BOUNDED ORBIT INJECTIONS AND SUSPENSION EQUIVALENCE FOR MINIMAL $\mathbb{Z}^2$-ACTIONS

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ABSTRACT. In this paper we prove that if two minimal $\mathbb{Z}^2$ actions of a Cantor set are related by a bounded orbit injection then the associated $\mathbb{R}^2$ suspension spaces are homeomorphic. We also prove a structural result about such suspension spaces. Namely, that they are a finite union of products of Cantor sets with polygons, $C_i \times P_i$, after an identification on boundary, $C_i \times \partial P_i$, with the action given by $\mathbb{R}^2$ on the polygon. The polygons P_i can be chosen to have properties associated with Voronoi or Delaunay tilings corresponding to a set of points located uniformly throughout the plane.

1. INTRODUCTION

In this paper we prove an equivalence between the existence of orbit preserving injective maps with continuous cocycles between minimal $\mathbb{Z}^d$ Cantor systems and the existence of a homeomorphism between the corresponding suspension spaces for $d = 1, 2$. Our main result can be considered as both a higher dimensional version of a theorem of Parry and Sullivan on flow equivalence of $\mathbb{Z}$ actions [PS] and a topological analog of measure-theoretic results on Kakutani equivalence for $\mathbb{Z}^d$ actions [dJR, ORW].

Let us first take the topological point of view. In [PS], Parry and Sullivan showed that two homeomorphisms of zero-dimensional sets are flow equivalent if and only if they are discrete cross-sections of a common system. Here we are concerned with a version of this theorem for higher dimensional, free minimal actions, i.e., $\mathbb{Z}^2$ actions of the Cantor set where all orbits are dense. Besides restricting attention to minimal actions, we substitute the two main notions in their statement to those which are more applicable to our setting of $\mathbb{Z}^d$ actions.

First, we substitute the notion of flow equivalence with homeomorphism of the suspension space, or *suspension equivalence*. The difference between these two notions for $\mathbb{Z}$ actions is that in a suspension equivalence one need not preserve the orientation on path components given by a flow. For minimal actions this is nearly a trivial consideration since a suspension equivalence must either preserve the orientation on all components or reverse all of them.

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Second, we replace the notion of a discrete cross-section with the existence of a bounded orbit injection. We define an orbit injection from one minimal $\mathbb{Z}^d$ actions of a Cantor set (X, T) into another (Y, S) to be a 1-1 map $h : X \to Y$ with the property that two points are in the same T-orbit if and only if their images under h are in the same S orbit. In particular, there is a map $\beta : X \times \mathbb{Z}^d \to \mathbb{Z}^d$ satisfying

$$S^{\beta(x,v)}h(x) = hT^v(x).$$

To say that h is a bounded orbit injection is essentially to require that β be continuous. On the surface, saying that there is a bounded orbit injection from T into S seems weaker than saying that T is a discrete cross-section of S, since the latter condition requires preservation of an order structure. For example, if S and T are $\mathbb{Z}$ actions, we would require $\beta(x, n) > 0$ for $n > 0$. However, if we restrict our attention to minimal actions, a result of Boyle [B] (see also [BT]) on bounded orbit equivalences makes the following formulation of the Parry-Sullivan theorem possible.

Theorem 1. *Two minimal $\mathbb{Z}$ actions S and T on a Cantor set are suspension equivalent if and only if there are bounded orbit injections from S and T into a common $\mathbb{Z}$ action R.*

Our goal in this paper is to prove the above statement for minimal $\mathbb{Z}^d$ actions. The forward direction, that suspension equivalence implies the existence of bounded orbit injections, is not difficult to prove for $\mathbb{Z}^d$ actions. The proof of the reverse direction for $\mathbb{Z}^2$ actions will occupy most of our attention in this paper. A significant step in our proof (the tiling results of Section 4) requires that we work with $\mathbb{Z}^2$ actions instead of the more general $\mathbb{Z}^d$ actions.

From the perspective of ergodic theory, this mimics the development of Kakutani equivalence. In [ORW], Ornstein, Rudolph and Weiss proved that if S and T are ergodic transformations which have isomorphic induced maps then S and T are measurable cross-sections to a common ergodic flow. The above equivalence relation in either formulation is called Kakutani equivalence. In [dJR], del Junco and Rudolph proved a higher dimesional version using orbit injections satisfying certain bounds and Katok cross-sections of n-dimensional flows. Thus we present our result also as a topological analog of the result of del Junco and Rudolph.

The proof of the main theorem is a constructive one, beginning with one map, we construct the other. Delaunay tilings of orbits of points play an important role here. We view these as a $\mathbb{Z}^2$ substitute for the Rohlin multitower picture for $\mathbb{Z}$ systems. The core of the proof is taking advantage of the special properties of the bounded orbit injection along with geometric properties of Delaunay tilings to create the map between the suspension spaces.

The outline of the paper is as follows. In the next section, we set out important definitions and give some proofs of simpler statements, leaving only the difficult direction of the main theorem. We outline the proof of the difficult direction at the end of the Section 2.

2. Preliminaries

2.1. **Definitions.** By a Cantor set we will mean a zero-dimensional compact metric space with no isolated points. (All such spaces are homeomorphic.) We consider $\mathbb{Z}^d$ actions of a Cantor set X (primarily $d = 2$). If T is such an action, we denote the action of a vector $v \in \mathbb{Z}^d$ on a point $x \in X$ by $T^v x$. In this paper we will exclusively consider the situation where T is *minimal* and *free*. To say that T is a free $\mathbb{Z}^d$ action is to say that T is aperiodic, or $T^v x = x$ implies $v = 0$. To say that T is minimal is to say that for all $x \in X$, the orbit of x (the set $\{T^v x : v \in \mathbb{Z}^d\}$) is dense in X. When X is a Cantor set and T is a minimal, free $\mathbb{Z}^d$ action of X we will call the pair (X, T) a $\mathbb{Z}^d$ *minimal Cantor system*, or simply, a *minimal Cantor system*. More definitions follow which we give in the setting of minimal Cantor systems, although in most cases identical or similar definitions would work for any free $\mathbb{Z}^d$ action of a Cantor set.

We describe suspension spaces and the relation of suspension equivalence for minimal Cantor systems.

Definition 2. For a $\mathbb{Z}^d$ minimal Cantor system (X, T), let X_T denote the space $X \times \mathbb{R}^d / \sim$ where $(x, u') \sim (y, v')$ if and only if $\exists u, v \in \mathbb{Z}^d$ such that $u' - u = v' - v$ and $T^u x = T^v y$. We call the topological space X_T the *suspension space* for (X, T).

Definition 3. Let (X, T) and (Y, S) be two minimal Cantor systems. We say that (X, T) and (Y, S) are *suspension equivalent* if the suspension spaces X_T and Y_S are homeomorphic.

There is a natural $\mathbb{R}^d$ action on $X \times \mathbb{R}^d$ given by $R^u : (x, v) \mapsto (x, v + u)$. One can check that this action respects the equivalence $\sim$, and thus induces a natural $\mathbb{R}^d$ action on X_T which we will refer to as $\widetilde{T}$.

Next we define bounded orbit injections. Let (X, T) and (Y, S) be minimal Cantor systems.

Definition 4. An *orbit injection* from (X, T) to (Y, S) is a continuous 1-1 map $h : X \to Y$ such that for all $x, x' \in X$ there is a vector $v \in \mathbb{Z}^d$ such that $T^v x = x'$ if and only if there is a vector $\beta(x, v) \in \mathbb{Z}^d$ such that $S^{\beta(x,v)} h(x) = h(x')$.

We say that an orbit injection h is an *orbit equivalence* if it is onto.

Whether or not the above definition of orbit injection should include the provision that $h(X)$ contain an open set is a debatable point, and this provision seems necessary for more general (non-minimal) versions of this theorem. In our case however, it turns out not to be necessary (see Theorem 7 below). Even more, it would not make the proof our main theorem any simpler.

Definition 5. We say that an orbit injection h from (X, T) to (Y, S) is *bounded* if there is a number $M > 0$ such that for all $v \in \mathbb{Z}^d$, $\|v\| = 1$ implies $\|\beta(x, v)\| < M$.

The primary purpose of this paper is to prove the following theorem.

Theorem 6 (Main Theorem)**.** *Let (X,T) and (Y,S) be $\mathbb{Z}^d$ minimal Cantor systems for $d = 1, 2$. The systems (X,T) and (Y,S) are suspension equivalent if and only if there is a $\mathbb{Z}^d$ minimal Cantor system (Z,R) and bounded orbit injections $h_1 : (X,T) \to (Z,R)$ and $h_2 : (Y,S) \to (Z,R)$.*

We have the following result, which follows from our main theorem, but for which we do not know a direct proof.

Theorem 7. *Let (X,T) and (Y,S) be $\mathbb{Z}^2$ minimal Cantor systems and suppose the continuous injection $h : X \to Y$ is a bounded orbit injection. Then $h(X)$ is of 2nd category in Y, i.e., $h(X)$ contains an open set.*

We begin with proving the simple cases of Theorem 6.

2.2. **The case when $d = 1$.** Let (X,T) and (Y,S) be $\mathbb{Z}$ minimal Cantor systems. To say that T and S are flow equivalent, is to say that there is an orientation-preserving homeomorphism from X_T to Y_S. Here, the orientation is determined by the natural $\mathbb{R}$ action on the two spaces. Let $\widetilde{x} \in X_T$. The path-connected component of $\widetilde{x}$ is the $\mathbb{R}$ orbit of $\widetilde{x}$ and is dense in X_T.

Suppose $g : X_T \to Y_S$ is a homeomorphism. Then g maps the $\mathbb{R}$-orbit of $\widetilde{x}$ bijectively to the $\mathbb{R}$-orbit of $g(\widetilde{x})$. The map g does so in either an orientation-preserving or orientation-reversing way. But since the $\mathbb{R}$-orbit of $\widetilde{x}$ is dense in X_T and its image is dense in Y_S, g has the same effect on all of the orientations of the $\mathbb{R}$ orbits in the space.

Proposition 8. *Two $\mathbb{Z}$ minimal Cantor systems (X,T) and (Y,S) are suspension equivalent if and only if S is flow equivalent to T or T^{-1}.*

Upon noting the above, the proof of the main theorem for $d = 1$ is a corollary of the following theorems.

Theorem 9 ([B])**.** *Let (X,T) and (Y,S) be $\mathbb{Z}$ minimal Cantor systems. There is a bounded orbit equivalence between T and S if and only if S is conjugate to T or T^{-1}.*

Theorem 10 ([PS])**.** *Let T and S be $\mathbb{Z}$ actions of a zero-dimensional compact metric space X. Then T and S are flow equivalent if and only if T and S are both conjugate to induced systems of a common $\mathbb{Z}$ action R.*

If R is a $\mathbb{Z}$ action of a zero-dimensional space X, then by an *induced system of R* we mean a clopen set $C \subset X$ along with the first return map $R_C : C \to C$ defined by $R_C(c) = R^n(c)$ where n is the smallest positive integer such that $R^n(c) \in C$. If S is conjugate to an induced system of R and R is minimal, then there is a bounded orbit injection from S into R.

Proof of Theorem 6 for $d = 1$. First suppose that T and S are suspension equivalent. This implies S is flow equivalent to T^e where $e \in \{-1, 1\}$. By the Parry-Sullivan Theorem this implies that T^e and S are both conjugate to induced systems of a common $\mathbb{Z}$ action of a third Cantor system (Z, R). Because the induced systems in R are minimal, so is (Z, R). It follows that there are bounded orbit injections from T and S into R.

Conversely, suppose there is a bounded orbit injection h from a $\mathbb{Z}$ minimal Cantor system (X, T) into another (Y, S). Let us assume here that $h(X)$ contains a clopen set $D \subset Y$. One can show this by modifying the proof of Boyle to our circumstance or by following through the argument in this paper which works equally well for $d = 1, 2$. Let $D = h^{-1}(C)$ and consider the induced dynamical systems (D, T_D) and (C, S_C). The map $h : D \to C$ is a bounded orbit *equivalence* between T_D and S_C.

Therefore, by Boyle's Theorem, one of T_D, T_D^{-1} is conjugate to S_C. By the Parry-Sullivan Theorem, this happens if and only if T or T^{-1} is flow equivalent to S. Therefore, T and S are suspension equivalent.

Similarly, if bounded orbit injections exist from T and S into a third system R then since suspension equivalence is an equivalence relation, T and S are suspension equivalent. □

2.3. **One direction of the main theorem when $d > 1$.** Here we provide a proof of the simpler direction of main theorem for any $d \geq 2$.

Proof of $\Rightarrow$ in Theorem 6. Assume that the suspension spaces X_T and Y_S are homeomorphic.

We begin by noting that there is a simple way to construct a minimal action with a bounded orbit injection from T into it by the following construction. Fix any $m \geq 0$ and let $Q_m = \{0, 1, 2, \ldots, m-1\}^d$ with the discrete topology. Let $X_m = X \times Q_m$, a Cantor set. For any vector $u \in \mathbb{Z}^d$, we can write $u = mv + w$ where $v \in \mathbb{Z}^d$ and a $w \in Q_m$. Define an action T_m on X_m by $T_m^u(x, 0) = (T^v x, w)$. There is a bounded orbit injection from T into T_m defined by $x \mapsto (x, 0)$.

We will refer to the action T_m above as a tower of size m over T. Note that there is a natural homeomorphism from X_T into $(X_m)_{T_m}$ given by $g_m : (x, u) \mapsto ((x, 0), mu)$.

Now assume that $\widetilde{h}$ is a homeomorphism from Y_S to X_T. Then in each set of the form $\left\{(x, v) : x \in X, \|v\| < \frac{2}{3}\right\} \subset X_T$ there is a finite number of points which are images of points of the form $(y, 0) \in Y_S$ where $y \in Y$. Further, there is a lower bound b on the size of a vector v such that both (x, u) and $(x, u + v)$ are images of points of the form $(y, 0) \in Y_S$. Therefore, if m is sufficiently large, the composition of maps $g_m\widetilde{h}$ is a homeomorphism from Y_S to $(X_m)_{T_m}$ with the property that each set of the form $\{(x, v) : x \in X_m, \|v\| < \frac{2}{3}\} \subset (X_m)_{T_m}$ contains exactly 0 or 1 point which is an image of a point of the form $(y, 0) \in Y_S$. The idea is to create a bounded orbit injection from S into T_m by mapping $y \in Y$ to the point $x \in X_m$ such that $g_m\widetilde{h}\,(y, 0) \in \left\{(x, v) : x \in X_m, |v| < \frac{2}{3}\right\}$. The only problem is that the point $g_m\widetilde{h}\,(y, 0)$ can be in the intersection of up to d^2 of these sets. Thus we will have the choice of up to d^2 points of the form $(x, \sum \delta_i e_i)$ where $\delta_i = 0$ or 1. Nevertheless, we can partition the set Y into finitely many clopen sets and make a consistent choice of δ_i's on each one. The resulting map will be a bounded orbit injection from S into T_m. □

The next five sections of the paper are dedicated to proving the theorem below, the converse direction of the main theorem when $d = 2$.

Theorem 11. *Let (X, T) and (Y, S) be $\mathbb{Z}^2$ minimal Cantor systems and suppose there is a bounded orbit injection from (X, T) to (Y, S). Then (X, T) and (Y, S) are suspension equivalent.*

Here we give a brief outline of the proof. In Section 3, we establish some important properties of bounded orbit injections and define a continuous map from $g : X_T \to Y_S$ which is an injection of $\mathbb{R}^2$ orbits. The main issue to resolve is that g is not necessarily 1-1 on X_T.

Note that the $\mathbb{R}^2$ orbit of a point $\widetilde{x} \in X_T$ is the continuous, 1-1 image of $\mathbb{R}^2$. Through this correspondence, we define tilings of X_T. Roughly, a tiling of X_T is a continuous, translation-commuting assignment of a family of tilings of $\mathbb{R}^2$ to points in X_T. The tilings we use are Delaunay tilings with some special properties. All of this is made precise in Section 4, the main lemma of which is the Tiling Lemma 22.

By restricting g to the union of the boundaries of the tiles, we may consider g as simply a continuous map on a planar graph. The heart of the proof is to establish the Graph Isomorphism Lemma in Section 5. In this section, we show that the properties of the bounded orbit injection, along with the geometric properties of the Delaunay tilings are strong enough to repair the lack of injectivity of g on the graph.

Finally, we show that it is a simple matter to extend the graph isomorphisms to a homeomorphism $g : X_T \to Y_S$. The details are contained in Section 6.

3. Bounded Orbit Injections

Assume that h is a bounded orbit injection from one $\mathbb{Z}^2$ minimal Cantor system (X, T) to another $\mathbb{Z}^2$ minimal Cantor system (Y, S). Let β be as in Definition 4. The function β satisfies the following cocycle equation for any $x \in X$ and $v, w \in \mathbb{Z}^2$.

$$\beta(x, v + w) = \beta(x, v) + \beta(T^v x, w) \tag{3.1}$$

That h is bounded means for some M, $\|v\| = 1$ implies $\|\beta(x, v)\| < M$.

Proposition 12. *For any $v \in \mathbb{Z}^2$, $\|\beta(x, v)\| < 2\|v\|M$.*

Proof. Let $0 = w_0, w_1, w_2, \ldots, w_n = v$ be a sequence of vectors in $\mathbb{Z}^2$ such that $\|w_{i+1} - w_i\| = 1$ and $n < 2\|v\|$. Then since $\beta(x, v) = \sum_{i=0}^{n-1} \beta(T^{w_i} x, w_{i+1})$, we have $\|\beta(x, v)\| < Mn < 2\|v\|M$. □

Proposition 13. *For $v \in \mathbb{Z}^2$ fixed, there exists a (finite) clopen partition of X such that $\beta(x, v)$ is constant on each element of the partition.*

Proof. For each $u \in \mathbb{Z}^2$, let $E_u^v = \{x \in X : \beta(x,v) = u\}$. We will be done if we can show that each set E_u^v is clopen.

Suppose that $\{x_n\}$ is a sequence of points in E_u^v with $\lim_{n\to\infty} x_n = x$. Then we have for all n, $h(T^v x_n) = S^{\beta(x_n,v)}h(x_n) = S^u h(x_n)$. Since h, T^v and S^u are all continuous, taking $\lim_{n\to\infty}$ of both sides yields $h(T^v x) = S^u h(x)$, which implies $x \in E_u^v$. Therefore E_u^v is closed. By the previous proposition, there can only be finitely many u for which E_u^v is nonempty. Since $X = \bigcup_u E_u^v$ each set E_u^v is clopen. □

Corollary 13.1. *The function $\beta : X \times \mathbb{Z}^2 \to \mathbb{Z}^2$ is continuous.*

The following is a key observation about continuous bounded orbit injections. From the definition we have an upper bound on $\|\beta(x,v)\|$ depending on $\|v\|$, this gives a lower bound. Let $E_u = \cup_{v\in\mathbb{Z}^2} E_u^v$.

Proposition 14. *E_u is clopen.*

Proof. E_u is clearly open, so we need to show it is closed. Take a sequence $x_n \in E_u$ which converges to $x \in X$. Because S^u and h are continuous $\lim_{n\to\infty} S^u h(x_n) = S^u h(x)$. On the other hand each $x_n \in E_u^{v_n}$ for some unique v_n. There exists a subsequence $T^{v_{n(k)}} x_{n(k)}$ which converges, to say z. So $\lim_{n(k)\to\infty} h(T^{v_{n(k)}}(x_{n(k)})) = h(z)$. But $h(T^{v_{n(k)}}(x_{n(k)})) = S^u h(x_{n(k)}) \to S^u h(x)$. So $h(z) = S^u h(x)$, thus $S^u h(x) \in h(X)$. Because h is an orbit injection there exists $v \in \mathbb{Z}^2$ such that $h(T^v x) = S^u h(x)$. Thus $x \in E_u^v \subset E_u$. So E_u is closed. □

Corollary 14.1. *For any $M > 0$ there is a number $N > 0$ such that for all $x \in X$ and $v \in \mathbb{Z}^2$, if $\|v\| > N$ then $\|\beta(x,v)\| > M$.*

Proof. By Proposition 14, there exists an N_u such that for all $x \in X$ and $v \in \mathbb{Z}^2$, if $\|v\| > N$ then $\|\beta(x,v)\| \neq u$, because E_u^v is empty for v sufficiently large. So the same is true for the finite collection of $u \in \mathbb{Z}^2$ with $\|u\| < M$. □

3.1. **Bounded orbit injections, minimality and 2nd category sets.** At the moment we do not know whether $h(X)$ contains an open set in Y or not. Establishing that it does will be important for showing that the map we construct from X_T to Y_S is onto. In particular, if $h(X)$ contains an open set in Y then we know that $h(X)$ *occurs syndetically in* Y (with respect to S), i.e. $\bigcup_{\|v\|<n} S^v h(X) = Y$. It will only be at the very end of the proof of the main theorem that we will know that $h(X)$ contains an open set. Here we lay the groundwork for that statement.

Let us say that for $C \subset X$ clopen, the image $h(C)$ *occurs syndetically in its own orbit* if there exists $n > 0$ such that for any $w \in \mathbb{Z}^2$ and $y \in h(C)$, there exists $v \in \mathbb{Z}^2$ such that $\|v\| < n$ and $S^{w-v} y \in h(C)$. That is, there exists an $n > 0$ such that

$$\bigcup_{\|v\|<n} S^v h(C) = \bigcup_{v\in\mathbb{Z}^2} S^v h(C). \tag{3.2}$$

Lemma 15. *Let $C \subset X$ be clopen. The following are equivalent.*

(1) *$h(C)$ contains an open set in Y,*
(2) *$h(C)$ occurs syndetically in Y,*
(3) *the image $h(C)$ occurs syndetically in its own orbit.*

Proof. (1) implies (2) because S is minimal. (2) implies (3) is clear. Assume (3), that $h(C)$ occurs syndetically in its own orbit. The set $h(C)$ is closed and therefore $\bigcup_{v\in\mathbb{Z}^2} S^v h(C) = \bigcup_{\|v\|<n} S^v h(C)$ is closed. But every S-orbit is dense which means that $\bigcup_{v\in\mathbb{Z}^2} S^v h(C)$ is dense and closed in Y, i.e., $\bigcup_{v\in\mathbb{Z}^2} S^v h(C) = Y$. If $h(C)$ contains no open sets, then $h(C)$ is nowhere dense in Y, and the same is true of $S^v h(C)$ for any $v \in \mathbb{Z}^2$. This would mean that Y is a countable union of nowhere dense sets, contradicting the Baire Category Theorem. □

3.2. **Constructing a map from X_T to Y_S.** Recall that X_T is the space $X \times \mathbb{R}^2$ modulo the relation $(x, u') \sim (y, v')$, if and only if $\exists u, v \in \mathbb{Z}^d$ such that $u' - u = v' - v$ and $T^u x = T^v y$. Let $\Pi_T : X \times \mathbb{R}^2 \to X_T$ denote the quotient map. We define the $\mathbb{R}^2$ *flow* action R on $X \times \mathbb{R}^2$ by $R^t(x, r) = (x, r + t)$ for $t \in \mathbb{R}^2$. Note that the flow action R respects the equivalence relation and thus projects down to induce the flow action $\widetilde{T}$ on X_T, where $\widetilde{T}^v \Pi_T = \Pi_T R^v$ for all $v \in \mathbb{R}^2$.

Because $T^v x = x$ implies $v = 0$ (T is free), there is a bijection between $\mathbb{R}^2$ and each $\widetilde{T}$ orbit. Namely, for $\widetilde{x} \in X_T$ there exists for each $v \in \mathbb{R}^d$ a unique element $\widetilde{T}^v \widetilde{x}$. Conversely, for each $\widetilde{y}$ in the $\widetilde{T}$ orbit of $\widetilde{x}$ there exists a unique $v \in \mathbb{R}^d$ such that $\widetilde{y} = \widetilde{T}^v \widetilde{x}$.

Observe, if a function $f : X \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies an equation similar to Equation 3.1,

$$f(x, v + w) \quad = \quad \beta(x, v) + f(T^v x, w)$$

(which we call a *β mediated cocycle* equation), then there is a well-defined function $\tilde{f} : X_T \to Y_S$ defined by $\tilde{f}(\widetilde{x}) = \Pi_S(h(x), f(x, u))$, for any $(x, u) \in \Pi_T^{-1}(\widetilde{x})$. This idea will be used several times beginning with the following.

Given the bounded orbit injection $h : X \to Y$, we now construct a continuous map $g : X_T \to Y_S$ by (essentially) extending the map h linearly. We do this as follows (see Figure 1). Let G be the graph with vertex set $\mathcal{V} = \mathbb{Z}^2$, and edge set $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ where $\mathcal{E}_i = \{v + te_i : v \in \mathbb{Z}^2, t \in [0, 1]\}$ for $i = 1, 2$ and $\mathcal{E}_3 = \{v + t(e_1 - e_2) : v \in \mathbb{Z}^2\}$. Let T_1 (and T_{-1}) be the triangles formed by the closed convex hull(s) of 0, e_1, e_2 (and 0, $-e_1$, $-e_2$).

For each $x \in X$, the map $v \mapsto \beta(x, v)$ is defined on $\mathbb{Z}^2$. We can extend this map to the edges of the triangles and their interiors linearly. That is, if $p \in \mathbb{R}^2$ then p is a convex combination of vertices v, w, u of (at least) one of the triangles, i.e., $p = \alpha_1 v + \alpha_2 w + \alpha_3 u$ where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$. We then define $\Gamma(x, p) = \alpha_1 \beta(x, v) + \alpha_2 \beta(x, w) + \alpha_3 \beta(x, u)$. In addition to being a well-defined map from

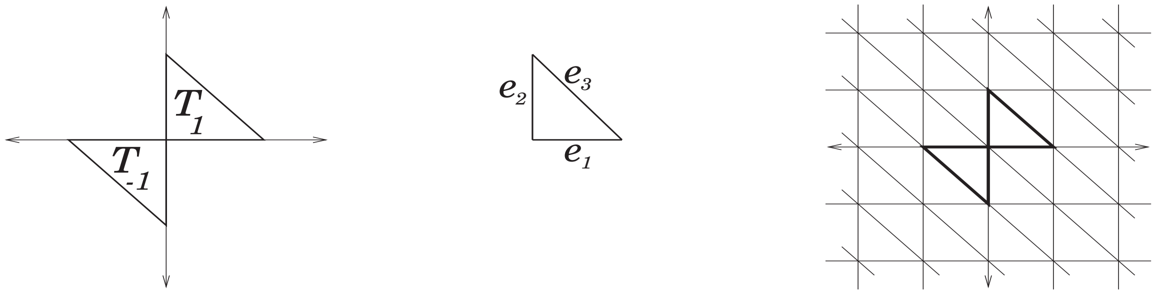


FIGURE 1. Extending h with a graph.

$X \times \mathbb{R}^2 \to \mathbb{R}^2$, Γ satisfies a β mediated cocycle equation,

$$\Gamma(x, r + v) = \beta(x, v) + \Gamma(T^v x, r)$$

for all $x \in X$, $v \in \mathbb{Z}^2$, and $r \in \mathbb{R}^2$. Moreover, analogs of propositions 12, 13, and 14.1 and their corollaries exist. For the most part, we leave the proofs to the reader.

Proposition 12′. *For any $v \in \mathbb{R}^2$, $|\Gamma(x, v)| \le 2\|v\|M$.*

Proposition 13′. *There exists a clopen partition $\{X_j\}$ of X such that $\beta(x, 0)$, $\beta(x, \pm e_1)$, $\beta(x, \pm e_2)$ are constant on each X_j. And hence the images $\Gamma(x, t)$ for $t \in T_i$ and $i = \pm 1$ are constant on each X_j, as well.*

Corollary 13.1′. The function $\Gamma : X \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.

Corollary 14.1′. *Let $u, v \in \mathbb{R}^2$. For any $M > 0$ there is an $N > 0$ such that if $\|u - v\| > N$ then for any $x \in X$, $\|\Gamma(x, u) - \Gamma(x, v)\| > M$.*

Proof. Fix $x \in X$, and let u be inside the triangle with vertices u_1, u_2 and u_3 and v inside the triangle with vertices v_1, v_2 and v_3. Corollary 14.1 shows that if $\|v_i - u_j\|$ is sufficiently large, then so is $\|\Gamma(x, v_i) - \Gamma(x, v_j)\|$ for all $i, j \in \{1, 2, 3\}$. Now it follows from Proposition 12 that any point in convex hull of the vertices $\Gamma(x, u_1), \Gamma(x, u_2), \Gamma(x, u_3)$ is far away from any point in the convex hull of $\Gamma(x, v_1), \Gamma(x, v_2), \Gamma(x, v_3)$. All bounds are uniform in x since this was true in Propositions 12 and Corollary 14.1. □

As mentioned earlier, because Γ satisfies a β mediated cocycle equation there is well-defined map $g : X_T \to Y_S$ defined as follows. For any $(x, r) \in \Pi_T^{-1}(\widetilde{x})$ let

$$g(\widetilde{x}) = \Pi_S(h(x), \Gamma(x, r)).$$

The central difficulty here is that one can not expect g to be injective. It is the purpose of this paper to show that there is a perturbation of g which is injective.

Let $\widetilde{T}$ and $\widetilde{S}$ denote the $\mathbb{R}^2$ actions on X_T and Y_S, respectively. While g is not an injection, g is a injection on the orbits of these $\mathbb{R}^2$ actions. That is, $\widetilde{x}$ and $\widetilde{x}'$ are in the same $\widetilde{T}$ orbit, if and only if the images $g(\widetilde{x})$ and $g(\widetilde{x}')$ are in the same $\widetilde{S}$ orbit. Thus, associated to g there must be an $\mathbb{R}^2$ cocycle $\alpha : X_T \times \mathbb{R}^2 \to \mathbb{R}^2$ such that for $(\widetilde{x}, r) \in X_T \times \mathbb{R}^2$,

$$g(\widetilde{T}^r(\widetilde{x})) = \widetilde{S}^{\alpha(\widetilde{x}, r)} g(\widetilde{x}).$$

Let us describe α in terms of Γ.

Proposition 16. *For $\widetilde{x} \in X_T$, $r \in \mathbb{R}^2$,*

$$\alpha(\widetilde{x}, r) = \Gamma(x, s+r) - \Gamma(x, s)$$

for any $(x, s) \in \Pi_T^{-1}(\widetilde{x})$.

Proof. First observe $(x, s+r) \in \Pi_T^{-1}(\widetilde{R^T}\widetilde{x})$. Next, $\Gamma(x, s+r) - \Gamma(x, s)$ is the vector in $\mathbb{R}^2$ which takes the Γ image of (x, s) to the Γ image of $(x, s+r)$. Consider any other preimage $(T^v x, s-v) \in \Pi_T^{-1}(\widetilde{x})$, then $(T^v x, s+r-v) \in \Pi_T^{-1}(R^r \widetilde{x})$, and it is easy to check that

$$\Gamma(x, s+r) - \Gamma(x, s) = \Gamma(T^v x, s+r-v) - \Gamma(T^v x, s-v)$$

because $\Gamma(T^v x, s-v) = \Gamma(x, s) - \beta(x, v)$. So $\alpha(\widetilde{x}, r)$ is well defined and $S^{\alpha(\widetilde{x}, r)}$ takes $g(\widetilde{x})$ to $g(\widetilde{T}^r \widetilde{x})$. It is also easy to check that α, so defined, is a cocycle. □

The cocycles α and β share some properties. For example, for $\widetilde{x} = \Pi_T(x, 0)$ and $v \in \mathbb{Z}^2$, $\alpha(\widetilde{x}, v) = \beta(x, v)$.

Proposition 12″. *For any $r \in \mathbb{R}^2$ and $\widetilde{x} \in X_T$, $\|\alpha(\widetilde{x}, r)\| \leq 2\|r\|M$.*

Let $A = \Pi_T(X \times \mathbb{Z}^2)$. In fact, A is homeomorphic to X because $\Pi_T(X \times \mathbb{Z}^2) = \Pi_T(X \times \{0\})$ and Π_T is injective on the latter. For any $\widetilde{x} \in X_T$ there exist $\widetilde{a} \in A$ and $r \in T_1 \cup T_2$ such that $\widetilde{T}^r \widetilde{a} = \widetilde{x}$ (because this holds above in $X \times \mathbb{R}^2$).

Proposition 13″. *Given $n > 0$, there exists a clopen partition $\{A_j\}$ of A such that if $\widetilde{x}, \widetilde{y} \in A_j$ and $\|r\| < n$ then $\alpha(\widetilde{x}, r) = \alpha(\widetilde{y}, r)$.*

Corollary 13.1″. The function $\alpha : X_T \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.

Corollary 14.1″. *For any $M > 0$ there is an $N_M > 0$ such that if $u, v \in \mathbb{R}^2$ and $\|u - v\| > N_M$ then for any $\widetilde{x} \in X_T$, $\|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > M$.*

The construction in this section gives more of an idea of the proof. Like we have done here, we will construct a tiling of X_T by action polygons. The map g constructed in this section gives a well-defined map ∂g from the graph defined by the boundaries of the tiles to Y_S. The map ∂g will not be 1-1, but the properties described in this section will be strong enough to allow us to repair ∂g to a graph isomorphism. Then we will be just left with the simple matter of extending ∂g to all of X_T.

4. Suspension space tilings

There is a great deal of literature on tilings, Voronoi and Delaunay tilings (e.g. [A, OBS] for introductions and physical science applications, [L1, L2, P] for some recent applications to dynamics). We will review some of the essentials for our construction.

Notation. *Set $B(n) = \{v \in \mathbb{R}^2 : \|v\| < n\}$ and $\overline{B}(n) = \{v \in \mathbb{R}^2 : \|v\| \leq n\}$.*

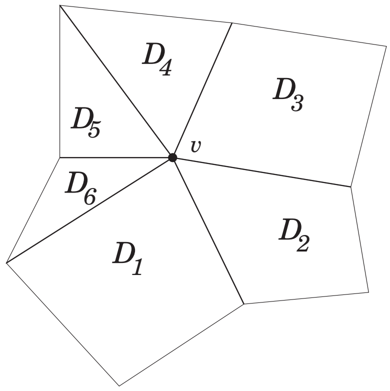


FIGURE 2. $Star(v)$

4.1. **Tilings of $\mathbb{R}^2$ by polygons.** We will consider finite polygonal tilings and the (infinite planar) graph formed by considering the union of boundaries of tiles in a tiling. By *a tiling of* $\mathbb{R}^2$ *by polygons* we mean a countable collection $\mathcal{D}$ of convex polygons which cover $\mathbb{R}^2$, have disjoint interiors, with polygons meeting edge to edge. By a *finite* tiling of $\mathbb{R}^2$ by polygons we mean that the set of polygons up to equivalence by translation only consists of a finite number of polygons. So, there exists a list $\{P_i\}_{i=1}^K$ of *prototiles* such that for each $D \in \mathcal{D}$, $D = P_i + v$ for some $i \in 1, .., K$ and $v \in \mathbb{R}^2$. A *full tiling space for* $\{P_i\}_{i=1}^K$ is the set of all finite tilings of $\mathbb{R}^2$ which can be made with prototiles $\{P_i\}_{i=1}^K$. This space is compact with the appropriate topology, and we have $\mathbb{R}^2$ acting on full tiling spaces by translation, $\sigma^v \mathcal{D} = \mathcal{D} - v$.

For any finite tiling of $\mathbb{R}^2$ by convex polygons $\mathcal{D}$, there is an associated boundary $\partial\mathcal{D} = \cup\{\partial D : D \in \mathcal{D}\}$. Each prototile P_i is the convex hull of a finite set of vertices, and each tile D in a finite tiling is the convex hull of translates of these vertices. Let $\mathcal{V}(D)$ be these vertices of D and let $\mathcal{E}(D)$ be the set of line segments that connect the vertices of D without intersecting the interior of D. We associate to the tiling $\mathcal{D}$ the graph $(\mathcal{V}, \mathcal{E})$ whose vertex set $\mathcal{V} = \cup\{\mathcal{V}(D) : D \in \mathcal{D}\}$ and whose (undirected) edge set $\mathcal{E} = \cup\{\mathcal{E}(D) : D \in \mathcal{D}\}$ is the union of its polygons' edges. So $\partial\mathcal{D} = \cup\{e : e \in \mathcal{E}\}$. We will use the notation $[v, w]$ (or $[w, v]$) to denote the edge with endpoints v and w. For each $v \in \mathcal{V}$ there is a subset $\mathcal{E}_v \subset \mathcal{E}$ of edges which have v as an endpoint ($\mathcal{E}_v = \{e \in \mathcal{E} : v \in e\}$). Let $Star(v) = \cup\{e : e \in \mathcal{E}_v\}$. See Figure 2.

4.2. **Tilings by action polygons.** The map Π_T from $X \times \mathbb{R}^2$ to X_T is a continuous surjection. We will often consider Π_T restricted to a single path-connected component $\{x\} \times \mathbb{R}^2$, and this is a continuous bijection which respects geometry. For example, suppose $l \subset \mathbb{R}^2$ is a straight line, and consider the set $\Pi_T(x, l)$. Because $\Pi_T^{-1}(\Pi_T(x, l)) = \cup_{v \in \mathbb{Z}^2}(T^v x, l - v)$, it follows that if any subset $l' \subset \mathbb{R}^2$ and $x' \in X$ are such that $\Pi_T(x', l') = \Pi_T(x, l)$, then l' is also a straight line. Thus, we may refer to the set $\Pi_T(x, l) \subset X_T$ unambiguously as a straight line. Moreover, because Π_T

intertwines the actions of R and $\widetilde{T}$ for $\widetilde{x} \in X_T$, a set $L \subset \{\widetilde{T}^v \widetilde{x} : v \in \mathbb{R}^2\}$ may be said to be a line segment if and only if there exists a line segment l in $\mathbb{R}^2$ such that $L = \{\widetilde{T}^v \widetilde{x} : v \in l\}$. Every notion of Euclidean geometry in $\mathbb{R}^2$ is well-defined in the $\widetilde{T}$ orbits of X_T. Thus we will be able to speak about tilings and their associated graphs in the $\widetilde{T}$ orbits of X_T, as well as notions such as the distance between two vertices and angle between two edges without ambiguity.

Let us define an *orbital metric* $d_T : X_T \times X_T \to \mathbb{R}^* = \mathbb{R} \cup \infty$ (and d_S on $Y_S \times Y_S$). For $\widetilde{x}, \widetilde{y} \in X_T$, if $\widetilde{x}$ and $\widetilde{y}$ are not in the same $\widetilde{T}$ orbit, then $d_T(\widetilde{x}, \widetilde{y}) = \infty$. If $\widetilde{x}$ and $\widetilde{y}$ are in the same $\widetilde{T}$ orbit, then $\widetilde{x} = \widetilde{T}^v \widetilde{y}$ for some $v \in \mathbb{R}^2$ (because X is free, v is unique). We define $d_T(\widetilde{x}, \widetilde{y}) = \|v\|$, where $\|v\|$ is the Euclidean norm of v. For $E, F \subset X_T$, subsets of the same $\widetilde{T}$ orbit, it follows that

$$d_T(E, F) \equiv \inf\{\|v\| : E \cap \widetilde{T}^v F \neq \emptyset\}.$$

Definition 17. Suppose we have a Cantor subset $\mathbf{V}_T \subset X_T$, a clopen partition $\mathcal{V}_T = \{\mathbf{V}_i\}_{i=1}^I$ of $\mathbf{V}_T$ and set of polygons $\{P_i\}_{i=1}^I$ such that

1. the collection of sets $\mathbf{P}_i = \{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in P_i\}$ for $i \in 1, .., I$ are closed and cover X_T,
2. the sets $\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in Interior(P_i)\}$ are open and pairwise disjoint
3. $\mathbf{P}_i \cap \mathbf{P}_j = C \times e$ where C is a Cantor set and e is either a point, or a translate of an edge that occurs in both P_i and P_j.

We call the triple $\mathbf{P}_T = (\mathbf{V}_T, \mathcal{V}_T, \{P_i\})$ a *tiling of X_T by action polygons.*

We will deal with two related tilings of X_T by action polygons, denoted $\mathbf{P}_T = (\mathbf{V}_T, \mathcal{V}_T, \{P_i\})$ and $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \widehat{\mathcal{V}}_T, \{D_i\})$.

Suppose $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \widehat{\mathcal{V}}_T, \{D_i\})$ is a tiling of X_T by action polygons, then we define the boundary $\partial \mathbf{D}_T$ of $\mathbf{D}_T$ to be

$$\partial \mathbf{D}_T = \cup\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \widehat{\mathbf{V}}_i, r \in \partial D_i\}.$$

We associate a graph $G(\mathbf{D}_T) = (\widetilde{\mathbf{V}}_T, \mathbf{E}_T)$ to $\mathbf{D}_T$ that tiles $\partial \mathbf{D}_T$ as follows. Let $Q(D)$ denote the extreme points of the convex polygon D. Let $\widetilde{\mathbf{V}}_T = \{\widetilde{T}^q \widetilde{x} : \widetilde{x} \in \widehat{\mathbf{V}}_i, q \in Q(D_i)\}$. Each D_i is a convex polygon, and so there is a collection E_i of line segments (in $\mathbb{R}^2$) whose endpoints are the extreme points of D_i and such that $\partial D_i = \cup\{e : e \in E_i\}$. Thus, if $\widetilde{x} \in \widehat{\mathbf{V}}_i$ and $e \in E_i$ one can see that the set $\{\widetilde{T}^r \widetilde{x} : r \in e\}$ is a line segment (in X_T) with endpoints in $\widetilde{\mathbf{V}}_T$. The set of edges, $\mathbf{E}_T$ is the set of all such line segments,

$$\mathbf{E}_T = \{\{\widetilde{T}^r b : r \in e\} : e \in E_i, b \in \widehat{\mathbf{V}}_i\},$$

and it should be clear that $\partial \mathbf{D}_T = \cup\{\mathbf{e} : \mathbf{e} \in \mathbf{E}_T\}$. One can check that $(\widetilde{\mathbf{V}}_T, \mathbf{E}_T)$ is a graph (with an uncountable number of vertices) and that the restriction of $(\widetilde{\mathbf{V}}_T, \mathbf{E}_T)$ to any $\widetilde{T}$ orbit is a planar graph (with a countable number of vertices). We will use the notation $e = [\widetilde{x}, \widetilde{y}]$ for an edge in $\mathbf{E}_T$ which has endpoints $\widetilde{x}$ and $\widetilde{y}$ in $\widetilde{\mathbf{V}}_T$. For each $\widetilde{x} \in \widetilde{\mathbf{V}}_T$ we define $star(\widetilde{x}) = \cup\{e : \widetilde{x} \in e \in \mathbf{E}_T\}$, that is, $star(\widetilde{x})$ is the union of the set of edges which end at $\widetilde{x}$. We will see that in our tilings $\mathbf{D}_T$ and $\mathbf{P}_T$ the sets $\widetilde{\mathbf{V}}_T = \mathbf{V}_T$.

4.3. **Voronoi and Delaunay Tilings of $\mathbb{R}^2$.** Let $\mathcal{V}$ be a subset of $\mathbb{Z}^2$. A subset $\mathcal{V} \subset \mathbb{Z}^2$ is said to be m *separated* if for every pair $u, v \in \mathcal{V}$ of distinct elements, $\|u - v\| \geq m$. We say $\mathcal{V} \subset \mathbb{Z}^2$ is m *syndetic* in $\mathbb{Z}^2$ if for every $u \in \mathbb{Z}^2$ there exists $v \in \mathcal{V}$ such that $\|u - v\| \leq m$. If $\mathcal{V} \subset \mathbb{Z}^2$ is both m separated and m syndetic in $\mathbb{Z}^2$ then we will say $\mathcal{V}$ is m *regular* in $\mathbb{Z}^2$. Let $\mathfrak{M}_m = \{\mathcal{V} \subset \mathbb{Z}^2 : \mathcal{V}$ is m regular in $\mathbb{Z}^2\}$. We will construct tilings based on m regular subsets of $\mathbb{Z}^2$.

For $v \in \mathcal{V} \subset \mathbb{R}^2$ the *Voronoi tile* corresponding to v with respect to $\mathcal{V}$ is the set $V_v \equiv \{r \in \mathbb{R}^2 : d(r, v) \leq d(r, \mathcal{V})\}$ (where $d(u, v) = \|u - v\|$ is the Euclidean metric). Let $\mathfrak{V}(\mathcal{V}) = \{V_v : v \in \mathcal{V}\}$. Fix $m \geq 1$ and suppose that $\mathcal{V} \subset \mathbb{Z}^2$ and $\mathcal{V}$ is m regular. Then the collection $\mathcal{P} = \{V_v - v : v \in \mathcal{V}$ for some $\mathcal{V} \in \mathfrak{M}_m\}$ is a finite set of convex polygonal prototiles $\mathcal{P} = \{P_i\}_{i=1}^K$ and $\overline{B}(0, m/2) \subset P_i \subset \overline{B}(0, m + 1)$ [L1]. Thus the collection $\mathfrak{V}(\mathcal{V})$ is a finite tiling of $\mathbb{R}^2$ which we refer to as the *Voronoi tiling* corresponding to $\mathcal{V}$. For $m \geq 1$ fixed, all tilings of the form $\mathfrak{V}(\mathcal{V})$ where $\mathcal{V} \in \mathfrak{M}_m$ are elements in the full tiling space for the set of prototiles $\{P_i\}_{i=1}^K$.

For each $p \in \mathbb{R}^2$ we define $\mathcal{V}(p) = \{v \in \mathcal{V} : d(p, v) = d(p, \mathcal{V})\}$ and let $r_p = d(p, \mathcal{V})$ (thus, $\mathcal{V}(p) \subset \partial B(p, r_p)$ and $\mathcal{V}(p) \cap B(p, r_p) = \emptyset$) where d denotes the Euclidean metric. We let $\widehat{\mathcal{V}} = \{p \in \mathbb{R}^2 : |\mathcal{V}(p)| \geq 3\}$ (See Figure 3). Thus, for each element $p \in \widehat{\mathcal{V}}$, the set of elements in $\mathcal{V}$ nearest to p and equidistant from p is denoted $\mathcal{V}(p)$. If we let $\mathcal{H}(\mathcal{V}(p))$ denote the closed convex hull of $\mathcal{V}(p)$, then $\mathcal{H}(\mathcal{V}(p))$ is a convex polygon inscribed in the circle $\partial B(p, r_p)$ which we refer to as the *Delaunay tile* corresponding to p. The collection $\mathfrak{D}(\mathcal{V}) = \{\mathcal{H}(\mathcal{V}(p)) : p \in \widehat{\mathcal{V}}\}$ is a tiling of $\mathbb{R}^2$ known as the *Delaunay tiling of* $\mathbb{R}^2$ *corresponding to* $\mathcal{V}$. Again for fixed $m \geq 1$, $\{\mathcal{H}(\mathcal{V}(p)) - p : p \in \widehat{\mathcal{V}}$ for some $\mathcal{V} \in \mathfrak{M}_m\}$ is a finite set of convex polygonal prototiles.

We will work with the graph associated with the boundary of Delaunay tilings. Note that with the above construction, the vertex set is simply $\mathcal{V}$ and the edge set $\mathcal{E}$ is comprised of the collection of line segments connecting vertices which are adjacent on the circle $\partial B(p, r_p)$ where $p \in \widehat{\mathcal{V}}$. Thus we are able to establish the following facts, important to our construction (see [L2] for details).

Fact 18. *There is a $\delta > 0$ such that for any $m \geq 1$, if $\mathcal{V}$ is an m-regular set then the minimum angle formed by two Delaunay edges meeting at a Delaunay vertex is at least δ.*

Fact 19. *(Cor 6.4.2* [L2]*) Given $K > 0$ there exists $m_K \geq 2K$ such that for $m \geq m_K$, if $\mathcal{V}$ is m regular then for any pair of Delaunay edges $l, l' \in \mathcal{E}$, either $l \cap l' \neq \emptyset$ or $d(l, l') > 2K$.*

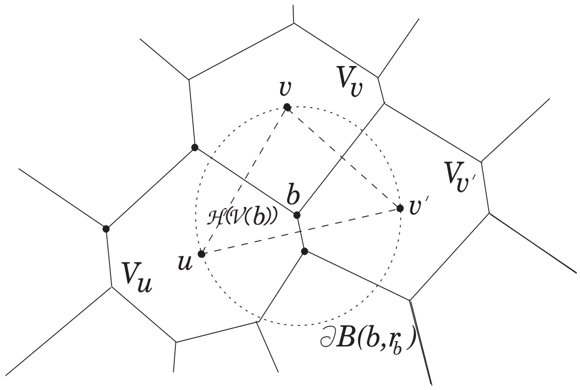


FIGURE 3. Voronoi and Delaunay tiles: $u, v, w \in \mathcal{V}$ and $b \in \widehat{\mathcal{V}}$

Corollary 19.1. *For $m \geq m_K$, if $\mathcal{V}$ is m regular then $\mathcal{V}$ is a $2K$ separated subset of $\mathbb{Z}^2$, and (in particular) every Delaunay edge $l \in \mathcal{E}$ has $\|l\| \geq 2K$.*

4.4. **Voronoi and Delaunay tilings by action polygons.** Our goal is to construct Delaunay tilings of X_T by action polygons maintaining some of the regularity properties of the vertices. In particular, we wish to establish the following, modestly extending a variation of a well-known lemma [K1, LM, L1].

Lemma 20. *Let (X,T) be a $\mathbb{Z}^2$ minimal Cantor system. For any m there exists a clopen set $C \subset X$ such that for each $x \in X$ the set $\mathcal{V}_x = \{v \in \mathbb{Z}^2 : T^v x \in C\}$ is an m regular subset of $\mathbb{Z}^2$.*

Further, the map $x \mapsto \mathcal{V}_x$ is continuous and shift-commuting, i.e., $\mathcal{V}_{T^v x} = \sigma^v \mathcal{V}_x$,

Proof. Our goal is to construct a clopen set $C \subset X$ with the following properties.

(1) for any $x \in X$, $\{T^v x : \|v\| \leq m\} \cap C \neq \emptyset$,
(2) for any $x \in C$, $\{T^v x : 0 < \|v\| \leq m\} \cap C = \emptyset$.

Since $T^v x = x$ implies $v = 0$ and the space X is compact, we can let

$$\delta = \min\{d(x, T^v x) : \|v\| \leq m\}.$$

For each $x \in X$, let E_x be a clopen set containing x with diameter less than δ. Since the space X is compact there is a finite subcover $E_1, E_2, \dots, E_n$ of clopen sets. We may assume that the sets E_j are pairwise disjoint, otherwise substitute E_j with $E_j - \bigcup_{i<j} E_i$. Thus we have a finite collection of pairwise disjoint clopen sets $\{E_j\}$ such that $X = \bigcup E_j$ and $diam(E_j) < \delta$ for all j. Let $F_1 = E_1$ and for $j > 1$, let $F_j = F_{j-1} \cup \left(E_j - \bigcup_{|v| \leq m} T^v F_{j-1}\right)$. Let $C = F_n$.

We now prove property (1). Let $x \in X$. Then $x \in E_j$ for some $1 \leq j \leq n$. Either $x \in F_j$, in which case $x \in C$, or $x \in \bigcup_{|v| \leq m} T^v F_{j-1}$. In the first case we are done, so assume the latter case. Then there is a $x' \in F_{j-1}$ and a vector v with $|v| \leq m$ such that $x = T^v x'$, or $T^{-v} x = x'$. This implies $\{T^v x : |v| \leq m\} \cap C \neq \emptyset$.

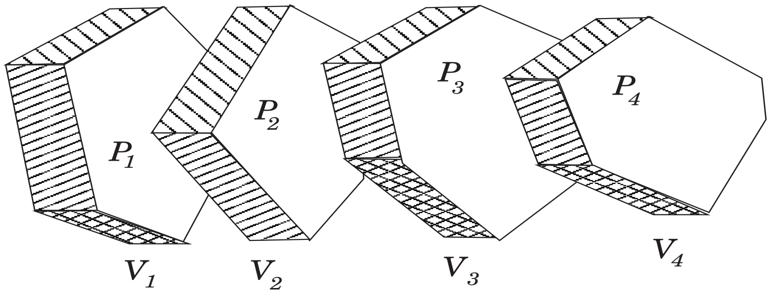


FIGURE 4. The structure of X_T, a flow in a finite set of polygons.

To prove property (2) assume $x \in C$. There is a $j > 0$ such that $x \notin F_k$ for $k < j$ and $x \in F_k$ for all $k \geq j$. Let $x' = T^v x$ for some vector v with $|v| \leq m$. Then $x' \notin F_k$ for $k < j$ since that would imply $x' \in F_{j-1}$ which would imply that $x \notin F_j$. Further, $x' \notin F_j$ since this would imply $x' \in E_j$, but $diam(E_j) < \delta = \min\{d(x, T^v x) : |v| \leq m\}$, a contradiction. Finally, if $k > j$ then $x' \notin F_k$ since $x' \in \bigcup_{|v|\leq m} T^v F_{k-1}$ for all $k > j$. Therefore, $\{T^v x : 0 < |v| \leq m\} \cap C = \emptyset$. This proves the set $\mathcal{V}_x = \{v \in \mathbb{Z}^2 : T^v x \in C\}$ is an m regular subset of $\mathbb{Z}^2$.

For $r > 0$, define $\mathcal{V}_x(r) = \{v \in \mathcal{V}_x : \|v\| < r\}$. To say the map $x \to \mathcal{V}_x$ is continuous is to say that for any $r > 0$, the map $x \to \mathcal{V}_x(r)$ is locally constant. But $\mathcal{V}_x(r) = \{v : \|v\| < r \text{ and } T^v x \in C\}$ and C is clopen. Hence, the map $x \mapsto \mathcal{V}_x$ is continuous.

Finally, it follows from the definition of $\mathcal{V}_x$ that $\mathcal{V}_{T^v x} = \mathcal{V}_x - v = \sigma^v \mathcal{V}_x$. □

The following result is related to the Tiling Lemma 22 and is of interest in its own right as a description of the structure of X_T. (See Figure 4.)

Lemma 21. *(A Voronoi tiling of X_T) Let (X, T) be a $\mathbb{Z}^2$ minimal Cantor system. Given m, there exist a Cantor set $\mathbf{V}_T \subset X_T$, a clopen partition $\{\mathbf{V}_i\}_{i=1}^I$ of $\mathbf{V}_T$, and a finite set of convex polygons $P_i \subset \mathbb{R}^2$, $i = 1, .., I$ such that $(\mathbf{V}_T, \{\mathbf{V}_i\}, \{P_i\})$ is a finite tiling of X_T by convex action polygons. Moreover,*

1. *(for $\widetilde{x} \in V_T$) the sets $\{r \in \mathbb{R}^2 : \widetilde{T}^r \widetilde{x} \in \mathbf{V}_T\}$ are m regular in $\mathbb{Z}^2$.*
2. *For each $\widetilde{x} \in \mathbf{V}_i$, $\{\widetilde{T}^r \widetilde{x} : r \in P_i\} = \{\widetilde{z} \in X_T : d_T(\widetilde{x}, \widetilde{z}) \leq d_T(\widetilde{x}, \mathbf{V}_T)\}$.*
3. *$\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in Interior(P_i)\} = Interior(\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in P_i\})$.*

Proof. Consider the subsets $(x, \mathcal{V}_x) = \{(x, v) : v \in \mathcal{V}_x\}$ of $X \times \mathbb{R}^2$. Because the map $x \mapsto \mathcal{V}_x$ commutes with the shift, the Π_T image of $(x, \mathcal{V}_x)$ is not only well-defined, but we have $\{v \in \mathbb{R}^2 : \widetilde{T}^v \Pi_T(x, 0) \in \Pi_T(x, \mathcal{V}_x)\} = \mathcal{V}_x$.

Define $\mathbf{V}_T = \Pi_T(C \times 0) = \{\Pi_T(x, \mathcal{V}_x) : x \in X\}$. Let $\{P_i\}_{i=1}^I = \{V_v - v : v \in \mathcal{V}_x, x \in X\}$ and $\mathbf{U}_i = \Pi_T\{x \in C : V_0 = P_i\}$. Setting $\mathbf{P}_i = \{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in P_i\}$ we have

1. the collection of sets $\mathbf{P}_i$ for $i \in 1, .., I$ are closed and cover X_T,

(2) the sets $\{\widetilde{T}^r \widetilde{x} : \widetilde{x} \in \mathbf{V}_i, r \in Interior(P_i)\}$ are open and pairwise disjoint,

the first two conditions for a tiling by action polygons. The third condition, that $\mathbf{P}_i \cap \mathbf{P}_j$ be a Cantor subset of $\mathbf{V}_T$ cross an edge or a point does not necessarily hold. But this can easily be arranged by refining the partition $\mathcal{U} = \{\mathrm{U}_i\}_{i=1}^I$. For any $m, n > 0$, we define a clopen partition $\{\mathbf{V}_i\}_{i=1}^K$ of $\mathbf{V}_T$ which refines the partition $\bigvee_{v \in \mathbb{Z}^2, \|v\| < m} T^v \mathcal{U}$, and satisfies $T^v \mathbf{V}_i \cap \mathbf{V}_i \neq \emptyset$ implies $\|v\| > n$. By choosing m, n sufficiently large, we have the desired properties. □

Similarly, we have the Delaunay version of this lemma, which carries the geometric facts we desire in our construction.

Lemma 22 (The Tiling Lemma)**.** *We have the following.*

1. *There exist a tiling of X_T by action polygons $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \widehat{\mathcal{V}}_T, \{D_i\})$ and an associated graph $G(\mathbf{D}_T) = (\mathbf{V}_T, \mathbf{E}_T)$ that tiles $\partial \mathbf{D}_T$ in which $\mathbf{V}_T \subset \Pi_T(X \times \mathbb{Z}^2)$.*
2. *There exist a clopen partition $\{\mathbf{V}_i\}$ of $\mathbf{V}_T$, such that if $\widetilde{x}, \widetilde{y} \in \mathbf{V}_i$ then $\widetilde{T}^v \widetilde{x} \in star(\widetilde{x})$ if and only if $\widetilde{T}^v \widetilde{y} \in star(\widetilde{y})$.*
3. *There exists a $\delta > 0$ such that for any $K > 0$ there exists $m_K \geq 2K$ such that the following hold. For each $m > m_K$ Parts 1 and 2 hold and in addition so do the following*
   (a) $\forall \widetilde{x} \in \mathbf{V}_T$ *the set* $\mathcal{V}_{\widetilde{x}} = \{v \in \mathbb{R}^2 : \widetilde{T}^v \widetilde{x} \in \mathbf{V}_T\} \subset \mathbb{Z}^2$ *and is m-regular in* $\mathbb{Z}^2$.
   (b) *for each edge* $e = [\widetilde{x}, \widetilde{y}] \in \mathbf{E}_T$, $m \leq d_T(\widetilde{x}, \widetilde{y}) \leq 2(m+1)$,
   (c) *for each pair of distinct edges* $e, f \in \mathbf{E}_T$, $e \cap f \neq \emptyset$ *implies* $\angle(e, f) > \delta$,
   (d) *for any pair of Delaunay edges* $l, l' \in \mathbf{E}_T$, *either* $l \cap l' \neq \emptyset$ *or* $d_T(l, l') > 2K$.

This lemma follows from considering the Delaunay tilings corresponding to the Voronoi tilings constructed in the previous lemma, along with a possible refinement of the partition $\mathbf{V}_i$ here to satisfy Part (2). The extra geometric properties in Part (3) stem from Facts 18 and 19 and Corollary 19.1.

5. Graph Isomorphism Lemma

If $G = (\mathcal{V}, \mathcal{E})$ is a planar graph, then a *planar graph isomorphism* on G is a continuous, injective function $\phi : G \to \mathbb{R}^2$. We define the image graph $\phi(G)$ as the planar graph with vertices $\phi(\mathcal{V})$ and edges $\phi(\mathcal{E})$. By *orbit injective* we mean, if for any $\widetilde{x}, \widetilde{y} \in X_T$ and $v \in \mathbb{R}^2$, $\phi(\widetilde{x}) = \widetilde{S}^v \phi(\widetilde{y})$, then $\widetilde{x} = \widetilde{T}^u \widetilde{y}$ for some $u \in \mathbb{R}^2$, and conversely, $\widetilde{x} = \widetilde{T}^u \widetilde{y}$ implies $\phi(\widetilde{x}) = \widetilde{S}^v \phi(\widetilde{y})$ for some $v \in \mathbb{R}^2$.

The main goal of this section is to prove the Graph Isomorphism Lemma, below.

Lemma 23 (Graph Isomorphism Lemma)**.** *Given a continuous bounded orbit injection $h : X \to Y$, there exist; an $m > m_K$, a tiling of X_T by action polygons $\mathbf{D}_T$, and a continuous, injective, orbit injective map $\phi : \partial \mathbf{D}_T \to Y_S$ such that*

1. *$\mathbf{D}_T$ satisfies the conditions of the Tiling Lemma (22),*

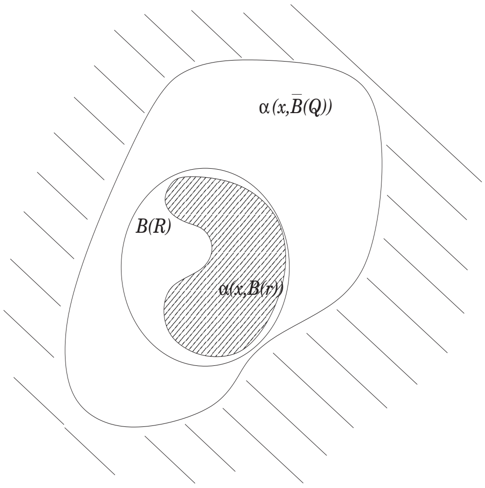


FIGURE 5. Choosing m

(2) *ϕ is a graph isomorphism on* $(\mathbf{V}_T, \mathbf{E}_T)$,
(3) $\phi(\mathbf{V}_T) \subset \Pi_S(Y \times \mathbb{Z}^2)$,
(4) *For each* $\mathbf{V}_i$ *there is a map* $\widehat{\alpha}_i : \{v : \widetilde{T}^v \widetilde{x} \in star(\widetilde{x}), \widetilde{x} \in \mathbf{V}_i\} \to \mathbb{R}^2$ *such that for all* $\widetilde{y} \in \mathbf{V}_i$,

$$\phi(\widetilde{T}^v \widetilde{y}) = \widetilde{S}^{\widehat{\alpha}_i(v)} \phi(\widetilde{y}).$$

Let ∂g be the restriction of g (from Section 3.2) to $\partial \mathbf{D}_T$ viewed as a map from $(\mathbf{V}_T, \mathbf{E}_T)$ to Y_S, where $\mathbf{D}_T$ is from the Tiling Lemma. It (∂g) satisfies all the desired properties in Lemma 23 except Parts (2) and (4). In light of Proposition 13$''$, property (4) can easily be arranged by further refinement of the partition $\{\mathbf{V}_i\}$. However, ∂g is not a graph homomorphism, much less a graph isomorphism. Our goal in this section is to perturb ∂g so that it is injective on $\partial \mathbf{D}_T$. We will call the perturbation ϕ and it will be the graph isomorphism sought in the Graph Isomorphism Lemma 23.

5.1. **Choosing the size of m.** In order to have all the desired machinery up and running for the perturbation, we need an appropriate value for m. We now describe how to find m (see Figure 5).

By Part 3c of the Tiling Lemma (and Fact 18), if $m \geq 1$ the angle at which edges in a Delaunay tiling based on an m regular vertex set has uniform lower bound, $\delta > 0$. Also, Corollary 14.1$''$ with $M = 0$ gives us a number N_0 such that if $d_T(\widetilde{x}, \widetilde{y}) > N_0$ then $g(\widetilde{x}) \neq g(\widetilde{y})$. Let R_1 and R_2 be (any) two rays originating from the origin such that the angle between them is bounded below by δ. Define $r > N_0$ to be such that if $u_1 \in R_1$ and $u_2 \in R_2$ are any two points with $\|u_i\| \geq r$, then $\|u_1 - u_2\| \geq N_0$.

Because of Proposition 12$''$, $\alpha(\widetilde{x}, \overline{B}(r))$ is uniformly bounded for $\widetilde{x} \in X_T$ so there exists an $R > 0$ such that $\alpha(\widetilde{x}, \overline{B}(r)) \subset B(R)$ for all $\widetilde{x} \in X_T$. By Corollary 14.1$''$ there exists $Q = N_R \geq r$ such that for all $\widetilde{x} \in X_T$ and all v with $\|v\| > Q$, $\|\alpha(\widetilde{x}, v)\| > R$. Note the by-product $Q = N_R \geq r \geq N_0$.

We **choose** $m > m_Q$ where m_Q is from Part 3 of the Tiling Lemma (22), (thus $m > 2r$).

5.2. **Preliminary results dependent on the choice of m.** Let $e \in \mathbf{E}_T$ be an edge with endpoints $\widetilde{x}$ and $\widetilde{y}$, $e = [\widetilde{x}, \widetilde{y}]$. Since $d_T(\widetilde{x}, \widetilde{y}) \geq m > 2r$, the following set is nonempty. Define

$$e(r) = \{\widetilde{z} \in e : d_T(\widetilde{x}, \widetilde{z}), d_T(\widetilde{z}, \widetilde{y}) \geq r\}.$$

The following two propositions are used in the proof of Lemma 28, they follow from our choice of m.

Proposition 24. *For any distinct $e, e' \in \mathbf{E}_T$ in the same $\widetilde{T}$ orbit, $d_T(e(r), e'(r)) \geq N_0$, and hence $g(e(r)) \cap g(e'(r)) = \emptyset$.*

Proof. If e and e' are in the same $\widetilde{T}$ orbit and do not intersect, then because $m \geq m_Q$ and $Q \geq N_0$, Part (3d) in the Tiling Lemma (22) tells us $d_T(e, e') \geq N_0$. Because $e(r) \subset e$ and $e'(r) \subset e'$ it follows that $d_T(e(r), e'(r)) \geq N_0$. That $g(e(r)) \cap g(e'(r)) = \emptyset$ follows by Corollary 14.1$''$ and the choice of N_0.

If e and e' do intersect, then they do so at a common endpoint, an element of $\mathbf{V}_T$, and the result follows from our choice of r. □

Corollary 24.1. *For any distinct $e, e' \in \mathbf{E}_T$, $g(e(r)) \cap g(e'(r)) = \emptyset$.*

Proposition 25. *For any $\widetilde{x} \in X_T$ and $u, v \in \mathbb{R}^2$,*

$$\|u - v\| > 2Q \textit{ implies } \|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > 2R.$$

In particular, $\|u - v\| \geq m$ implies $\|\alpha(\widetilde{x}, u) - \alpha(\widetilde{x}, v)\| > 2R$.

Proof. By assumption $u + \overline{B}(Q) \cap v + \overline{B}(Q) = \emptyset$. If $w \notin u + \overline{B}(Q)$, then $w - u \notin \overline{B}(Q)$ and, by our choice of Q, $\alpha(\widetilde{T}^u\widetilde{x}, w - u) \notin \overline{B}(R)$. By the cocycle equation $\alpha(\widetilde{T}^u\widetilde{x}, w - u) = \alpha(\widetilde{x}, w) - \alpha(\widetilde{x}, u)$, so $\alpha(\widetilde{x}, w) \notin \alpha(\widetilde{x}, u) + \overline{B}(R)$. Consequently, $\alpha(\widetilde{x}, \mathbb{R}^2 \backslash u + \overline{B}(Q)) \cap \alpha(\widetilde{x}, u) + \overline{B}(R) = \emptyset$ and $\alpha(\widetilde{x}, u + \overline{B}(Q)) \supset \alpha(\widetilde{x}, u) + \overline{B}(R)$. Now, $v + \overline{B}(Q) \subset \mathbb{R}^2 \backslash u + \overline{B}(Q)$ so $\alpha(\widetilde{x}, v + \overline{B}(Q)) \cap \alpha(\widetilde{x}, u) + \overline{B}(R) = \emptyset$. And $\alpha(\widetilde{x}, v) + \overline{B}(R) \subset \alpha(\widetilde{x}, v + \overline{B}(Q))$, so $\alpha(\widetilde{x}, u) + \overline{B}(R)$ and $\alpha(\widetilde{x}, v) + \overline{B}(R)$ are disjoint. □

5.3. **A Planar Graph Isomorphism: the Perturbation of ∂g.** In this section we will perturb the restriction ∂g to get a map ϕ from $\partial \mathbf{D}_T$ to Y_S such that ϕ is a planar graph isomorphism from $(\mathbf{V}_T, \mathbf{E}_T)$ in each $\widetilde{T}$ orbit of a point into the appropriate $\mathbb{R}^2$ orbit of Y_S. This will take two steps, first perturbing ∂g to ϕ_1 and then perturbing ϕ_1 to $\phi_2 = \phi$, the map sought in the Graph Isomorphism Lemma.

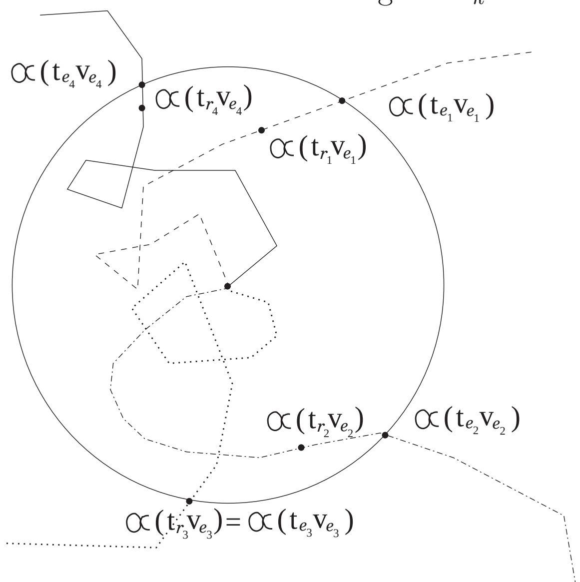
5.4. **Near $\mathbf{V}_T$: the first perturbation.** Let us assume that the following items are such that the Tiling Lemma holds: $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \widehat{\mathcal{V}}_T, \{D_i\})$ is the tiling of X_T by action polygons with the associated graph $G(\mathbf{D}_T) = (\mathbf{V}_T, \mathbf{E}_T)$ and partition $\{\mathbf{V}_i\}_{i=1}^K$ of $\mathbf{V}_T$, and $m \geq m_Q$ where m_Q is the parameter from Part 3 of the Tiling Lemma with $K = Q$. We have established that $\partial g : \mathbf{D}_T \to Y_S$ is a continuous, orbit injective map. By its construction $\partial g(\mathbf{V}_T) \subset \Pi_S(Y \times \mathbb{Z}^2)$. Finally we have the maps $\alpha_i : \{v : \widetilde{T}^v \widetilde{x} \in star(\widetilde{x}), \widetilde{x} \in \mathbf{V}_i\} \to \mathbb{R}^2$ such that for all $\widetilde{T}^v \widetilde{y} \in star(\widetilde{y})$ and $\widetilde{y} \in \mathbf{V}_i$,

$$\partial g(\widetilde{T}^v \widetilde{y}) = \widetilde{S}^{\alpha_i(v)} \partial g(\widetilde{y}).$$

Our first perturbation addresses the issue that it may happen that $\partial g(\widetilde{x}) = \partial g(\widetilde{y})$ for distinct $\widetilde{x}, \widetilde{y} \in star(\widetilde{c})$, $\widetilde{c} \in \mathbf{V}_T$.

Let $k \in 1, .., K$ and $\widetilde{c} \in \mathbf{V}_k$. Each edge $e \subset star(\widetilde{c})$ is a set of the form $\widetilde{T}^{[0,v_e]}(\widetilde{c})$ for some $v_e \in \mathbb{R}^2$. The ∂g image of this set $\partial g \widetilde{T}^{[0,v_e]}(\widetilde{c})$ is equal to the set $\widetilde{S}^{\alpha_k([0,v_e])} \partial g(\widetilde{x})$. The map α_k restricted to $[0, v_e]$ is a piecewise linear map by the definition of g. For each edge $e \subset star(\widetilde{c})$, let $t_r > 0$ be such that $t_r \|v_e\| = r$. The choice of r ensures that the image points $\alpha_k(t_r v_e)$ are distinct for the different edges $e \subset star(\widetilde{c})$. However, we also want the images to lie on a circle centered about the origin, 0. Let t_e be the largest value of t such that $\|\alpha_k(t_e v_e))\| = R$. Our choice of R ensures $t_r \leq t_e$ and $\|t_e v_e\| \leq Q$. Because $t_e \geq t_r$, again the image points $\alpha_k(t_e v_e)$ are distinct for the different edges $e \subset star(\widetilde{c})$. The situation is accurately reflected in Figure 6.

FIGURE 6. The image of α_k.



We **define** ϕ_1 as follows. For $\widetilde{x} \in \partial\mathbf{D}_T$ such that $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ for some $1 \le i \le K$, $\widetilde{c} \in \mathbf{V}_i$, edge $e \subset star(\widetilde{c})$, and $0 \le t \le t_e$, define

$$\phi_1(\widetilde{x}) = \phi_1(\widetilde{T}^{tv_e}(\widetilde{c})) \equiv \widetilde{S}^{t\alpha_i(t_e v_e)} g(\widetilde{c}).$$

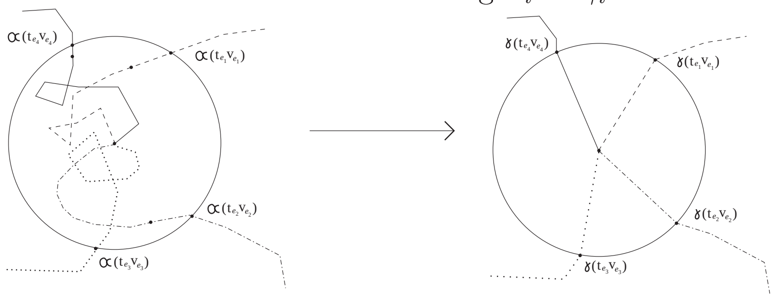
Let $\phi(\widetilde{x}) = \partial g(\widetilde{x})$ otherwise (for $\widetilde{x} \in \partial\mathbf{D}_T$ for which there is no such $\widetilde{c}$, e, and t).

We may also regard this as a perturbations of the maps α_i. Namely, for $\widetilde{c} \in \mathbf{V}_i$, $e \in \mathbf{E}_T$ with $e \subset star(\widetilde{c})$, and $0 \le t \le t_e$ define

$$\gamma_i(tv_e) = (t/t_e)\alpha_i(t_e v_e).$$

Otherwise, let $\gamma_i(tv_e) = \alpha_i(tv_e)$. Figure 7 depicts the effect of the perturbation $\alpha_i \mapsto \gamma_i$.

FIGURE 7. Perturbing α_i to γ_i



Claim 26. *$\phi_1 : \partial\mathbf{D}_T \to Y_S$ is well-defined.*

Proof. There is only one way ϕ_1 could fail to be well-defined. Namely, for each edge $e \in \mathbf{E}_T$ there are precisely two points $\widetilde{c}, \widetilde{c}' \in \mathbf{V}_T$ such that $e \subset star(\widetilde{c})$ and $e \subset star(\widetilde{c}')$. Therefore every point $\widetilde{x} \in e$ is equal to $\widetilde{T}^{tv}(\widetilde{c}) = T^{t'v'}(\widetilde{c}')$ where $t' = 1-t$, $v' = -v$ and $\widetilde{T}^{v}(\widetilde{c}) = \widetilde{c}'$. Thus we have a potential conflict in the definition of ϕ_1 on e if t_e is greater than $1/2$. However, note that $d_T(\widetilde{c}, \widetilde{c}') = \|v\| > 2Q$ and $0 < t_e\|v\|,\ t'_e\|v'\| \le Q$, forcing $0 < t_e < 1/2$. □

Claim 27. *$\phi_1 : \partial\mathbf{D}_T \to Y_S$ is continuous.*

Proof. For each $\widetilde{c} \in \mathbf{V}_k$ and edge $e \subset star(\widetilde{c})$ the values t_e are constant, as are the set of images $\{\alpha_k(t_e v_e) : e \subset star(\widetilde{c})\}$. (A similar map exists for each $\mathbf{V}_k$.) Thus for each $t \in [0,1]$ the map $\widetilde{c} \mapsto \widetilde{S}^{\gamma_k(tv_e)}(\widetilde{c})$ is achieved by the same map $\widetilde{S}^{\gamma_k(tv_e)}$ for all $\widetilde{c} \in \mathbf{V}_k$, and as a map from $[0,1] \to Y_S$, the map $t \mapsto \widetilde{S}^{\gamma_i(tv_e)}(\widetilde{c})$ is continuous, thus we have a continuous map from each edge $e \subset \mathbf{E}_T$ to Y_S. These maps agree on the intersections (in $\mathbf{V}_T$), thus we have a continuous map from $\partial\mathbf{D}_T$ to Y_S. □

The next lemma states that ϕ_1 is virtually the graph isomorphism ϕ in the Graph Isomorphism Lemma (23). It says that the ϕ_1 images of disjoint edges are disjoint, and if two edges meet at a vertex then their images intersect at the image of that vertex. We are verifying that Figure 8 presents an essentially correct picture of the image $\phi_1(\partial \mathbf{D}_T)$ in the $\widetilde{S}$ orbit of a single point.

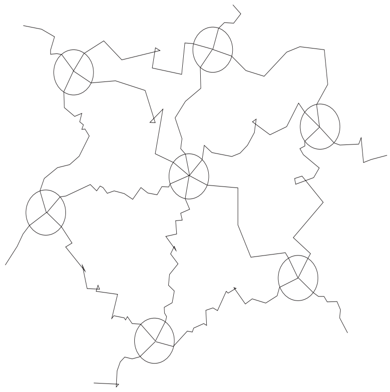


FIGURE 8. The ϕ_1 image of $\partial \mathbf{D}_T$

Lemma 28. *Let e, f be two distinct edges in $\mathbf{E}_T$. Suppose $\widetilde{x} \in e$ and $\widetilde{y} \in f$. Then $\phi_1(\widetilde{x}) = \phi_1(\widetilde{y})$ implies $\widetilde{x} = \widetilde{y}$.*

Proof. If e, f are not elements of the same $\mathbb{R}^2$ orbit, then it is clear that $e \cap f = \phi_1(e) \cap \phi_1(f) = \emptyset$.

If $e \cap f = \emptyset$ then by the Tiling Lemma 22, part 3d, we have $d_T(e, f) > 2Q$ which means, by Proposition 25, that $d_S(g(e), g(f)) > 2R$. In particular, if $\widetilde{c} \in \mathbf{V}_T$ is an endpoint of e then $d_S(g(\widetilde{c}), g(f)) > 2R$. Therefore, even after the perturbation to the map ϕ_1, we still maintain $\phi_1(e) \cap \phi_1(f) = \emptyset$.

Now suppose $e \cap f = \{\widetilde{c}\} \in \mathbf{V}_k \subset \mathbf{V}_T$, then $e, f \subset star(\widetilde{c})$. Let $t_1, t_2 \in [0, 1]$, and assume $\phi_1\left(\widetilde{T}^{t_1 v_e}(\widetilde{c})\right) = \phi_1\left(\widetilde{T}^{t_2 v_f}(\widetilde{c})\right)$, or equivalently, $\gamma_k(t_1 v_e) = \gamma_k(t_2 v_f)$.

First assume $t_1 \in [0, t_e]$ and $t_2 \in [0, t_f]$. Then since γ_k is linear on $[0, t_e v_e] \cup [0, t_f v_f]$, $\gamma_k(t_1 v_e) = \gamma_k(t_2 v_f)$ can only happen if $t_1 = t_2 = 0$ or the ray starting at the origin and passing through $\gamma_k(t_e v_e)$ contains $\gamma_k(t_f v_f)$. But $\|\gamma_k(t_e v_e)\| = \|\gamma_k(t_f v_f)\| = R$ by definition, so if $\gamma_k(t_f v_f)$ lies on this ray then $\gamma_k(t_e v_e) = \gamma_k(t_f v_f)$. But since $t_e\|v_e\|, t_f\|v_f\| \geq r$, this is a contradiction.

Now assume $t_1 \in (t_e, 1]$ and $t_2 \in [0, t_f]$. Since t_e was the maximal value of $t \in [0, 1]$ with $\|\gamma_k(t v_e)\| \leq R$, we have $\|\gamma_k(t_1 v_e)\| > R$ but $\|\gamma_k(t_2 v_f)\| \leq R$.

Finally, assume $t_1 \in (t_e, 1]$ and $t_2 \in (t_f, 1]$. The edge intersection $e \cap f = \widetilde{c}$ is a single point. That means the other ends of e and f are distinct elements of $\mathbf{V}_T$, $\widetilde{a} = \widetilde{T}^{v_e}(\widetilde{c}) \in \mathbf{V}_i$, $\widetilde{b} = \widetilde{T}^{v_f}(\widetilde{c}) \in \mathbf{V}_j$ for some $i, j \in 1, .., K$ (so $d_S(\widetilde{a}, \widetilde{b}) \geq m > 2Q$). There exist minimal values of t'_e and t'_f such that $t_e < t'_e < 1$ and $t_f < t'_f < 1$ and $\|\gamma_i((t'_e - 1)v_e)\| = R = \|\gamma_j((t'_f - 1)v_f)\|$. Because $d_T(e, \widetilde{b}) > 2Q$ (similarly,

$d_T(\widetilde{a}, f) > 2Q$), it follows by Proposition 25, for all $t_1 \in [0,1]$ and $t_2 \in [1-t'_f, 1]$ (similarly, $t_1 \in [1-t'_e, 1]$ and $t_2 \in [0,1]$) that $\phi_1\left(\widetilde{T}^{t_1 v_e}(\widetilde{c})\right) \neq \phi_1\left(\widetilde{T}^{t_2 v_f}(\widetilde{c})\right)$. Lastly, then for $t_1 \in [t_e, 1-t'_e]$ and $t_2 \in [t_f, 1-t'_f]$, both $\widetilde{T}^{t_1 v_e}(\widetilde{c}) \in e(r)$ and $\widetilde{T}^{t_2 v_f}(\widetilde{c}) \in f(r)$. So, by Proposition 24, $\phi_1 \widetilde{T}^{t_1 v_e}(\widetilde{c}) = g\widetilde{T}^{t_1 v_e}(\widetilde{c}) \neq g\widetilde{T}^{t_2 v_f}(\widetilde{c}) = \phi_1 \widetilde{T}^{t_2 v_f}(\widetilde{c})$. □

5.5. **Far from vertices: the second perturbation.** In order for ϕ_1 to be the graph homomorphism we want, ϕ_1 must injective on each edge, which at the moment, is not necessarily the case. However, this is easily remedied. To do so we will focus on the γ_i, $i = 1, .., K$.

We have constructed γ_i so that the following holds. For $\widetilde{x} \in \partial \mathbf{D}_T$ such that $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ for some $1 \leq i \leq K$, $\widetilde{c} \in \mathbf{V}_i$, $e \in \mathbf{E}_T$, $e \subset star(\widetilde{c})$, and $0 \leq t \leq 1$,

$$\phi_1(\widetilde{x}) = \widetilde{S}^{\gamma_i(tv_e)} g(\widetilde{c}).$$

The map $t \mapsto \gamma_i(tv_e)$ is a piecewise linear function from $[0,1]$ to $\mathbb{R}^2$ which is not necessarily injective. We now refer to the following.

Lemma 29. *Let $H : [0,1] \to \mathbb{R}^2$ be a piecewise linear function. There exists an injective piecewise linear function $F : [0,1] \to \mathbb{R}^2$ such that $F(0) = H(0)$, $F(1) = H(1)$ and $F([0,1]) \subset H([0,1])$.*

Working with one map of the form $t \mapsto \gamma_i(tv_e)$ at a time (and there are only finitely many of these maps), we apply the above lemma to each γ_i with each edge. We obtain a finite set of injective maps $t \mapsto \widehat{\gamma}_i(tv_e)$ which are also continuous and piecewise linear and have the same image. For $\widetilde{x} \in \partial \mathbf{D}_T$ such that $\widetilde{x} = \widetilde{T}^{tv_e}(\widetilde{c})$ for some $1 \leq i \leq K$, $\widetilde{c} \in \mathbf{V}_i$, $e \in \mathbf{E}_T$, $e \subset star(\widetilde{c})$ and $0 \leq t \leq 1$, we define ϕ as follows

$$\phi(\widetilde{x}) = \widetilde{S}^{\widehat{\gamma}_i(tv_e)} g(\widetilde{c}).$$

Thus we have a continuous injective, orbit injective mapping $\phi : \partial \mathbf{D}_T$ into Y_S and therefore a graph injection $\phi : (\mathbf{V}_T, \mathbf{E}_T) \to Y_S$, proving the Graph Isomorphism Lemma (23).

6. Extending Graph Isomorphisms

All that remains in order to prove the main theorem then, is to extend the graph isomorphism ϕ from the previous section to a homeomorphism on X_T.

In order to show ϕ extends to an injection of X_T, we must prove Lemma 30 below, which asserts that the ϕ-image of every 2-cell is a 2-cell. That is, if Γ is a collection of edges that form a closed loop and the interior of the polygon with boundary Γ contains no vertices or edges, then the interior of the polygon formed by $\phi(\Gamma)$ contains no image of a vertex or an edge. General results of this type are surely known. What makes our situation simple is that tiles in a Delaunay tiling are convex and for $\widetilde{x} \in X_T$, both sets $\{v : \widetilde{T}^v(\widetilde{x}) \in \mathbf{V}_T\}$ and $\{v : \widetilde{S}^v(\phi(\widetilde{x})) \in \phi(\mathbf{V}_T)\}$ are *uniformly discrete* subsets of

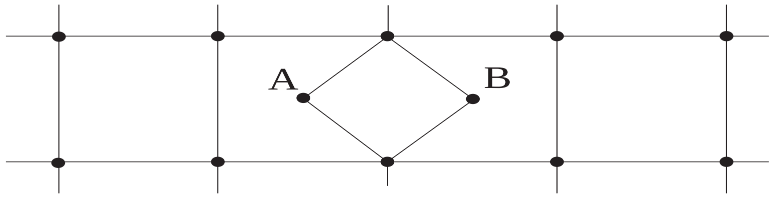


FIGURE 9. A graph whose graph isomorphism does not extend to $\mathbb{R}^2$.

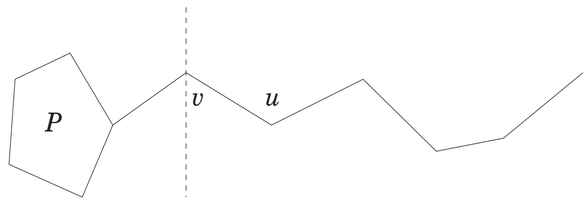


FIGURE 10. The cardinality of $\partial\mathbf{T}\backslash P$ is not finite.

$\mathbb{R}^2$. A set $U \subset \mathbb{R}^2$ is uniformly discrete if there is an $n > 0$ such that $u, v \in U$ implies $u + B(n) \cap v + B(n) = \emptyset$.

If, for example, tiles in the domain are not convex, there can be a counter-example as is demonstrated by the example depicted in Figure 9. Consider the graph isomorphism which is the identity on all the vertices except for A and B; the isomorphism exchanges them. The edges are mapped by extension of the the vertex map. Here we see 2-cells bounding tiling polygons whose images contain vertices. One can also construct counterexamples when the image of the set of vertices is not discrete. For example, one could map the entire plane inside a single tile boundary.

Lemma 30. *Let $\mathcal{D}$ be a finite tiling of $\mathbb{R}^2$ by convex polygons. Consider the boundary graph $(\mathcal{V}, \mathcal{E})$ of $\mathcal{D}$. Suppose $\phi : \partial\mathcal{D} \to \mathbb{R}^2$ is a graph isomorphism such that $\phi(\mathcal{V})$ is uniformly discrete. Then the ϕ-image of every 2-cell in $\mathcal{D}$ is a 2-cell. Moreover, there is a continous injection $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\psi|_{\partial\mathcal{D}} = \phi$.*

Proof. Let D be a polygonal tile and consider any vertex $v \in \partial\mathcal{D}\backslash\partial P$. We first consider the possibility that $\phi(v)$ is in the interior of $\phi(\partial D)$. Because every tile is convex, it not hard to see that the vertex cardinality of the graph component in $\partial\mathcal{D}\backslash\partial P$ containing v is countably infinite. To wit, there exists a half-plane containing P whose boundary contains v. The tile convexity property implies the complementary half-plane contains an edge e which ends at v (so $e = [u, v]$). Repeating this argument at u we construct an infinite sequence of edges which don't intersect P. (See Figure 10.)

Because ϕ is injective, the ψ image of any non-self-intersecting loop is also a non-self-intersecting. The continuity of ϕ ensures the image is a bounded loop. The Jordan Curve Theorem tells us the image bounds a bounded set (which we refer to as the *interior*) and that the image curve together with the interior is homeomorphic to a disk. If $\phi(v)$ is contained in the interior of $\phi(\partial P)$, then the interior of $\phi(\partial P)$ must also contain the ϕ image of the countable connected component of v. That is, the interior contains a countable number of vertices. This contradicts the uniform discreteness of the image $\phi(\mathcal{V})$ and the boundedness of the interior. Thus, the interior $\phi(\partial P)$ does not contain the ϕ image of any vertices.

Next we consider the possibility that an edge in $\partial\mathcal{D}$ has an image in the interior of $\phi(\partial P)$. If so, then the image of that edge begins and ends at vertices in $\phi(\partial P)$ or else we are in the case considered previously. For a set of edges $\{e_1, e_2, .., e_n\}$, let $\mathcal{V}(\{e_1, e_2, .., e_n\}) = \{v \in \mathcal{V} : v \in e_i \text{ for some } 1 \leq i \leq n\}$, i.e., the set of vertices contained in $\mathcal{E}$. Let Γ and Γ' be non-self-intersecting loops of edges bounding interiors P and P'. If $\mathcal{V}(\Gamma') \subset \mathcal{V}(\Gamma)$ and if P and P' are convex, then $P' \subset P$. So, if $\phi(\partial D)$ does contain an edge $\phi(e)$ for some edge, e, then we can construct Γ' with $\mathcal{V}(\Gamma') \subset \mathcal{V}(\Gamma)$ where Γ is the list of edges bounding D. The curve Γ' is non-self-intersecting and bounds a convex region P', hence $P' \subset D$, contradicting the property that D is a tile.

Therefore, for each tile $D \in \mathcal{D}$, its boundary ∂D is a 2-cell which ϕ then takes to the 2-cell $\phi(\partial D)$. Both D and the interior of the image loop $\phi(\partial D)$ are homeomorphic to disks, thus there exists a homeomorphic extension ψ from D to the interior of $\phi(\partial D)$. Thus, because ϕ maps distinct 2-cells to distinct 2-cells and $\mathcal{D}$ tiles the plane, ϕ can be extended to a continuous injection $\psi : \mathbb{R}^2 \to \mathbb{R}^2$. □

Lemma 31. *There is a homeomorphism $\Phi : X_T \to Y_S$ such that $\Phi|_{\partial \mathbf{D}_T} = \phi$.*

Proof. The preceeding statement tells us how to proceed. In this case, we have a tiling $\mathbf{D}_T = (\widehat{\mathbf{V}}_T, \widehat{\mathcal{V}}_T, \{D_i\})$ of X_T by action polygons. By restricting our attention to the $\widetilde{T}$ orbit of a single point $\widetilde{x} \in X_T$, we have a tiling of $\mathbb{R}^2$. We use the above lemma to extend the graph isomorphism to a continuous injection, but we must take some care. Again, by further partitioning of $\{\widehat{\mathbf{V}}_i\}$, we may assume that if $\widetilde{x}, \widetilde{y} \in \widehat{\mathbf{V}}_i$ and $v \in \partial D_i$ then $\widetilde{S}^w(\phi(\widetilde{x})) = \phi(\widetilde{T}^v(\widetilde{x}))$ if and only if $\widetilde{S}^w(\phi(\widetilde{y})) = \phi(\widetilde{T}^v(\widetilde{y}))$ for all $w \in \mathbb{R}^2$. That is, that an extension of ϕ can be done in a way which is the same for each prototile. This insures the continuity of the extended map on X_T.

We have not yet seen that this map is surjective, this is our last task. Recall Lemma 15 of Section 3 in which we showed that $h(C)$ is syndetic in its own orbit if and only if $h(C)$ is syndetic in Y. In that same spirit, note that if $\Phi(X_T)$ contains an entire $\widetilde{S}$-orbit then $\Phi(X_T) = Y_S$.

Let $\widetilde{y} \in \Phi(X_T)$ and suppose $\widetilde{S}^v(\widetilde{y})$ is not in $\Phi(X_T)$. Consider the set $[\widetilde{y}, \widetilde{S}^v(\widetilde{y})] = \{\widetilde{S}^{tv}(\widetilde{y}) : t \in [0, 1]\} \subset Y_S$. Set $t_* = \max\{t \in [0, 1] : \widetilde{S}^{tv}(\widetilde{y}) \in \Phi(X_T)\}$ and $\widetilde{z} = \widetilde{S}^{t_*v}(\widetilde{y})$. Then $\widetilde{z} = \Phi(\widetilde{x})$ for $\widetilde{x} \in X_T$. For any $\epsilon > 0$ there is a $\delta > 0$ sufficiently small such that $\{\widetilde{T}^w(\widetilde{x}) : \|w\| = \delta\}$ maps to a loop around $\widetilde{z}$ which lies inside $\{\widetilde{S}^w(\widetilde{x}) : \|w\| < \epsilon\}$. If $\epsilon > 0$ is sufficiently small, this latter set must intersect $[\widetilde{y}, \widetilde{S}^v(\widetilde{y})]$ at a point $\widetilde{S}^{tv}(\widetilde{y})$ where $t_* < t < 1$, a contradiction. □

In particular, the latter part of the above proof demonstrates that $h(X)$ is of 2nd category in Y, proving Theorem 7.

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