NON-DISTURBANCE FOR FUZZY QUANTUM MEASUREMENTS

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Abstract

We consider three criteria for describing non-disturbance between quantum measurements. While previous discussions of these criteria considered sharp measurements, we shall treat more general measurements that may be unsharp (or fuzzy). It has been shown that in the sharp case, the three criteria are equivalent to compatibility of the measurements. We shall show that only the third criterion is equivalent to compatibility for the general fuzzy case. Moreover, the first two criteria are not symmetric relative to the two measurements.

1 Introduction

K. A. Kirkpatrick [9] has recently discussed three ways of describing non-disturbance between quantum measurements. The first two are due to Lüders [10] and the third to Davies [5] and Kirkpatrick [8]. Letting X and Y be two quantum measurements, the three non-disturbance criteria are given roughly as follows.

(1) The probability of an established value of Y is unchanged by the later occurrence of a value of X.

(2) The probability of occurrence of a Y value is unchanged by a preceding execution of X.

(3) If p and q are X and Y values, respectively, then the probability of p followed by q coincides with the probability of q followed by p.

We said that the criteria are given roughly because the English language can be imprecise and ambiguous. We shall later translate these criteria into precise mathematical language and prove results about them. It is appropriate that these criteria are phased in terms of probabilities because statistical results are what are observed in the laboratory. Kirkpatrick showed that (1), (2) and (3) are equivalent to the compatibility (commutativity) of X and Y and hence are equivalent to each other. However, Kirkpatrick only considered sharp quantum measurements which are described by projection-valued measures. In the present paper we shall generalize his results to measurements that may be unsharp (or fuzzy) and these are described by positive operator-valued measures. Unsharp measurements are quite important in quantum measurement theory [2, 3, 5] and in applications such as quantum computation and quantum information [11].

Unlike the sharp case, we shall show that (1), (2) and (3) are not equivalent for general measurements. Only Criterion (3) is equivalent to the compatibility of X and Y. Although Criterion (1) implies compatibility it is not appropriate for the general case because it also implies that Y is sharp. Compatibility implies (2) but the converse does not hold. Although they are not explicitly symmetric relative to X and Y, the three criteria are implicitly symmetric in the sharp case because they are equivalent to the symmetric relation of compatibility. In the general fuzzy case, only (3) is symmetric in X and Y. This further justifies Kirkpatrick's choice of (3) for his discussion of noncompatibility in classical probability theory [9]. As in Kirkpatrick [9] and Lüders [10], we shall only consider discrete quantum measurements.

2 Measurements and Probabilities

Let H be a complex Hilbert space representing the state space of a physical system S. Let $\mathcal{P}(H)$ be the set of projections on H and $\mathcal{E}(H)$ the set of operators on H satisfying $0 \le A \le I$. The elements of $\mathcal{E}(H)$ are called **effects** and represent yes-no measurements on S that may be unsharp while the elements of $\mathcal{P}(H)$ represent sharp effects [4, 6, 7]. Let $\mathcal{D}(H)$ be the set of positive trace class operators on H with unit trace. The elements of $\mathcal{D}(H)$

are called **density operators** and represent the et of **states** of S. When S is in state $W \in \mathcal{D}(H)$, the probability that the effect $A \in \mathcal{E}(H)$ is observed (or has value yes) is given by $p_W(A) = \mathrm{tr}(AW)$ and if A is observed then the post-measurement state becomes

$$A^{1/2}WA^{1/2}/\mathrm{tr}(AW) \tag{2.1}$$

whenever $\mathrm{tr}(AW) \neq 0$. Of course, if $\mathrm{tr}(AW) = 0$ then with certainty A will not be observed in state W so the fact that (2.1) is meaningless in this case is of no consequence.

For $A, B \in \mathcal{E}(H)$, the **conditional probability** that B is observed given that A has been observed is defined by

$$p_W(B \mid A) = \mathrm{tr}(BA^{1/2}WA^{1/2})/\mathrm{tr}(AW) \tag{2.2}$$

whenever $\mathrm{tr}(AW) \neq 0$ [4, 7]. Notice that (2.2) follows in a natural way from (2.1). For $A, B \in \mathcal{E}(H)$ we use the notation $A\&B$ to denote the effect in which A is performed first and then B is performed next. We call $A\&B$ "A and then B." Then in a natural way we have that

$$p_W(A\&B) = p_W(A)p_W(B \mid A) = \mathrm{tr}(BA^{1/2}WA^{1/2})$$

and

$$p_W(C \mid A\&B) = \mathrm{tr}(CB^{1/2}A^{1/2}WA^{1/2}B^{1/2})/\mathrm{tr}(BA^{1/2}WA^{1/2}) \tag{2.3}$$

whenever $\mathrm{tr}(BA^{1/2}WA^{1/2}) \neq 0$

A measurement with more than two values is given by a positive operator-valued (POV) measure $X \colon \mathcal{B}(\mathbb{R}) \to \mathcal{E}(H)$ where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of $\mathbb{R}$. In particular, X satisfies (a) $X(\mathbb{R}) = 1$, (b) $X(\cup\Delta_i) = \sum X(\Delta_i)$ whenever $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$, and the summation converges in the strong operator topology. The special case of a sharp measurement is given by a projection-valued (PV) measure $X \colon \mathcal{B}(\mathbb{R}) \to \mathcal{P}(H)$.

In the sequel we shall only consider **discrete measurements** and these have only countably many values. Suppose X has values (or outcomes) a_i and A_i is the effect that a_i is observed, $i = 1, 2, \ldots$. It follows from (a) and (b) that $\sum A_i = I$. We call $\{A_i\}$ the set of **measurement operators** for X. If the measurement X is performed in the state W, then the probability that the ith outcome is observed is $p_W(A_i) = \mathrm{tr}(A_iW)$ and if this outcome is observed then by (2.1) the post-measurement state becomes

$$A_i^{1/2}WA_j^{1/2}/\mathrm{tr}(A_iW) \tag{2.4}$$

whenever $\mathrm{tr}(A_iW) \neq 0$. Moreover, if the measurement is performed in state W and no observation is made, then the post-measurement state is given by

$$\sum A_i^{1/2} W A_i^{1/2} \tag{2.5}$$

Notice that (2.5) is a natural consequence of (2.4) because (2.5) is a convex combination of the elements in (2.4) with coefficients $\mathrm{tr}(A_iW)$. In the case of a sharp discrete measurement, the measurement operators are projections $P_i \in \mathcal{P}(H)$ satisfying $\sum P_i = I$. In this case (2.4) and (2.5) reduce to the usual von Neumann-Lüders formulas $P_iWP_i/\mathrm{tr}(P_iW)$ and $\sum P_iWP_i$, respectively.

In the sequel we shall identify a (discrete) measurement with its set of measurement operators $\{A_i\}$ without specifying its values. We say that two effects A, B are **compatible** if they commute, $AB = BA$. Physically, this corresponds to A and B being simultaneously measurable. More generally, two measurements $\{A_i\}$ and $\{B_j\}$ are **compatible** if their measurement operators are compatible, $A_iB_j = B_jA_i$ for all i and j.

3 Non-Disturbance and Compatibility

We now translate the three non-disturbance criteria into mathematical forms and then draw conclusions concerning compatibility of the measurements. As in Section 2, a measurement is given by a (finite or infinite) sequence $\{A_i\}$ where $A_i \in \mathcal{E}(H)$ and $\sum A_i = I$.

If $X = \{A_i\}$ and $Y = \{B_j\}$ are measurements, then the mathematical form of Criterion (1) for non-disturbance becomes

$$p_W(B_j \mid B_j \& A_k) = 1 \tag{3.1}$$

for all j, k and W. Equation (3.1) says that if a Y-measurement results in outcome j and a later X-measurement results in outcome k, then a subsequent Y-measurement will again result in outcome j with certainty in state W. Our first result shows that (3.1) holds if and only if X and Y are compatible and Y is sharp.

Theorem 3.1. $p_W(B_j \mid B_j \& A_k) = 1$ *for every* j, k *and* W *if and only if* $A_iB_j = B_jA_i$ *and* $B_i \in \mathcal{P}(H)$ *for every* i *and* j.

Proof. Since conditional probability is countably additive in its first argument, we have that (3.1) implies

$$p_W(B_i \mid B_j \& A_k) = 0 \tag{3.2}$$

for $i \neq j$. Applying the definition of conditional probability (2.3), (3.2) becomes

$$\frac{\mathrm{tr}(B_i A_k^{1/2} B_j^{1/2} W B_j^{1/2} A_k^{1/2})}{\mathrm{tr}(A_k B_j^{1/2} W B_j^{1/2})} = 0 \tag{3.3}$$

for all $i \neq j$ whenever $\mathrm{tr}(A_k B_j^{1/2} W B_j^{1/2}) \neq 0$. We can write (3.3) as

$$\mathrm{tr}(B_j^{1/2} A_k^{1/2} B_i A_k^{1/2} B_j^{1/2} W) = 0 \tag{3.4}$$

Now (3.4) holds even if $\mathrm{tr}(A_k B_j^{1/2} W B_j^{1/2}) = 0$ because in this case $A_k^{1/2} B_j^{1/2} W B_j^{1/2} A_k^{1/2} = 0$. Since (3.4) holds for every W we conclude that

$$B_j^{1/2} A_k^{1/2} B_i A_k^{1/2} B_j^{1/2} = 0 \tag{3.5}$$

for all $i \neq j$. We then obtain

$$(B_i^{1/2} A_k^{1/2} B_j^{1/2})^* (B_i^{1/2} A_k^{1/2} B_j^{1/2}) = B_j^{1/2} A_k^{1/2} B_i A_k^{1/2} B_j^{1/2} = 0$$

Hence, $B_i^{1/2} A_k^{1/2} B_j^{1/2} = 0$ so that $B_i A_k^{1/2} B_j = 0$ for all $i \neq j$. Summing over $i \neq j$ and applying $\sum B_i = I$ we have

$$0 = (I - B_j) A_k^{1/2} B_j = A_k^{1/2} B_j - B_j A_k^{1/2} B_j$$

Thus,

$$A_k^{1/2} B_j = B_j A_k^{1/2} B_j = (B_j A_k^{1/2} B_j)^* = B_j A_k^{1/2}$$

We conclude that $A_k B_j = B_j A_k$ for all j and k. Now (3.5) becomes

$$A_k B_j^{1/2} B_i B_j^{1/2} = 0 \tag{3.6}$$

for all $i \neq j$. Summing (3.6) over k gives $B_j^{1/2} B_i B_j^{1/2} = 0$ for all $i \neq j$ and now summing over $i \neq j$ we have that

$$B_j^{1/2} (I - B_j) B_j^{1/2} = 0$$

Hence, $B_j = B_j^2$ so $B_j \in \mathcal{P}(H)$ for all j. The converse is straightforward. □

It is easy to see that Criterion (1) does not imply that $A_i \in \mathcal{P}(H)$ so X need not be sharp. This shows that Criterion (1) is not symmetric in X and Y.

The mathematical form of Criterion (2) becomes

$$p_W(B_j) = \sum_i p_W(A_i \& B_j) \tag{3.7}$$

for every j and W. Equation (3.7) says the probability that a Y-measurement results in outcome j coincides with the sum of the probabilities that an X-measurement results in its possible outcomes and a later Y-measurement results in outcome j.

Theorem 3.2. *If $A_iB_j = B_jA_i$ for every i and j then (3.7) holds for every j and W.*

Proof. In terms of traces, (3.7) becomes

$$\operatorname{tr}(B_jW) = \sum_i \operatorname{tr}(B_jA_i^{1/2}WA_i^{1/2}) = \sum_i \operatorname{tr}(A_i^{1/2}B_jA_i^{1/2}W) \tag{3.8}$$

If $A_iB_j = B_jA_i$, then the right side of (3.8) becomes

$$\textstyle\sum_i \operatorname{tr}(B_jA_jW) = \operatorname{tr}\left(B_j \sum A_iW\right) = \operatorname{tr}(B_jW) \qquad \square$$

The converse of Theorem 3.2 does not hold except for some special cases. For example, the converse holds when $\dim H < \infty$ or when X has just two measurement operators. In general, suppose that (3.7) holds for every j and W. Then by (3.8) we have that

$$B_j = \sum_i A_i^{1/2}B_jA_i^{1/2}$$

for every j. Now it is shown in [1] that for any infinite dimensional Hilbert space H there exists a measurement $\{A_i \in \mathcal{E}(H) \colon 1 \le i \le 5\}$ and a $B \in \mathcal{E}(H)$ such that $B = \sum A_i^{1/2}BA_i^{1/2}$ and yet $BA_i \ne A_iB$ for all i. Letting $X = \{A_i\}$ and $Y = \{B, I - B\}$ we see that X and Y satisfy Criterion (2) (Equation (3.7)) but are not compatible. This example also shows that Criterion (2) is not symmetric in X and Y. Indeed, writing $Y = \{B_1, B_2\}$ where $B_1 = B$,

$B_2 = I - B$, it follows from the work in [4, 7] that $A_i \neq \sum_j B_j^{1/2} A_i B_j^{1/2}$ for some $1 \leq i \leq 5$.

Finally, the mathematical form of Criterion (3) becomes

$$p_W(A_i \& B_j) = p_W(B_j \& A_i) \tag{3.9}$$

for all i, j and W. The simple proof of the next result is repeated here for completeness.

Theorem 3.3. [7] *Equation (3.9) holds for all W if and only if $A_i B_j = B_j A_i$.*

Proof. In terms of traces, (3.9) becomes

$$\mathrm{tr}(B_j A_i^{1/2} W A_i^{1/2}) = \mathrm{tr}(A_i B_j^{1/2} W B_j^{1/2})$$

which can be written

$$\mathrm{tr}(A_i^{1/2} B_j A_i^{1/2} W) = \mathrm{tr}(B_j^{1/2} A_i B_j^{1/2} W) \tag{3.10}$$

Since (3.10) holds for all W we have that

$$A_i^{1/2} B_j A_i^{1/2} = B_j^{1/2} A_i B_j^{1/2} \tag{3.11}$$

Writing (3.11) as

$$(A_i^{1/2} B_j^{1/2})(A_i^{1/2} B_j^{1/2})^* = (A_i^{1/2} B_j^{1/2})^* (A_i^{1/2} B_j^{1/2})$$

we see that $A_i^{1/2} B_j^{1/2}$ is a normal operator. Since

$$(A_i^{1/2} B_j^{1/2}) A_i^{1/2} = A_i^{1/2} (B_j^{1/2} A_i^{1/2})$$

by Fuglede's theorem [12] we have that

$$B_j^{1/2} A_i = (A_i^{1/2} B_j^{1/2})^* A_i^{1/2} = A_i^{1/2} (B_j^{1/2} A_i^{1/2})^* = A_i B_j^{1/2}$$

It follows that $B_j A_i = A_i B_j$. The converse is straightforward. □

We conclude that Criterion (3) for non-disturbance of measurements is equivalent to their compatibility. This gives further justification for Kirkpatrick's choice of this criterion for his study of noncompatibility in classical probability theory [9].

4 Classical Probability Theory

We have seen in Section 3 that Criterion (3) gives the most appropriate definition of non-disturbance for measurements. Following Kirkpatrick [8, 9], we now briefly show that there are even disturbing measurements in classical probability theory.

Suppose we have a deck of ordinary playing cards and we consider two measurements, FACE and SUIT. For the FACE measurement we draw a card at random and observe its face value, ace, two, ..., king and for the SUIT measurement we draw a card at random and observe its suit, club, diamond, heart or spade. Let K be the value "king" for FACE and H the value "heart" for SUIT. Most of the usual probability problems for cards are one of two kinds, we either always replace the chosen card or always discard the chosen card. In either of these two cases, two card draws are independent. We then have for any standard probability distribution p that

$$p(K\&H) = p(K)p(H) = p(H\&K)$$

Of course, this holds for any values of FACE and SUIT so these two measurements do not disturb each other in these two cases for the usual probability distributions.

We now consider the situation in which we draw a card at random, replace it if its suit is spades and discard it otherwise. For simplicity, suppose we have a deck with just four cards, the kings and queens of spades and hearts KH, KS, QH, QS. Letting p be the uniform distribution, we obtain

$$\begin{aligned} p(K\&H) &= p(KS)p(H \mid KS) + p(KH)p(H \mid KH) \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{3} = \frac{5}{24} \end{aligned}$$

and

$$\begin{aligned} p(H\&K) &= p(KH)p(K \mid KH) + p(QH)p(K \mid QH) \\ &= \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{4} \end{aligned}$$

In accordance with Criterion (3), FACE and SUIT disturb each other. A strange property of this example is that the measurement SUIT disturbs itself and this does not happen for sharp quantum measurements. To verify this disturbance, we have

$$p(H\&S) = p(H)p(S \mid H) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

and

$$p(S\&H) = p(S)p(H \mid S) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Another problem with the previous example is that it does not have the important property of repeatability possessed by sharp quantum measurements [8, 9]. That is, if a sharp measurement is immediately repeated then the same outcome should be obtained. We now give an example that satisfies repeatability.

We again have the two measurements FACE and SUIT. But now if a FACE measurement is performed and value F is obtained then all cards with other face values are discarded and we next draw from a deck all of whose cards have face value F. A similar procedure is followed for a SUIT measurement. For simplicity we assume that the deck contains only six cards, KS, QS, KH, QH, KD, QD and that p is the uniform probability distribution. We shall consider three preparation states depending on the cards available in the initial drawing. These states can be prepared by performing a sequence of measurements. The first two states may be considered to be "mixed states" and the third a "pure state."

We first start with all six cards. We then obtain

$$\begin{aligned} p(K\&H) &= p(KS)p(H \mid KS) + p(KH)p(H \mid KH) + p(KD)p(H \mid KD) \\ &= \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{6} \end{aligned}$$

and

$$\begin{aligned} p(H\&K) &= p(KH)p(K \mid KH) + p(QH)p(K \mid QH) \\ &= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{6} \end{aligned}$$

We next start with KS, QS, KH, QH. We now obtain

$$\begin{aligned} p(K\&H) &= p(KS)p(H \mid KS) + p(KH)p(H \mid KH) \\ &= \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{6} \end{aligned}$$

and

$$p(H\&K) = p(KH)p(K \mid KH) + p(QH)p(K \mid QH)$$
$$= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}$$

Finally, we start with KS, QS. We now obtain

$$p(K\&H) = p(K)p(H \mid K) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

and

$$p(H\&K) = 0$$

We again conclude that FACE and SUIT disturb each other.

5 Concluding Remarks

It has been known since the early development of quantum mechanics that quantum measurements can disturb (or interfere with) each other. One of the first manifestations of this phenomenon was the Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar$ which is a statistical inequality involving the standard deviations (error uncertainty) in position x and momentum p measurements. Disturbance and the uncertainty principle are closely related to the fact that the operators representing x and p do not commute which is described by saying that x and p are incompatible or not simultaneously measurable. This provides a circle of ideas involving the concepts of disturbance, compatibility and probability for quantum measurements. It appears that the relationships between these three concepts are not clearly understood even today.

The present paper attempts to clarify these relationships in a mathematically rigorous way. We considered three possible definitions for non-disturbance between quantum measurements that may be unsharp. We then showed that only one of the definitions is equivalent to compatibility and is symmetric relative to the measurements. Following Kirkpatrick [8, 9] it is also pointed out that the disturbance of measurements even occurs in classical probabilistic situations. However, since there is no Hilbert space structure available, the role of compatibility is not at all clear. This indicates that disturbance in quantum mechanics may be due to its intrinsic stochastic nature and not to its Hilbert space operator-theoretic framework.

References

[1] A. Arias, A. Gheondea and S. Gudder, Fixed Points of quantum operations *J. Math. Phys.* **43** (2002), 5872–5881.

[2] P. Busch, P. J. Lahti and P. Middlestaedt, *The Quantum Theory of Measurements*, Springer-Verlag, Berlin, 1991.

[3] P. Busch, M. Grabowski and P. J. Lahti, *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995.

[4] P. Busch and J. Singh, Lüders theorem for unsharp measurements, *Phys. Lett. A* **249** (1998), 10–24.

[5] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, New York, 1976.

[6] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures*, Kluwer, Dordrecht, 2000.

[7] S. Gudder and G. Nagy, Sequential quantum measurements, *J. Math. Phys.* **42** (2001), 5212–5222.

[8] K. A. Kirkpatrick, "Quantal" behavior in classical probability, *Found. Phys. Lett.* **16** (2003), 199–224.

[9] K. A. Kirkpatrick, Compatibility and probability, arXiv: quant-ph/0403021, (2004).

[10] G. Lüders, Überdie zustandsänderung durch den Messprozess, *Ann. Phsik* **8** (1951), 322–328.

[11] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* Cambridge University Press, Cambridge, 2000.

[12] W. Rudin, *Functional Analysis* McGraw-Hill, New York, 1991.