), then e_G is an isomorphism, which means $\widehat{G} = ES(G)$. If (2), and $G \xrightarrow{\phi} H$ is epic, then in the equation $e_H\phi = ES(\phi)\, e_G$, e_G and $ES(\phi)$ are surjections by hypothesis and 14.2, respectively. So $e_H\phi$ is a surjection, with e_H one-to-one; thus ϕ is a surjection. □

In 14.4, the equivalence of (1) and (3) is originally from [4], and the equivalence with (4) is in [5]. [4] contains an abstract proof that epicompleteness is monoreflective in **W**; that result was obtained slightly earlier by Madden and Vermeer in [23], using frames. [5] contains the explicit description of $ES(G)$ as a quotient

$\mathcal{B}(YG)/J$, $\mathcal{B}(\ )$ being the **W**-object of real-valued Baire functions. This description is visible in our description here $ES(G) = D\left((SG)^{\#}\right)$: $(SG)^{\#} = St\frac{\mathcal{B}(YG)}{I}$, with $I = I\left(G^{-1}R\right)$ (§9 and following). Therefore

$$D\left((SG)^{\#}\right) = D\left(St\mathcal{B}(YG) \smallsetminus \bigcup_{E \in I} E\right) = \frac{D\left(St\mathcal{B}(YG)\right)}{J}$$

for $J = \{f : \operatorname{coz} f \in I\}$, while $D\left(St\mathcal{B}(YG)\right) \approx B(YG)$. We omit the details.

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(R. Ball) Department of Mathematics, University of Denver, Denver, Colorado 80208, U.S.A.

E-mail address, Ball: `rball@du.edu`

URL: `http://www.math.du.edu/~rball`

(Hager) Department of Mathematics, Wesleyan University, Middletown, Connecticut 06459, U.S.A.

E-mail address, Hager: `ahager@wesleyan.edu`