ON THE INFIMUM OF QUANTUM EFFECTS

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Abstract. The quantum effects for a physical system can be described by the set $\mathcal{E}(\mathcal{H})$ of positive operators on a complex Hilbert space $\mathcal{H}$ that are bounded above by the identity operator. While a general effect may be unsharp, the collection of sharp effects is described by the set of orthogonal projections $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{H})$. Under the natural order, $\mathcal{E}(\mathcal{H})$ becomes a partially ordered set that is not a lattice if $\dim \mathcal{H} \geq 2$. A physically significant and useful characterization of the pairs $A, B \in \mathcal{E}(\mathcal{H})$ such that the infimum $A \wedge B$ exists is called the infimum problem. We show that $A \wedge P$ exists for all $A \in \mathcal{E}(\mathcal{H})$, $P \in \mathcal{P}(\mathcal{H})$ and give an explicit expression for $A \wedge P$. We also give a characterization of when $A \wedge (I - A)$ exists in terms of the location of the spectrum of A. We present a counterexample which shows that a recent conjecture concerning the infimum problem is false. Finally, we compare our results with the work of T. Ando on the infimum problem.

1. Introduction

A quantum mechanical measurement with just two values 1 and 0 (or yes and no) is called a *quantum effect*. These elementary measurements play an important role in the foundations of quantum mechanics and quantum measurement theory [3, 4, 5, 7, 13, 15, 17]. We shall follow the Hilbert space model for quantum mechanics in which effects are represented by positive operators on a complex Hilbert space $\mathcal{H}$ that are bounded above by the identity operator I. In this way the set of effects $\mathcal{E}(\mathcal{H})$ becomes

$$\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \colon \quad 0 \leq A \leq I\}$$

The set of orthogonal projections $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{H})$ corresponds to sharp effects while a general $A \in \mathcal{E}(\mathcal{H})$ may be unsharp (fuzzy, imprecise). Employing the usual order $A \leq B$ for the set of bounded self-adjoint operators $\mathcal{S}(\mathcal{H})$ on $\mathcal{H}$, we see that $(\mathcal{E}(\mathcal{H}), \leq)$ is a partially ordered set. It is well known that $(\mathcal{E}(\mathcal{H}), \leq)$ is not a lattice if $\dim \mathcal{H} \geq 2$. However, if the infimum $A \wedge B$ of $A, B \in \mathcal{E}(\mathcal{H})$ exists then $A \wedge B$ has the important property of being the largest effect that physically implies both A and B. It would thus be of interest to give a physically significant and useful characterization of when $A \wedge B$ exists. This so-called infimum problem has been considered for at least 10 years [2, 10, 11, 12, 16, 18].

Before discussing the progress that has been made toward solving the infimum problem, let us compare the situation with that of the partially ordered set $(\mathcal{S}(\mathcal{H}), \leq)$. Of course, if $A, B \in \mathcal{S}(\mathcal{H})$ are comparable, that is, $A \leq B$ or $B \leq A$, then $A \wedge B$ exists and is the smaller of the two. A surprising result of R. Kadison [14] states that the converse holds. Thus, for $A, B \in \mathcal{S}(\mathcal{H})$, $A \wedge B$ exists in $\mathcal{S}(\mathcal{H})$ if and only if A and B are comparable. We conclude that $(\mathcal{S}(\mathcal{H}), \leq)$ is an antilattice which is as far from being a lattice as possible. The situation is quite different in $(\mathcal{E}(\mathcal{H}), \leq)$. In fact it is well known that $P \wedge Q$ exists in $\mathcal{E}(\mathcal{H})$ for all $P, Q \in \mathcal{P}(\mathcal{H})$. More generally, we shall show that $A \wedge P$ exists in $\mathcal{E}(\mathcal{H})$ for all $A \in \mathcal{E}(\mathcal{H})$, $P \in \mathcal{P}(\mathcal{H})$ and give an explicit expression for $A \wedge P$.

For $A, B \in \mathcal{E}(\mathcal{H})$ let $P_{A,B}$ be the orthogonal projection onto the closure of $\operatorname{Ran}(A^{1/2}) \cap \operatorname{Ran}(B^{1/2})$. It is shown in [18] that if $\dim \mathcal{H} < \infty$ then $A \wedge B$ exists in $\mathcal{E}(\mathcal{H})$ if and only if $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$ are comparable and in this case $A \wedge B$ is the smaller of the two. This was considered to be a solution to the infimum problem for the case $\dim \mathcal{H} < \infty$ and it was conjectured in [18] that this result also holds in general. One of our main results is that this conjecture is false. We shall present an example of an $A, B \in \mathcal{E}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$ in which $A \wedge B$ exists in $\mathcal{E}(\mathcal{H})$ but $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$ are not comparable. In addition, we prove that, assuming $A \wedge B$ exists, $P_{A,B}$ is the smallest of all orthogonal projections P having the property that $(A \wedge P) \wedge (B \wedge P)$ exists and $(A \wedge P) \wedge (B \wedge P) = A \wedge B$. Combined with the counter-example as described before, this means that, in the infinite dimensional case, there is no orthogonal projection to replace $P_{A,B}$ and have a positive solution to the infimum problem.

The *negation* A' of an effect A is defined to be the effect $A' = I - A$. Physically, A' is the effect A with its values 1 and 0 reversed. We also present a simple spectral characterization of when $A \wedge A'$ exists in $\mathcal{E}(\mathcal{H})$. The result is essentially the same with Theorem 2 in [2], with the difference that we express the condition in terms of the location of the spectrum of A and the proof is based on the matrix representations obtained in the previous section.

T. Ando has given a solution to the infimum problem in terms of a generalized shorted operator [2]. However, in our opinion, these shorted operators do not have a physical significance in contrast to the operationally defined operators $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$. Finally, we discuss the relationship between our work and that of T. Ando. First, we show that the shorted operator of A by B is always smaller than $A \wedge P_{A,B}$. Actually, it is the fact, that in the infinite dimensional case, the shorted operator of A by B can be strictly smaller than $A \wedge P_{A,B}$, that is responsible for the failure of the infimum problem. This can be viewed from the counter-example as before, but we record also a simpler one that illustrates this situation.

2. Infimum of a Quantum Effect and a Sharp Effect

We first record a parameterization of bounded positive 2×2 matrices with operator entries, in terms of operator balls.

In the following we make use of the *Frobenius-Schur factorization*: for T, X, Y, Z bounded operators on appropriate spaces and T boundedly invertible, we have

$$\begin{bmatrix} T & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ YT^{-1} & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & Z - YT^{-1}X \end{bmatrix} \begin{bmatrix} I & T^{-1}X \\ 0 & I \end{bmatrix}. \tag{2.1}$$

For instance, by using Frobenius-Schur factorizations and a perturbation argument one can obtain the following classical result of Yu. Shmulyan [20].

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be selfadjoint and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ an orthogonal decomposition of $\mathcal{H}$. Then $A \geq 0$ if and only if it has a matrix representation of the following form:*

$$A = \begin{bmatrix} A_1 & A_1^{1/2} \Gamma A_2^{1/2} \\ A_2^{1/2} \Gamma^* A_1^{1/2} & A_2 \end{bmatrix}, \quad \text{w.r.t. } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{2.2}$$

where $A_1 \in \mathcal{B}(\mathcal{H}_1)^+$, $A_2 \in \mathcal{B}(\mathcal{H}_2)^+$, and $\Gamma \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ is contractive.

In addition, the operator Γ *can be chosen in such a way that* $\mathrm{Ker}(\Gamma) \supseteq \mathrm{Ker}(A_2)$ *and* $\mathrm{Ker}(\Gamma^*) \supseteq \mathrm{Ker}(A_1)$*, and in this case it is unique.*

For two effects $A, B \in \mathcal{E}(\mathcal{H})$ we denote by $A \wedge B$, the *infimum*, equivalently, the *greatest lower bound*, of A and B over the partially ordered set $(\mathcal{E}(\mathcal{H}), \leq)$, if it exists. To be more precise, $A \wedge B$ is an operator in $\mathcal{E}(\mathcal{H})$ uniquely determined by the following properties: $A \wedge B \leq A$, $A \wedge B \leq B$, and an arbitrary operator $D \in \mathcal{E}(\mathcal{H})$ satisfies both $D \leq A$ and $D \leq B$ if and only if $D \leq A \wedge B$. Characterizations of the existence of infimum for positive operators have been obtained for the finite-dimensional case in [18], and in general in [2].

In Theorem 4.4 of [18] it is proved that the infimum $A \wedge P$ exists for any $A \in \mathcal{E}(\mathcal{H})$ and $P \in \mathcal{P}(\mathcal{H})$. As a consequence of Theorem 2.1 we can obtain an explicit description of $A \wedge P$, together with another proof of the existence.

Theorem 2.2. *For any* $A \in \mathcal{E}(\mathcal{H})$ *and* $P \in \mathcal{P}(\mathcal{H})$ *the infimum* $A \wedge P$ *exists, more precisely, if* A *has the matrix representation as in* (2.2) *with respect to the orthogonal decomposition* $\mathcal{H} = \mathrm{Ran}(P) \oplus \mathrm{Ker}(P)$*, where* $A_1 \in \mathcal{E}(\mathrm{Ran}(P))$*,* $A_2 \in \mathcal{E}(\mathrm{Ker}(P))$*, and* $\Gamma \in \mathcal{B}(\mathrm{Ker}(P), \mathrm{Ran}(P))$*, with* $\|\Gamma\| \leq 1$*,* $\mathrm{Ker}(\Gamma) \supseteq \mathrm{Ker}(A_2)$ *and* $\mathrm{Ker}(\Gamma^*) \supseteq \mathrm{Ker}(A_1)$*, then*

$$A \wedge P = \begin{bmatrix} A_1^{1/2}(I - \Gamma\Gamma^*)A_1^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad w.r.t.\ \mathcal{H} = \mathrm{Ran}(P) \oplus \mathrm{Ker}(P). \tag{2.3}$$

Proof. Let $A \in \mathcal{E}(\mathcal{H})$ and $P \in \mathcal{P}(\mathcal{H})$. In the following we consider the orthogonal decomposition $\mathcal{H} = \mathrm{Ran}(P) \oplus \mathrm{Ker}(P)$. By Theorem 2.1 A has a matrix representation as in (2.2), with $A_1 \in \mathcal{B}(\mathrm{Ran}(P))^+$, $A_2 \in \mathcal{B}(\mathrm{Ker}(P))^+$, and $\Gamma \in \mathcal{B}(\mathrm{Ker}(P), \mathrm{Ran}(P))$, with $\|\Gamma\| \leq 1$, $\mathrm{Ker}(\Gamma) \supseteq \mathrm{Ker}(A_2)$ and $\mathrm{Ker}(\Gamma^*) \supseteq \mathrm{Ker}(A_1)$. Since $A \leq I$ it follows that $A_1 \leq I_{\mathrm{Ran}(P)}$ and $A_2 \leq I_{\mathrm{Ker}(P)}$. Consider the operator $D \in \mathcal{B}(\mathcal{H})$, defined by the matrix in (2.3). Clearly $0 \leq D \leq P$, in particular $D \in \mathcal{E}(\mathcal{H})$. In addition,

$$A - D = \begin{bmatrix} A_1^{1/2}\Gamma\Gamma^* A_1^{1/2} & A_1^{1/2}\Gamma A_2^{1/2} \\ A_2^{1/2}\Gamma^* A_1^{1/2} & A_2 \end{bmatrix} = \begin{bmatrix} \Gamma^* A_1^{1/2} & A_2^{1/2} \end{bmatrix}^* \begin{bmatrix} \Gamma^* A_1^{1/2} & A_2^{1/2} \end{bmatrix} \geq 0,$$

hence $A \geq D$.

Let $C \in \mathcal{E}(\mathcal{H})$ be such that $C \leq A, P$. From $C \leq P$ it follows that $CP = PC = C$ and hence

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{w.r.t. } \mathcal{H} = \mathrm{Ran}(P) \oplus \mathrm{Ker}(P).$$

Then

$$0 \leq A - C = \begin{bmatrix} A_1 - C_1 & A_1^{1/2}\Gamma A_2^{1/2} \\ A_2^{1/2}\Gamma^* A_1^{1/2} & A_2 \end{bmatrix}. \tag{2.4}$$

The matrix with operator entries in (2.4) can be factored as

$$\begin{bmatrix} I_{\mathrm{Ran}(P)} & 0 \\ 0 & A_2^{1/2} \end{bmatrix} \begin{bmatrix} A_1 - C_1 & A_1^{1/2}\Gamma \\ \Gamma^* A_1^{1/2} & I_{\mathrm{Ker}(P)} \end{bmatrix} \begin{bmatrix} I_{\mathrm{Ran}(P)} & 0 \\ 0 & A_2^{1/2} \end{bmatrix}. \tag{2.5}$$

Note that by $\mathrm{Ker}(\Gamma) \supseteq \mathrm{Ker}(A_2)$ or, equivalently, $\overline{\mathrm{Ran}(\Gamma^*)} \subseteq \overline{\mathrm{Ran}(A_2)}$, $A - C$ and each of the factors of (2.5) map the subspace $\mathcal{H}' = \mathrm{Ran}(P) \oplus \overline{\mathrm{Ran}(A_2)}$ into itself. Since $\mathrm{diag}(I_{\mathrm{Ran}(P)} \quad A_2^{1/2})$ regarded as an operator on $\mathcal{H}'$, is symmetric and has dense range, $A - C \geq 0$ implies that the middle term in (2.5) regarded as an operator in $\mathcal{H}'$ is nonnegative. By performing a Frobenius-Schur factorization of this middle term, we find $A_1^{1/2}\Gamma\Gamma^* A_1^{1/2} \leq A_1 - C_1$, that is, $C_1 \leq A_1^{1/2}(I_{\mathrm{Ran}(P)} - \Gamma\Gamma^*)A_1^{1/2}$, or, equivalently, $C \leq D$.

We thus proved that $A \wedge P$ exists and has the matrix representation as in (2.3). □

Remark 2.3. If $A \in \mathcal{E}(\mathcal{H})$, E_A is the spectral function of A and Δ is a Borel subset of $[0, 1]$, then $A \wedge E_A(\Delta) = AE_A(\Delta)$. This is an immediate consequence of Theorem 2.2.

Let $A, B \in \mathcal{E}(\mathcal{H})$. By $P_{A,B}$ we denote the orthogonal projection onto the closure of $\mathrm{Ran}(A^{1/2}) \cap \mathrm{Ran}(B^{1/2})$. As mentioned in the introduction, the infimum problem for a finite dimensional space $\mathcal{H}$ was solved in [18] by showing that $A \wedge B$ exists if and only if $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$ are comparable, and that $A \wedge B$ is the smaller of $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$. The following proposition shows that for $\dim \mathcal{H} = \infty$ the infimum problem for A and B can be reduced to the same problem for the "smaller" operators $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$. In Section 4 we will see that in this case the infimum problem cannot be solved in the same fashion, as conjectured in [18].

Proposition 2.4. *Let $A, B \in \mathcal{E}(\mathcal{H})$. Then $A \wedge B$ exists if and only if $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$ exists. In this case $A \wedge B = (A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$.*

Proof. Note first that the operators $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$ exist, e.g. by Theorem 2.2.

Let us assume that $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$ exists and let $C \in \mathcal{E}(\mathcal{H})$ be such that $C \leq A, B$, thus we have $\mathrm{Ran}(C^{1/2}) \subseteq \mathrm{Ran}(A^{1/2}) \cap \mathrm{Ran}(B^{1/2}) \subseteq \mathrm{Ran}(P_{A,B})$ and hence $C \leq P_{A,B}$. Therefore, $C \leq A \wedge P_{A,B}$ and $C \leq B \wedge P_{A,B}$ and hence, by the majorization theorem as in [6], $C \leq (A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$. Taking into account that $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B}) \leq A, B$ it follows that $A \wedge B$ exists and equals $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$.

Conversely, let us assume that $A \wedge B$ exist. Then, $A \wedge B \leq P_{A,B}$. This relation and $A \wedge B \leq A, B$ give $A \wedge B \leq A \wedge P_{A,B}$, $A \wedge B \leq B \wedge P_{A,B}$. Let $C \in \mathcal{E}(\mathcal{H})$ be such that $C \leq A \wedge P_{A,B}, B \wedge P_{A,B}$. Then $C \leq A, B, P_{A,B}$ and, in particular, $C \leq A \wedge B$. □

One may ask for which orthogonal projections P except $P_{A,B}$ the statement of Proposition 2.4 is true. It turns out that $P_{A,B}$ is the infimum of the set of those projections P.

Theorem 2.5. *Let $A, B \in \mathcal{E}(\mathcal{H})$ such that $A \wedge B$ exists. Let $\Pi_{A,B}$ be the set of all orthogonal projections subject to the properties that $(A \wedge P) \wedge (B \wedge P)$ exists and $(A \wedge P) \wedge (B \wedge P) = A \wedge B$. Then*

$$\Pi_{A,B} = \{P \in \mathcal{P}(\mathcal{H}) \mid P_{A,B} \leq P\}.$$

In order to prove the above stated proposition, we first consider the connection of parallel sum with the infimum of quantum effects (see also [2]). To see this, instead of giving the original definition as in [8], we prefer to introduce the parallel sum of two quantum effects by means of the characterization of Pekarev-Shmulyan [19]:

$$\langle (A : B)h, h\rangle = \inf\{\langle Aa, a\rangle + \langle Bb, b\rangle \mid h = a + b\}, \text{ for all } h \in \mathcal{H}. \tag{2.6}$$

Theorem 2.6. ([8] and [19]) *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. Then:*

(i) $0 \leq A\colon B \leq A, B$;
(ii) $A\colon B = B\colon A$;
(iii) $\mathrm{Ran}((A\colon B)^{1/2}) = \mathrm{Ran}(A^{1/2}) \cap \mathrm{Ran}(B^{1/2})$;
(iv) *If $A_1, B_1 \in \mathcal{B}(\mathcal{H})^+$ are such that $A \leq A_1$ and $B \leq B_1$, then $A\colon B \leq A_1\colon B_1$;*
(v) *If $A, B \neq 0$ then $\|A\colon B\| \leq (\|A\|^{-1} + \|B\|^{-1})^{-1}$;*
(vi) *If $A_n \searrow A$ and $B_n \searrow B$ strongly, then $A_n : B_n \searrow A : B$ strongly.*

In view of the properties of the parallel sum listed above, a moment of thought shows that if $P, Q \in \mathcal{P}(\mathcal{H})$, that is, P and Q are orthogonal projections in $\mathcal{H}$, then $P \wedge Q$ over $\mathcal{E}(\mathcal{H})$ always exists (this is the orthogonal projection onto $\mathrm{Ran}(P) \cap \mathrm{Ran}(Q)$) and $P \wedge Q = 2(P\colon Q)$, cf. Theorem 4.3 in [8].

Lemma 2.7. *Let $A, B \in \mathcal{E}(\mathcal{H})$ be such that $A \wedge B$ exists. Then*

(i) $\mathrm{Ran}((A \wedge B)^{1/2}) = \mathrm{Ran}((A\colon B)^{1/2})$;
(ii) $(A \wedge B)^{1/2} = (A\colon B)^{1/2} V$ *for some boundedly invertible operator $V \in \mathcal{B}(\mathcal{H})$;*
(iii) $A\colon B \leq A \wedge B \leq \gamma(A\colon B)$, *for some $\gamma > 0$.*

Proof. Since $A \wedge B \leq A$ it follows that $\mathrm{Ran}((A \wedge B)^{1/2}) \subseteq \mathrm{Ran}(A^{1/2})$. Similarly we have $\mathrm{Ran}((A \wedge B)^{1/2}) \subseteq \mathrm{Ran}(B^{1/2})$, hence $\mathrm{Ran}((A \wedge B)^{1/2}) \subseteq \mathrm{Ran}(A^{1/2}) \cap \mathrm{Ran}(B^{1/2}) = \mathrm{Ran}((A\colon B)^{1/2})$.

For the converse inclusion, note that $A\colon B \leq A$ and $A\colon B \leq B$; since $A\colon B \leq A\colon I = A(A+I)^{-1} \leq A$. Thus, by the definition of $A \wedge B$, it follows that $A\colon B \leq A \wedge B$. In particular, this proves that $\mathrm{Ran}((A \wedge B)^{1/2}) \supseteq \mathrm{Ran}((A\colon B)^{1/2})$, and hence (i) is proved.

The assertions (ii) and (iii) are consequences of (i) and the majorization theorem as in [6]. □

Lemma 2.8. *If $A, B \in \mathcal{E}(\mathcal{H})$ and $A \wedge B$ exists, then $A \wedge B \leq P_{A,B}$ and $\mathrm{Ran}(A \wedge B)$ is dense in $\mathrm{Ran}(P_{A,B})$.*

Proof. This is a consequence of Theorem 2.6 and Lemma 2.7. □

We now come back to Theorem 2.5.

Proof of Theorem 2.5. Let $P \in \Pi_{A,B}$. Then $A \wedge B \leq P$ and hence $\overline{\mathrm{Ran}(A \wedge B)} \subseteq \mathrm{Ran}(P)$. Therefore, by Lemma 2.8 $\mathrm{Ran}(P_{A,B}) \subseteq \mathrm{Ran}(P)$, that is, $P_{A,B} \leq P$.

Assume that $P \geq P_{A,B}$. We claim that then $(A \wedge P) \wedge (B \wedge P)$ exists and it coincides with $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$. Evidently, $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B}) \leq A \wedge P, B \wedge P$. Let $C \in \mathcal{E}(\mathcal{H})$ with $C \leq A \wedge P, B \wedge P$. Then $C \leq A \wedge B \leq P_{A,B}$ and hence,

$$C \leq (A \wedge P_{A,B}) \wedge (B \wedge P_{A,B}).$$

Therefore, $(A \wedge P) \wedge (B \wedge P)$ exists and, by Proposition 2.4 it coincides with $A \wedge B$. □

3. Infimum of a Quantum Effect and its Negation

The *negation* A' of an effect A is defined to be the effect $A' = I - A$. Physically, A' is the effect A with its values 1 and 0 reversed. In the following we present a characterization of when $A \wedge A'$ exists in $\mathcal{E}(\mathcal{H})$ in terms of the location of the spectrum of A. The theorem essentially coincides with the result of T. Ando ([2], Theorem 2): the difference consists on that we express the condition with the help of the spectrum of A and the proof is based on the matrix representations as in Section 2.

Theorem 3.1. *Let A be a quantum effect on the Hilbert space $\mathcal{H}$. Then the following assertions are equivalent:*

(i) $A \wedge (I - A)$ *exists;*
(ii) $\sigma(A)$*, the spectrum of* A*, is contained either in* $\{0\} \cup [\frac{1}{2}, 1]$ *or in* $[0, \frac{1}{2}] \cup \{1\}$*;*
(iii) $A \wedge P_{A,I-A}$ *and* $(I - A) \wedge P_{A,I-A}$ *are comparable, that is, either* $A \wedge P_{A,I-A} \leq (I - A) \wedge P_{A,I-A}$ *or* $(I - A) \wedge P_{A,I-A} \leq A \wedge P_{A,I-A}$.

In addition, if either of the above holds, letting $g \in C([0, 1])$ *be the function*

$$g(t) = \min(t, 1 - t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2}, \\ 1 - t, & \frac{1}{2} \leq t \leq 1, \end{cases} \tag{3.1}$$

we have, by continuous functional calculus, $A \wedge (I - A) = g(A)$.

Proof. Let E_A denote the spectral function of A. In view of Proposition 2.4, $A \wedge (I - A)$ exists if and only if $(A \wedge P_{A,I-A}) \wedge ((I - A) \wedge P_{A,I-A})$ exists. A moment of thought shows that $P_{A,I-A} = E_A((0, 1))$ and hence, by Remark 2.3, we have that $A \wedge P_{A,I-A} = AE_A((0, 1))$ and $(I - A) \wedge P_{A,I-A} = (I - A)E_A((0, 1))$. Thus, without restricting the generality, we can and will assume in the following that 0 and 1 are not eigenvalues of A. Now, the equivalence of (ii) with (iii) is a matter of elementary spectral theory for selfadjoint operators, hence we will prove only the equivalence of (i) and (ii).

To prove that (ii) implies (i), let us assume that $\sigma(A)$ is contained either in $\{0\} \cup [\frac{1}{2}, 1]$ or in $[0, \frac{1}{2}] \cup \{1\}$. To make a choice, let us assume that $\sigma(A) \subseteq \{0\} \cup [\frac{1}{2}, 1]$. Since, by assumption, 0 is not an eigenvalue of A, it follows that $\sigma(A) \subseteq [\frac{1}{2}, 1]$. Then $A \geq I - A$ and clearly $A \wedge (I - A) = I - A = g(A)$, where the function g is defined as in (3.1). A similar argument holds in case we assume $\sigma(A) \subseteq [0, \frac{1}{2}] \cup \{1\}$; in this case $A \wedge (I - A) = A = g(A)$.

Conversely, let us assume that $A \wedge (I - A) = D$, the infimum of A and $I - A$ over $\mathcal{E}(\mathcal{H})$, exists. Using the spectral measure E_A of A, let $E_1 = E_A([0, 1/2]$, $A_1 = A|E_1\mathcal{H}$, $E_2 = E_A((1/2, 1])$, $A_2 = A|E_2\mathcal{H}$. We write D as an operator matrix with respect to the decomposition $\mathcal{H} = E_1\mathcal{H} \oplus E_2\mathcal{H}$

$$D = \begin{bmatrix} D_1 & D_1^{1/2}\Gamma D_2^{1/2} \\ D_2^{1/2}\Gamma^* D_1^{1/2} & D_2 \end{bmatrix},$$

with contractive $\Gamma \in \mathcal{B}(E_2\mathcal{H}, E_1\mathcal{H})$, cf. Theorem 2.1. Since $g(A) \leq A, I - A$, by the definition of D we have

$$0 \leq D - g(A) = \begin{bmatrix} D_1 - A_1 & D_1^{1/2}\Gamma D_2^{1/2} \\ D_2^{1/2}\Gamma^* D_1^{1/2} & D_2 - (I_2 - A_2) \end{bmatrix}. \tag{3.2}$$

Therefore, $0 \leq D_1 - A_1$ while taking into account that $D \leq A$ it follows that $D_1 \leq A_1$, hence $D_1 = A_1$. Similarly, $0 \leq D_2 - (I_2 - A_2)$ and, since $D \leq I - A$ it follows $D_2 \leq I_2 - A_2$, hence $D_2 = I_2 - A_2$. Thus, the main diagonal of the matrix in (3.2) is null, hence (e.g. by Theorem 2.1) it follows that $D = g(A)$.

Further, let $\varepsilon \in (0, 1/4)$, and consider the operators

$$E_{\varepsilon,1} = E_A((\varepsilon, -\varepsilon + 1/2)), \quad E_{\varepsilon,2} = E_A((\varepsilon + 1/2, 1 - \varepsilon)). \tag{3.3}$$

Denote $E_\varepsilon = E_{\varepsilon,1} + E_{\varepsilon,2}$ and $A_\varepsilon = A|E_\varepsilon\mathcal{H}$. We show that $A_\varepsilon \wedge (I - A_\varepsilon)$ exists. To see this, we remark that, as proven before, $g(A) = A \wedge (I - A)$, so we actually show that $D_\varepsilon = D|E_\varepsilon\mathcal{H} = g(A_\varepsilon)$ coincides with $A_\varepsilon \wedge (I - A_\varepsilon)$. Indeed, assume that for some $C_\varepsilon \in \mathcal{E}(E_\varepsilon\mathcal{H})$ we have $C_\varepsilon \leq A_\varepsilon, I - A_\varepsilon$. Then, letting $C = C_\varepsilon E_\varepsilon \in \mathcal{E}(\mathcal{H})$ it follows that $C \leq A, I - A$. Since $D = A \wedge (I - A)$ this implies $C \leq D$ and hence $C_\varepsilon \leq D_\varepsilon$. Therefore, D_ε coincides with $A_\varepsilon \wedge (I - A_\varepsilon)$.

We finally prove that (i) implies (ii). Assume that (i) holds and (ii) is not true. Then there exists $\varepsilon \in (0, 1/4)$ such that $E_{\varepsilon,1} \neq 0$ and $E_{\varepsilon,2} \neq 0$, where we use the notation as in (3.3). Letting

$$A_{\varepsilon,1} = A|E_{\varepsilon,1}\mathcal{H}, \quad A_{\varepsilon,2} = A|E_{\varepsilon,2}\mathcal{H},$$

and $d = \varepsilon(1 + \sqrt{3})^{-1}$, consider an arbitrary contraction $T \in \mathcal{B}(E_{\varepsilon,2}\mathcal{H}, E_{\varepsilon,1}\mathcal{H})$. In the following all operator matrices are understood with respect to the decomposition $E_{\varepsilon,1}\mathcal{H} \oplus E_{\varepsilon,2}\mathcal{H}$. Then, letting

$$\begin{aligned} C &= \begin{bmatrix} A_{\varepsilon,1} - dI_{\varepsilon,1} & \sqrt{3}dT \\ \sqrt{3}dT^* & I_{\varepsilon,2} - A_{\varepsilon,2} - dI_{\varepsilon,2} \end{bmatrix} \\ &= \begin{bmatrix} A_{\varepsilon,1} - \varepsilon I_{\varepsilon,1} + \sqrt{3}dI_{\varepsilon,1} & \sqrt{3}dT \\ \sqrt{3}dT^* & I_{\varepsilon,2} - A_{\varepsilon,2} - \varepsilon I_{\varepsilon,2} + \sqrt{3}dI_{\varepsilon,2} \end{bmatrix} \\ &= \begin{bmatrix} A_{\varepsilon,1} - \varepsilon I_{\varepsilon,1} & 0 \\ 0 & I_{\varepsilon,2} - A_{\varepsilon,2} - \varepsilon I_{\varepsilon,2} \end{bmatrix} + \sqrt{3}d \begin{bmatrix} I_{\varepsilon,1} & T \\ T^* & I_{\varepsilon,2} \end{bmatrix} \geq 0, \end{aligned}$$

we have

$$\begin{aligned} A_\varepsilon - C &= \begin{bmatrix} dI_{\varepsilon,1} & -\sqrt{3}dT \\ -\sqrt{3}dT^* & 2A_{\varepsilon,2} - I_{\varepsilon,2} + dI_{\varepsilon,2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 2A_{\varepsilon,2} - I_{\varepsilon,2} - 2dI_{\varepsilon,2} \end{bmatrix} + d \begin{bmatrix} I_{\varepsilon,1} & -\sqrt{3}T \\ -\sqrt{3}T^* & 3I_{\varepsilon,2} \end{bmatrix} \geq 0, \end{aligned}$$

and

$$\begin{aligned} I - A_\varepsilon - C &= \begin{bmatrix} I_{\varepsilon,1} - 2A_{\varepsilon,1} + dI_{\varepsilon,1} & -\sqrt{3}T \\ -\sqrt{3}dT^* & dI_{\varepsilon,2} \end{bmatrix} \\ &= \begin{bmatrix} I_{\varepsilon,1} - 2A_{\varepsilon,1} - 2dI_{\varepsilon,1} & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 3I_{\varepsilon,1} & -\sqrt{3}T \\ -\sqrt{3}dT^* & I_{\varepsilon,2} \end{bmatrix} \geq 0. \end{aligned}$$

But, the operator

$$\begin{aligned}(A_\varepsilon \wedge (I_\varepsilon - A_\varepsilon)) - C &= g(A_\varepsilon) - C \\ &= d\begin{bmatrix} I_{\varepsilon,1} & -\sqrt{3}T \\ -\sqrt{3}T^* & I_{\varepsilon,2}\end{bmatrix}\end{aligned}$$

is not nonnegative for some choices of T, unless at least one of the spectral projections $E_{\varepsilon,1}$ and $E_{\varepsilon,2}$ is trivial. Since ε is arbitrarily small, it follows that A cannot simultaneously have spectral points in $(0,1/2)$ and $(1/2,1)$. Therefore, (i) implies (ii). □

4. Two Examples

In this section we answer in the negative a question raised in [18]. Let $A, B \in \mathcal{E}(\mathcal{H})$ and consider the operators $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$ that exist by Theorem 2.2. By scaling both operators with the same positive constant $\|A+B\|^{-1}$, without restricting the generality we can assume that $A+B$ is contractive, and hence, that $A+B \in \mathcal{E}(\mathcal{H})$. Then we can use the affine (that is, linear on convex combinations) mapping f_{A+B} as defined in [9],

$$f_{A+B}\colon \{C \mid 0 \leq C \leq A+B\} \to \{D \mid 0 \leq D \leq P_{A+B}\}, \tag{4.1}$$

with $C = (A+B)^{1/2} f_{A+B}(C)(A+B)^{1/2}$. By Theorem 2.2 in [9], f_{A+B} is well-defined. In addition, since $\{D \mid 0 \leq D \leq P_{A+B}$ is affine isomorphic with $\mathcal{E}(\mathcal{H} \ominus \operatorname{Ker}(A+B))$ (cf. Theorem 2.5 in [9], without restricting the generality we can consider f_{A+B} having values in $\mathcal{E}(\mathcal{H} \ominus \operatorname{Ker}(A+B))$. Thus, considering now the function f_{A+B}, $A \wedge B$ exists if and only if $f_{A+B}(A) \wedge f_{A+B}(B)$ exists and, in this case, we have

$$f_{A+B}(A \wedge B) = f_{A+B}(A) \wedge f_{A+B}(B).$$

Since

$$f_{A+B}(A) + f_{A+B}(B) = f_{A+B}(A+B) = I_{\mathcal{H} \ominus \operatorname{Ker}(A+B)}, \tag{4.2}$$

we are in the situation of Theorem 3.1 and it remains only to compute $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$; recall that, by Theorem 2.2, these infima always exist. However, we now prove that the finite dimensional result obtained in [18] does not extend to the infinite dimensional case, and hence answering in the negative a question raised in that paper. Recall that, by Proposition 2.4, for two quantum effects A and B on the same Hilbert space, $A \wedge B$ exists if and only if $(A \wedge P_{A,B}) \wedge (B \wedge P_{A,B})$ exists, and in this case the two infima do coincide.

Actually, this comes from a more general fact:

Lemma 4.1. *Let $A \in \mathcal{E}(\mathcal{H})$, $C, D \in [0,1]$, and consider the mapping f_A as defined in (4.2). Then $C \wedge D$ exists if and only if $f_A(C) \wedge f_A(D)$ exists and, in this case, we have*

$$f_A(C \wedge D) = f_A(C) \wedge f_A(D).$$

Proof. This is a consequence of Theorem 2.5 in [9]. □

The next example shows that, contrary to the finite dimensional case, we may have two quantum effects B_1 and B_2 for which $B_1 \wedge B_2$ exists, but $(B_1 \wedge P_{B_1,B_2})$ $(B_2 \wedge P_{B_1,B_2})$ are not comparable.

Example 4.2. Let $\mathcal{H} = L^2[-1,1]$ and A be the operator of multiplication with the square of the independent variable on $\mathcal{H}$, $(Ax)(t) = t^2x(t)$, for all $x \in L^2[-1,1]$. Then A is a nonnegative contraction on $\mathcal{H}$, that is, a quantum effect, and the same is its square root $A^{1/2}$, that is, $(A^{1/2}x)(t) = |t|x(t)$, $x \in L^2[-1,1]$. Note that A, and hence $A^{1/2}$, are injective.

Let $\mathbf{1}$ be the constant function equal to 1 on $[-1,1]$, $\theta(t) := \operatorname{sgn}(t)$, and $\chi_\pm := \frac{1}{2}(\mathbf{1}\pm\theta)$, the characteristic functions of $[0,1]$ and, respectively, $[-1,0]$. All these functions are in $L^2[-1,1]$. Note that $\mathbf{1}$ and θ span the same two dimensional space as $\chi_\pm$. Denote

$$\mathcal{H}_0 = \mathcal{H} \ominus \operatorname{span}\{\mathbf{1},\theta\} = \mathcal{H} \ominus \operatorname{span}\{\chi_+,\chi_-\}.$$

With respect to the decomposition

$$\mathcal{H} = \mathbb{C}\mathbf{1} \oplus \mathbb{C}\theta \oplus \mathcal{H}_0$$

consider two quantum effects C_1 and C_2 on $\mathcal{H}$ defined by

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}I_0 \end{bmatrix} \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}I_0 \end{bmatrix},$$

where I_0 is the identity operator on $\mathcal{H}_0$. Clearly we have $C_1 + C_2 = I$ and letting

$$B_1 = A^{1/2}C_1A^{1/2}, \quad B_2 = A^{1/2}C_2A^{1/2},$$

we have

$$B_1 + B_2 = A.$$

Comparing the spectra of C_1 and C_2 and using Theorem 3.1, it follows that $C_1 \wedge C_2$ exists, but C_1 and C_2 are not comparable. Therefore, using Lemma 4.1, it follows that $B_1 \wedge B_2$ exists, but B_1 and B_2 are not comparable. In the following we will prove that $P_{B_1,B_2} = I$, that is, $\operatorname{Ran}(B_1^{1/2}) \cap \operatorname{Ran}(B_2^{1/2})$ is dense in $\mathcal{H}$. We divide the proof in several steps.

Step 1. *$A^{1/2}\mathcal{H}_0$ is dense in $\mathcal{H}$.*

Indeed, let $f \in \mathcal{H} = L^2[-1,1]$ be a function such that for all $h_0 \in \mathcal{H}_0$ we have

$$0 = \langle A^{1/2}h_0, f\rangle = \langle h_0, A^{1/2}f\rangle.$$

Then $A^{1/2}f$ is a linear combination of the functions $\mathbf{1}$ and θ, that is, there exist scalars α and β such that

$$|t|f(t) = \alpha + \beta\operatorname{sgn}(t), \quad t \in [-1,1]$$

and hence

$$f(t) = \frac{\alpha + \beta\operatorname{sgn}(t)}{|t|} = \begin{cases} \dfrac{\alpha+\beta}{t}, & 0 < t \leq 1 \\ \dfrac{\beta-\alpha}{t}, & -1 \leq t < 0. \end{cases}$$

Since $f \in L^2[-1,1]$ this shows that $f = 0$ and the claim is proven.

Let us consider the following linear manifolds in $\mathcal{H}$:

$$\mathcal{F} := \{f \in L^2[-1,1] \mid f \text{ piecewise constant}\}$$

$$\mathcal{F}_0 := \{f \in \mathcal{F} \mid \exists\, \varepsilon > 0 \text{ s.t. } f|(-\varepsilon,\varepsilon) = 0, \\ \langle f, \chi_-\rangle = \langle f, \chi_+\rangle = 0\}.$$

Step 2. *$\mathcal{F}_0$ is dense in $\mathcal{H}_0$.*

Indeed, to see this, let us first note that $\mathcal{F}_0 \subset \mathcal{H}_0$. If h_0 is an arbitrary vector in $\mathcal{H}_0$ and $\varepsilon > 0$, there exists $f_1 \in \mathcal{F}$ such that

$$\|h_0 - f_1\| \leq \frac{\varepsilon}{8} \text{ hence } |\langle h_0 - f_1, \chi_\pm\rangle| \leq \frac{\varepsilon}{8}. \tag{4.3}$$

Moreover, there exists $f_2 \in \mathcal{F}$ such that it is zero in a neighbourhood of zero and

$$\|f_1 - f_2\| \leq \frac{\varepsilon}{8}. \tag{4.4}$$

Consequently,

$$\|h_0 - f_2\| \leq \frac{\varepsilon}{4} \text{ and hence } |\langle h_0 - f_2, \chi_\pm\rangle| \leq \frac{\varepsilon}{4}. \tag{4.5}$$

Let

$$f_3 = f_2 + 2\chi_{[1/2,1]}\langle h_0 - f_2, \chi_+\rangle + 2\chi_{[-1,-1/2]}\langle h_0 - f_2, \chi_-\rangle.$$

Then, from the choice of f_2 it follows

$$\langle f_3, \chi_+\rangle = \langle f_2, \chi_+\rangle + \langle h_0 - f_2, \chi_+\rangle = \langle h_0, \chi_+\rangle = 0,$$

and

$$\langle f_3, \chi_-\rangle = \langle f_2, \chi_-\rangle + \langle h_0 - f_2, \chi_-\rangle = \langle h_0, \chi_-\rangle = 0,$$

hence $f_3 \in \mathcal{F}_0$. Finally, from (4.3), (4.4), and (4.5) we get

$$\|h_0 - f_3\| \leq \|h_0 - f_1\| + \|f_1 - f_2\| + \|f_2 - f_3\| \leq \varepsilon,$$

and the claim is proven.

Finally, we prove that

Step 3. *$P_{B_1,B_2} = I$, that is,* $\operatorname{Ran}(B_1^{1/2}) \cap \operatorname{Ran}(B_2^{1/2})$ *is dense in $\mathcal{H}$.*

In the following we are using the inverse operator $A^{-1/2}$ on its range. By the preceding claim, $A^{1/2}(A^{-1/2}\mathcal{F}_0)$ is a linear submanifold in $\mathcal{H}_0$ and dense in it. Since the restrictions of C_1 and C_2 to $\mathcal{H}_0$ coincide with $\frac{1}{2}I_0$, it follows that the linear manifolds $C_1A^{1/2}(A^{-1}\mathcal{F}_0)$ and $C_2A^{1/2}(A^{-1}\mathcal{F}_0)$ coincide and are dense in $\mathcal{H}_0$. Consequently, the linear manifolds $A^{1/2}C_1A^{1/2}(A^{-1}\mathcal{F}_0)$ and $A^{1/2}C_2A^{1/2}(A^{-1}\mathcal{F}_0)$ coincide and, by Step 1 and Step 2, they are dense in $\mathcal{H}$. Thus, the linear manifold

$$\mathcal{L} = B_1(A^{-1/2}\mathcal{F}_0) = B_2(A^{-1/2}\mathcal{F}_0) \subseteq \operatorname{Ran}(B_1) \cap \operatorname{Ran}(B_2) \subseteq \operatorname{Ran}(B_1^{1/2}) \cap \operatorname{Ran}(B_2^{1/2}),$$

is dense in $\mathcal{H}$. This concludes the proof of the last step, and the example.

In order to explain the connection with the characterization of the existence of infimum obtained by Ando in [2] we consider the comparison of $A \wedge P_{A,B}$ with the *generalized shorted* operator, as considered in [2].

Lemma 4.3. *Let $A, B \in \mathcal{E}(\mathcal{H})$. Then, for any sequence α_n of positive numbers that converge increasingly to infinity, we have*

$$\text{so-}\lim_{n\to\infty}(A : \alpha_n B) \le A \wedge P_{A,B}, \tag{4.6}$$

and the limit does not depend on the sequence (α_n).

Proof. First note that the sequence of positive operators $A : \alpha_n B$ is nondecreasing and bounded by A, cf. [8]. Consequently, the strong operator limit exists and does not depend on the sequence α_n increasing to infinity. We thus can take $\alpha_n = n$. Since the parallel sum is strongly continuous in the second variable with respect to nondecreasing sequences, cf. Theorem 2.6, we have $A : nB \le A$ and, since $\operatorname{Ran}((A : nB)^{1/2}) = \operatorname{Ran}(A^{1/2}) \cap \operatorname{Ran}(B^{1/2})$ it follows $A : nB \le P_{A,B}$ and hence (4.6) holds. □

Given two positive operators A and B, the *generalized shorted operator* $[B]A$ is defined (see [1]) by

$$[B]A = \lim_{n\to\infty} A : (nB).$$

The main result in [2] states that the infimum $A \wedge B$ exists if and only $[B]A$ and $[A]B$ are comparable and, in this case, $A \wedge B$ is the smaller of $[A]B$ and $[B]A$. In view of this result and our Example 4.2, it follows that, in general, (4.6) cannot be improved to equality. Here we have a simpler example emphasizing this fact.

Example 4.4. Let $\mathcal{H} = L^2[0, 1]$ and A the operator of multiplication with the function t^2. Then A is bounded, contractive, and positive. In addition, $A^{1/2}$ is the operator of multiplication with the independent variable t. Note that both A and $A^{1/2}$ are injective.

Further, let $\mathbf{1}$ be the function constant 1 in $L^2[0, 1]$ and note that it does not belong to the range of either A or $A^{1/2}$. Let C be a nonnegative contraction in $\mathcal{H}$ with kernel $\mathbb{C}\mathbf{1}$ and define $B = A^{1/2}CA^{1/2}$. Then the operator B is injective and hence its range is dense in $\mathcal{H}$. Since $\operatorname{Ran}(B) \subseteq \operatorname{Ran}(B^{1/2})$ and, by construction, $\operatorname{Ran}(B) \subseteq \operatorname{Ran}(A^{1/2})$ as well, it follows that $\operatorname{Ran}(A^{1/2}) \cap \operatorname{Ran}(B^{1/2})$ is dense in $\mathcal{H}$, hence $P_{A,B} = I$.

For each $n \ge 1$ consider the function $v_n \in L^2[0, 1]$ defined by

$$v_n(t) = \begin{cases} 0, & 0 \le t \le 1/n \\ 1/t, & 1/n < t \le 1 \end{cases}$$

Note that $A^{1/2}v_n = \chi_{(1/n,1]}$, the characteristic function of the interval $(1/n, t]$. Taking into account that the sequence of functions $\chi_{(1/n,1]}$ converges in norm to the function $\mathbf{1}$, it follows that

$$\langle Bv_n, v_n\rangle = \langle CA^{1/2}v_n, A^{1/2}v_n\rangle = \langle C\chi_{(1/n,1]}, \chi_{(1/n,1]}\rangle \to \langle C\mathbf{1}, \mathbf{1}\rangle = 0.$$

Let α_n be a sequence of positive numbers increasing to $+\infty$ and such that $\alpha_n\langle Bv_n, Bv_n\rangle$ converges to 0. It is easy to see that this is always possible. Then using the characterization of the parallel sum as in Theorem 2.6.(vi), for arbitrary $n \geq m > 2$ we have

$$\begin{aligned}\langle (A:\alpha_n B)v_m v_m\rangle &= \inf\{\langle Au,u\rangle + \alpha_n\langle Bv,v\rangle \mid v_m = u+v\}\\ &= \inf\{\langle A(v_m - v), v_m - v\rangle + \alpha_n\langle Bv,v\rangle \mid v\in\mathcal{H}\}\\ &= \inf\{\langle Av_m,v_m\rangle - 2\mathrm{Re}\langle Av_m, v\rangle + \langle Av,v\rangle + \alpha_n\langle Bv,v\rangle \mid v\in\mathcal{H}\}\\ &\leq \langle Av_m,v_m\rangle - 2\mathrm{Re}\langle Av_m,v_n\rangle + \langle Av_n,v_n\rangle + \alpha_n\langle Bv_n,v_n\rangle\\ &= 1-\frac{1}{m}-2+\frac{2}{m}+1-\frac{1}{n}+\alpha_n\langle Bv_n,v_n\rangle\\ &= \frac{1}{m}-\frac{1}{n}+\alpha_n\langle Bv_n,v_n\rangle \to \frac{1}{m} < \frac{1}{2} \text{ as } n\to\infty.\end{aligned}$$

On the other hand

$$\langle Av_m, Av_m\rangle = 1-\frac{1}{m} \geq \frac{1}{2}.$$

Hence, we have strict inequality in (4.6).

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